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Solution of the Equations of Motion in Classical and Quantum Theories*

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ABSTRACT

The purpose of the present paper is to elucidate the relationship between the time dependence of quantum operators in the Heisenberg picture and the time dependence of the corresponding dynamical variables in the underlying classical theory. This problem is studied in the non-relativistic particle mechanics and in the field theory. It is shown how the operator solutions of the quantum equations of motion are related to the corresponding solutions of the classical equations of motion. An explicit formula is given, which expresses quantum operators in the Heisenberg picture in terms of their classical counterparts. This formula is particularly useful in the study of the classical limit of the quantum theory. The dependence on $\hbar$ of the matrix elements of the coordinate and field operators is explicitly given, which enables one to study quantum corrections to the classical theory in all orders. Coherent states of the quantum system play an essential role in the formalism.

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1. Introduction

In the standard method of the quantization of the classical system a certain subset of dynamical variables, which must contain the canonical variables and the generators of all the relevant canonical transformations (translations, rotations, etc.), plays a distinguished role. This subset must be closed under the Poisson bracket operation i.e. it must form a Lie subalgebra. In the process of the quantization one represents this Poisson bracket Lie algebra in terms of the commutator Lie algebra of linear operators in the Hilbert space. In addition one postulates, that all the operators which represent the dynamical variables in the quantum theory have the same form, when they are expressed in terms of the canonical operators, as their classical counterparts. The consistency of this postulate with the assumed commutator structure must be checked separately in every case.

This approach to the quantization stresses the role of the operator algebra at a fixed time and it is best suited for the formulation of the quantum theory in the Schrödinger picture. The Heisenberg picture is obtained usually from the Schrödinger picture by applying the time dependent unitary automorphism to the operator algebra. The Schrödinger picture description is not very convenient in relativistic theories, since it does not take the full advantage of the space-time symmetry leading, in particular, to a complicated formalism for the scattering processes. The Heisenberg picture is better suited for the explicitly relativistic considerations, but its relation to the underlying classical theory was so far only indirect, via the Schrödinger picture.

In this paper we shall directly compare the time dependent quantum operators in the Heisenberg picture with the corresponding classical functions - the solutions of the classical equations of motion. In order to make such a direct comparison possible, we introduce the expectation values of quantum operators in the coherent states. We show, that these expectation values become the solutions of the classical equations in the limit $\hbar \to 0$. In the study of these expectation values we will make constant use of the explicit formula for the Heisenberg operators. The derivation of this formula in the non-relativistic particle mechanics is presented in Sec. 2. In Sec. 4 we extend our discussion to the scalar field theory and in Sec. 5 to quantum electrodynamics. The classical limits of the quantum mechanics and the quantum field theory are studied in Sec. 3 and Sec. 6 respectively. As an application of the general formalism, we study in Sec. 7 the gauge transformations of the field operators in the Heisenberg picture.

An explicit formula for the Heisenberg field operators was derived also by Symanzik,[1] but his formalism, based on the external source method, is less convenient for the study of the classical limit than our formalism based on the classical fields.

The classical limit of the quantum field theory has been studied recently by several authors[2] in connection with tree diagrams. The most thorough discussion of the tree diagrams was given by DeWitt.[3] However, all these papers were devoted to the study of the classical limit of the transition amplitudes, whereas we determine the classical limit of the matrix elements of field operators in the Fock space of the incoming particles.

2. Non-relativistic particle mechanics

This section is devoted to the study of the non-relativistic mechanical system having only one degree of freedom. The generalization to systems having many degrees of freedom is obvious.

Ordinarily one uses completely different mathematical objects to describe the states of the dynamical system in the classical and in the quantum theories. In classical mechanics the state of the system is fully described by giving the trajectory $x(t)$. In quantum mechanics we describe the state of the system by giving the state vector Ψ, but the information contained in the state vector is extracted with the help of various operators like $\hat{x}(t)$, $\hat{p}(t)$ etc. For example, the average trajectory is determined by the expectation value $\langle x(t)\rangle = (\Psi|\hat{x}(t)\Psi)$. In order to compare directly the classical and the quantum descriptions we must first formulate both theories in terms of the mathematical objects of the same type. We shall show, that the convenient objects in the quantum theory, which become in the limit the classical solutions of the equations of motion, are the expectation values of the quantum operators in the Heisenberg picture evaluated in the coherent state. These expectation values will be in the center of our discussion. We shall begin our exposition with a brief discussion of the coherent states in quantum mechanics.

The coherent states in quantum mechanics will be defined as usually as the eigenstates of the annihilation operator. The annihilation and creation operators will be defined in terms of the position and the momentum operators in the following general manner.

$$a = (2\hbar)^{-1/2}(\gamma^{-1}\hat{x} + i\delta^{-1}\hat{p}), \qquad (1a)$$

$$a^{+} = (2\hbar)^{-1/2}(\gamma^{*-1}\hat{x} - i\delta^{*-1}\hat{p}), \qquad (1b)$$

where γ and δ are complex constants with dimensions $(\text{mass/time})^{1/2}$ and $(\text{mass/time})^{-1/2}$ respectively, which obey the following condition

$$\mathrm{Re}(\gamma^{*}\delta) = 1 . \qquad (2)$$

The coherent state vectors will be denoted by Ψ_α, where α is the (complex) eigenvalue of the annihilation operator (1a),

$$a\Psi_\alpha = \alpha\Psi_\alpha \quad . \qquad (3)$$

The average values of the coordinate $\bar{x}$ and the momentum $\bar{p}$ of the particle in the state described by Ψ_α are related to the real and imaginary parts of $\alpha\gamma$ and $\alpha\delta$ through the formulas

$$\bar{x} = (2\hbar)^{1/2}\,\mathrm{Re}(\alpha\gamma) , \qquad (4a)$$

$$\bar{p} = (2\hbar)^{1/2}\,\mathrm{Im}(\alpha\delta) . \qquad (4b)$$

The wave functions of the coherent state in the coordinate and momentum representations have the following symmetric forms

$$\psi_\alpha(x) = |\gamma|^{-1/2}(\pi\hbar)^{-1/4}\exp[-\frac{1}{2\hbar|\gamma|^2}(1 + i\mathrm{Im}(\gamma^{*}\delta))(x - \bar{x})^2$$
$$+ \frac{i}{\hbar}\bar{p}x - \frac{i}{2\hbar}\overline{p}\,\overline{x}] , \qquad (5a)$$

$$\tilde{\psi}_\alpha(p) = |\delta|^{-1/2}(\pi\hbar)^{-1/4} \exp[-\frac{1}{2\hbar|\delta|^2}(1 + i\mathrm{Im}(\delta^*\gamma))(p - \bar{p})^2 \tag{5b}$$

$$- \frac{i}{\hbar}\bar{x}p + \frac{i}{2\hbar}\bar{p}\bar{x}] .$$

The central object under study in this section will be the expectation value of the position operator $\hat{x}(t)$ in the Heisenberg picture evaluated in the coherent state. This expectation value will be a function of the time parameter t and of the parameters $\bar{x}$ and $\bar{p}$, which characterize the coherent state. It will also depend implicitly on the choice of the canonically conjugate operators $\hat{x}$ and $\hat{p}$ in the definition (1a) of the annihilation operator. These can be chosen, for example, to be equal to the operators $\hat{x}(t)$ and $\hat{p}(t)$ at some fixed time t_0, but since we want to apply our formalism later to field theory, we will rather choose them to be the asymptotic (incoming) position and momentum operators $\hat{x}_{in}$ and $\hat{p}_{in}$ evaluated at $t = 0$,

$$\hat{p}_{in} \underset{\mathrm{df}}{\equiv} \lim_{t\to-\infty} (m \frac{d\hat{x}(t)}{dt}) , \tag{6a}$$

$$\hat{x}_{in} \underset{\mathrm{df}}{\equiv} \lim_{t\to-\infty} (\hat{x}(t) - \hat{p}_{in}t/m) . \tag{6b}$$

We shall also need the time dependent asymptotic position operator $\hat{x}_{in}(t)$, which can be defined in the usual manner through the equation

$$\hat{x}(t) = \hat{x}_{in}(t) + \int dt' \, G_R(t,t')m \frac{d^2\hat{x}(t')}{dt'^2} , \tag{7}$$

where G_R ,

$$G_R(t,t') = m^{-1}\theta(t-t')(t-t'), \tag{8}$$

is the retarded solution of the equation

$$m \frac{\partial^2}{\partial t^2} G_R(t,t') = \delta(t-t') . \tag{9}$$

It follows from Eqs. (6)-(8), that

$$\hat{x}_{in}(t) = \hat{x}_{in} + \hat{p}_{in} t/m . \tag{10}$$

The coherent states defined with the use of the in operators will be denoted by Ψ_α^{in}. Thus, we shall be dealing in this section with functions $\mathfrak{X}_q(t;\, x_{in},\, p_{in})$ defined in the following manner

$$\mathfrak{X}_q(t;\, x_{in},\, p_{in}) \underset{df}{=} (\Psi_\alpha^{in}|\hat{x}(t)\Psi_\alpha^{in}) . \tag{11}$$

Assuming the normal form of the Hamiltonian H for the quantum system,

$$H = p^2/2m + V(x) , \tag{12}$$

we can express the position operator in the Heisenberg picture in terms of the incoming position operator

$$\hat{x}(t) = U^\dagger(t, -\infty)\, \hat{x}_{in}(t)\, U(t, -\infty) , \tag{13}$$

where

$$U(t, t_0) = T \exp[-\frac{i}{\hbar} \int_{t_0}^{t} dt' V(\hat{x}_{in}(t'))] . \tag{14}$$

Due to the unitarity of the U operator, we can rewrite Eq. (13) in the form

$$\hat{x}(t) = \frac{1}{2} S^{\dagger} T(\hat{x}_{in}(t) S) + \frac{1}{2} \bar{T}(\hat{x}_{in}(t) S^{\dagger}) S , \tag{15}$$

where

$$S = U(\infty, -\infty) \tag{16}$$

and $\bar{T}$ denotes the antichronological ordering. The expression (15) will be used as the starting point in our study of the function $\mathcal{X}_q(t; x_{in}, p_{in})$.

In order to evaluate the expectation value (11) of the position operator we must convert the r.h.s. of Eq. (15) into the normally ordered product of the $\hat{x}_{in}(t)$ operators. The normal ordering is obtained, as usually, by expressing $\hat{x}_{in}(t)$ in terms of the annihilation and creation operators,

$$\hat{x}_{in}(t) = a_{in}(\hbar/2)^{1/2}(\gamma - i\delta t/m) + a^{\dagger}_{in}(\hbar/2)^{1/2}(\gamma^{*} + i\delta^{*} t/m), \tag{17}$$

and then bringing all the annihilation operators to the right. The expectation values of the normally ordered products of the $\hat{x}_{in}(t)$ operators in the coherent states can be easily evaluated with the following result

$$(\Psi^{in}_{\alpha}| : \hat{x}_{in}(t_1) \ldots \hat{x}_{in}(t_n) : \Psi^{in}_{\alpha}) = x_{in}(t_1) \ldots x_{in}(t_n) , \tag{18}$$

where

$$x_{in}(t) = x_{in} + p_{in} t/m = (2\hbar)^{1/2}(\mathrm{Re}(\alpha\gamma) + t\,\mathrm{Im}(\alpha\delta)/m) . \tag{19}$$

The conversion of the chronologically and antichronologically ordered products of the $\hat{x}_{in}(t)$ operators appearing in (15) into the normally ordered products will be accomplished with the help of Wick's lemma, which will be used here in a compact functional version given by Hori.[4] For an arbitrary functional $\mathfrak{F}[x]$ of $x(t)$ we have

$$T\mathfrak{F}[\hat{x}_{in}(\cdot)]$$

$$= :\exp\left(\int \hat{x}_{in} \frac{\delta}{\delta x}\right): \exp\left(-\frac{\hbar}{2i} \int \frac{\delta}{\delta x} G_F \frac{\delta}{\delta x}\right) \mathfrak{F}[x]\Big|_{x=0} \tag{20}$$

where we used the following abbreviations

$$\int \hat{x}_{in} \frac{\delta}{\delta x} = \int_{-\infty}^{\infty} dt\; \hat{x}_{in}(t) \frac{\delta}{\delta x(t)} \quad , \tag{21a}$$

$$\int \frac{\delta}{\delta x} G_F \frac{\delta}{\delta x} = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \; \frac{\delta}{\delta x(t)} G_F(t, t') \frac{\delta}{\delta x(t')} \quad , \tag{21b}$$

and G_F is another solution of Eq. (9) defined in complete analogy with the quantum field theory,

$$\frac{\hbar}{i} G_F(t, t') \underset{\overline{df}}{=} (\Psi_\alpha^{in} | \, T(\hat{x}_{in}(t)\, \hat{x}_{in}(t'))\Psi_\alpha^{in}) \; . \tag{22}$$

The coherent state, which is centered around the origin in both the coordinate and the momentum space, described by the state vector Ψ_0 ,

$$\Psi_0^{in} = \Psi_\alpha^{in}\Big|_{\alpha=0} \tag{23}$$

therefore, plays the role of the vacuum state in our formalism. The function G_F can be easily found to be

$$G_F(t, t') = \theta(t-t')(i/2)(\gamma - i\delta t/m)(\gamma^* + i\delta^* t'/m)$$

$$+ \theta(t'-t)(i/2)(\gamma^* + i\delta^* t/m)(\gamma - i\delta t'/m) \tag{24}$$

$$= (2m)^{-1}\varepsilon(t-t')(t-t') + (i/2)(|\gamma|^2 + |\delta|^2 tt'/m + \mathrm{Im}(\gamma^*\delta)(t+t')).$$

With the help of the formula (20) and its hermitian conjugate, we can rewrite the expression (15) in the form

$$\hat{x}(t) = :\exp\left(\int \hat{x}_{in} \frac{\delta}{\delta x_1}\right): \; :\exp\left(\int \hat{x}_{in} \frac{\delta}{\delta x_2}\right):$$

$$\times \exp\left(-\frac{\hbar}{2i}\int \frac{\delta}{\delta x_1} \bar{G}_F \frac{\delta}{\delta x_1}\right) \exp\left(\frac{\hbar}{2i}\int \frac{\delta}{\delta x_2} G_F \frac{\delta}{\delta x_2}\right) \tag{25}$$

$$\times \exp\left(\frac{i}{\hbar}\int [V(x_1) - V(x_2)]\right) \frac{1}{2}(x_1(t) + x_2(t))\Big|_{x_1=0=x_2} .$$

In order to cast this expression into a more manageable form, we will introduce a pair of new variables x and $\tilde{x}$ which are linear combinations of x_1 and x_2,

$$x_1(t) = x(t) + (\hbar/2)\,\tilde{x}(t) , \tag{26a}$$

$$x_2(t) = x(t) - (\hbar/2)\,\tilde{x}(t) . \tag{26b}$$

We can simplify then the formula (25) with the help of the following relation

$$
\begin{aligned}
&: \exp\left(\int \hat{x}_{in} \frac{\delta}{\delta x_1}\right) : \; : \exp\left(\int \hat{x}_{in} \frac{\delta}{\delta x_2}\right) : \\
&= \; : \exp\left(\int \hat{x}_{in}\left(\frac{\delta}{\delta x_1} + \frac{\delta}{\delta x_2}\right)\right) : \exp\left(\frac{\hbar}{i}\int \frac{\delta}{\delta x_1} G^{(+)} \frac{\delta}{\delta x_2}\right) \quad ,
\end{aligned}
\tag{27}
$$

where

$$
\frac{\hbar}{i} G^{(+)}(t,t') \underset{\overline{df}}{=} (\Psi_0^{in} |\; \hat{x}_{in}(t)\; \hat{x}_{in}(t')\; \Psi_0^{in}) \;. \tag{28}
$$

The final expression for $\hat{x}(t)$ reads

$$
\begin{aligned}
\hat{x}(t) \; &= \; : \exp\left(\int \hat{x}_{in} \frac{\delta}{\delta x}\right) : \exp\left(\frac{\hbar}{4}\int \frac{\delta}{\delta x} G^{(1)} \frac{\delta}{\delta x}\right) \exp\left(i\int \frac{\delta}{\delta x} G_R \frac{\delta}{\delta \tilde{x}}\right) \\
&\times \exp\left(\frac{i}{\hbar}\int [V(x + \hbar\tilde{x}/2) - V(x - \hbar\tilde{x}/2)]\right) x(t) \Big|_{x(t)=0=\tilde{x}(t)} \quad ,
\end{aligned}
\tag{29}
$$

where

$$
\begin{aligned}
\frac{\hbar}{i} G^{(1)}(t,\, t') \; &= \; (\Psi_0^{in} | \left\{ \hat{x}_{in}(t),\; \hat{x}_{in}(t') \right\} \Psi_0^{in}) \\
&= \; 2\mathrm{Im}\; G_F(t,\, t') \;.
\end{aligned}
\tag{30}
$$

Inserting (29) into the definition of $\mathcal{X}_q(t;\, x_{in},\, p_{in})$ and using the formula (18), we obtain finally

$$
\mathcal{X}_q(t;\, x_{in},\, p_{in}) \; = \; \mathcal{X}_q[t|x] \Big|_{x(t)=x_{in}+p_{in}t/m} \quad , \tag{31}
$$

where

$$\mathcal{X}_q[t|x] \underset{\mathrm{df}}{=} \exp(\frac{\hbar}{4}\int \frac{\delta}{\delta x} G^{(1)} \frac{\delta}{\delta x}) \exp(i\int \frac{\delta}{\delta x} G_R \frac{\delta}{\delta \tilde{x}})$$

$$\times \exp(\frac{i}{\hbar}\int [V(x + \hbar\tilde{x}/2) - V(x - \hbar\tilde{x}/2)]) \; x(t)\Big|_{\tilde{x}(t)=0} \quad . \tag{32}$$

The position operator in the Heisenberg picture can be reconstructed from the function $\mathcal{X}_q$ with the use of the formula

$$\hat{x}(t) = :\exp(\hat{x}_{in} \frac{\partial}{\partial x} + \hat{p}_{in} \frac{\partial}{\partial p}) : \mathcal{X}_q(t;\, x,\, p)_{\;x=0=p} \quad . \tag{33}$$

The expression (32) is particularly useful in the study of the classical limit, because all the dependence on the Planck constant is explicitly shown there.

3. The classical limit of the quantum mechanics

The transition to the classical limit can be very easily performed by letting $\hbar \to 0$ in the formula (32). The resulting function of t, x_{in} and p_{in} will be called $\mathfrak{X}_{c\ell}$.

$$\mathfrak{X}_{c\ell}(t;\, x_{in},\, p_{in}) \;\underset{\mathrm{df}}{\equiv}\; \mathfrak{X}_q(t;\, x_{in},\, p_{in})\Big|_{\hbar=0}$$

$$= \exp\left(i\int\frac{\delta}{\delta x}\, G_R\, \frac{\delta}{\delta\tilde{x}}\right) \exp\left(-i\int F(x)\tilde{x}\right) x(t)\Big|_{\substack{x(t)=x_{in}(t)\\ \tilde{x}(t)=0}} \quad , \tag{34}$$

where F is the force exerted on the particle,

$$F(x) = -\frac{\partial V(x)}{\partial\, x} \quad . \tag{35}$$

We shall show now, that $\mathfrak{X}_{c\ell}$ is the solution of the classical equation of motion,

$$m\,\frac{d^2x(t)}{dt^2} = F(x) \quad , \tag{36}$$

obeying the asymptotic condition

$$\lim_{t\to-\infty} \left(\mathfrak{X}_{c\ell}(t;\, x_{in},\, p_{in}) - x_{in}(t)\right) = 0 \;. \tag{37}$$

To prove this we shall make use of the following lemma.

Lemma:

The operation K,

$$K = \exp(i\int \frac{\delta}{\delta x} G_R \frac{\delta}{\delta \tilde{x}}) \exp(-i\int F(x)\tilde{x}\Big|_{\tilde{x}=0} \quad , \tag{38}$$

applied to two functionals of x(t), say $\mathfrak{F}[x]$ and $\mathfrak{G}[x]$, and the multiplication of these two functionals are interchangeable i.e.

$$K(\mathfrak{F}\cdot\mathfrak{G}) = K(\mathfrak{F})\cdot K(\mathfrak{G}) \ . \tag{39}$$

The proof of this lemma is given in the Appendix. Since K is clearly also a linear operation, we can prove by induction, that for every polynomial p(z)

$$K(p(\mathfrak{F})) = p(K(\mathfrak{F})) \ . \tag{40}$$

This relation will also hold for sufficiently regular functions f(z),

$$K(f(\mathfrak{F})) = f(K(\mathfrak{F})) \quad . \tag{41}$$

We shall employ now the following relation

$$\begin{aligned} &\exp(i\int \frac{\delta}{\delta x} G_R \frac{\delta}{\delta x}) \ x(t) \ \exp(-i\int \frac{\delta}{\delta x} G_R \frac{\delta}{\delta \tilde{x}}) \\ &= x(t) + i\int dt' \ G_R(t,t') \frac{\delta}{\delta \tilde{x}(t')} \end{aligned} \tag{42}$$

in order to rewrite the formula (34) for $\mathfrak{X}_{c\ell}$ in the form

$$\mathfrak{X}_{c\ell}(t;\, x_{in},\, p_{in}) = x_{in}(t)K(1) + K(\int dt' \ G_R(t,t')F(x(t')))\Big|_{x(t)=x_{in}(t)} \tag{43}$$

Finally, with the help of the lemma, we obtain the following integral equation for $\mathfrak{X}_{c\ell}$

$$\not{x}_{c\ell}(t) = x_{in}(t) + \int dt' \, G_R(t,t') \, F(\not{x}_{c\ell}(t')) \, . \tag{44}$$

The solution of this equation satisfies Newton's equation (36) and the asymptotic condition[5] (37).

Thus we have shown, that the function $\not{x}_q(t;\, x_{in},\, p_{in})$ describes the quantum analogue of the classical trajectory and that it becomes exactly the classical trajectory in the limit when $\hbar \to 0$. Since the function $G^{(1)}$ depends on the parameters γ and δ which appear in the definition of the coherent state Ψ_α, the quantum analogue of the trajectory is not uniquely defined. For example, the lowest order quantum correction to the classical trajectory has the form

$$\begin{aligned} \not{x}_q(t;\, x_{in},\, p_{in}) &\simeq \not{x}_{c\ell}(t;\, x_{in},\, p_{in}) \\ &+ \frac{\hbar}{4}\left(|\gamma|^2 \frac{\partial^2}{\partial x_{in}^2} + |\delta|^2 \frac{\partial^2}{\partial p_{in}^2} + 2\mathrm{Im}(\gamma^*\delta)\frac{\partial^2}{\partial x_{in}\partial p_{in}}\right) \not{x}_{c\ell}(t;\, x_{in},\, p_{in}) \, , \end{aligned} \tag{45}$$

where the dependence on γ and δ is explicitly shown. As will be seen in the next section, there is no ambiguity of this type in the field theory, where the particle interpretation makes it possible to separate unambiguously the field into the annihilation and creation parts.

4. Scalar field theory

All the results obtained in Secs. 2 and 3 in the nonrelativistic quantum mechanics can be generalized without any difficulty to the relativistic field theory. In this section we shall consider a general relativistic field theory of one real scalar field. The Lagrangian density in this case has the form

$$\mathcal{L}(\phi) = \tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi - \kappa^2\phi^2) - \mathcal{H}(\phi) , \tag{46}$$

where $\kappa = mc/\hbar$ is the inverse of the Compton wave length and the interaction energy density $\mathcal{H}$ can be chosen to be any function of ϕ. The classical field equation derived from this Lagrangian has the form

$$(\Box + \kappa^2)\phi(x) = j(\phi(x)) , \tag{47}$$

where

$$j(\phi) = -\frac{\partial\mathcal{H}(\phi)}{\partial\phi} . \tag{48}$$

The solution $\Phi_{c\ell}(x)$ of this classical field equation will be characterized by the Fourier transform $f(\underline{k})$ of the incoming field $\phi_{in}(x)$

$$\Phi_{c\ell}(x) = \Phi_{c\ell}[x|f] , \tag{49}$$

$$\phi_{in}(x) = \phi_{in}[x|f] \underset{\text{df}}{=} \hbar^{1/2}\int d\Gamma(f(\underset{\sim}{k})e^{-ik\cdot x} + f^*(\underset{\sim}{k})e^{ik\cdot x}) , \tag{50}$$

where $d\Gamma$ is the invariant volume element of the mass hyperboloid

$$d\Gamma = (2\pi)^{-3}(2\omega(\underset{\sim}{k}))^{-1}\, d^3k . \tag{51}$$

In complete analogy with the particle mechanics, the classical field $\Phi_{c\ell}$ will be the solution of the integral equation

$$\Phi_{c\ell}(x) = \phi_{in}(x) + \int dy\ \Delta_R(x-y)j(\Phi_{c\ell}(y)) \ . \tag{52}$$

The quantum field functional $\Phi_q[x|f]$ will be defined as the following expectation value

$$\Phi_q[x|f] = (\Psi_f^{in}|\hat{\phi}(x)\Psi_f^{in}) \quad , \tag{53}$$

where $\hat{\phi}(x)$ is the field operator in the Heisenberg picture and Ψ_f^{in} is the coherent state vector of the free incoming field $\hat{\phi}_{in}(x)$,

$$\Psi_f^{in} = \exp(-\frac{1}{2}\int d\Gamma|f(\underset{\sim}{k})|^2)\ \exp(\int d\Gamma f(\underset{\sim}{k})a_{in}^{\dagger}(\underset{\sim}{k}))\Omega \quad . \tag{54}$$

The function f will be normalized according to the condition

$$\int d\Gamma|f(\underset{\sim}{k})|^2 = N \quad , \tag{55}$$

where N is a dimensionless constant, which can be interpreted as the average number of the quanta present in the field.

In order to evaluate the quantum functional we should follow exactly the same procedure, which had led us before to the final formula (32) for $\bar{x}_q$ in the particle mechanics. We shall not repeat these calculations here, because it would amount only to changes in the notation. The final formulas read

$$\Phi_q[x|f] = \Phi_q[x|\phi]\Big|_{\phi=\phi_{in}[x|f]} \quad , \tag{56}$$

where

$$\Phi_q[x|\phi] = Q(\phi(x)) \tag{57}$$

and the operation Q is defined in the following way

$$Q = \exp(\frac{\hbar}{4}\int \frac{\delta}{\delta\phi} \Delta^{(1)} \frac{\delta}{\delta\phi}) \exp(i\int\frac{\delta}{\delta\phi} \Delta_R \frac{\delta}{\delta\tilde{\phi}})$$
$$\times \exp(\frac{i}{\hbar}\int[\mathcal{H}(\phi + \hbar\tilde{\phi}/2) - \mathcal{H}(\phi - \hbar\tilde{\phi}/2)]\Big|_{\tilde{\phi}(x)=0} . \tag{58}$$

The propagators $\Delta^{(1)}$ and Δ_R are defined as usually,

$$\frac{\hbar}{i}\Delta^{(1)}(x-y) = (\Omega|\left\{\hat{\phi}_{in}(x), \hat{\phi}_{in}(y)\right\}\Omega) \quad , \tag{59a}$$

$$\frac{\hbar}{i}\Delta_R(x-y) = \theta(t-t')(\Omega|[\hat{\phi}_{in}(x), \hat{\phi}_{in}(y)]\Omega) \quad . \tag{59b}$$

The formulas (57) and (58) define a natural off-mass-shell extension of the expectation value (53). We have used the same symbol $\Phi_q[x|\cdot]$ to denote both the expectation value $\Phi_q[x|f]$ and its extension $\Phi_q[x|\phi]$. This should not lead to a confusion, because the argument of the functional will always indicate which is the case.

We can easily obtain similar formulas for the symmetrized products of the field operators. The expectation value of such a product in the coherent state can be obtained from the n-point quantum functional $\Phi_q[x_1 \ldots x_n|\phi]$,

$$\Phi_q[x_1 \ldots x_n|\phi] = Q(\phi(x_1)\ldots\phi(x_n)) \quad . \tag{60}$$

For example,

$$(\Psi_f^{in}|\left\{\hat{\phi}(x), \hat{\phi}(y)\right\}\Psi_f^{in}) = \Phi_q[x,y|\phi]\Big|_{\phi=\phi_{in}[x|f]} \quad . \tag{61}$$

5. The classical limit, the tree diagrams and Dyson's double diagrams

In this section we shall restrict ourselves to the discussion of the two simplest models of the scalar field theory, namely, those described by the interaction Lagrangians $\lambda\phi^3$ and $\lambda\phi^4$. In these two cases there exists a simple procedure based on a diagramatic technique, which enables one to write down the perturbation expansion for the functional Φ_q. Similar, although relatively more complicated procedures can be developed for other interaction Lagrangians, but since the perturbation expansion is meaningful only for renormalizable interactions, we shall not go beyond the $\lambda\phi^4$ coupling.

In the simpler case of $\lambda\phi^3$, the formula (58) for Q can be rewritten in the form

$$Q = \exp\left(\frac{\hbar}{4}\int \frac{\delta}{\delta\phi}\, \Delta^{(1)}\, \frac{\delta}{\delta\phi}\right) \exp\left(i\int\frac{\delta}{\delta\phi}\, \Delta_R\, \frac{\delta}{\delta\tilde{\phi}}\right)$$
$$\times \exp\left(\frac{i\lambda\hbar^2}{4}\int\tilde{\phi}^3\right) \exp\left(3i\lambda\int \phi^2\tilde{\phi}\right)\Big|_{\tilde{\phi}=0} \quad . \tag{62}$$

The differentiation with respect to $\tilde{\phi}$ can be absorbed in this case into $\Phi_{c\ell}$ and the following formula for Φ_q results

$$\Phi_q[x|\phi] = \exp\left(\frac{\hbar}{4}\int \frac{\delta}{\delta\phi}\, \Delta^{(1)}\, \frac{\delta}{\delta\phi}\right)$$
$$\times \exp\left(\frac{i\lambda\hbar^2}{4}\int dz dx_1 dx_2 dx_3\, \frac{\delta}{\delta\phi(x_1)}\, \Delta_R(x_1-z)\, \frac{\delta}{\delta\phi(x_2)}\, \Delta_R(x_2-z)\, \frac{\delta}{\delta\phi(x_3)}\, \Delta_R(x_3-z)\Phi_{c\ell}[x|\phi]\right), \tag{63}$$

where, as before, $\Phi_{c\ell}$ is the limit of Φ_q when $\hbar \to 0$,

$$\Phi_{c\ell}[x|\phi] = \exp(i\int \frac{\delta}{\delta\phi}\, \Delta_R\, \frac{\delta}{\delta\tilde{\phi}})\, \exp(3i\lambda\int \phi^2\tilde{\phi})\phi(x)\Big|_{\tilde{\phi}=0} \quad . \tag{64}$$

In taking the limit $\hbar \to 0$ we have kept the Compton wave length κ fixed, so that the classical field theory described by Eqs. (47) and (48) was obtained. This classical limit can be thought as the one corresponding to the strong field regime, because the average number of field quanta must infinitely increase with decreasing $\hbar$ (cf. Eq. (55)), if the field is to survive the limiting process. This is obviously a different classical limit than the second one, in which the mass of the particle associated with the field is kept fixed, so that the classical theory of particles is obtained. In this second limit the field is completely destroyed.

In the case of the $\lambda\phi^4$ coupling the resulting expression for Φ_q reads

$$\Phi_q[x|\phi] = \exp(\frac{\hbar}{4}\int \frac{\delta}{\delta\phi}\, \Delta^{(1)}\, \frac{\delta}{\delta\phi})$$

$$\times\ L\, \exp(\frac{i\lambda\hbar^2}{2}\int dz dx_1 dx_2 dx_3 \phi(z)\, \frac{\delta}{\delta\phi(x_1)}\, \Delta_R(x_1-z)\, \frac{\delta}{\delta\phi(x_2)}\, \Delta_R(x_2-z)\, \frac{\delta}{\delta\phi(x_3)}\, \Delta_R(x_3-z))$$

$$\times\ \Phi_{c\ell}[x|\phi] \ , \tag{65}$$

Where L denotes the ordering operation, which places all the functional derivatives to the left of all ϕ's, so that all the field variables will be subject to the differentiation.

The general structure of both expressions (63) and (65) is very similar. In each case there appears the classical solution $\Phi_{c\ell}$ of the field equation and there are two exponential operations acting on this classical solutions and converting it into the quantum functional Φ_q. Even though these exponential operations in the formulas for Φ_q can not be carried out effectively, these formulas are very useful in the study of the perturbation expansion of Φ_q. In order to obtain such an expansion, we must first expand the classical solution into the power series in λ. This can be done either by solving Eq. (52) by iteration or by expanding the exponential function in the formula (64). In the n-th order of the expansion we obtain a collection of terms, each of them containing a product of n retarded propagators and n + 1 or 2n + 1 fields , depending on the type of the coupling. Every such term can be represented graphically by a tree diagram and vice versa every tree diagram constructed according to certain rules gives rise to one term in the expression for $\Phi_{c\ell}$. These rules vary with the form of the interaction Lagrangian. Every tree diagram consists of the trunk (corresponding to the $\Delta_R(x-x_1)$ function), the branches (corresponding to the $\Delta_R(x_i-x_j)$ functions) and the twigs (corresponding to the $\phi(x_i)$ fields). Depending on the form of the interaction Lagrangian ($\lambda\phi^3$ or $\lambda\phi^4$), three or four branches and/or twigs meet at every branching point. An example of the tree diagram for the $\lambda\phi^4$ coupling is given on Fig. 1. All possible topologically different tree diagrams with n branching points must be taken into account in order to obtain the total contribution to $\Phi_{c\ell}$ in the n-th order in λ. The product of the Δ_R functions and fields corresponding to a given diagram is to be integrated over 4n space-time variables x_i. There is also a numerical coefficient, say N_D; multiplying the contribution from every diagram. This numerical coefficient depends on the interaction Lagrangian. For the $\lambda\phi^4$ theory this coefficient is

$$N_D = (4)^{n_1} (4\cdot3)^{n_2} (4\cdot3\cdot2)^{n_3} (4\cdot3\cdot2\cdot1)^{n_4} , \tag{66}$$

where n_i is the number of the branching points, at which i branches and 4-i twigs meet.

Differential operations acting on $\Phi_{c\ell}$ also can be described in terms of certain graphical rules.

Let us consider first the operation $(\hbar/4)\int(\delta/\delta\phi)\Delta^{(1)}(\delta/\delta\phi)$. When acting on any product of the ϕ fields it removes a pair of fields, say $\phi(x_i)$ and $\phi(x_j)$, in all possible ways and replaces them by $(\hbar/2)\Delta^{(1)}(x_i-x_j)$. In the graphical language, two twigs are connected together and the line which is thus formed will represent the function $(\hbar/2)\Delta^{(1)}(x_i-x_j)$. This operation will be called the line contraction. The n-th order term of the exponential operation $\exp((\hbar/4)\int(\delta/\delta\phi)\Delta^{(1)}(\delta/\delta\phi))$ will give rise to n line contractions carried out in all possible ways. However, the twigs that emerge from the same branching point are not to be contracted. This restriction corresponds to the normal ordering prescription (removal of the zero point interaction energy) for the interaction energy density.

The second differential operation also can be given a simple graphical interpretation. For the $\lambda\phi^4$ theory the operation

$$(3i\lambda\hbar^2/4)\int\phi(\delta/\delta\phi)\Delta_R(\delta/\delta\phi)\Delta_R(\delta/\delta\phi)\Delta_R$$

removes three fields, say $\phi(x_i)$, $\phi(x_j)$ and $\phi(x_k)$ in all possible ways and replaces them by the expression

$$\int dz\Delta_R(x_i-z)\Delta_R(x_j-z)\Delta_R(x_k-z)\phi(z) .$$

This operation will be called the vertex contraction. In the graphical language, the vertex contraction adds one new branching point to the diagram, connects three existing twigs to this vertex replacing them at the same time by the branches and adds one new twig at this new branching point. The n-th order term of the exponential operation will give rise to n vertex contractions, which should be carried out in all possible ways. However, if two twigs, which emerge from the same branching point, are contracted in the same vertex contraction, the resulting contribution vanishes, because

$$\Delta_R(\vec{x}, t)\Big|_{t=0} = 0 . \tag{67}$$

Therefore, the vertex contraction contributes for the first time to Φ_q in the 4-th order of perturbation theory.

Similar rules for representing the perturbative expansion in terms of the diagrams can be given in the $\lambda\phi^3$ theory. One can also give analogous rules for the calculation of the n-point function $\Phi_q[x_1 \ldots x_n|\phi]$ in perturbation theory.

Our rules for representing the perturbative expansion of the field operators in terms of the diagrams differ from the rules given by Dyson,[6] but they are equivalent to the rules derived by Symanzik[1] with the use of the external current techniques.[7]

6. Quantum electrodynamics

In this section we shall discuss quantum electrodynamics within the framework which has been developed for the treatment of the scalar field. In order to apply our formalism to quantum electrodynamics, we only must generalize it to the case of the anticommuting spinor field operators.

First we define formal coherent states of the electron field with the help of anticommuting functions $g^{(\pm)}(\underset{\sim}{k}, r)$,

$$\Psi_g^{in} = \exp(-\frac{1}{2}\sum_r \int d\Gamma(g^{(+)*}(\underset{\sim}{k},r)g^{(+)}(\underset{\sim}{k}, r) + g^{(-)*}(\underset{\sim}{k},r)(g^{(-)}(\underset{\sim}{k},r))$$
$$\times \exp(\sum_r \int d\Gamma(g^{(+)}(\underset{\sim}{k}, r)a^{\dagger}_{in}(\underset{\sim}{k}, r) + g^{(-)}(\underset{\sim}{k}, r)b^{\dagger}_{in}(\underset{\sim}{k}, r))\Omega \ . \tag{68}$$

The functions $g^{(\pm)}$ and $g^{(\pm)*}$ are assumed to anticommute with themselves and with the creation and annihilation operators.

In order to avoid all known difficulties in the formulation of the theory, we will give photons a small mass. The coherent states of the photon field will have the form

$$\Psi_f^{in} = \exp(-\frac{1}{2}\sum_r \int d\Gamma|f(\underset{\sim}{k}, r)|^2)$$
$$\times \exp(\sum_r \int d\Gamma f(\underset{\sim}{k}, r)\ c^{\dagger}_{in}(\underset{\sim}{k}, r))\Omega \ . \tag{69}$$

Combining together the formulas (68) and (69) we can easily produce simultaneous coherent states of the electron and the photon fields. They will be denoted by $\Psi^{in}_{f,g}$.

We can proceed now in exactly the same way, as we did in the case of the scalar field, to derive the following formula for the quantum functionals Ψ_q, $\bar{\Psi}_q$ and A_q.

$$\begin{Bmatrix} \Psi_q[x|\psi,\ \bar{\psi},\ a] \\ \\ \bar{\Psi}_q[y|\psi,\ \bar{\psi},\ a] \\ \\ A_q[z|\psi,\ \bar{\psi},\ a] \end{Bmatrix} = Q \begin{Bmatrix} \Psi_{c\ell}[x|\psi,\ \bar{\psi},\ a] \\ \\ \bar{\Psi}_{c\ell}[y|\psi,\ \bar{\psi},\ a] \\ \\ A_{c\ell}[z|\psi,\ \bar{\psi},\ a] \end{Bmatrix}, \tag{70}$$

where the operation Q has now the form

$$Q = \exp\left(\frac{\hbar}{2}\int \frac{\delta}{\delta\psi}\, s^{(1)}\, \frac{\delta}{\delta\bar{\psi}}\right) \exp\left(\frac{\hbar}{4}\int \frac{\delta}{\delta a_\mu}\, \Delta^{(1)}_{\mu\nu}\, \frac{\delta}{\delta a_\nu}\right)$$
$$\times \exp\left(i\varepsilon\hbar^2\int dz\,dz'\,dx\,dy\, \frac{\delta}{\delta a_\mu(z')}\ \Delta^R_{\mu\nu}(z'-z)\, \frac{\delta}{\delta\psi(x)}\, s_R(x-z)\gamma^\nu s_A(z-y)\, \frac{\delta}{\delta\bar{\psi}(y)}\right) \tag{71}$$

and $\Psi_{c\ell}$, $\bar{\Psi}_{c\ell}$ and $A_{c\ell}$ are the solutions of "classical" field equations in the integral form.

$$\Psi_{c\ell}(x) = \psi(x) + \varepsilon\int dx'\ S_R(x-x')\gamma\cdot A_{c\ell}(x')\Psi_{c\ell}(x'), \tag{72a}$$

$$\bar{\Psi}_{c\ell}(y) = \bar{\psi}(y) + \varepsilon\int dy'\ \bar{\Psi}_{c\ell}(y')\gamma\cdot A_{c\ell}(y')\ S_A(y'-y), \tag{72b}$$

$$A_{c\ell}(z) = a(z) + \varepsilon\int dz'\ \Delta_R(z-z')\ \bar{\Psi}_{c\ell}(z')\gamma\Psi_{c\ell}(z'). \tag{72c}$$

In order to simplify the notation we have suppressed the vector and spinor indices. Functional derivatives with respect to the anticommuting fields have been widely used in the literature. We have adopted here the usual convention, that $\delta/\delta\bar{\psi}$ are the right derivatives (acting to the right) and $\delta/\delta\psi$ are the left derivatives (acting to the left).

The expectation values of the field operators in the coherent states are obtained from the quantum functionals Ψ_q, $\bar{\Psi}_q$ and A_q by evaluating them on the mass shell

$$(\Psi^{in}_{f,g}|\hat{\psi}(x)\ \Psi^{in}_{f,g}) = \Psi_q[x|\psi,\ \bar{\psi},\ a]\Big|_{\psi=\psi_{in},\ \bar{\psi}=\bar{\psi}_{in},\ a=a_{in}}\ , \tag{73}$$

etc.

We can also give closed formulas for the "classical" fields in the complete analogy with the scalar field case. The operation K, which converts the fields ψ, $\bar{\psi}$ and a onto the classical functionals $\Psi_{c\ell}$, $\bar{\Psi}_{c\ell}$ and $A_{c\ell}$ has the form.

$$\begin{aligned} K = {} & \exp(i\int \frac{\delta}{\delta\psi} S_R \frac{\delta}{\delta\tilde{\bar{\psi}}})\ \exp(i\int \frac{\delta}{\delta\tilde{\psi}} S_A \frac{\delta}{\delta\bar{\psi}})\ \exp(i\int \frac{\delta}{\delta a_\mu} \Delta^R_{\mu\nu} \frac{\delta}{\delta\tilde{a}_\nu}) \\ & \times \exp(i \int(\tilde{\bar{\psi}}\gamma\cdot a\psi + \tilde{\bar{\psi}}\gamma\cdot a\psi + \bar{\psi}\gamma\cdot a\tilde{\psi}))\Big|_{\tilde{\psi}=0=\tilde{\bar{\psi}}\ ,\ \tilde{a}=0}\ . \end{aligned} \tag{74}$$

The iterative solutions of Eqs. (72) can be again represented graphically with the help of the tree diagrams. The exponential operations, which appear in the definition (71) of the Q operation, give rise to the electron and photon line contractions and to the vertex contractions. These contractions are to be carried out in all possible ways on the tree diagrams. The appropriate rules relating the diagrams to the corresponding contributions to the quantum functionals Ψ_q, $\bar{\Psi}_q$ and A_q can be easily worked out.

Formal, as all these results may seem to be, they do enable us to write down easily the expressions for the field operators in terms of the incoming fields in perturbation theory and also to study various general properties of such expansions. One such application of this formalism will be given in the next section.

We have put the term "classical" in quotation marks, when we referred to the solutions of Eqs. (69), because apart from the formal analogies, there is very little there that would justify the use of these terms. Not only the Compton wave length, instead of the mass, but also $\varepsilon = e/\hbar$, instead of the charge e, are kept fixed in this limit. We were forced to choose this limiting procedure, since we have insisted on having smooth fields in the limit, whereas in the true classical limit, the expectation values of the fields must become highly singular. In particular the expectation values of the current operator and the energy-momentum operator must develop δ-function singularities

$$(\Psi|\hat{j}^{\mu}(x)\Psi) \xrightarrow[\hbar \to 0]{} e\int ds\, \frac{d\xi^{\mu}}{ds}\, \delta_{(4)}(x-\xi(s)) \;, \tag{75a}$$

$$(\Psi|\hat{T}^{\mu\nu}(x)\Psi) \xrightarrow[\hbar \to 0]{} m\int ds\, \frac{d\xi^{\mu}}{ds}\frac{d\xi^{\nu}}{ds}\, \delta_{(4)}(x-\xi(s)) \;. \tag{75b}$$

The study of these limits is beyond the scope of the present paper. We intend to return to this problem in a future publication.

7. Gauge transformations of the field operators in quantum electrodynamics

The explicit formulas given in the previous section for the field functionals will be now used to study the relations between the field operators in different gauges. The most interesting example of such a transformation is the one which leads from the Proca gauge to the Feynman gauge. The transformation formula for the fields will be derived from an identity satisfied by the solutions of the classical field equations. In order to derive this identity, we shall write first the field equations in the Proca gauge.

$$(D_x + \epsilon\gamma\cdot A_{c\ell})\Psi_{c\ell} = D_x\psi \quad , \tag{76a}$$

$$\bar{\Psi}_{c\ell}(\overleftarrow{D}_y + \epsilon\gamma\cdot A_{c\ell}) = \bar{\psi}\overleftarrow{D}_y \quad , \tag{76b}$$

$$K^{\mu\nu}A_\nu^{c\ell} - \bar{\Psi}_{c\ell}\gamma^\mu\Psi_{c\ell} = K^{\mu\nu}a_\nu \quad , \tag{76c}$$

where

$$D_x \underset{\mathrm{df}}{=} -i\gamma\cdot\partial + \kappa \quad , \tag{77a}$$

$$\overleftarrow{D}_y \underset{\mathrm{df}}{=} i\gamma\cdot\overleftarrow{\partial} + \kappa \quad , \tag{77b}$$

$$K^{\mu\nu} \underset{\mathrm{df}}{=} (\partial_\lambda\partial^\lambda + \mu^2)g^{\mu\nu} - \partial^\mu\partial^\nu \quad . \tag{77c}$$

The inhomogeneous terms appearing on the r.h.s. of Eqs. (76) are present there, because we can not set ψ, $\bar{\psi}$ and a_μ to be free fields before all the functional derivatives with respect to those fields are evaluated.

Next, let us suppose that the fields ψ, $\bar{\psi}$ and a_μ are subject to the following changes

$$\psi(x) \to {}'\psi(x) = e^{-i\varepsilon\Lambda(x)}\psi(x) + \varepsilon\int dy\ S_R(x-y)\gamma\cdot\partial\Lambda(y)e^{-i\varepsilon\Lambda(y)}\psi(y), \tag{78a}$$

$$\bar{\psi}(x) \to {}'\bar{\psi}(x) = \bar{\psi}(x)e^{i\varepsilon\Lambda(x)} + \varepsilon\int dy\ \bar{\psi}(y)e^{i\varepsilon\Lambda(y)}\gamma\cdot\partial\Lambda(y)\ S_A(y-x), \tag{78b}$$

$$a(z) \to {}'a(z) = a(z) + \partial\Lambda(z), \tag{78c}$$

where $\Lambda(z)$ is an arbitrary function of z. With the help of the field equations (76) we can derive the following transformation properties for the classical field functionals under the transformation (78) of their arguments.

$$\Psi_{c\ell}[x|{}'\psi,{}'\bar{\psi},{}'a] = e^{-i\varepsilon\Lambda}\Psi_{c\ell}[x|\psi,\bar{\psi},a], \tag{79a}$$

$$\bar{\Psi}_{c\ell}[y|{}'\psi,{}'\bar{\psi},{}'a] = e^{i\varepsilon\Lambda}\bar{\Psi}_{c\ell}[y|\psi,\bar{\psi},a], \tag{79b}$$

$$A_{c\ell}[z|{}'\psi,{}'\bar{\psi},{}'a] = A_{c\ell}[z|\psi,\bar{\psi},a] + \partial\Lambda(z). \tag{79c}$$

Differentiating functionally both sides of these relations with respect to $\Lambda(z)$, we obtain the following equation for $\Psi_{c\ell}$

$$\int dy\left[\frac{\delta}{\delta{}'\psi(y)}\frac{\delta{}'\psi(y)}{\delta\Lambda(z)} + \frac{\delta{}'\bar{\psi}(y)}{\delta\Lambda(z)}\frac{\delta}{\delta{}'\bar{\psi}(z)} + \frac{\delta{}'a(y)}{\delta\Lambda(z)}\frac{\delta}{\delta{}'a(y)}\right]\Psi_{c\ell}[x|{}'\psi,{}'\bar{\psi},{}'a]$$
$$= -i\epsilon\delta(x-z)\ e^{-i\epsilon\Lambda(x)}\ \Psi_{c\ell}[x|\psi,\bar{\psi},a] \tag{80}$$

and two analogous equations for $\bar{\Psi}_{c\ell}$ and $A_{c\ell}$.

In this way we arrive at the final set of identities which are obeyed by the classical fields.

$$\left(G(z) - \partial_\mu \frac{\delta}{\delta a_\mu(z)}\right) \Psi_{c\ell}(x) = -i\epsilon\delta(x-z)\Psi_{c\ell}(x) , \tag{81a}$$

$$\left(G(z) - \partial_\mu \frac{\delta}{\delta a_\mu(z)}\right) \bar{\Psi}_{c\ell}(y) = i\epsilon\delta(y-z)\, \bar{\Psi}_{c\ell}(y) , \tag{81b}$$

$$\left(G(z) - \partial_\mu \frac{\delta}{\delta a_\mu(z)}\right) A_\lambda^{c\ell}(z') = \partial_\lambda \delta(z-z') , \tag{81c}$$

where

$$\begin{aligned} G(z) \underset{\overline{\overline{df}}}{} \; & -i\epsilon\int dy \frac{\delta}{\delta\psi(y)} S_R(y-z)D_z\psi(z) \\ & + i\epsilon\int dy\; \bar{\psi}(z)\overset{\leftarrow}{D}_z S_A(z-y)\frac{\delta}{\delta\bar{\psi}(y)} \quad . \end{aligned} \tag{82}$$

With the use of these identities we can relate the quantum field functionals evaluated in one gauge to the functionals evaluated in another gauge; the gauge being determined in this case by the form of the photon propagators. The photon propagators can be written in the form

$$\Delta_{\mu\nu}^{(\ )}(z-z') = d_{\mu\nu}\, \Delta^{(\ )}(z-z') , \tag{83}$$

where the tensor $d_{\mu\nu}$ specifies the gauge and $\Delta^{(\ )}$ is the corresponding propagator for the scalar field. In the Proca gauge

$$\text{Proca:} \quad d_{\mu\nu} = -g_{\mu\nu} + \mu^{-2}\partial_\mu\partial_\nu \tag{84}$$

and in the Feynman gauge

$$\text{Feynman:} \quad d_{\mu\nu} = -g_{\mu\nu} \quad . \tag{85}$$

The transformation formula for the field functionals off-mass-shell, which is given below, is rather complicated, but it simplifies significantly when the fields ψ and $\bar{\psi}$ are put on the mass shell.

$$Q(\text{Proca})\begin{Bmatrix} \Psi_{c\ell}(x) \\ \\ \bar{\Psi}_{c\ell}(x) \end{Bmatrix} = \exp((\hbar\varepsilon^2/4\mu^2)\Delta^{(1)}(0))$$

$$\times \quad \exp(-(\hbar/4\mu^2)\int G\Delta^{(1)}G) \; \exp(\mp(i\varepsilon\hbar/2\mu^2)\int dz\Delta^{(1)}(x-z)G(z)) \tag{86a}$$

$$\times \quad \exp(\pm(i\varepsilon^2\hbar^2/\mu^2)\int dz\Delta_R(x-z)H(z))$$

$$\times \quad \exp((\hbar^2/\mu^2)\int G\Delta_R H)Q(\text{Feynman})\begin{Bmatrix} \Psi_{c\ell}(x) \\ \bar{\Psi}_{c\ell}(x) \end{Bmatrix} \quad ,$$

$$Q(\text{Proca})\; A_{c\ell}(z) = Q(\text{Feynman})\; A_{c\ell}(z) \quad , \tag{86b}$$

where

$$H(z) = \int dy\Big[\frac{\delta}{\delta\psi(y)}S_R(y-z)\frac{\delta}{\delta\bar{\psi}(z)} - \frac{\delta}{\delta\psi(z)}S_A(z-y)\frac{\delta}{\delta\bar{\psi}(y)}\Big] . \tag{87}$$

Three exponential functions out of five present in Eq. (86a) disappear when ψ and $\bar{\psi}$ satisfy the free Dirac equation, because then $G = 0$. Eqs. (86) describe changes in the form of the field functionals, which are due to the change in the photon propagators. In addition we should also use different free photon fields when we evaluate these functionals on the mass shell. In the Proca gauge free photon field satisfies the equation

$$K^{\mu\nu}a_{\nu}(z) = 0 \tag{88}$$

whereas in the Feynman gauge we should use the solution of the Klein-Gordon equation

$$(\Box + \mu^2)a_{\mu}(z) = 0 \ . \tag{89}$$

The most interesting feature of the transformation formula (86) is that the relationship between the asymptotic field operators in and out, which can be derived from it by taking the (weak) limits $t \to \pm\infty$, does not depend on the gauge. The first of two exponentials which contribute on the mass shell produces just change in the renormalization constant and the second does not contribute in the limit $t \to \pm\infty$, because of the dependence on x in the exponent. This implies the invariance of the S operator under the change of the gauge.

One can also use the transformation formulas (86) to show, that if an expression constructed from classical fields is gauge invariant, like for example $\bar{\psi}(x)\gamma\psi(x)$ or $\bar{\psi}(x)(\partial + i\varepsilon A(x))\psi(x)$, then the corresponding quantum operator is formally invariant under the gauge transformation of the photon propagator (apart from the change in the renormalization constants). However, due to the singular behavior of the product of the field operators at coinciding points, in the correct proof of the gauge invariance we should rather start from non-local gauge invariant classical expressions of the type

$$j^{\mu}(x, \eta) = \bar{\psi}(x + \eta)\gamma^{\mu} \exp(-i\epsilon \int_{x}^{x+\eta} d\xi^{\lambda}A_{\lambda}(\xi))\psi(x) \tag{90}$$

in order to obtain finite operators in the limit $\eta \to 0$.

Appendix

In order to prove the lemma we shall first expand the operation K into a double series and carry out all the differentiations with respect to $\tilde{x}$.

$$K(\mathfrak{F}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int dt_1 \ldots dt_n dt_1' \ldots dt_n'$$

$$\times \frac{\delta}{\delta x(t_1)} G_R(t_1, t_1') \ldots \frac{\delta}{\delta x(t_n)} G_R(t_n, t_n') F(x(t_1')) \ldots F(x(t_n'))\mathfrak{F}[x] \ , \tag{A.1}$$

or symbolically

$$K(\mathfrak{F}) = L \exp(\int \frac{\delta}{\delta x} G_R F)\mathfrak{F}[x] \ , \tag{A.2}$$

where L denotes the ordering operation, which places all the functional derivatives to the left of all the x's. The r.h.s. of the relation (39) can be written symbolically in the form

$$\begin{aligned} K(\mathfrak{F})K(\mathfrak{G}) &= L \exp(\int \frac{\delta}{\delta x_1} G_R F(x_1) + \int \frac{\delta}{\delta x_2} G_R F(x_2)) \\ &\times \mathfrak{F}[x_1]\mathfrak{G}[x_2]\Big|_{x_1=x=x_2} = L \Big\{ \exp(\int \frac{\delta}{\delta x} G_R F_+) \\ &\times \exp(\int \frac{\delta}{\delta \bar{x}} G_R F_-) \mathfrak{F}[x + \bar{x}] \mathfrak{G}[x - \bar{x}] \Big\} \Big|_{\bar{x}=0} \ , \end{aligned} \tag{A.3}$$

where

$$\begin{aligned} x &= (x_1 + x_2)/2 \ , \\ \bar{x} &= (x_1 - x_2)/2 \ , \end{aligned} \tag{A.4}$$

and

$$F_{\pm}(x, \bar{x}) = \frac{1}{2}(F(x + \bar{x}) \pm F(x - \bar{x})). \qquad (A.5)$$

After expanding the last exponential in (A.3) into a series, we will find that each term will contain the following object

$$\int dt_1 \ldots dt_n dt_1' \ldots dt_n' \frac{\delta}{\delta\bar{x}(t_1)} G_R(t_1, t_1') \ldots \frac{\delta}{\delta\bar{x}(t_n)} G_R(t_n, t_n')$$
$$\times \; F(x(t_1'), \bar{x}(t_1')) \ldots F(x(t_n'), \bar{x}(t_n')) \; . \qquad (A.6)$$

Since $F_-(x, \bar{x})$ vanishes when $\bar{x} = 0$, there will be no contribution from (A.6) in (A.3), unless every derivative $\delta/\delta\bar{x}$ in (A.6) acts on one function F_- . With the use of the property of the force that it is local in time i.e.,

$$\frac{\delta F_-(x(t'), \bar{x}(t'))}{\delta\bar{x}(t)} = \delta(t-t') \frac{\partial F_-(x(t'), \bar{x}(t'))}{\partial\bar{x}(t')} \quad , \qquad (A.7)$$

we can reduce then the expression (A.6) to $n!$ terms each having the following form

$$\int dt_1 \ldots dt_n G_R(t_1, t_2) G_R(t_2, t_3) \ldots G_R(t_k, t_1) \ldots G_R(t_\ell, t_{\ell+1}) \ldots G_R(t_n, t_\ell)$$
$$\times \; \frac{\partial F_-(x(t_1), \bar{x}(t_1))}{\partial\bar{x}(t_1)} \quad \ldots \quad \frac{\partial F_-(x(t_n), \bar{x}(t_n))}{\partial\bar{x}(t_n)} \quad . \qquad (A.8)$$

All the propagators appearing in this formula are divided into several groups, the time arguments in every group forming a closed cycle. Due to the retarded character of these products such products always vanish:

$$G_R(t_1, t_2) \ldots G_R(t_k, t_1) = 0 \tag{A.9}$$

Thus we have proven that the exponential operation $\exp(\int(\delta/\delta\bar{x})G_R F_-)$ can be omitted from the formula (A.3), so that we can immediately set $\bar{x} = 0$ and identify the r.h.s. as $K(\mathfrak{F}\cdot\mathfrak{G})$.

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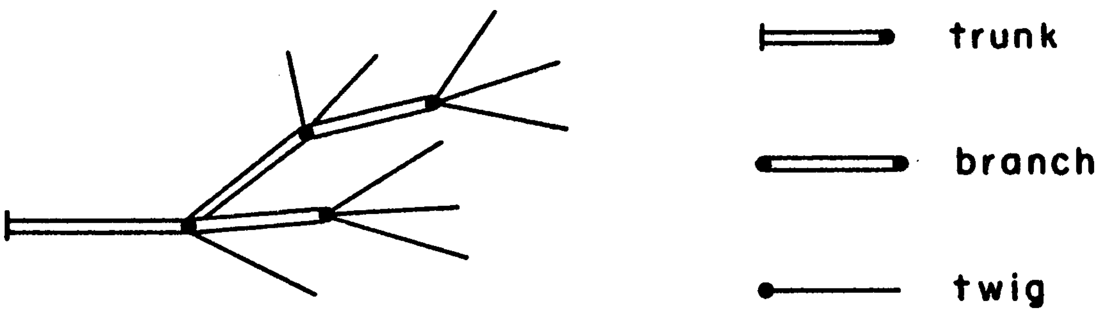

Fig. 1