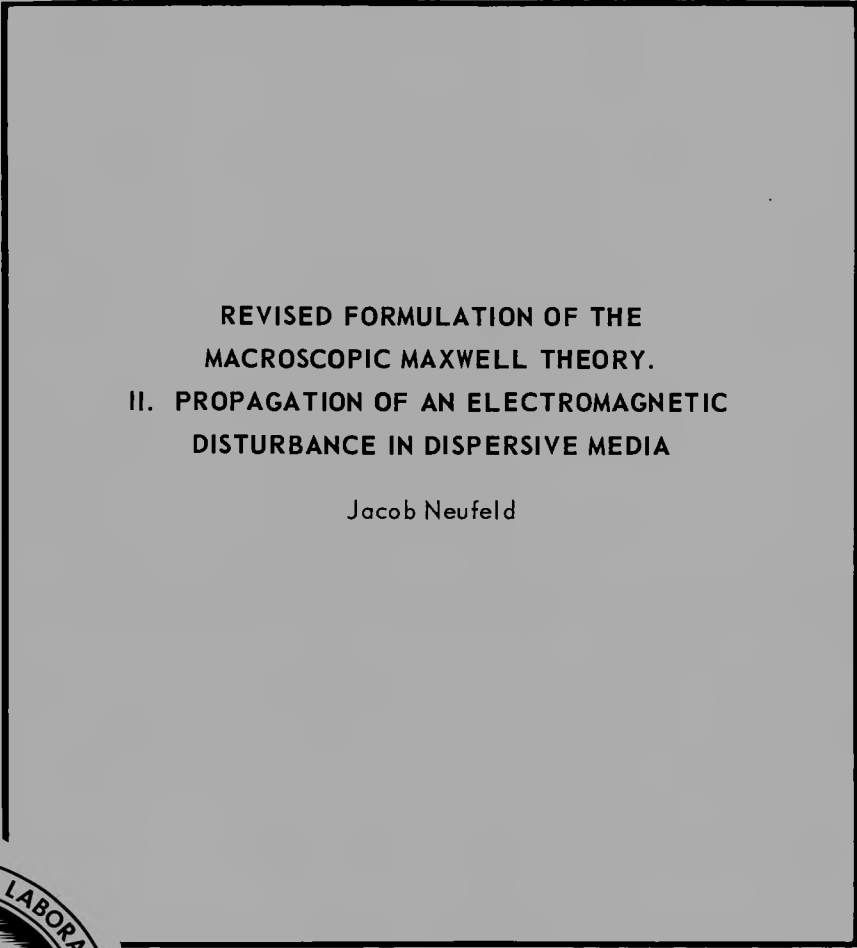


275
11-18-69

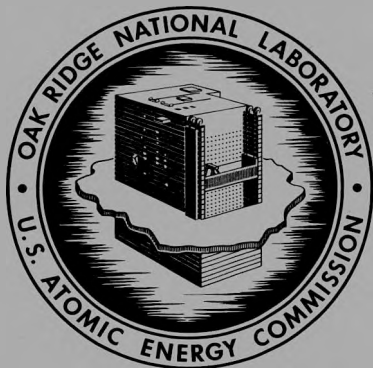
1089
MASTER

ORNL-4406, Part II
UC-34 - Physics
UC-20 - Controlled Thermonuclear Processes



REVISED FORMULATION OF THE
MACROSCOPIC MAXWELL THEORY.
II. PROPAGATION OF AN ELECTROMAGNETIC
DISTURBANCE IN DISPERSIVE MEDIA

Jacob Neufeld



OAK RIDGE NATIONAL LABORATORY
operated by
UNION CARBIDE CORPORATION
for the
U.S. ATOMIC ENERGY COMMISSION

DISTRIBUTION STATEMENT

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

Printed in the United States of America. Available from Clearinghouse for Federal
Scientific and Technical Information, National Bureau of Standards,
U.S. Department of Commerce, Springfield, Virginia 22151
Price: Printed Copy \$3.00; Microfiche \$0.65

— LEGAL NOTICE —

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

- A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or
- B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

Contract No. W-7405-eng-26

HEALTH PHYSICS DIVISION

REVISED FORMULATION OF THE MACROSCOPIC MAXWELL THEORY.
II. PROPAGATION OF AN ELECTROMAGNETIC DISTURBANCE
IN DISPERSIVE MEDIA

Jacob Neufeld

NOVEMBER 1

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

OAK RIDGE NATIONAL LABORATORY
Oak Ridge, Tennessee
operated by
UNION CARBIDE CORPORATION
for the
U. S. ATOMIC ENERGY COMMISSION

fy

CONTENTS

Abstract.....	1
I. Introduction.....	1
II. Kinematic Behavior of an Electromagnetic Disturbance in a Dispersive Medium	3
Velocity of a Disturbance.....	3
Unperturbed and Perturbed Wave Packet	5
Convective and Nonconvective Disturbance	7
Attenuation (or Growth) of an Electromagnetic Disturbance	8
Relaxation Length	9
Relaxation Time.....	10
III. Energetic Behavior of an Electromagnetic Disturbance in a Dispersive Medium.....	11
Kinematic and Energetic Velocity.....	11
Extension of the Brillouin Theory to Absorbing Media	12
IV. Reformulation of Ohm's Law	16
Energetic Version of Ohm's Law.....	16
Differential Expression for Ohm's Law	17
V. Reassessment of the Two Versions and Comparison of Results.....	18
Attenuation and Growth Parameters	18
Convective Disturbances	18
Nonconvection Disturbances	20
VI. Concluding Remarks	21
Validity of the Perturbation Treatment.....	21
Experiment and Theory	22
Microscopic and Macroscopic Aspects of Energy Dissipation	22
References	23

REVISED FORMULATION OF THE MACROSCOPIC MAXWELL THEORY. II. PROPAGATION OF AN ELECTROMAGNETIC DISTURBANCE IN DISPERSIVE MEDIA

Jacob Neufeld

ABSTRACT

The theory of energy propagation in dispersive media formulated by Brillouin is generalized and extended by taking into account energy attenuation or energy growth during the transmission process. Because of the exponential decay due to attenuation, the concept of propagation velocity of an electromagnetic disturbance can now be shown to have a physical meaning, even if there is absorption, and in such case the velocity of the disturbance is represented by the velocity of the point at which it attains its maximum intensity. Expressions are obtained for the energy density and the energy flow in an absorbing medium, and it is shown that the propagation velocity (a kinematic concept) is equal to the velocity of the energy transport. Such an equality is considered to be of fundamental interest, since it supports the validity of the kinematic approach from the energetic point of view. Other aspects of energy transmission are discussed and analyzed, with particular references to nonconvective and convective disturbances in absorbing and emitting media.

I. INTRODUCTION

By reducing the problem of an electrical-field theory to the corresponding problem in generalized dynamics, we reformulated in Part I¹ the expression for energy stored by an electromagnetic field in a dispersive medium and gave a systematic outline of a theory in which the energy density U_e was shown to be dependent only on the electrical intensity \mathbf{E} and not on the parameter γ , which defines dissipation.

We also revised the traditional interpretation of the dielectric constant and of conductivity and replaced the current version of the dispersion equation,

$$F_s^0(k_s, \omega_s) = (c^2 k_s^2 / \omega_s) - \omega_s \epsilon_s = 0, \quad (1-1)$$

for a nonabsorbing molecular medium, and the corresponding version,

$$F_s(k_s, \omega_s) - 4\pi i \sigma_s = 0, \quad (1-2)$$

for an absorbing medium by two characteristic equations which were expressed as

$$F = F(k, \omega) = (c^2 k^2 / \omega) - \omega \epsilon = 0, \quad (1-3)$$

$$\alpha \frac{\partial F}{\partial k} + \beta \frac{\partial F}{\partial \omega} + 4\pi i \sigma = 0. \quad (1-4)$$

We will now center our attention on the practical applications of the characteristic equations and will analyze the propagation of electromagnetic disturbances in dispersive media by taking into account energy decay or energy growth during the transmission process.

It is customary to express an electromagnetic disturbance in a nonabsorbing medium in the form of a group of elementary waves which are essentially mathematical functions and do not represent observable physical quantities. There is a curious situation concerning the relationship between a disturbance and an elementary wave in nonabsorbing and in absorbing media. If the medium is nonabsorbing, a clear distinction is made between a disturbance and an elementary wave. In such case an elementary wave is a homogeneous plane wave $\exp i(k_s x - \omega_s t)$, where k_s and ω_s are real quantities related to each other by (1-1). Then the disturbance is thought of as a *superposition* of such waves. If, however, there is absorption, the distinction between a disturbance and an elementary wave mysteriously disappears. In other words, a disturbance becomes an elementary wave. It assumes a form of a nonhomogeneous plane wave $\exp i(k_s x - \omega_s t)$ in which either k_s is real and ω_s is complex, or vice versa, and k_s and ω_s are related to each other by (1-2).

A nonhomogeneous plane wave, whether characterized by real k_s and complex ω_s , or vice versa, is not considered to be an appropriate vehicle for transmitting energy through space. I propose, therefore, to revise the current formulation and to consider a disturbance as a *superposition* of harmonic waves which are homogeneous when the medium is nonabsorbing, but become nonhomogeneous if absorption occurs. In this interpretation a disturbance is a wave packet both in absorbing and nonabsorbing media and never a single elementary wave. Therefore, the attenuation of the disturbance is expressed by the relaxation length characterizing a wave packet rather than by the extinction index characterizing a single nonhomogeneous plane wave.

The revised formulation is shown to provide precise answers to problems which have not been solved within the framework of the current macroscopic theory. By means of the procedures developed in Part I, expressions are obtained for the energy density, energy flow, velocity of energy transport, and propagation velocity in absorbing and emitting media. Because of the exponential decay of an electromagnetic disturbance, kinematic velocity can now be shown to have a physical meaning, even if there is absorption. It can also be shown to be the same as the velocity of energy transport.

II. KINEMATIC BEHAVIOR OF AN ELECTROMAGNETIC DISTURBANCE IN A DISPERSIVE MEDIUM

Velocity of a Disturbance

The concept of the “velocity of propagation” has a clear physical meaning provided there is no distortion in the shape of the disturbance. A distortionless disturbance can only occur if the medium is nonabsorbing; therefore, it is not entirely clear what is meant by the “velocity” if there is absorption.

In order to extend the meaning of propagation velocity to absorbing media, it will now be assumed that the disturbance can be localized at its point of maximum intensity, and the velocity of this point will be considered to be the velocity of the disturbance. Since absorption acts as a perturbation, the propagation velocity in a nonabsorbing medium will be referred to as the “unperturbed” velocity v_g^0 , whereas the velocity in an absorbing medium will be referred to as the “perturbed” velocity v_g . The unperturbed velocity v_g^0 is known to be constant, and we will now determine the necessary requirements for a perturbed velocity v_g to be a constant quantity.

Suppose that $S^0(x, t)$ is the intensity of the disturbance as it proceeds along the x axis in a nonabsorbing medium, and let $A(x = x_A)$ and $B(x = x_B)$ be any two points situated in the path of the disturbance. If there is no distortion the intensity measured at A at a time t_A is equal to the intensity measured at B at an appropriate subsequent time t_B . Consequently,

$$S^0(x_A, t_A) = S^0(x_B, t_B), \quad (2-1)$$

provided

$$t_B - t_A = (x_B - x_A)/v_g^0, \quad (2-2)$$

where v_g^0 is a constant. Then the disturbance observed at A and B varies with time in the same manner, provided the reference time for measuring the intensity at B is shifted with respect to the one at A by an amount proportional to the distance from A to B . We can, therefore, associate with the measurements at A and B a point $P(x_P, t_P)$ which is fixed with respect to the disturbance and is moving therewith so that if $x_P = x_A$ at $t_P = t_A$, then $x_P = x_B$ at the subsequent time $t_P = t_B$. It is clear from (2-2) that x_P and t_P satisfy the equality

$$x_P - v_g^0 t_P = \psi_P. \quad (2-3)$$

The quantity ψ_P remains constant at all times, that is, for all values of t_P , and it serves to identify the point P . Differentiating both sides of (2-3) with respect to t_P , one obtains

$$v_P = dx_P/dt_P = v_g^0, \quad (2-4)$$

which defines the velocity of P .

One can apply similar arguments to any other point fixed with respect to the disturbance and moving therewith. If $R(x_R, t_R)$ is such a point, then

$$x_R - v_g^0 t_R = \psi_R , \quad (2-5)$$

where ψ_R serves to identify the point R . Differentiating (2-5) with respect to t_R , one obtains

$$v_R = dx_R/dt_R = v_g^0 , \quad (2-6)$$

which is the velocity of R and is the same as the velocity of P . Since all the points move with the same velocity, we can look on the disturbance as moving as a whole with the velocity v_g^0 . Clearly, it is not necessary to identify the disturbance with any particular point, and instead of $S^0(x, t)$, one can write $S^0(\psi)$, where

$$\psi = x - v_g^0 t . \quad (2-7)$$

Suppose now that there is absorption, and let $S(x, t)$ represent the intensity of the disturbance as it progresses along the x axis. Because of distortion, one cannot infer that all the points in the disturbance move with the same velocity, and, therefore, the disturbance cannot be expressed in the form $S(x, t) = S(\psi)$, where ψ is defined as in (2-7).

Assume that the time of arrival of the disturbance $S(x, t)$ at any point along its path is the time at which one observes its maximum intensity. Then, if we denote the times of arrival at A and B as $t_A^{(m)}$ and $t_B^{(m)}$, respectively, one obtains

$$[\partial S(x_A, t)/\partial t]_{t=t_A^{(m)}} = 0 \text{ and } [\partial S(x_B, t)/\partial t]_{t=t_B^{(m)}} = 0 . \quad (2-8)$$

Suppose that there exists an equality

$$t_B^{(m)} - t_A^{(m)} = (x_A - x_B)/v_g , \quad (2-9)$$

where v_g is an appropriate constant. We can then associate with the measurements at A and B a point $M(x_M, t_M)$ which is fixed with respect to the disturbance and moving therewith in such a manner that if $x_M = x_A$ at $t_M = t_A^{(m)}$, then $x_M = x_B$ at a subsequent time $t_M = t_B^{(m)}$. The point M is the point at which the disturbance attains its maximum intensity. We have

$$x_M - v_g t_M = C_M , \quad (2-10)$$

where C_M is a constant at all times, that is, for all values of t_M ; hence

$$dx_M/dt_M = v_g \quad (2-11)$$

is the velocity of the disturbance which is constant at all times. Therefore, we can state that the equality

$$(\partial S/\partial t)_{\substack{x=x_M \\ t=t_M}} = 0 \quad (2-12)$$

together with the equality (2-10) represent the necessary requirement for a disturbance $S(x, t)$ to have a constant velocity v_g . Both the localization of the disturbance at its point of maximum intensity and the velocity concept are physically meaningful, since, operationally, the time t_M is well defined and can be observed experimentally for any t_M .

Unperturbed and Perturbed Wave Packet

If the medium is nonabsorbing the disturbance can be represented by a specified mixture of homogeneous plane waves of different frequencies centered around a frequency ω_0 (wave packet). The electrical intensity will then be expressed as

$$\mathbf{E}^0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\omega \mathbf{E}_{\mathbf{k}, \omega}^0 e^{i(kx - \omega t)} \delta[k - k(\omega)] , \quad (2-13)$$

where k and ω are related to each other by (1-3), and, consequently,

$$k(\omega) = \omega \epsilon^{1/2} / c . \quad (2-14)$$

Both ω and k are real quantities, and, therefore,

$$\epsilon > 0 , \quad (2-15)$$

which means that the waves are non-evanescent. It is also assumed that the phase velocity is a slowly varying function of frequency. Neglecting the derivatives of $k = k(\omega)$ of an order higher than 1, one can express k as

$$k = k_0 + (\omega - \omega_0) (dk/d\omega)_{\omega=\omega_0} , \quad (2-16)$$

where $k_0 = k(\omega_0)$.

Using (2-16) one can transform (2-13) into an expression such as

$$\mathbf{E}^0 = 0.5 [\mathcal{E}^0(\psi) e^{i(kx - \omega t)} + \text{c.c.}] , \quad (2-17)$$

which represents a modulated carrier wave (c.c. denotes a complex conjugate). The modulating function is $\mathcal{E}^0(\psi)$, where

$$\psi = x - v_g^0 t , \quad (2-18)$$

and it moves along the x axis with velocity

$$v_g^0 = \frac{d\omega}{dk} = - \frac{\partial F / \partial k}{\partial F / \partial \omega} , \quad (2-19)$$

known as the group velocity. [To simplify notation we have replaced in (2-17) $\exp i(k_0 x - \omega_0 t)$ by $\exp i(kx - \omega t)$.]

It should be made clear that the expressions such as (2-17) and (2-19), which are well known, are applicable to nonabsorbing media. Our discussion which follows deals with the formulation of the corresponding expressions for absorbing (and emitting) media.

We will now return to the perturbation analysis outlined in Part I and apply the operator \hat{K} defined by (I-5-31) to each harmonic wave in the integrand of (2-13). A wave which was previously homogeneous and was expressed by $\exp i(kx - \omega t)$ has become nonhomogeneous and is now expressed as $\exp (\alpha x - \beta t) \exp i(kx - \omega t)$. The total effect of absorption on a wave

packet can now be obtained by summing the nonhomogeneous plane waves in the same manner as the homogeneous waves were summed in (2-13). We arrive then at an expression for the wave packet which has the form

$$\mathbf{E} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\omega \mathbf{E}_{\mathbf{k}, \omega} e^{i[k - i\alpha]x - (\omega - i\beta)t} \delta[k - k(\omega)] . \quad (2-20)$$

Since α and β do not vary appreciably with ω , one obtains

$$\mathbf{E} = e^{\alpha x - \beta t} \mathbf{E}^0 , \quad (2-21)$$

and, therefore,

$$\mathbf{E} = 0.5 [e^{\alpha x - \beta t} \mathcal{E}^0(\psi) e^{i(kx - \omega t)} + \text{c.c.}] , \quad (2-22)$$

or

$$\mathbf{E} = 0.5 [\mathcal{E} e^{i(kx - \omega t)} + \text{c.c.}] , \quad (2-23)$$

where

$$\mathcal{E} = e^{\alpha x - \beta t} \mathcal{E}^0(\psi) . \quad (2-24)$$

It can be easily seen that the structures of the perturbed and the unperturbed wave packet are basically similar. In both cases the carrier wave is the same, but the difference is in modulating functions, and these are expressed as $\exp(\alpha x - \beta t) \mathcal{E}^0(\psi)$ and $\mathcal{E}^0(\psi)$ respectively.

The formulation of the magnetic induction is similar to the one of the electric intensity. If the wave packet is unperturbed, one has

$$\mathbf{B}^0 = 0.5 [\mathcal{B}^0(\psi) e^{i(kx - \omega t)} + \text{c.c.}] , \quad (2-25)$$

whereas in the case of a perturbed packet the relevant expression is

$$\mathbf{B} = 0.5 [\mathcal{B}(\psi) e^{i(kx - \omega t)} + \text{c.c.}] , \quad (2-26)$$

where

$$\mathcal{B} = e^{\alpha x - \beta t} \mathcal{B}^0 . \quad (2-27)$$

It will now be shown that the perturbed disturbance (2-22) moves with a constant velocity v_g which is the same as the velocity v_g^0 of the unperturbed disturbance (2-17).

Differentiating (2-24) with respect to time we obtain

$$\partial \mathcal{E} / \partial t = -e^{\alpha x - \beta t} [\beta \mathcal{E}^0 + v_g^0 (\partial \mathcal{E}^0 / \partial \psi)] . \quad (2-28)$$

Consequently, the disturbance attains its maximum intensity at such points x and at such times t which satisfy the equality

$$\beta \mathcal{E}^0 + v_g^0 \partial \mathcal{E}^0 / \partial \psi = 0 . \quad (2-29)$$

However, it should be noted that x and t do not appear explicitly in (2-29). The independent variable is ψ , and both x and t are contained in ψ by means of the relationship $\psi = x - v_g^0 t$. The root of the equality (2-29) is ψ_M , which defines the point M , and, therefore, both x_M and t_M are related to each other by

$$\psi_M = x_M - v_g^0 t \quad . \quad (2-30)$$

Differentiating both sides of (2-30) with respect to x_M , we obtain

$$dx_M/dt_M = v_g^0 \quad , \quad (2-31)$$

and, therefore, the point M moves with velocity v_g^0 . Consequently,

$$v_g = v_g^0 = \frac{d\omega}{dk} = - \frac{\partial F/\partial k}{\partial F/\partial \omega} \quad . \quad (2-32)$$

If the medium is molecular [Eq. (1-3)], then

$$v_g = v_g^0 = \frac{2c\epsilon^{1/2}}{\epsilon + d(\omega\epsilon)/d\omega} \quad . \quad (2-33)$$

Since ϵ and $d(\omega\epsilon)$ are always positive, we have $v_g > 0$, which means that in a stationary molecular medium the group velocity has the same direction as the phase velocity (a different result is not expected to occur).

Convective and Nonconvective Disturbance

The expression (2-32) can now be used to classify disturbances into convective and nonconvective. A disturbance is considered to be convective when $v_g \neq 0$, and this occurs when

$$\partial F/\partial k \neq 0 \quad \text{and} \quad \partial F/\partial \omega \neq 0 \quad . \quad (2-34)$$

On the other hand, a disturbance is considered to be nonconvective when $v_g = 0$, and this can happen when

$$\partial F/\partial k = 0 \quad \text{and} \quad \partial F/\partial \omega \neq 0 \quad . \quad (2-35)$$

If the medium is molecular, a convective disturbance can be shown to occur at any frequency for which ϵ is positive and finite, whereas a nonconvective disturbance occurs at two limiting frequencies which correspond to $\epsilon = 0$ (cutoff frequency) and to $\epsilon = \infty$ (resonance frequency). These two frequencies represent special cases for which either the electric or the magnetic field is nonexistent. This can be verified by expressing (1-5-15) in the form

$$c(\mathbf{k} \times \mathbf{E}_{\mathbf{k}, \omega}) = \omega \mathbf{B}_{\mathbf{k}, \omega} \quad , \quad (2-36)$$

from which we obtain

$$\mathbf{B}_{\mathbf{k}, \omega} = \frac{ck}{\omega} (\boldsymbol{\ell} \times \mathbf{E}_{\mathbf{k}, \omega}), \quad (2-37)$$

where $\boldsymbol{\ell}$ is a unit vector.

Combining (2-37) with (1-3), one obtains

$$\mathbf{B}_{\mathbf{k}, \omega} = \epsilon^{1/2} (\boldsymbol{\ell} \times \mathbf{E}_{\mathbf{k}, \omega}). \quad (2-38)$$

It can be easily seen that at the cutoff frequency ($\epsilon = 0$), one has $\mathbf{B}_{\mathbf{k}, \omega} = 0$ and $\mathbf{E}_{\mathbf{k}, \omega} \neq 0$; whereas at resonance frequency ($\epsilon = \infty$), one has $\mathbf{B}_{\mathbf{k}, \omega} \neq 0$ and $\mathbf{E}_{\mathbf{k}, \omega} = 0$. The mechanism of transmission can only be effective when $\mathbf{E} \neq 0$ and $\mathbf{B} \neq 0$, and it is, therefore, obvious that when $\epsilon = 0$ or $\epsilon = \infty$ the disturbance is nonconvective.

The criteria (2-34) and (2-35) can also be applied to a plasma. A plasma is, however, space dispersive. Therefore, in deriving the characteristic equations for a plasma, one should take into account that \mathbf{D} as well as \mathbf{H} depends on both \mathbf{E} and \mathbf{B} .² (In a molecular medium, \mathbf{D} depends solely on \mathbf{E} , and \mathbf{H} solely on \mathbf{B} .) As soon, however, as the characteristic equations are formulated, the procedure for analyzing disturbances is the same for plasmas and for molecular media.

It will be convenient, particularly in the study of plasma, to distinguish between convective and nonconvective disturbances which can decay or grow with time. Our classification will, therefore, be based on both v_g and σ . Thus, when $v_g \neq 0$, one has convective attenuation when $\sigma > 0$ and convective instability when $\sigma < 0$. Similarly, when $v_g = 0$, one has nonconvective attenuation when $\sigma > 0$ and nonconvective instability when $\sigma < 0$.

Attenuation (or Growth) of an Electromagnetic Disturbance

Let \mathcal{E}_M be the electrical intensity at M . Consequently,

$$\mathcal{E}_M = e^{\alpha x_M - \beta t_M} \mathcal{E}_M^0(\psi_M). \quad (2-39)$$

Since

$$t_M = (x_M - \psi_M)/v_g, \quad (2-40)$$

one can express (2-39) as

$$\mathcal{E}_M = e^{\beta \psi_M / v_g - \lambda x_M} \mathcal{E}_M^0(\psi_M), \quad (2-41)$$

where

$$\lambda = \alpha - \frac{\beta}{v_g}. \quad (2-42)$$

The zero point of the ψ scale can be fixed arbitrarily, and, therefore, assuming $\psi_M = 0$, one obtains

$$\xi_M = e^{\lambda x_M} \xi^0(0). \quad (2-43)$$

It is obvious that Eq. (2-40) can only be valid if $v_g \neq 0$, and, therefore, the disturbance expressed by (2-43) can only be convective.

Assume now that no restrictions are imposed on the character of the disturbance, that is, it can be convective or nonconvective. Then substituting

$$x_M = \psi_M + v_g t_M \quad (2-44)$$

in (2-39), one obtains

$$\xi_M = e^{\alpha \psi_M} e^{\delta t_M} \xi^0(\psi_M), \quad (2-45)$$

where

$$\delta = \alpha v_g - \beta, \quad (2-46)$$

and since $\psi = 0$, one has

$$\xi_M = e^{\delta t_M} \xi^0(0). \quad (2-47)$$

If the disturbance is convective, both (2-43) and (2-47) describe the same physical event. The variable x_M is the length of the path followed by the point M , and (2-43) tells how the maximum intensity decays or grows with x_M . On the other hand, t_M is the time at which the disturbance attains its maximum, and consequently (2-47) indicates how the maximum intensity decays or grows with time. On the other hand, a nonconvective disturbance can only be expressed in the form (2-47).

Relaxation Length

If $\lambda \neq 0$ the disturbance can only be convective, and then $\partial F / \partial k \neq 0$. Then dividing both sides of (1-4) by $\partial F / \partial k$, we obtain

$$\alpha + \beta \frac{\partial F / \partial \omega}{\partial F / \partial k} = - \frac{4\pi\sigma}{\partial F / \partial k}. \quad (2-48)$$

Taking into account (2-32), the above relationship gives

$$\alpha - \frac{\beta}{v_g} = - \frac{4\pi\sigma}{\partial F / \partial k}, \quad (2-49)$$

and combining (2-49) with (2-42), one obtains

$$\lambda = - \frac{4\pi\sigma}{\partial F/\partial k} . \quad (2-50)$$

Assuming that the medium is molecular, one obtains

$$\partial F/\partial k = 2c\epsilon^{1/2} , \quad (2-51)$$

and, therefore,

$$\lambda = - 2\pi\sigma/c\epsilon^{1/2} . \quad (2-52)$$

The equality (2-52) applies to both absorbing ($\sigma > 0$) and emitting ($\sigma < 0$) media. If the medium is absorbing, the attenuation can be expressed by the relaxation length

$$L_\lambda = 1/\lambda = c\epsilon^{1/2}/2\pi\sigma . \quad (2-53)$$

It should be noted that (2-50) to (2-53) have been derived on the assumption that $v_g \neq 0$, and, therefore, these expressions can only be applied to disturbances which are convective.

Relaxation Time

Assume now a more general case when the disturbance can be convective or nonconvective. Then dividing both sides of (1-4) by $\partial F/\partial\omega$, we obtain

$$\alpha \frac{\partial F/\partial k}{\partial F/\partial\omega} + \beta = - \frac{4\pi\sigma}{\partial F/\partial\omega} . \quad (2-54)$$

Taking into account (2-32), the above equality gives

$$- \alpha v_g + \beta = - \frac{4\pi\sigma}{\partial F/\partial\omega} , \quad (2-55)$$

and combining (2-55) with (2-46), one obtains

$$\delta = \frac{4\pi\sigma}{\partial F/\partial\omega} . \quad (2-56)$$

Assuming that the medium is molecular, one obtains

$$\frac{\partial F}{\partial\omega} = - \left[\epsilon + \frac{d(\omega\epsilon)}{d\omega} \right] , \quad (2-57)$$

and, therefore,

$$\delta = - \frac{4\pi\sigma}{\epsilon + d(\omega\epsilon)/d\omega} . \quad (2-58)$$

The sign minus in the above expression is consistent with the meaning of δ . Note that ϵ and $d(\omega\epsilon)/d\omega$ are always positive, and, therefore, when $\sigma > 0$ we have $\delta < 0$, and the disturbance is attenuated exponentially with time. If the medium is absorbing, the attenuation can be expressed by the relaxation time

$$T_{\delta} = \left| \frac{1}{\delta} \right| = \frac{\epsilon + d(\omega\epsilon)/d\omega}{4\pi\sigma}. \quad (2-59)$$

Both L_{λ} and T_{δ} are directly observable and measurable quantities. When L_{λ} is known, one can calculate conductivity from (2-53) and obtain

$$\sigma = c\epsilon^{1/2}/2\pi L_{\lambda}. \quad (2-60)$$

Similarly, using (2-59) one can determine conductivity from the measurement of T_{δ} and obtain

$$\sigma = \frac{\epsilon + d(\omega\epsilon)/d\omega}{4\pi T_{\delta}}. \quad (2-61)$$

III. ENERGETIC BEHAVIOR OF AN ELECTROMAGNETIC DISTURBANCE IN A DISPERSIVE MEDIUM

Kinematic and Energetic Velocity

The velocity of propagation of an electromagnetic disturbance is both a kinematic and an energetic concept. Kinematically, v_{ξ}^0 or v_{ξ} is defined as the rate of change of distance with respect to time. Energetically, the velocity can be defined by the ratio of the mean energy flow to the mean energy density. The energetic velocity, known as the velocity of energy transport,³ is

$$v_u^0 = \bar{S}^0/\bar{U}^0 \quad (3-1)$$

if there is no absorption, and

$$v_u = \bar{S}/\bar{U} \quad (3-2)$$

if there is absorption.

The kinematic and energetic velocities, although differently defined, must obviously be numerically equal. Brillouin⁴ has shown that there is an equality

$$v_{\xi}^0 = v_u^0 \quad (3-3)$$

which relates the two velocities in a nonabsorbing medium. However, a similar equation

$$v_{\xi} = v_u \quad (3-4)$$

for an absorbing medium has not been established to date. The current theory has failed in providing valid expressions for the energy density \bar{U} and energy flow \bar{S} , and these are necessary for defining v_u in an absorbing medium. Fortunately, the revised version of the theory adapts itself very easily to the analysis of the propagation effects from the energetic point of view. Therefore, it proves possible to obtain expressions for the energy density \bar{U} and energy flow \bar{S} , and then to determine v_u and to show that $v_u = v_g$. This equality is of some fundamental importance, since it supports the validity of the kinematic approach from an energetic point of view.

Extension of the Brillouin Theory to Absorbing Media

Energy Density. – The revised version of the theory developed in Part I is based on the assumption that the relationship between the energy density and the field intensity is the same for nonabsorbing and absorbing media. Consequently, if the intensity of the field is known, the procedure for calculating energy is the same whether or not the medium is absorbing. We are not confronted here with a frustrating situation as in the current theory, in which the same approach when applied to a nonabsorbing and to an absorbing medium gives valid results in the former case but invalid results in the latter case.

We should recall, however, that the disturbance has been defined differently for a nonabsorbing and for an absorbing medium. In the former case we have an unperturbed wave packet:

$$\mathbf{E}^0 = 0.5 [\mathcal{E}^0(\psi) e^{i(kx - \omega t)} + \text{c.c.}] , \quad (3-5)$$

$$\mathbf{B}^0 = 0.5 [\mathcal{B}^0(\psi) e^{i(kx - \omega t)} + \text{c.c.}] ; \quad (3-6)$$

whereas in the latter case the wave packet is perturbed and has the form

$$\mathbf{E} = 0.5 (\mathcal{E} e^{i(kx - \omega t)} + \text{c.c.}) , \quad (3-7)$$

$$\mathbf{B} = 0.5 (\mathcal{B} e^{i(kx - \omega t)} + \text{c.c.}) , \quad (3-8)$$

where

$$\mathcal{E} = e^{\alpha x - \beta t} \mathcal{E}^0(\psi) , \quad (3-9)$$

$$\mathcal{B} = e^{\alpha x - \beta t} \mathcal{B}^0(\psi) . \quad (3-10)$$

Therefore,

$$\mathbf{E} = e^{\alpha x - \beta t} \mathbf{E}^0 \quad \text{and} \quad \mathbf{B} = e^{\alpha x - \beta t} \mathbf{B}^0 . \quad (3-11)$$

The procedure currently used for nonabsorbing media is based on (3-5) and (3-6) and on the relationships

$$\frac{\partial U_e^0}{\partial t} = \frac{1}{4\pi} \mathbf{E}^0 \cdot \frac{\partial \mathbf{D}^0}{\partial t} , \quad \frac{\partial U_m^0}{\partial t} = \frac{1}{4\pi} \mathbf{B}^0 \cdot \frac{\partial \mathbf{H}^0}{\partial t} , \quad (3-12)$$

and $\mathbf{D}_{\mathbf{k},\omega}^0 = \epsilon^0 \mathbf{E}_{\mathbf{k},\omega}^0$ and it leads to the well-known expression⁴

$$\frac{\partial \bar{U}^0}{\partial t} = \frac{1}{16\pi} \frac{\partial}{\partial t} \left[\frac{\partial(\omega \epsilon^0)}{\partial \omega} \mathcal{E}^0 \cdot \mathcal{E}^{0*} + \mathcal{B}^0 \cdot \mathcal{B}^{0*} \right], \quad (3-13)$$

from which one obtains

$$\bar{U}^0 = \bar{U}^0(\psi) = \frac{1}{16\pi} \left[\frac{\partial(\omega \epsilon^0)}{\partial \omega} \mathcal{E}^0 \cdot \mathcal{E}^{0*} + \mathcal{B}^0 \cdot \mathcal{B}^{0*} \right]. \quad (3-14)$$

In order to calculate energy density in an absorbing medium, we use (3-7) and (3-8) and the relationships

$$\frac{\partial U_e}{\partial t} = \frac{1}{4\pi} \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}, \quad \frac{\partial U_m}{\partial t} = \frac{1}{4\pi} \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t}, \quad (3-15)$$

and $\mathbf{D}_{\mathbf{k},\omega} = \epsilon \mathbf{E}_{\mathbf{k},\omega}$. Since $\epsilon = \epsilon^0$, the calculations of $\partial U/\partial t$ in terms of \mathbf{E} are the same as the ones currently used to obtain $\partial U^0/\partial t$ in terms of \mathbf{E}^0 . We obtain

$$\frac{\partial \bar{U}}{\partial t} = \frac{1}{16\pi} \frac{\partial}{\partial t} \left[\frac{\partial(\omega \epsilon)}{\partial \omega} \mathcal{E} \cdot \mathcal{E}^* + \mathcal{B} \cdot \mathcal{B}^* \right]. \quad (3-16)$$

Taking into account (3-9) and (3-10) and the equality $\epsilon = \epsilon^0$, we obtain from (3-16)

$$\frac{\partial \bar{U}}{\partial t} = e^{2(\alpha x - \beta t)} \frac{\partial \bar{U}^0}{\partial t}, \quad (3-17)$$

and assuming that

$$|\beta U^0| \ll \left| \frac{\partial U^0}{\partial t} \right|, \quad (3-18)$$

we have

$$\bar{U} = e^{2(\alpha x - \beta t)} \bar{U}^0. \quad (3-19)$$

We assume that the disturbance is located at $M(x_M, t_M)$. Consequently, substituting in (3-19) $t = t_M$ and $x = x_M$, we obtain

$$\bar{U}_M = \bar{U}(x_M, t_M) e^{2(\alpha x_M - \beta t_M)} U^0(\psi_M), \quad (3-20)$$

and, therefore,

$$\bar{U}_M = e^{2\lambda x_M} \bar{U}_0^0, \quad (3-21)$$

where $\bar{U}_0^0 = \bar{U}_{\psi=0}^0$.

An alternate way to calculate energy density is to express x_M in terms of t_M and ψ_M . Then

$$\bar{U}_M = e^{2\delta t_M} \bar{U}_0^0. \quad (3-22)$$

Both (3-21) and (3-22) describe what actually happens to the energy of the disturbance both in space and in time. The variable x_M gives the successive positions of M , and, therefore, (3-21) shows how the energy decays or grows along the path of the disturbance. Similarly, t_M are the times at which the point M reaches successive positions along its path, and, consequently, (3-22) shows how the energy density decays or grows with time.

Energy Flow and the Velocity of the Energy Transport. – If the medium is nonabsorbing, the energy flow is known to be

$$\text{div } \mathbf{S}^0 = \frac{1}{4\pi} (\mathbf{E}^0 \times \mathbf{B}^0), \quad (3-23)$$

and since the relationship between energy and intensity is the same for nonabsorbing and absorbing media, we obtain in the latter case

$$\text{div } \mathbf{S} = \frac{1}{4\pi} (\mathbf{E} \times \mathbf{B}). \quad (3-24)$$

Taking \mathbf{E} and \mathbf{B} as given by (3-11), we express (3-24) as

$$\text{div } \mathbf{S} = e^{2(\alpha x - \beta t)} \text{div } \mathbf{S}^0, \quad (3-25)$$

and assuming

$$|\alpha S^0| \ll \left| \frac{\partial S^0}{\partial x} \right|, \quad (3-26)$$

we obtain

$$\bar{\mathbf{S}}_M = e^{2(\alpha x - \beta t)} \bar{\mathbf{S}}^0. \quad (3-27)$$

Expressing t_M as a function of x_M and assuming $\psi_M = 0$, we can represent the energy flow as

$$\bar{\mathbf{S}}_M = e^{2\lambda x_M} \bar{\mathbf{S}}_0^0, \quad (3-28)$$

where $\bar{\mathbf{S}}_0^0 = \bar{\mathbf{S}}_{\psi=0}^0$. Similarly, taking x_M as a function of t_M and assuming $\psi_M = 0$, we obtain

$$\bar{\mathbf{S}}_M = e^{2\delta t_M} \bar{\mathbf{S}}_0^0. \quad (3-29)$$

By dividing (3-28) or (3-29) by (3-21) or (3-22), respectively, we can express the velocity of energy transport as

$$v_u = \bar{S}_M / \bar{U}_M = \bar{S}_0^0 / \bar{U}_0^0 . \quad (3-30)$$

It is well known that

$$v_u^0 = \bar{S}_0^0 / \bar{U}_0^0 = d\omega / dk , \quad (3-31)$$

and, therefore,

$$v_u^0 = v_u = d\omega / dk . \quad (3-32)$$

Consequently, the relationship which has been shown by Brillouin to be valid for nonabsorbing media has now been generalized to all dispersive media, both nonabsorbing and absorbing.

Rate of Energy Transfer. – Using

$$\bar{Q} = \sigma \bar{E}^2 \quad (3-33)$$

and substituting in (3-33) E as defined by (3-7), we calculate the average rate of energy transfer at the point M and obtain

$$\bar{Q}_M = \bar{Q}(x_M, t_M) = e^{2\lambda x_M} Q_0^0 , \quad (3-34)$$

$$\bar{Q}_M = \bar{Q}(x_M, t_M) = e^{2\delta t_M} Q_0^0 , \quad (3-35)$$

where

$$\bar{Q}_0^0 = 0.5 \sigma (\mathcal{E}^0 \cdot \mathcal{E}^{0*}) . \quad (3-36)$$

The expressions (3-34) and (3-35) show how \bar{Q}_M decay or grow with x_M and t_M respectively.

Convective and Nonconvective Disturbances. – The above results will now be summarized.

If the disturbance is convective, the three quantities \bar{U}_M , \bar{S}_M , and \bar{Q}_M have been shown to “move” with the velocity v_u and decay or grow as $\exp 2\lambda x$ and $\exp 2\delta t$. When $\lambda > 0$ and $\delta > 0$, there is an exponential growth and, consequently, a convective instability. Conversely, when $\lambda < 0$ and $\delta < 0$, there is a convective attenuation. If, however, the disturbance is nonconvective, then v_u and, consequently, \bar{S}_M are zero. Then the only significant quantity is \bar{U}_M , and its decay or growth can only be expressed with reference to time as $\exp 2\delta t$.

We have pointed out previously that if there is a nonconvective disturbance in a molecular medium, then either \mathbf{E} or \mathbf{B} is zero. The mechanism of energy growth or decay is dependent on Ohm’s law ($Q = \sigma E^2$) and can, therefore, occur when $E \neq 0$. This can take place at resonance and not at the cutoff frequency. Therefore, at resonance, \bar{U}_M increases or decreases as $\exp 2\delta t$, where

$$\delta = - \frac{4\pi\sigma}{d(\omega\epsilon)/d\omega} . \quad (3-37)$$

IV. REFORMULATION OF OHM'S LAW

Energetic Version of Ohm's Law

The relationship

$$\bar{Q} = \sigma \bar{E}^2 \quad (4-1)$$

is expressed in terms of the electric-field intensity \mathbf{E} , which is not an energetic concept. Therefore, from the energetic point of view, Ohm's law is not defined adequately. We will now reformulate Ohm's law on the assumption that energy, and not the field intensity, is a fundamental concept, and replace (4-1) by a relationship in which the field intensity \mathbf{E} is not directly involved.

The first attempt to describe energetically Ohm's law was made by Planck.⁵ Planck pointed out that there is an analogy between the generation of Joule's heat and "the transformation of elastic energy into heat, such as occurs in a deformed incompletely elastic body when the elastic forces of tension gradually subside." Consequently, "the energy which in the time dt and in the volume $d\tau$ becomes transformed into heat is proportional to the momentary electrical energy density $dt \cdot d\tau \cdot \frac{\epsilon}{8\pi} \mathbf{E}^2 \cdot \text{const.}$ " Thus, according to Planck, Ohm's law represents a mechanism in which the amount of energy transformed in a small element of volume during an appropriately short time interval is proportional to the amount of energy stored. He assumed, however, that the medium is nondispersive and, therefore, that ϵ is not dependent on frequency. We will extend Planck's formulation to dispersive media.

Consider the two expressions (I-5-3) and (I-5-10), which show energy conservation in a non-absorbing and in an absorbing medium respectively. Since the electromagnetic processes are oscillatory, it will be useful to reformulate these expressions by taking into account the mean rather than the instantaneous energy. The conservation of energy for a nonabsorbing or for an absorbing medium can then be expressed by

$$\frac{\partial U^0}{\partial t} + \text{div } \mathbf{S}^0 = 0 \quad (4-2)$$

and as

$$\frac{\partial U}{\partial t} + \text{div } \mathbf{S} + Q = 0 \quad (4-3)$$

respectively.

Taking $\bar{U} = \exp 2(\alpha x - \beta t)\bar{U}^0$ and $\bar{S} = \exp 2(\alpha x - \beta t)\bar{S}^0$, one obtains

$$\left(\frac{\partial \bar{U}^0}{\partial t} + \text{div } \bar{\mathbf{S}}^0 \right) e^{-2(\alpha x - \beta t)} - 2(\beta \bar{U} - \alpha \bar{S}) + Q = 0, \quad (4-4)$$

and considering (4-2), one has

$$2(\beta \bar{U} - \alpha \bar{S}) = \bar{Q}. \quad (4-5)$$

Since the disturbance is localized at the point M , the relationship between \bar{U}_M , \bar{S}_M , and \bar{Q}_M can be expressed as

$$2(\beta\bar{U}_M - \alpha\bar{S}_M) = \bar{Q}_M. \quad (4-6)$$

Substituting in (4-6) $\bar{U}_M = \bar{S}_M/v_g$ and using the relationship $\lambda = \alpha - \beta/v_g$, one obtains

$$\bar{Q}_M = -2\lambda\bar{S}_M, \quad (4-7)$$

and similarly, substituting in (4-6) $\bar{S}_M = \bar{U}_M v_g$ and using the equality $\delta = \alpha v_g - \beta$, one has

$$\bar{Q}_M = -2\delta\bar{U}_M. \quad (4-8)$$

The above relationships represent the energetic version of Ohm's law. They state that the rate of energy conversion is proportional to the energy density and to the energy flow in any dispersive medium, whether emitting or absorbing. If the disturbance is convective, Ohm's law can be expressed by either (4-7) or (4-8). On the other hand, in case of a nonconvective disturbance, Ohm's law can only be expressed by (4-8). A formulation similar to (4-8) was obtained in a previous investigation.⁶

Differential Expression for Ohm's Law

It may be useful to express the energetic version of Ohm's law in a differential rather than integral form. The differential form will show explicitly the rate of change of energy density or energy flow and its dependence on the rate of energy conversion.

Differentiating (3-28), we have

$$d\bar{S}_M/dx_M = 2\lambda\bar{S}_M, \quad (4-9)$$

and combining (4-9) with (4-7), we obtain

$$-d\bar{S}_M/dx_M = \bar{Q}_M. \quad (4-10)$$

Similarly, differentiating (3-22), we obtain

$$d\bar{U}_M/dt_M = 2\delta\bar{U}_M, \quad (4-11)$$

and combining (4-11) with (4-8), we have

$$-d\bar{U}_M/dt_M = \bar{Q}_M. \quad (4-12)$$

Other useful relationships are as follows. Differentiating (3-34), we have

$$d\bar{Q}_M/dx_M = 2\lambda\bar{Q}_M. \quad (4-13)$$

Combining (4-13) with (4-7), we obtain

$$\bar{S}_M (d\bar{Q}_M/dx_M) = -\bar{Q}_M^2. \quad (4-14)$$

Similarly, we can obtain a relationship

$$\bar{U}_M (d\bar{Q}_M/dt) = -\bar{Q}_M^2. \quad (4-15)$$

V. REASSESSMENT OF THE TWO VERSIONS AND COMPARISON OF RESULTS

Attenuation and Growth Parameters

The two versions of the electromagnetic theory differ from each other in the selection of parameters which are assumed to represent energy growth or energy attenuation. In the current version, an electromagnetic disturbance is a nonhomogeneous plane wave $\exp i(k_s x - \omega_s t)$, and the parameters which define transmission and attenuation are obtained from the dispersion equation (1-2) by taking either ω_s real and $k_s = k_s^{(r)} + ik_s^{(i)}$ complex, or k_s real and $\omega_s = \omega_s^{(r)} - i\omega_s^{(i)}$ complex. The choice depends partly on expediency and partly on the formulation of the problem. On the other hand, in the revised version the disturbance has the form of an attenuated wave packet and is represented by a superposition of nonhomogeneous plane waves $\exp(\alpha x - \beta t) \exp i(kx - \omega t)$, and the parameters k , ω , α , and β , which characterize the nonhomogeneous waves, are related to each other by means of the characteristic equations (1-3) and (1-4). These two descriptions are physically dissimilar and will now be analyzed in connection with convective and nonconvective disturbances.

Convective Disturbances

Current Formulation. – Consider an electromagnetic wave expressed by

$$\mathbf{E}_s = 0.5 [(\mathbf{E}_{\mathbf{k}, \omega})_s e^{i(k_s x - \omega_s t)} + \text{c.c.}], \quad (5-1)$$

$$\mathbf{B}_s = 0.5 [(\mathbf{B}_{\mathbf{k}, \omega})_s e^{i(k_s x - \omega_s t)} + \text{c.c.}], \quad (5-2)$$

where k_s and ω_s are related to each other by the dispersion equation (1-2). Assuming that ω_s is real and that $k_s = k_s^{(r)} + ik_s^{(i)}$ is complex, Eqs. (5-1) and (5-2) represent a nonhomogeneous plane wave which decays exponentially as $\exp(-k_s x)$. It represents, therefore, a convective disturbance. The parameters which define the transmission and the attenuation are expressed by a complex quantity $n_c = ck_s/\omega_s = n + i\kappa$ known as the complex index of refraction. The real component of n_c has the form

$$n = ck_s^{(r)}/\omega_s = \{0.5 \epsilon_s [(1 + 16\pi^2 \sigma_s^2/\omega_s^2 \epsilon_s^2)^{1/2} + 1]^{1/2}\}^{1/2} \quad (5-3)$$

and is the refraction index, whereas the imaginary component is expressed as⁷

$$\kappa = ck_s^{(i)}/\omega_s = \{0.5 \epsilon_s [(1 + 16\pi^2 \sigma_s^2/\omega_s^2 \epsilon_s^2)^{1/2} - 1]^{1/2}\}^{1/2} \quad (5-4)$$

and is known as the extinction index. Assuming $\sigma_s \ll \omega \epsilon_s$, Eq. (5-4) can be expressed as

$$k_s^{(i)} = 2\pi\sigma_s / c\epsilon_s^{1/2}. \quad (5-5)$$

Equation (5-3) represents the transmission, whereas Eqs. (5-4) and (5-5) represent the attenuation effect.

Revised Formulation. — Since the disturbance is convective, it is characterized by both the energy density

$$\bar{U}_M = e^{2\lambda x} \bar{U}_0^0 \quad (5-6)$$

and the energy flow

$$\bar{S}_M = e^{2\lambda x} \bar{S}^0. \quad (5-7)$$

The transmission is represented by the propagation velocity

$$v_u = \frac{d\omega}{dk} = - \frac{\partial F / \partial k}{\partial F / \partial \omega} = \frac{2c\epsilon^{1/2}}{\epsilon + d(\omega\epsilon)/d\omega}, \quad (5-8)$$

and the attenuation can be expressed either by the attenuation constant,

$$\lambda = - 2\pi\sigma / c\epsilon^{1/2}, \quad (5-9)$$

or by the relaxation length,

$$L_\lambda = c\epsilon^{1/2} / 2\pi\sigma. \quad (5-10)$$

Comparison of the Two Formulations. — It should be recalled that ϵ_s and σ_s cannot be energetically defined. Therefore, although (5-5) and (5-9) are formally similar, one cannot consider (5-5) or (5-4) as physically meaningful quantities. From a physical, observational point of view, the extinction index (5-4) is a fictitious quantity which cannot be measured. It will now be shown that a theory based on the extinction index leads to the paradoxical conclusion that if there is no absorption, the medium cannot transmit energy.

Consider a disturbance represented by

$$\mathbf{E}_s = 0.5 [\mathbf{E}_0 e^{-\kappa k_0 x} e^{i(nk_0 x - \omega t)} + \text{c.c.}], \quad (5-11)$$

$$\mathbf{B}_s = 0.5 [(n + i\kappa)\mathbf{E}_0 e^{-\kappa k_0 x} e^{i(nk_0 x - \omega t)} + \text{c.c.}], \quad (5-12)$$

where $k_0 = \omega/c$. The average energy flow can then be expressed as

$$\bar{S} = \frac{nc}{8\pi} \mathbf{E}_0 \cdot \mathbf{E}_0^* e^{-2\kappa k_0 x}. \quad (5-13)$$

Therefore, it is seen that \bar{S} is attenuated as $\exp(-2\kappa k_0 x)$, and the attenuation decreases with α . When $\sigma \rightarrow 0$ one has $\kappa \rightarrow 0$, and in the limiting case of $\sigma = 0$, one obtains

$$\bar{S} = \frac{nc}{8\pi} \mathbf{E}_0 \cdot \mathbf{E}_0^* . \quad (5-14)$$

It would appear at first sight that (5-14) is a physically meaningful quantity which represents the energy flow when $\sigma = 0$. It is, however, difficult to find a logical connection between (5-14) and the expressions

$$\mathbf{E}_s = 0.5 [\mathbf{E}_0 e^{i(nk_0 x - \omega t)} + \text{c.c.}] , \quad (5-15)$$

$$\mathbf{B}_s = 0.5 [n\mathbf{E}_0 e^{i(nk_0 x - \omega t)} + \text{c.c.}] , \quad (5-16)$$

which give the intensity of the field when $\sigma = 0$. Both (5-15) and (5-16) represent strictly monochromatic waves, and obviously in such case there can be no energy transmission and no energy flow. In order to transmit energy, there must be a notice of the beginning of the end of the transmission event. However, this does not occur in (5-15) and (5-16). We have a wave which is uniform in intensity and extends without limits, both in space and in time, and nowhere does it have a beginning or an end.

Therefore, there is a *prima facie* contradiction. We obtain a result which supposedly represents energy transmission for any σ , however small it can be, but at the same time it precludes the possibility of the energy transmission when σ becomes vanishingly small. Therefore, one would conclude that absorption is a necessary requirement for energy transmission. In other words, if there is no absorption, energy cannot be transmitted through space.

The above paradox results from the assumption that a disturbance can be expressed by $\exp i(k_s x - \omega_s t)$, where k_s is real and ω_s is complex. This assumption is universally used, and its validity has not been questioned to date. However, it is clear that a nonhomogeneous plane wave cannot be considered to be a suitable vehicle for carrying energy. Therefore, the concept of extinction index is physically misleading. In order to have a physically meaningful result it would be necessary to represent the disturbance by a group of nonhomogeneous plane waves rather than by a single wave. This is essentially what we have done. We have obtained an expression for a wave packet which is valid for all values of σ including $\sigma = 0$. When $\sigma = 0$, the intensity of the wave packet is constant, but nonetheless there is a notice of the beginning and of the end of the transmission event, and the energy transmission does occur. We are not confronted here with a situation in which there is energy flow when $\sigma \neq 0$, but no energy flow when $\sigma = 0$.

Nonconvective Disturbances

Consider now the alternative approach in which k_s is real and $\omega = \omega_s^{(r)} - i\omega_s^{(i)}$ is complex. In such case the disturbance is attenuated exponentially as $\exp [-\omega_s^{(i)} t]$. Since there is no exponential decay with x , the disturbance can only be nonconvective.

Substituting $\omega_s = \omega_s^{(r)} - i\omega_s^{(i)}$ in (1-2) and solving for $\omega_s^{(i)}$, one obtains an expression which defines $\omega_s^{(i)}$ in terms of σ_s . When σ_s tends to zero, the quantity $\omega_s^{(i)}$ becomes vanishingly small. In the limiting case, when $\sigma_s = 0$, one has $\omega_s^{(i)} = 0$, and then Eqs. (5-1) and (5-2) give a strictly monochromatic wave. There is, therefore, no energy flow. This is a plausible situation for a disturbance which is nonconvective. Nevertheless, such a description is not physically valid, since it involves the parameters ϵ_s and σ_s which cannot be energetically defined.

In the revised formulation an expression for a nonconvective disturbance is obtained by assuming $v_g = 0$. It can then be shown that it decays as $\exp \beta t$, where

$$\beta = \frac{4\pi\sigma}{\partial F / \partial \omega}. \quad (5-17)$$

VI. CONCLUDING REMARKS

A few remarks on the following topics will conclude the paper: validity of the perturbation treatment, discrepancies between theory and experiment, and microscopic and macroscopic aspects of the mechanism of energy dissipation in dispersive media.

Validity of the Perturbation Treatment

We have performed the calculations on the assumption that the perturbing parameter σ is small. Obviously, there must be an answer to the question, "Small compared with what?" We assume that σ should be such that the amount of energy dissipated in an absorbing medium during a physically significant time interval is small when compared with the total energy store.

The question is, "What is the significant time interval?" Since the processes are oscillatory, the time scale for measuring energy change is that of one period of oscillation, and, therefore, $T = 1/\omega$ is the significant time interval. Considering that the rate of the energy attenuation is $\bar{Q} = 0.5 \mathcal{E} \cdot \mathcal{E}^*$, we can express the amount of the energy dissipated during one period as

$$\bar{Q}T \sim \omega \mathcal{E} \cdot \mathcal{E}^*. \quad (6-1)$$

We know that the average amount of the energy stored is

$$\bar{U} \sim \frac{d(\omega\epsilon)}{d\omega} \mathcal{E} \cdot \mathcal{E}^*, \quad (6-2)$$

and since (6-1) is small when compared with (6-2), we obtain

$$\sigma \ll \omega \frac{d(\omega\epsilon)}{d\omega}. \quad (6-3)$$

The expression (6-3) represents the necessary condition for the validity of the perturbation treatment. It is consistent with the general intent and purpose of Maxwell's theory. Obviously,

the primary purpose of Maxwell's theory is to describe energy propagation, and, consequently, the process of dissipation of energy is only incidental. The conductivity current must, therefore, be small when compared with the displacement current, and in such case the inequality (6-3) must be fulfilled. If one assumes that

$$\sigma \gg \omega \frac{d(\omega \epsilon)}{d\omega}, \quad (6-4)$$

then the displacement current is relatively small, and the basic features of the Maxwellian hypothesis become of secondary importance.

Experiment and Theory

We have emphasized throughout our analysis that there is an inconsistency in the current theory which is entirely of a theoretical and conceptual nature. It is, therefore, questionable whether any discussion on experiments could throw light on the problem we have been trying to clarify. It can be easily seen that if there is an agreement between the theory and the experiment, no meaningful conclusion can be arrived at concerning the logical structure of the theory, since in such case the conceptual difficulties will not be automatically resolved.

The literature on the experimental aspects of the theory is obviously immense. It has been found, however, that in some instances the experiments do not confirm the theory. This has been clearly shown by Landsberg⁸ in his discussion on the current status of the theory dealing with the absorption of light. He pointed out that measurements do not agree precisely with results obtained by the theory. The agreement can be considered at its best as only qualitative, and, therefore, "we are compelled to recognize that these (absorption) processes exist, but we are not able to form a comprehensive representation of these processes."

There is extensive literature on measurements of the refraction and extinction index in various chemical compositions. There is, however, a question as to whether the refraction index and particularly the extinction index have any direct physical significance. These measurements per se do not represent, therefore, an agreement or disagreement between the theory and the experiment.

Microscopic and Macroscopic Aspects of Energy Dissipation

The brief discussion which follows contains additional details regarding the scope of this investigation. Our primary objective has been to clarify the concept of energy and to reformulate the macroscopic theory in accordance with the requirements of generalized dynamics. We considered, therefore, as sufficient and adequate the interpretation of conductivity as a parameter which is *phenomenologically* defined. No attempts have been made to pass from microscopic to macroscopic theory and to investigate the intricate interplays on the microscopic scale which enter into the mechanism of energy loss, both by radiation emission and by collisional effects.

Our primary concern was the formulation of the dielectric constant, which in our interpretation is independent of the effects of absorption. Therefore, the passage from the microscopic to the macroscopic point of view, as far as the dielectric constant is concerned, is the same as the one outlined by Rosenfeld,⁹ provided one assumes that there is no absorption, that is, that the atomic polarizability is real. We previously pointed out that a complex polarizability (assumed by Rosenfeld on the basis of the Lorentz theory) is not consistent with the energetic point of view.

Incidentally, it should be noted that the current version of the theory does not provide a satisfactory account of energy losses on a microscopic scale. The usual formulation is to express the parameter γ as a sum of two terms, $\gamma = \gamma^{(P)} + \gamma^{(L)}$, where $\gamma^{(P)}$ represents the radiative and $\gamma^{(L)}$ the collisional effects.¹⁰ The inclusion of the radiative effect is not consistent with the Kramers-Kronig causality requirements.¹¹ Therefore, a certain arbitrary assumption has been made that $\gamma^{(P)}$ is not equal to $2e^2\omega^2/3mc^2$ as it should be, but is equal to $2e^2\omega_e^2/3mc$, where $\omega_e^2 = 4\pi Ne^2/m$.¹² Moreover, as mentioned previously,⁸ the estimate of $\gamma^{(L)}$ does not account quantitatively for experimental results.

Although the energy losses have not been considered from the microscopic point of view, we have provided an outline of some ideas which may serve as a starting point for another investigation to accomplish in this particular area what one has not been able to obtain within the framework of the current Maxwell-Lorentz theory.

REFERENCES

¹Jacob Neufeld, *Revised Formulation of the Macroscopic Maxwell Theory. I. Fundamentals of the Proposed Formulation*, ORNL-4406, Part I (in press). Reference to equations in Part I will be written in the typical form (I-2-3).

²Jacob Neufeld, *Phys. Rev.* **123**, 1 (1961); *Phys. Rev.* **127**, 346 (1962); *J. Appl. Phys.* **34**, 2549 (1963).

³The concept of the velocity of energy transport as applied in the electromagnetic theory is discussed by T. H. Havelock in *The Propagation of Disturbances in Dispersive Media*, Cambridge University Press, Cambridge, 1914; and by Th. DeDonder in *Compt. Rend.* **158**, 687 (1914).

⁴L. Brillouin, *Congres International d'Electricite, Paris, 1932*, Vol. 2, pp. 739–88, Gauthier-Villars, Paris, 1932; *Wave Propagation and Group Velocity*, Academic Press, New York, 1960; also L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, pp. 254–56, Pergamon, Oxford, 1960.

⁵Max Planck, *Theory of Electricity and Magnetism*, pp. 19–20, Macmillan, London, 1932.

⁶Jacob Neufeld, *Phys. Rev.* **152**, 708 (1966).

⁷See, for instance, Max Born and Emil Wolf, *Principles of Optics*, p. 610, Pergamon, New York, 1959.

⁸G. S. Landsberg, *Optika (Optics)*, pp. 475–76, Gosizdat, Moscow, 1952.

⁹L. Rosenfeld, *Theory of Electrons*, pp. 59–110, Interscience, New York, 1955.

¹⁰K. L. Wolf and K. F. Herzfeld, pp. 520–25 in *Handbuch der Physik*, Vol. XX, Ed. by S. Flügge, Springer, Berlin, 1928.

¹¹H. A. Kramers, *Atti Congr. Intern. Fis. Como 2*, 545 (1927); R. de L. Kronig, *J. Opt. Soc. Am.* **12**, 547 (1926).

¹²J. Hamilton, p. 156 in *Progress in Nuclear Physics*, Vol. 8, Ed. by O. R. Frisch, Pergamon, New York, 1960.

INTERNAL DISTRIBUTION

- | | |
|-------------------------------------|---------------------------------|
| 1. Biology Library | 59. R. J. Lauer |
| 2-4. Central Research Library | 60. J. L. Liverman |
| 5-6. ORNL - Y-12 Technical Library | 61. H. G. MacPherson |
| Document Reference Section | 62. K. Z. Morgan |
| 7-48. Laboratory Records Department | 63-262. J. Neufeld |
| 49. Laboratory Records, ORNL R.C. | 263. J. S. Olson |
| 50. D. S. Billington | 264. M. J. Skinner |
| 51. R. D. Birkhoff | 265. D. A. Sundberg |
| 52. W. D. Brown | 266. J. E. Turner |
| 53. W. Davis, Jr. | 267. A. M. Weinberg |
| 54. R. B. Evans III | 268. J. S. Cantlon (consultant) |
| 55. M. H. Fontana | 269. J. C. Frye (consultant) |
| 56. R. W. Hendricks | 270. J. B. Hursh (consultant) |
| 57. R. F. Hibbs | 271. H. O. Wyckoff (consultant) |
| 58. H. B. Hupf | |

EXTERNAL DISTRIBUTION

272. J. A. Swartout, Union Carbide Corporation, New York, N.Y.
273. Laboratory and University Division, AEC, ORO
274-533. Given distribution as shown in TID-4500 under Physics category (25 copies - CFSTI)