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Collins-type fragmentation energy correlator in semi-inclusive deep inelastic lepton-hadron scattering

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ABSTRACT: We initiate a systematic study of fragmentation energy correlators (FECs), which generalize traditional fragmentation functions and encode non-perturbative information about transverse dynamics in parton fragmentation processes. We define boost-invariant, non-perturbative FECs and derive a corresponding collinear factorization formula. A spin decomposition of the FECs is carried out, analogous to that of transverse-momentum-dependent fragmentation functions. In this work we focus particularly on the Collins-type quark FEC, which is sensitive to chiral symmetry breaking and characterizes the azimuthal asymmetry in the fragmentation of a transversely polarized quark. We perform a next-to-leading-order calculation of the corresponding hard coefficient in semi-inclusive deep-inelastic scattering for the quark non-singlet component, thereby validating the consistency of our theoretical framework.

KEYWORDS: Deep Inelastic Scattering or Small-x Physics, Factorization, Renormalization Group, Properties of Hadrons

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1 Introduction

Energy-energy correlators (EEC) have emerged as powerful infrared and collinear (IRC) safe observables for probing microscopic details in particle collisions [1–4]. In modern language, it measures the correlation $\langle \mathcal{E}(n_i) \mathcal{E}(n_j) \rangle$ between energy deposits in two detectors along two directions, $n_i = (1, \mathbf{n}_i)$ and $n_j = (1, \mathbf{n}_j)$ with angular separation $\cos \theta_{ij} = \mathbf{n}_i \cdot \mathbf{n}_j$. The asymptotic energy flow operator is

$$\mathcal{E}(n) = \lim_{r \rightarrow \infty} r^2 \int_{-\infty}^{\infty} dt T_{0\vec{n}}(t, \vec{n}r), \quad (1.1)$$

where $T_{\mu\nu}$ is the energy-stress tensor [4–6]. In the collinear region where $\theta_{ij} \rightarrow 0$, the EEC exhibits a perturbative scaling law $\lim_{n_j \rightarrow n_i} \langle \mathcal{E}(n_i) \mathcal{E}(n_j) \rangle \sim 1/\theta_{ij}^{2-\gamma}$, as derived from the factorization theorem and perturbative collinear resummation [7] or light-ray operator product expansion (OPE) [4, 8, 9], where γ is the anomalous dimension of twist-2 spin-3 operator and the constant 2 is from classical scaling [7, 10–13]. Non-perturbative effects can then be investigated from analyzing deviations from the perturbative scaling behavior [14–16].

In recent years, there has been a resurgence of interests in the phenomenology applications of energy correlators. These include, for instance, a set of novel event shape observables

called transverse energy-energy correlators to investigate deep inelastic scattering [17, 18], a precision determination of the strong coupling constant [19] by extracting the scaling behavior near the collinear limit and calculating the ratio of three-point and two-point projected energy correlators [20–22], a potentially precise extraction of the top quark mass through the measurement of multi-point energy correlators on decaying massive states [23–26], resolving spin structures of electroweak bosons or fragmenting gluons [27, 28], and revealing the hadronization dynamics in quarkonium production by analyzing neighboring radiation patterns [29]. We refer to ref. [30] for a complete review of recent applications of energy correlators.

Combining the concepts of energy correlator and parton distribution function (PDF), refs. [31, 32] introduced the nucleon energy correlator (NEC), offering a new probe for the internal structure of nucleons. In this paper, building upon the previous study [33], we extend NECs to fragmentation processes and introduce fragmentation energy correlators (FECs). Unlike the well-established EEC, which sums over asymptotic energy fluxes from all particles in some given directions, the FEC represents an “exclusive” extension, with one of the energy fluxes in the original EEC replaced by an observed hadron, and thus becomes a nonperturbative object. Compared to the standard PDF or fragmentation function (FF), the NEC or FEC tags an additional energy flow around the incoming or outgoing hadron, and gives an extra handle to probe the nonperturbative transverse-momentum structure intrinsic to the hadron or in the fragmentation process. As established in refs. [33–35], NECs and FECs can be related to transverse-momentum-dependent (TMD) PDFs and FFs [36], respectively, in terms of their angular integrals. They can therefore serve as novel tools to uncover complementary information of QCD tomography.

While TMD PDFs and FFs have played a primary role in the investigation of non-perturbative QCD aspects such as confinement and chiral symmetry breaking [36–42], their phenomenological extractions from experimental data present significant difficulties. A major challenge stems from the transverse momentum convolutions between pairs of TMD distributions, making a unique one-to-one determination rather difficult.¹ Furthermore, the measured transverse momentum distributions are largely dominated by the perturbative showering, which greatly dilutes the intrinsic confining details [46]. In contrast, NECs and FECs bypass these obstacles by entering the observables through collinear factorization. The non-perturbative information encoded therein can be directly extracted from the correlation of the measured energy flow with the scattered or produced hadron, without any Sudakov factor contamination. The physical processes and kinematics for their measurements are no different from those for the normal PDFs and FFs, and thus no new experiments need to be carried out; one just needs to analyze the same events with an additional energy flow weight.

There have already been intriguing explorations of NECs applied to gluon saturation [47] and spin physics [48–52]. In this paper, we focus on the phenomenological study of one-point quark FECs in the semi-inclusive deep inelastic lepton-hadron scattering (SIDIS). The SIDIS has been extensively measured at facilities such as the HERA at DESY [53–55], Jefferson Lab (with CEBAF) [56–59], and COMPASS at CERN [60–62], and will be further studied at

¹There are efforts to remove such pairwise convolutions by using jets instead of hadrons as final-state observables [43, 44], which has been further improved to be recoiling-free [45].

future high-luminosity facilities such as the EIC [63–66] and EicC [67]. At leading twist, the quark FEC has an analogous spin structure to the quark TMD FF, which, for unpolarized hadrons, yields two fundamental non-perturbative functions: an unpolarized FEC, $\mathcal{D}_1$, and a transversely polarized FEC, $\mathcal{H}_1^\perp$. It is the latter that is of particular interest to us as it is associated with the quark helicity flipping during the fragmentation process and thus it serves as a sensitive probe of chiral symmetry breaking effect in QCD. This feature is shared by the well-known Collins function [39] for TMD fragmentation, so we term it Collins-type FEC. The FEC observable is analogous, in a sense, to the one-point energy correlator in ref. [68]; both yield an azimuthal asymmetry that relies on a transversely polarized nucleon target and enables access to the quark transversity PDF. The difference, however, is that by centering around the observed hadron, the FEC observable eliminates the need for jet reconstruction, thereby removing any dependence on jet algorithms. Formally, FECs can be interpreted as an interpolation in the observable space between EECs and dihadron fragmentation function. Very recently there have been active researches on the connection of EECs and dihadron fragmentation through moment integrals [69–74]. We expect similar analysis to be performed for FECs, for which the NLO QCD correction to dihadron fragmentation is available [75].

Since the transverse momentum (p_{hT}) of the final-state hadron produced in the SIDIS process is typically small, we introduce a novel FEC observable in this paper. This observable is defined by integrating over (p_{hT}), yielding a more inclusive quantity. To construct this novel FEC observable, we begin by demonstrating the utility of FEC functions within a rigorous theoretical framework, establishing all-order collinear factorization in energy flow-tagged SIDIS. Following the Trento convention [76], we work in the Breit frame and distinguish a number of kinematic regions according to the magnitude of p_{hT} . The factorization is mostly easily formulated for a large p_{hT} . To preserve the collinear factorization structure when integrating out the kinematic variable p_{hT} (in defining the proposed FEC observable), we must modify the original FEC definition — which exhibits explicit p_{hT} dependence — to derive a new formalism. This new formalism, which uses the particles' plus momenta (p_i^+) instead of energies as the weight factor in the energy-flow observable, is Lorentz boost-invariant along the direction of the moving hadron (h). Moreover, it allows for precise predictions of various azimuthal distributions that are essential for measuring Collins-type FEC observables associated with energy-tagged SIDIS cross sections, after properly addressing the apparent azimuthal singularity that arises as $p_{hT} \rightarrow 0$ during the p_{hT} integration.

The remainder of this paper is structured as follows. In section 2, we first provide the formal definition of the FEC and elaborate on its relationship to TMD FFs, and then introduce the modified boost-invariant definition which is advantageous for phenomenological studies. We give a detailed discussion of the collinear factorization in section 3 for measuring FECs in SIDIS at a large p_{hT} . We organize the observables according to the azimuthal modulations in the Breit frame, mimicking the well-studied SIDIS formalism, and present the leading-order (LO) partonic hard coefficients. Section 4 is devoted to addressing the subtleties in the integration over p_{hT} as $p_{hT} \rightarrow 0$. We perform a next-to-leading-order (NLO) calculation of the hard coefficients and demonstrate the cancellation of soft and collinear divergences, verifying the consistency of the factorization and our prescription for dealing with the azimuthal singularity. Finally, in section 5, we provide a summary of our findings and a discussion of future prospects.

2 Fragmentation energy correlators

2.1 Basic definition

As introduced in ref. [33], the fragmentation energy correlator (FEC) modifies the normal fragmentation function (FF) by inserting an extra energy flow operator,

$$\mathcal{D}_{h/q,1}^{[\Gamma]}(z, \mathbf{n}; p_h) = \frac{z}{2N_c} \sum_X \int \frac{dy^-}{2\pi} e^{ip_h^+ y^- / z} \text{Tr} \left[\Gamma \langle 0 | W(\infty, y^-; w) \psi(y^-) \mathcal{E}(\mathbf{n}) | h, X; \text{out} \rangle \right. \\ \left. \times \langle h, X; \text{out} | \bar{\psi}(0) W^\dagger(\infty, 0; w) | 0 \rangle \right], \quad (2.1)$$

where the tagged hadron h has its momentum p_h boosted along the z -axis direction, carrying a fraction z of the longitudinal momentum of the fragmenting quark. Γ represents a gamma matrix for some spin structure. We define the lightfront coordinate system using two auxiliary lightlike vectors,

$$w = (1, 0, 0, -1)/\sqrt{2}, \quad \bar{w} = (1, 0, 0, 1)/\sqrt{2}, \quad (2.2)$$

on which any Lorentz vector V^μ can be decomposed as

$$V = (V^+, V^-, \mathbf{V}_T) \equiv (V \cdot w, V \cdot \bar{w}, V^1, V^2). \quad (2.3)$$

As for the FF, the FEC quantifies the nonlocal parton correlation on the backward light-cone along w . The two quark fields, $\psi(y^-)$ and $\bar{\psi}(0)$, are connected by the Wilson lines along w for gauge invariance. In eq. (2.1), we defined

$$W(b, x; w) \equiv (W_{ij}(b, x; w)) = \mathcal{P} \exp \left\{ -ig \int_0^b d\lambda w^\mu A_\mu^a(x + \lambda w) (t_{ij}^a) \right\} \quad (2.4)$$

for the straight Wilson line in the fundamental color representation going from x to $x + bw$ along the direction w . We will focus on quark FECs in this paper, with $N_c = 3$ in eq. (2.1). Similar definitions can be given for antiquark and gluon FECs.

We have marked the hadron states in eq. (2.1) as “out” states as required for FFs [36, 37, 39]. Any hadrons X alongside h from the fragmentation process are inclusively summed, with their phase space integrated over. However, we anchor in this sum an energy flow measurement along a fixed direction $\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$,

$$\mathcal{E}(\mathbf{n}) |X; \text{out}\rangle = \sum_{i \in X} E_i \delta^{(2)}(\mathbf{n} - \mathbf{n}_i) |X; \text{out}\rangle, \quad (2.5)$$

which counts the energy deposit along the $\mathbf{n}$ direction, and where $\delta^{(2)}(\mathbf{n} - \mathbf{n}_i) = \delta(\cos \theta - \cos \theta_i) \delta(\phi - \phi_i)$. The polar and azimuthal angles, θ and ϕ , are defined with respect to the explicitly measured hadron h ; the two transverse directions x and y will be specified in the discussion of factorization in SIDIS process in section 3.1.

The energy flow insertion extracts extra information from the fragmentation process in addition to the longitudinal momentum fraction z of the hadron. This is encoded in the θ and ϕ dependence of the FEC in eq. (2.1), which obviously maps out the energy deposition structure in the transverse directions, similarly to the TMD FFs [33]. We will review this

aspect in section 2.2. In particular, it is the ϕ dependence that is of our main interest in this paper. By the rotational symmetry around p_h , a nontrivial ϕ dependence occurs only when the fragmenting quark carries a transverse spin.

The spinor projector Γ in eq. (2.1) defines the spin structure of the quark. At the twist-2 level, we have three possibilities, $\Gamma = \gamma^+/2 \equiv \gamma \cdot w/2$ for an unpolarized quark, $\gamma^+\gamma_5/2$ for a longitudinally polarized quark, or $\gamma^+\gamma^i\gamma_5/2$ for a transversely polarized quark, with $i = 1$ or 2 being the transverse Lorentz index. These define various FECs with different azimuthal dependence and for different polarization states of h , in a way similar to the TMD FFs [42]. Since the mostly measured hadrons in practice are pions, kaons, proton, and antiproton, we only consider unpolarized h in this paper, leaving studies beyond to future works. With this constraint, the number of FECs is much reduced by parity, giving only two possible FECs.

Using similar notations to the TMD FFs [42], we have the unpolarized FEC,

$$\mathcal{D}_{h/q,1}^{[\gamma^+/2]}(z, \mathbf{n}; p_h) = \mathcal{D}_{1,h/q}(z, \theta; p_h), \quad (2.6)$$

for $\Gamma = \gamma^+/2$, and the transversity FEC,

$$\mathcal{D}_{h/q,1}^{[\gamma^+\gamma^i\gamma_5/2]}(z, \mathbf{n}; p_h) = \frac{(\hat{z} \times \mathbf{n}_T)^i}{|\mathbf{n}_T|} \mathcal{H}_{1,h/q}^\perp(z, \theta; p_h) = (-\sin \phi, \cos \phi)^i \mathcal{H}_{1,h/q}^\perp(z, \theta; p_h), \quad (2.7)$$

for $\Gamma = \gamma^+\gamma^i\gamma_5/2$. The longitudinally polarized FEC $\mathcal{D}_{h/q,1}^{[\gamma^+\gamma_5/2]}$ vanishes by parity. In eq. (2.7), we defined $\hat{z} = \mathbf{P}_h/|\mathbf{P}_h| = (0, 0, 1)$, and $\mathbf{n}_T = \sin \theta (\cos \phi, \sin \phi, 0)$ for the transverse component of $\mathbf{n}$. The $\mathcal{H}_{1,h/q}^\perp(z, \theta; p_h)$ is the FEC analogue of the Collins function. It induces an azimuthal asymmetry in the energy deposit of the fragmentation of a quark that carries a transverse spin $\mathbf{s}_T = s_T(\cos \phi_S, \sin \phi_S)$,

$$\begin{aligned} \mathcal{D}_{h/q,1}^{[\gamma^+/2]}(z, \mathbf{n}; p_h) + s_T^i \mathcal{D}_{h/q,1}^{[\gamma^+\gamma^i\gamma_5/2]}(z, \mathbf{n}; p_h) \\ = \mathcal{D}_{1,h/q}(z, \theta; p_h) - s_T \mathcal{H}_{1,h/q}^\perp(z, \theta; p_h) \sin(\phi - \phi_S), \end{aligned} \quad (2.8)$$

with the ratio $\mathcal{H}_{1,h/q}^\perp/\mathcal{D}_{1,h/q}$ playing the role of the spin analyzing power.

While this paper mainly focuses on the one-point FEC in eq. (2.1), we note that the definition can be easily generalized to N -point FECs [33],

$$\begin{aligned} \mathcal{D}_{h/q,N}^{[\Gamma]}(z, \{\mathbf{n}_i\}; p_h) = \frac{z}{2N_c} \sum_X \int \frac{dy^-}{2\pi} e^{ip_h^+ y^- / z} \text{Tr} \left[\Gamma \langle 0 | W(\infty, y^-; w) \psi(y^-) \right. \\ \left. \times \mathcal{E}(\mathbf{n}_1) \cdots \mathcal{E}(\mathbf{n}_N) | h, X; \text{out} \rangle \langle h, X; \text{out} | \bar{\psi}(0) W^\dagger(\infty, 0; w) | 0 \rangle \right], \end{aligned} \quad (2.9)$$

where the dependence on $\{\mathbf{n}_i\}_{i=1}^N$ is symmetrized. These would certainly allow more complicated spin and azimuthal dependence. For instance, at $N = 2$, the longitudinally polarized FEC $\mathcal{D}_{h/q,2}^{[\gamma^+\gamma_5/2]}(z, \{\mathbf{n}_1, \mathbf{n}_2\}; p_h) = \hat{z} \cdot (\mathbf{n}_1 \times \mathbf{n}_2) G_{2,h/q}(z, \theta_1, \theta_2; p_h)$ would not vanish [77].

2.2 Relation between TMD FF and FEC

The relation between TMD FFs and FECs has been constructed in ref. [33] in such a way that the N -th transverse momentum moments of the former are directly equal to the angular weighted integral of N -point FECs. The basic observation is that in the so-called *hadron*

frame where the measured hadron h is along the z direction, the transverse momentum $\mathbf{k}_T$ of the fragmenting quark in the TMD FF is identical to the sum of transverse momenta of all the remaining particles (in X) in the final state. This motivates one to rewrite the TMD FF as

$$d_{h/q}^{[\Gamma]}(z, -z\mathbf{k}_T) = \frac{1}{2zN_c} \sum_X \int \frac{dy^-}{2\pi} e^{ip_h^+ y^- / z} \text{Tr} \left[\Gamma \langle 0 | W(\infty, y^-; w) \psi(y^-) \right. \\ \left. \times \delta^{(2)} \left(\mathbf{k}_T - \int d\Omega \mathcal{E}(\mathbf{n}) \mathbf{n}_T \right) | h, X; \text{out} \rangle \langle h, X; \text{out} | \bar{\psi}(0) W^\dagger(\infty, 0; w) | 0 \rangle \right], \quad (2.10)$$

where the operator definition on the right-hand side is in the hadron frame with h along the z direction, but the arguments of the TMD FF $d_{h/q}^{[\Gamma]}(z, -z\mathbf{k}_T)$ on the left-hand side are defined in the *parton frame* such that the fragmenting quark is moving along the z direction. We follow the standard normalization that $d_{h/q}^{[\Gamma]}(z, \mathbf{p}_T) dz d^2\mathbf{p}_T$ stands for the hadron number in the phase space $[z, z + dz] \times [\mathbf{p}_T, \mathbf{p}_T + d^2\mathbf{p}_T]$. The connection between the two frames is given by $\mathbf{p}_T = -z\mathbf{k}_T$. Clearly, the operator $\int d\Omega \mathcal{E}(\mathbf{n}) \mathbf{n}_T \equiv \int d\cos\theta d\phi \mathcal{E}(\mathbf{n}) \sin\theta(\cos\phi, \sin\phi)$ measures the total transverse momentum of the particles in X , giving meaning to the $\mathbf{k}_T$ in this new representation of the TMD FF.

Comparing eqs. (2.10) and (2.9), one can see that the $\mathbf{k}_T$ moments of eq. (2.10) bring out the energy flow factors and become the angular integrals of eq. (2.9),

$$z^2 \int d^2\mathbf{k}_T k_T^{i_1} \cdots k_T^{i_N} d_{h/q}^{[\Gamma]}(z, -z\mathbf{k}_T) = \int d\Omega_1 \cdots d\Omega_N n_{1T}^{i_1} \cdots n_{NT}^{i_N} \mathcal{D}_{h/q,N}^{[\Gamma]}(z, \{\mathbf{n}_i\}; p_h), \quad (2.11)$$

where $i_j = 1, 2$ for $j = 1, \dots, N$ are the transverse index. We note that this relation is based on the bare definitions of the TMD FFs and FECs and is susceptible to renormalization. However, neither side contains rapidity divergences; the $\mathbf{k}_T$ integral on the left-hand side cancels the rapidity divergences of the bare TMD FFs, while the N -point FECs on the right-hand side are free of rapidity divergences in the first place.

Eq. (2.11) also indicates the difference between TMD FFs and FECs in the transverse momentum structure of the fragmentation. Taking both in the hadron frame, the $\mathbf{k}_T$ of the TMD FF sums over all the hadrons globally in the phase space, whereas the energy flow in an FEC operates locally along a certain direction $\mathbf{n}$. On the other hand, because of the linear dependence of $\mathcal{E}(\mathbf{n})$ on the momentum — due to the constraint of IRC safety, the k_T moments of the same TMD FF at different orders map out a whole tower of FECs of different points, N , of energy correlators. For example, taking $N = 2$, we see that the k_T^2 moment of a TMD FF corresponds to a 2-point FEC,

$$z^2 \int d^2\mathbf{k}_T k_T^2 d_{h/q}^{[\Gamma]}(z, -z\mathbf{k}_T) = \int d\Omega_1 d\Omega_2 (\mathbf{n}_{1T} \cdot \mathbf{n}_{2T}) \mathcal{D}_{h/q,2}^{[\Gamma]}(z, \{\mathbf{n}_1, \mathbf{n}_2\}; p_h), \quad (2.12)$$

which cannot be constructed out of a 1-point FEC.

If a TMD FF could be reconstructed from the complete set of its $\mathbf{k}_T$ moments, then combining the right-hand side of eq. (2.11) for all values of N would fully recover the TMD FF. In this sense, the whole set of FECs encapsulates the transverse momentum structure encoded in TMD FFs. However, in practice, it is impossible to obtain FECs for all N . With only a limited number of finite-point FECs available, the constraints imposed by eq. (2.11) are

inherently restricted. Consequently, FECs and TMD FFs provide complementary information of the fragmentation process.

The $N = 0$ case of eq. (2.11) gives a trivial normalization condition of the TMD FF. For $N = 1$, there is only one relation,

$$z^2 \int d^2 \mathbf{k}_T k_T^j d_{h/q}^{[\Gamma]}(z, -z \mathbf{k}_T) = \int d\Omega n_T^j \mathcal{D}_{h/q,1}^{[\Gamma]}(z, \mathbf{n}; p_h). \quad (2.13)$$

Both sides are nonvanishing only for $\Gamma = \gamma^+ \gamma^i \gamma_5 / 2$. Thus it relates the Collins-type FEC to the Collins function, which is defined as

$$d_{h/q}^{[\gamma^+ \gamma^i \gamma_5 / 2]}(z, -z \mathbf{k}_T) = H_{1,h/q}^\perp(z, z k_T) \frac{\epsilon_{\perp}^{ij} k_T^j}{m_h}, \quad (2.14)$$

with m_h being the mass of h . Combining this and eq. (2.7), we then get the relation implied by eq. (2.13),

$$\int_0^\pi d\theta \sin^2 \theta \mathcal{H}_{1,h/q}^\perp(z, \theta; p_h) = -z^2 \int \frac{dk_T^2}{2m_h} k_T^2 H_{1,h/q}^\perp(z, z k_T). \quad (2.15)$$

In this work, the FECs we define are distinguished from TMD FF by using a script-style symbol. Readers can also differentiate between them based on their respective arguments. The right-hand side can also be matched onto a twist-3 FF [78]. The definition in the l.h.s. of eq. (2.15) is frame dependent. In the next section we shall present a frame-independent definition.

2.3 Modified definition with boost invariance

As first pointed out in ref. [31], an energy flow can be measured individually in place of a jet or together with a hadron nearby in the angular region. In the former case, the operator $\mathcal{E}(\mathbf{n})$ defined originally in eq. (1.1) is best suited for e^+e^- collisions in the CM frame, where energy and angle are suitable kinematic observables [79, 80]. When applied to lepton-hadron or hadron-hadron collisions, it is convenient to replace energy by transverse momentum p_T with respect to the collision axis, and use pseudo-rapidity η instead of θ , as adopted in refs. [19, 81–83]. In this case, the p_T weight suppresses soft-hadron contributions and justifies the use of perturbative QCD.

In this paper, we consider the FEC as measured in semi-inclusive deep inelastic scattering (SIDIS) of an electron off a proton,

$$e(\ell) + p(p) \rightarrow e(\ell') + h(p_h) + \mathcal{E}(\mathbf{n}) + X(p_X). \quad (2.16)$$

We require the energy flow to be measured near the observed hadron h , so that it can no longer be described by perturbative QCD, but more closely related to nonperturbative fragmentation dynamics. Then, this process can be factorized [31, 33] into an FEC as defined in eq. (2.1), which combines the hadron and energy flow in a single nonperturbative object, as will be described in section 3.2.

A disadvantage of the original FEC definition in eq. (2.1) is its reliance on the energy E and polar angle θ , which introduces an explicit dependence on the magnitude of the

hadron momentum p_h , as noted in eq. (2.1), in addition to its dependence on z and angular variables. This complicates the extraction of the nonperturbative FEC function from a global analysis across experiments conducted at different collider energies. In SIDIS, the reference frame used for analysis typically varies — from the lab frame to the Breit frame — each subject to arbitrary longitudinal boosts. As a result, the FECs differ even when evaluated at the same set of z and θ values. Furthermore, this definition requires full knowledge of the hadron’s kinematics; partial integration over its transverse momentum $\mathbf{p}_{hT}$ at the cross-section level would cause mixing among different FECs, thereby obstructing a clean extraction of individual FEC function.

To resolve this issue, we propose a modified definition of the FEC with a nice boost property along the p_h direction, by replacing the energy flow operator in eq. (2.5) with

$$\mathcal{E}_L(\eta, \phi) |X; \text{out}\rangle = \sum_{i \in X} p_i^+ \delta(\eta - \eta_i) \delta(\phi - \phi_i) |X; \text{out}\rangle, \quad (2.17)$$

where $\eta_i = -\ln[\tan(\theta_i/2)]$, ϕ_i , and $p_i^+ = (E_i + p_i^z)/\sqrt{2}$ are the pseudo-rapidity, azimuthal angle, and plus momentum of the i -th particle, respectively, with respect to the h direction. The reason for using p_i^+ as the momentum weighting, instead of the transverse momentum p_{iT} as used in refs. [84, 85], is that in the collinear region that we are considering, p_{iT} can be of the same order as the hadronic mass scale so that nonperturbative corrections might not be negligible. In contrast, the p_i^+ weight suppresses contributions from soft momenta and hadron masses, justifying the use of pseudo-rapidity η in place of rapidity y in the leading-power accuracy.² Therefore, by neglecting the masses of hadrons in the energy flow measurement, the $\mathcal{E}_L(\eta, \phi)$ transforms under a longitudinal boost $U(\alpha)$ along the h direction as

$$U(\alpha) \mathcal{E}_L(\eta, \phi) U^{-1}(\alpha) = e^{-\alpha} \mathcal{E}_L(\eta + \alpha, \phi), \quad (2.18)$$

where α is defined such that any momentum p_i transforms under $U(\alpha)$ by $p_i^+ \rightarrow e^\alpha p_i^+$.

In this way, we redefine the FEC as

$$\begin{aligned} \mathcal{D}_{h/q,1}^{[\Gamma]}(z, \Lambda_F, \phi) = \frac{z}{2N_c} \sum_X \int \frac{dy^-}{2\pi p_h^+} e^{ip_h^+ y^- / z} \text{Tr} \left[\Gamma \langle 0 | W(\infty, y^-; w) \psi(y^-) \right. \\ \left. \times \mathcal{E}_L(\eta, \phi) |h, X; \text{out}\rangle \langle h, X; \text{out}| \bar{\psi}(0) W^\dagger(\infty, 0; w) |0\rangle \right], \end{aligned} \quad (2.19)$$

where an additional factor $1/p_h^+$ is inserted to compensate for the factor $e^{-\alpha}$ in eq. (2.18); it also makes the FEC dimensionless. Eq. (2.19) is boost invariant along the h direction and therefore depends on η only through its difference from the rapidity y_h of h . This dependence is explicitly captured in the newly defined variable,

$$\Lambda_F = m_h e^{y_h - \eta} = \sqrt{2} p_h^+ e^{-\eta}. \quad (2.20)$$

Analogous to eqs. (2.6) and (2.7), we have new definitions of the unpolarized FEC, $\mathcal{D}_{1,h/q}(z, \Lambda_F)$, and the Collins-type FEC, $\mathcal{H}_{1,h/q}^\perp(z, \Lambda_F)$, corresponding to $\Gamma = \gamma^+/2$ and $\gamma^+ \gamma^i \gamma_5/2$, respectively. Crucially both functions are now boost invariant and depend only on z and Λ_F .

²Of course, this requires h to be highly boosted, a condition easily satisfied in practice.

As noted in eq. (2.17), the light-cone momentum component p_i^+ is related to energy E_i via $p_i^+ = E_i(1 + \tanh \eta_i)/\sqrt{2}$, and $\mathcal{E}_L(\eta, \phi)$ can be expressed in terms of the original energy flow operator $\mathcal{E}(\mathbf{n})$ in eq. (1.1) as

$$\mathcal{E}_L(\eta, \phi) = \frac{1 + \tanh \eta}{\sqrt{2} \cosh^2 \eta} \mathcal{E}(\theta, \phi), \quad (2.21)$$

where the denominator arises from the Jacobian factor associated with the change of variables from $\delta(\cos \theta - \cos \theta_i)$ in eq. (2.5) to $\delta(\eta - \eta_i)$ in eq. (2.17). Then the $\mathcal{D}_{h/q,1}^{[\Gamma]}(z, \Lambda_F, \phi)$ in eq. (2.19) can be related to $\mathcal{D}_{h/q,1}^{[\Gamma]}(z, \mathbf{n}; p_h)$ in eq. (2.1)³ by

$$\mathcal{D}_{h/q,1}^{[\Gamma]}(z, \Lambda_F, \phi) = \frac{1 + \tanh \eta}{\sqrt{2} p_h^+ \cosh^2 \eta} \mathcal{D}_{h/q,1}^{[\Gamma]}(z, \mathbf{n}; p_h). \quad (2.22)$$

It is well known that in the collinear limit, the product of energy and angle, $E_i \theta_i$, is approximately Lorentz invariant. Thus, we can gain a better insight of the variable Λ_F by examining the small- θ limit, which corresponds to large η . This gives

$$\theta \simeq \frac{1}{\cosh \eta}, \quad \Lambda_F \simeq \frac{p_h^+ \theta}{\sqrt{2}}, \quad \mathcal{D}_{h/q,1}^{[\Gamma]}(z, \Lambda_F, \phi) \simeq \frac{\theta^3}{\Lambda_F} \mathcal{D}_{h/q,1}^{[\Gamma]}(z, \mathbf{n}; p_h). \quad (2.23)$$

Therefore, Λ_F serves as an effective transverse momentum scale between h and the energy flow, taking over the role of θ in probing the transverse structure of parton fragmentation. The FEC originally defined in eq. (2.1) transforms as $1/\theta^3$ under a longitudinal boost, while the new definition in eq. (2.19) is boost invariant.

Following the same spirit of section 2.2, we can connect the boost-invariant FECs to TMD FFs. This can be simply done by using the relation,

$$\int d \cos \theta = \int \frac{d \Lambda_F}{|\mathbf{n}_T|} \frac{1 + \tanh \eta}{\sqrt{2} p_h^+ \cosh^2 \eta}, \quad (2.24)$$

where $|\mathbf{n}_T| = \sin \theta = 1/\cosh \eta$. Applying this to eq. (2.15) then gives

$$\int d \Lambda_F \mathcal{H}_{1,h/q}^\perp(z, \Lambda_F) = -z^2 \int \frac{d k_T^2}{2m_h} k_T^2 H_{1,h/q}^\perp(z, z k_T), \quad (2.25)$$

where the left-hand side is the boost-invariant Collins-type FEC. This relation constitutes one of the central results of our work.

3 FECs from energy-tagged SIDIS: finite- p_{hT} case

The FECs can be measured in any inclusive hadron production process at lepton-lepton, lepton-hadron, or hadron-hadron collisions, simply by recording the energy flow surrounding the measured hadron. In this section, we consider such new observables in SIDIS at lepton-hadron collision. We will derive a collinear factorization formalism that differs from that of SIDIS only by replacing the normal FF by the FEC. The boost-invariant FEC definitions in section 2.3 will be employed in our analysis. Similar studies can be performed for lepton-lepton and hadron-hadron collisions.

³Although the same symbol is used, the quantities in eq. (2.19) and eq. (2.1) are distinct definitions, related by the non-trivial factor in eq. (2.22) rather than a simple change of variables. They are distinguished by their arguments.

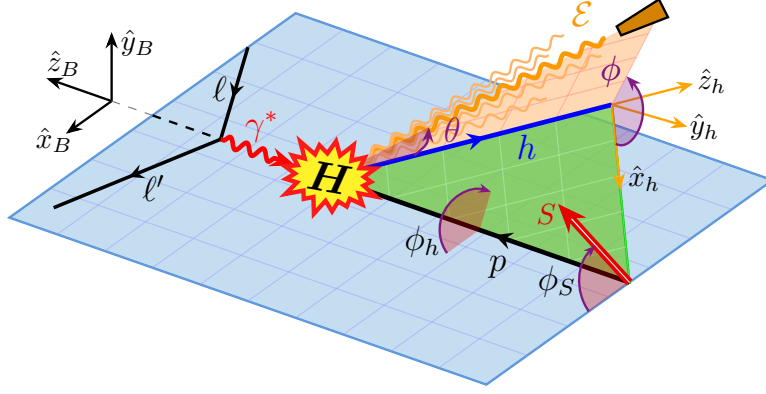


Figure 1. Breit frame for the energy-tagged SIDIS in eq. (2.16).

3.1 Kinematics

To the leading order in QED, the electron scatters with the hadron in eq. (2.16) by exchanging a virtual photon $\gamma^*(q = \ell - \ell')$,

$$e(\ell) \rightarrow e(\ell') + \gamma^*(q), \quad (3.1a)$$

$$p(p) + \gamma^*(q) \rightarrow h(p_h) + \mathcal{E}(\mathbf{n}) + X(p_X), \quad (3.1b)$$

where the γ^* carries a hard virtuality $Q^2 = -q^2$ much beyond the hadronic mass scale, and X denotes anything else in the final state. We consider the most general polarization scenario for both the beam and target, and describe the kinematics in the Breit frame, as displayed in figure 1, which is defined as the collision frame between the proton p and the γ^* , with p boosted in the $\hat{z}_B$ direction. The electron scattering happens in the $\hat{x}_B$ - $\hat{z}_B$ plane, in which we fix the $\hat{x}_B$ axis such that the initial-state electron momentum has a positive x component, $\ell^x > 0$.

Since we work only to the leading power of $1/Q$, we may neglect electron and hadron masses for simplicity and parametrize the initial-state kinematics as

$$\begin{aligned} p &= (p^+, 0^-, \mathbf{0}_T), \quad q = \left(-x p^+, \frac{Q^2}{2x p^+}, \mathbf{0}_T\right), \\ \ell &= \left(\frac{x(1-y)}{y} p^+, \frac{Q^2}{2xy p^+}, \frac{Q\sqrt{1-y}}{y}, 0\right), \end{aligned} \quad (3.2)$$

which are written in terms of lightfront coordinates as defined in eq. (2.3) and where

$$x = \frac{Q^2}{2p \cdot q}, \quad y = \frac{p \cdot q}{p \cdot \ell} = \frac{Q^2}{x s}. \quad (3.3)$$

By neglecting the proton mass, the transformation of the azimuthal angle ϕ_S of the target transverse spin $\mathbf{s}_T$ becomes trivial [86]; the ϕ_S in the Breit frame varies from event to event and is equivalent to the azimuthal angle of the electron scattering in the lab frame. The momentum p_h of the final-state hadron h is parametrized using its transverse momentum $\mathbf{p}_{hT} = p_{hT}(\cos \phi_h, \sin \phi_h)$, with $p_{hT} = |\mathbf{p}_{hT}|$ and ϕ_h being its azimuthal angle, and

$$z = \frac{p \cdot p_h}{p \cdot q} = \frac{p_h^-}{q^-}. \quad (3.4)$$

These are the same as the SIDIS kinematics. Our analysis introduces a new observable: the measurement of the energy flow surrounding the final-state hadron h . The azimuthal distribution of the energy flow is of primary importance, as it directly probes the polarization of the fragmenting parton. To quantify this observable, the cross section is weighted by the factor $p_i^+ \delta(\eta - \eta_i) \delta(\phi - \phi_i)$ in eq. (2.17), summed over each final-state particle i . As mentioned there, the (p_i^+, η_i, ϕ_i) are defined relative to the h direction, so we define the hadron coordinate system,

$$\hat{z}_h = \frac{\mathbf{p}_h}{|\mathbf{p}_h|}, \quad \hat{y}_h = \frac{\hat{z}_B \times \hat{z}_h}{|\hat{z}_B \times \hat{z}_h|}, \quad \hat{x}_h = \hat{y}_h \times \hat{z}_h, \quad (3.5)$$

and the energy flow measurements are also performed with respect to this system. Within this frame, the azimuthal angle ϕ is therefore the central variable. It is crucial to note the distinction in reference frames that this introduces. The light-cone component p_i^+ used for the energy flow is defined within this hadron frame. This contrasts with the light-cone coordinates used for the global SIDIS kinematics (e.g., q^- in eq. (3.4)), which are defined in the Breit frame.

Without measuring the energy flow, the SIDIS cross section can be expressed in terms of $(x, Q^2, \phi_S, z, p_{hT}^2, \phi_h)$,

$$\frac{d\sigma}{dx dQ^2 d\phi_S dz dp_{hT}^2 d\phi_h} = \frac{y^2}{zQ^2} \frac{\alpha_e^2}{8(2\pi)^4} \overline{|\mathcal{M}|^2}, \quad (3.6)$$

where $\alpha_e = e^2/(4\pi)$. Let us denote $\mathcal{M}^X = \mathcal{M}_{\lambda_e \lambda_p}^X$ as the amplitude of

$$e(\ell, \lambda_e) + p(p, \lambda_p) \rightarrow e(\ell') + h(p_h) + X(p_X), \quad (3.7)$$

with the QED coupling e^2 factored out and with λ_e and λ_p being the helicities of the initial-state electron and proton. The $\overline{|\mathcal{M}|^2}$ is obtained by integrating out the X phase space in the spin-averaged amplitude squared,

$$\overline{|\mathcal{M}|^2} = \sum_X \sum_{\lambda_e, \lambda_p, \bar{\lambda}_e, \bar{\lambda}_p} \int d\Pi_X (2\pi)^4 \delta^{(4)}(p + \ell - \ell' - p_h - p_X) \left[\rho_{\lambda_e \bar{\lambda}_e}^e \rho_{\lambda_p \bar{\lambda}_p}^p \mathcal{M}_{\lambda_e \lambda_p}^X \mathcal{M}_{\bar{\lambda}_e \bar{\lambda}_p}^{X*} \right], \quad (3.8)$$

where the density matrices ρ^e and ρ^p for the beam and target, respectively, are defined in the helicity basis and can be written in the Breit frame as

$$\rho^e = \rho^e(P_e) = \frac{1}{2} \begin{pmatrix} 1 + P_e & 0 \\ 0 & 1 - P_e \end{pmatrix}, \quad \rho^p = \rho^p(P_N, s_T, \phi_S) = \frac{1}{2} \begin{pmatrix} 1 + P_N & s_T e^{-i\phi_S} \\ s_T e^{i\phi_S} & 1 - P_N \end{pmatrix}. \quad (3.9)$$

The P_e refers to the longitudinal polarization degree of the beam, and P_N and s_T the longitudinal and transverse polarization degrees, respectively, of the target; their numerical values are the same between the lab and the Breit frames in the massless approximation. We have omitted the helicity indices of final-state particles in eq. (3.8), which are implicitly summed over.

Now with the energy flow measured, we replace the $\overline{|\mathcal{M}|^2}$ in eq. (3.6) by $\overline{|\mathcal{M}_\mathcal{E}|^2}$, which modifies eq. (3.8) with an additional factor

$$\sum_{i \in X} (p_i^+ / p_h^+) \delta(\eta - \eta_i) \delta(\phi - \phi_i) \quad (3.10)$$

inserted in the Π_X integral, where p_i^+ and p_h^+ are defined as in eq. (2.17). This introduces two additional kinematic observables (η, ϕ) into eq. (3.6), and results in an energy flow-weighted cross section,

$$\frac{d\Sigma}{dx dQ^2 d\phi_S dz dp_{hT}^2 d\phi_h d\eta d\phi} = \frac{y^2}{zQ^2} \frac{\alpha_e^2}{8(2\pi)^4} |\overline{\mathcal{M}_\mathcal{E}}|^2, \quad (3.11)$$

where we note that using eq. (2.20), the $d\eta$ can be also written as $d\ln \Lambda_F$.

3.2 Factorization

We consider SIDIS in the current fragmentation region, with z being of order unity, $z \sim \mathcal{O}(1)$. When we measure the hadron's transverse momentum p_{hT} , as indicated in eq. (3.11), it needs to be of the same order as Q in order to allow collinear factorization of eq. (2.16) into the FEC. We focus on this kinematic region in this section and will consider integrating over $\mathbf{p}_{hT}$ in section 4.

Without measuring an energy flow in the final state, the standard factorization of SIDIS cross section in eq. (3.6) has been established as the convolution of parton distribution functions (PDFs), FFs, and hard coefficients [37],

$$\begin{aligned} |\overline{\mathcal{M}}|^2 \simeq \sum_{a,b} \int \frac{d\xi_1}{\xi_1} \int \frac{d\xi_2}{\xi_2^2} D_{h/b}(\xi_2, \mu) \left[f_{a/p}(\xi_1, \mu) C_{ab} \left(\frac{x}{\xi_1}, \frac{z}{\xi_2}, \frac{\mathbf{p}_{hT}}{Q}, \frac{Q^2}{\mu^2} \right) \right. \\ \left. + P_N g_{a/p}(\xi_1, \mu) \Delta C_{ab} \left(\frac{x}{\xi_1}, \frac{z}{\xi_2}, \frac{\mathbf{p}_{hT}}{Q}, \frac{Q^2}{\mu^2} \right) \right] \\ + \mathcal{O}(\Lambda_{\text{QCD}}/Q, \Lambda_{\text{QCD}}/p_{hT}), \end{aligned} \quad (3.12)$$

where the sums over a and b run over parton flavors. The unpolarized and polarized quark PDFs, $f_{q/p}$ and $g_{q/p}$, are defined as

$$f_{q/p}(\xi_1) = \int_{-\infty}^{\infty} \frac{d\lambda}{4\pi} e^{-i\lambda \xi_1 p \cdot u} \langle p, S | \bar{\psi}(\lambda u) \gamma \cdot u \psi(0) | p, S \rangle, \quad (3.13a)$$

$$P_N g_{q/p}(\xi_1) = \int_{-\infty}^{\infty} \frac{d\lambda}{4\pi} e^{-i\lambda \xi_1 p \cdot u} \langle p, S | \bar{\psi}(\lambda u) \gamma \cdot u \gamma_5 \psi(0) | p, S \rangle, \quad (3.13b)$$

where $u = (1, -\hat{z}_B)/\sqrt{2}$ is the lightlike vector opposite to the proton momentum direction (cf. figure 1), and S is the proton spin vector. The FF $D_{h/q}$ of a quark is defined similarly to eq. (2.1), just with the energy flow operator $\mathcal{E}(\mathbf{n})$ removed. Since we consider unpolarized hadron production, only the unpolarized FF is allowed, which sets $\Gamma = \gamma^+/2$ in eq. (2.1). Definitions of gluon PDFs and FFs are similarly given in ref. [37]. The hard coefficients C_{ab} and ΔC_{ab} correspond to the electron scattering off an unpolarized and polarized parton a , respectively, and inclusively producing an unpolarized parton b ,

$$e(\ell) + a(\xi_1 p) \rightarrow e(\ell') + b(p_h/\xi_2) + X(p_X). \quad (3.14)$$

Now we examine the factorization structure when an energy flow is measured around the hadron h . Most of the analysis is identical to the SIDIS [37] so will not be repeated. In the following, we only remark on the differences caused by the energy flow factor in eq. (3.10).

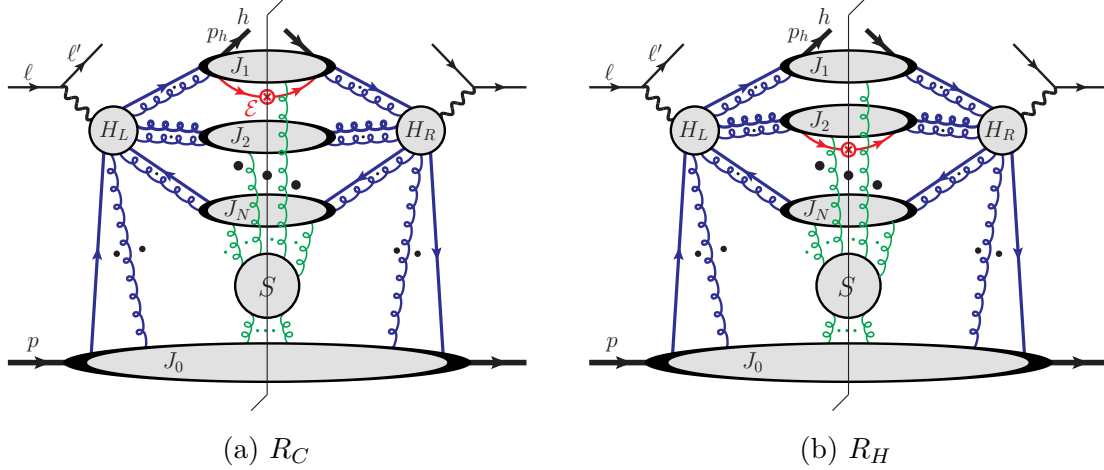


Figure 2. Leading regions (a) R_C and (b) R_H of the energy-tagged SIDIS in eq. (3.1). The cross vertex ($\otimes$) indicates the energy flow, which is in the same collinear subgraph J_1 as the measured hadron h in R_C , but is in a separate collinear subgraph J_2 in R_H . We are using cut diagram notation here: to the left of the cut is the amplitude $\mathcal{M}^X$, and to the right is its complex conjugate. Lines crossed by the cut stand for the final-state particles; they are put on shell with phase space integrated out, except for the observed hadron h and energy flow weighting factor.

The proof of all-order factorization is based on the Libby-Sterman analysis [87–90]. The methodology has been reviewed in many contexts, e.g., refs. [37, 91, 92] and is based on the simple observation that in the limit of $Q \sim p_{hT} \gg \Lambda_{\text{QCD}}$, dominant contributions to $|\overline{\mathcal{M}_E}|^2$ from loop and phase space integrations come from the momentum space regions that would give collinear and soft divergences in the corresponding massless limit. The latter occur if and only if the Feynman integration contour is pinched on shell in multidimensional complex space by two or more coalescing poles from some of the propagators. These pinch singularities are identified by the Landau criterion [93–95]. Each of them can be represented by a reduced diagram formed by those lines that are pinched on shell. Contours of unpinched loop momenta can be deformed such that the associated lines have hard virtualities and correspond to local interactions; they are shrunk into hard vertices in the reduced diagram. After equipping each vertex with a spacetime coordinate, a reduced diagram must correspond to a classical scattering process.

Power counting around each pinch-singular region, or *pinch surface*, quantifies the degree of the collinear or soft divergence in the corresponding massless theory, which in turn quantifies the power of Λ_{QCD}/Q or $\Lambda_{\text{QCD}}/p_{hT}$ of the contributions around the pinch surface in the confined or massive QCD theory. Regions that give the most dominant power are called the leading regions. In the current case, they are given in figure 2 by their corresponding reduced diagrams. The virtual photon $\gamma^*(q)$ initiates a hard interaction subgraph H_L (H_R) to the left (right) of the cut, which is joined to the collinear subgraph J_0 associated with the proton by a physically polarized parton line on either side of the cut, along with an arbitrary number of longitudinally polarized gluon lines. Out of the hard subgraph are produced one or more set of collinear lines, collected by the collinear subgraphs $J_1, \dots, J_N$ ($N \geq 1$), where we choose the observed hadron h to emerge from J_1 , whose direction is defined by p_h . Depending on

whether the energy flow is measured in the same collinear subgraph as h , we distinguish the leading regions into types R_C and R_H . On top of these, there are arbitrarily many soft gluons, collected by the soft subgraph S , connecting all these collinear subgraphs.

3.2.1 Factorization of R_C

The factorization of region R_C is exactly the same as of SIDIS and also shares the similar structure as the dihadron production [96]. For each collinear subgraph, we define two auxiliary lightlike vectors $\bar{w}_i$ and w_i ,

$$\bar{w}_i = (1, \mathbf{n}_i)/\sqrt{2}, \quad w_i = (1, -\mathbf{n}_i)/\sqrt{2}, \quad \bar{w}_i \cdot w_i = 1, \quad (3.15)$$

where $\mathbf{n}_i$ is the unit vector along the direction of J_i , for $i = 0, \dots, N$. [Compared with eq. (3.13), we have defined $w_0 = u$ and $\bar{w}_0 = \bar{u}$.] Using eq. (3.15), we approximate each collinear momentum k_i of J_i by its large component $\hat{k}_i$ when it flows in the hard subgraph,

$$k_i^\mu \mapsto \hat{k}_i^\mu = (k_i \cdot w_i) \bar{w}_i^\mu. \quad (3.16)$$

At the leading power, each J_i is connected to the hard subgraph H_L or H_R by only one quark line or physically polarized gluon. We insert the spinor projector

$$\mathcal{P}_i = \frac{(\gamma \cdot w_i)(\gamma \cdot \bar{w}_i)}{2} \quad \left[\text{or} \quad \bar{\mathcal{P}}_i = \frac{(\gamma \cdot \bar{w}_i)(\gamma \cdot w_i)}{2} \right] \quad (3.17)$$

for each quark line leaving (or entering) the hard subgraph into (or from) the collinear subgraph J_i . This extracts the leading spinor component and projects on shell the quark lines external to the hard part. Similar projectors are defined for physically polarized gluons [37].

Along with these can be arbitrarily many collinear gluons, but with scalar polarizations proportional to their momenta. This fact allows the use of Ward identity to factorize them out of the hard subgraph onto a Wilson line along w_i (on either side of the cut) for each collinear subgraph J_i . Similarly, the leading polarization components of all soft gluons attached to J_i are proportional to $\bar{w}_i$, which allows them to be factorized from the collinear subgraphs onto Wilson lines along $\bar{w}_i$. A simple unitarity argument then shows that the soft Wilson lines are reduced to unity, while all the collinear pinch singularities associated with $J_2, \dots, J_N$ are canceled, such that their integration contours can be deformed to make themselves into the hard region (We refer to refs. [37, 97] for more details).

This results in a factorization formula that differs from eq. (3.12) by replacing the FF with the FEC,

$$\begin{aligned} |\overline{\mathcal{M}_\mathcal{E}}|^2 &\simeq \sum_{a,b} \int \frac{d\xi_1}{\xi_1} \int \frac{d\xi_2}{\xi_2^2} \left\{ \mathcal{D}_{1,h/b}(\xi_2, \Lambda_F, \mu) \left[f_{a/p}(\xi_1, \mu) C_{ab} \left(\frac{x}{\xi_1}, \frac{z}{\xi_2}; \frac{\mathbf{p}_{hT}}{Q}, \frac{Q^2}{\mu^2} \right) \right. \right. \\ &\quad \left. \left. + P_N g_{a/p}(\xi_1, \mu) \Delta C_{ab} \left(\frac{x}{\xi_1}, \frac{z}{\xi_2}; \frac{\mathbf{p}_{hT}}{Q}, \frac{Q^2}{\mu^2} \right) \right] \right. \\ &\quad \left. + \mathcal{H}_{1,h/b}^\perp(\xi_2, \Lambda_F, \mu) h_{a/p}(\xi_1, \mu) \sum_{i,j=1}^2 \frac{(\hat{p}_h \times \mathbf{n}_T)^i}{|\mathbf{n}_T|} T_{ab}^{ij} \left(\frac{x}{\xi_1}, \frac{z}{\xi_2}; \frac{\mathbf{p}_{hT}}{Q}, \frac{Q^2}{\mu^2} \right) s_T^j \right\} \\ &\quad + \mathcal{O}(\Lambda_F/Q, \Lambda_F/p_{hT}). \end{aligned} \quad (3.18)$$

Because of the existence of Collins-type FEC, cf. eq. (2.7), there is an extra term in the third line of eq. (3.18) which allows the occurrence of the transversity PDF $h_{a/p}$ even for unpolarized hadron production. It is defined by

$$s_T^j h_{q/p}(\xi_1) = \int_{-\infty}^{\infty} \frac{d\lambda}{4\pi} e^{-i\lambda \xi_1 p \cdot u} \langle p, S | \bar{\psi}(\lambda u) \gamma \cdot u \gamma^j \gamma_5 \psi(0) | p, S \rangle. \quad (3.19)$$

The hard coefficient is now a transverse spin transfer matrix T_{ab}^{ij} that takes the initial-state quark spin along s_T^j to the final-state quark spin along $(\hat{p}_h \times \mathbf{n}_T)^i$. We note, however, that because the PDF and FEC are defined using different sets of lightcone vectors — $(u, \bar{u})$ and $(w_1, \bar{w}_1)$, respectively — the transverse indices i and j are taken in different coordinate systems [96]: j is in the Breit coordinate system $\hat{x}_B\text{--}\hat{y}_B\text{--}\hat{z}_B$ while i is in the hadron coordinate system $\hat{x}_h\text{--}\hat{y}_h\text{--}\hat{z}_h$, as shown in figure 1.

The hard coefficients C_{ab} , ΔC_{ab} , and T_{ab}^{ij} in eq. (3.18) are defined with amputated partonic cut diagrams traced by certain spinor or Lorentz matrices. For the quark-to-quark channel, $(a, b) = (q, q)$, they are explicitly given in appendix A. By a simple Dirac algebra, they can be re-expressed using partonic helicity amplitudes. The C_{ab} and ΔC_{ab} in eq. (3.18) are the same as those in eq. (3.12) for the SIDIS. The T_{ab}^{ij} is the same as that for dihadron production in SIDIS [96] and would also be the same as that for transversely polarized hadron production in SIDIS.

3.2.2 Factorization of R_H

Since R_C has the hadron and energy flow in the same collinear subgraph, eq. (3.18) holds only when Λ_F is small, i.e., it is accurate up to power corrections of Λ_F/p_{hT} and Λ_F/Q . As Λ_F becomes larger, the region R_H becomes more important, where the energy flow emerges from a different collinear subgraph J_2 , as shown in figure 2(b).

Now that the energy flow and hadron h are independent, it appears unnatural to define the former with respect to h as in eq. (3.10). Rather than changing the definition back to eq. (2.5), we argue that as long as we keep J_2 away from the back-to-back region with J_1 , the energy weight factor in eq. (3.10) can be approximated as

$$\frac{p_i^+}{p_h^+} = \frac{p_i \cdot w_1}{p_h \cdot w_1} \simeq \frac{(p_i \cdot w_2)(\bar{w}_2 \cdot w_1)}{p_h \cdot w_1}, \quad (3.20)$$

up to power suppressed corrections of Λ_{QCD}/Q . This requires $\bar{w}_2 \cdot w_1 = \cos^2(\theta/2)$ to be of order 1, a condition easily satisfied in practice.

Since $p_i \cdot w_2$ is the leading momentum component in the collinear subgraph J_2 , collinear and soft factorization in R_H works in the same way as in R_C , and then unitarity renders the soft factor unity and the unobserved collinear subgraphs $J_3, \dots, J_N$ into the hard part. The nontrivial point here is that we are left with two collinear factors for the final state, one giving the normal FF associated with the hadron h , and the other giving a jet function with the energy flow tagging,

$$J^{\mathcal{E}}(k_2, \eta, \phi) = \frac{1}{N_c} \frac{\bar{w}_2 \cdot w_1}{p_h \cdot w_1} \sum_X \int d^4y e^{ik_2 \cdot y} \langle 0 | \psi(y) | X \rangle \sum_{i \in X} (p_i \cdot w_2) \delta(\eta - \eta_i) \delta(\phi - \phi_i) \\ \times \langle X | \bar{\psi}(0) | 0 \rangle, \quad (3.21)$$

where we suppress the spinor indices and omit the spinor projectors $\mathcal{P}_2$ and $\bar{\mathcal{P}}_2$. Without the energy flow factor, $J^\mathcal{E}$ would reduce to the same jet function as $J_3, \dots, J_N$, and could be absorbed into the hard part. In the following, we show that this fact is unaltered by the energy flow factor.

First, on the exact collinear pinch, every line in the collinear subgraph J_2 is along the direction specified by η and ϕ . Then we can write the energy flow factor in eq. (3.21) as

$$\sum_{i \in X} (p_i \cdot w_2) \delta(\eta - \eta_i) \delta(\phi - \phi_i) \rightarrow (k_2 \cdot w_2) \sum_{i \in X} \delta(\eta - \eta_i) \delta(\phi - \phi_i), \quad (3.22)$$

where i counts every particle in X . This allows us to sum over all possible cuts in eq. (3.21) to cancel the collinear pinch.

The exact collinear limit, however, exists only in perturbative theory, but neither in the full QCD nor experimental measurement. For this purpose, it is more appropriate to consider the delta function as being integrated with an angular function peaked around (η, ϕ) . That is, we replace the energy flow factor in eq. (3.21) as

$$\sum_{i \in X} (p_i \cdot w_2) \delta(\eta - \eta_i) \delta(\phi - \phi_i) \rightarrow \sum_{i \in X} (p_i \cdot w_2) f_R(\eta - \eta_i, \phi - \phi_i), \quad (3.23)$$

where f_R can be chosen as a Gaussian distribution with a width R ,

$$f_R(\eta - \eta_i, \phi - \phi_i) = \frac{1}{2\pi R^2} \exp \left[-\frac{(\eta - \eta_i)^2 + (\phi - \phi_i)^2}{2R^2} \right]. \quad (3.24)$$

Using eq. (3.23) avoids the difficulty of a sharp delta function that involves only particles exactly lying along (η, ϕ) and makes the sum over cuts more convenient.

Now, as we move the cut in eq. (3.21), all the particle lines crossed by the cut line are included in the energy flow weight. Their angular distances from the center (η, ϕ) of the collinear subgraph J_2 are of the order $\Delta R \sim \ell_T / \ell^+ \sim \Lambda_{\text{QCD}} / Q$, with ℓ_T and ℓ^+ being the transverse and plus momenta of any typical line in J_2 . As long as $\Delta R / R \ll 1$, we can approximate eq. (3.24) by

$$f_R(\eta - \eta_i, \phi - \phi_i) \simeq \frac{1}{2\pi R^2}, \quad (3.25)$$

with corrections suppressed by $(\Delta R / R)^2 \sim \Lambda_{\text{QCD}}^2 / (QR)^2$. Then eq. (3.23) becomes independent of the state X ,

$$\sum_{i \in X} (p_i \cdot w_2) f_R(\eta - \eta_i, \phi - \phi_i) \simeq \frac{k_2 \cdot w_2}{2\pi R^2}, \quad (3.26)$$

and the sum-over-cuts argument can be applied to J_2 in the same way as $J_3, \dots, J_N$ [37, 97]. All the collinear singularities associated with them are cancelled, so that they become part of the hard subgraph.

The only left collinear singularities are those associated with the observed hadron h , and are collected by the fragmentation function. The hard coefficient differs from those of eq. (3.12) having an energy tagging in the direction around (η, ϕ) .

Taking the limit $R \rightarrow 0$ then reduces eq. (3.23) back to the delta function. This will not produce any perturbative divergence, but, as we have mentioned, can induce some nonperturbative effects which are cancelled only upon a local average of the angular distribution.

3.2.3 Subtraction of double counting

In sections 3.2.1 and 3.2.2, we have only considered factorizations for the regions R_C and R_H individually. Each of the two regions contains smaller regions within themselves and they also overlap with each other. To achieve an all-order factorization, one needs to add up contributions from all leading regions of all graphs, with necessary subtractions made to remove double counting between the different regions. The subtraction formalism was established systematically in refs. [37, 98]. A detailed discussion of this is beyond the scope of this paper. We only make a few remarks that are important to our specific situation.

First, by virtue of the soft cancellation in any of the leading regions, we only need to consider subtractions associated with collinear regions. Subsequently, any leading region will take the form of R_C or R_H in figure 2, with the soft subgraph S along with its connections to the collinear subgraphs removed.

For a certain region L , a smaller region $L' < L$ is obtained by either (1) moving lines in the hard subgraph to one of the collinear subgraphs, or (2) merging any two or more of the collinear subgraphs into a single one by decreasing their angular separation. The subtraction term is obtained at the integrand level for a fixed (loop and phase space) momentum configuration by successively applying the approximations $\hat{T}_{L'}$ and $\hat{T}_L$, designed for L' and L , respectively. That is, for any graph G , the contribution from these two regions, G_L and $G_{L'}$, is given by

$$G_L + G_{L'} = \hat{T}_L G + \hat{T}_{L'} G - \hat{T}_L \hat{T}_{L'} G. \quad (3.27)$$

In more general case, each region would contain chains of smaller regions nested in an ordered way, and the contribution from this region will be given by a recursive subtraction formula [37].

The type-(1) subtraction does not change the number or structure of the collinear subgraphs. It occurs only in gauge theory and involves longitudinally polarized gluons. Each of these subtraction terms can be factorized in the same way. The result only affects the collinear factor J_1 in both R_C and R_H , modifying the corresponding hard coefficient so as to remove its collinear singularities.

The type-(2) subtraction is more nontrivial and mixes regions of different N as well as the two types R_C and R_H . First, the subtraction associated with the merging of unobserved jets $J_3, \dots, J_N$ in either R_C or R_H is simply zero by use of the same unitarity argument as that for the individual collinear cancellation of these jets themselves. Second, in regions of type R_H , merging an unobserved jet with J_2 gives a subtraction term that is power suppressed by $1/(QR)^2$ in a way similar to eq. (3.26). Third, in regions of type R_C or R_H , merging an unobserved jet with J_1 gives a new collinear subgraph associated with h . The subtraction therefore corresponds to removing the collinear singularity from the hard part.

The most nontrivial subtraction term in our case involves merging J_1 and J_2 in regions of type R_H . This can be perturbatively understood as the following. While the region R_H is factorized into an FF with a hard coefficient in which all collinear singularities associated with $J_2, \dots, J_N$ are canceled, the hard coefficient can contain enhancement associated with small Λ_F or θ as J_2 becomes close to J_1 . However, the latter has already been accounted for by the J_1 factor in a corresponding region of type R_C . The subtraction therefore removes this double counting and modifies the hard coefficient in the factorization of region R_H .

To summarize, the cross section in eq. (3.11) is comprised of two parts. The first is given by a factorization formula in eq. (3.18), which is a good approximation at small Λ_F . The second part takes a factorization form that is similar to eq. (3.12), but the hard coefficient contains an energy tag along with a subtraction of the double counted region with eq. (3.18). It is more accurate at moderate Λ_F values. The subtraction term serves as a matching between the two parts. This factorization structure is similar to the transverse momentum dependent factorization in refs. [98, 99].

Since our aim is to extract the FEC from SIDIS, we will focus on the small Λ_F region in the rest of this paper, and take eqs. (3.11) and (3.18) as our starting point. A further discussion of the two-part factorization structure will be left for future work.

3.3 Leading-twist observables for Collins-type quark FEC

The energy-tagged SIDIS cross section in eq. (3.11) contains a number of azimuthal observables associated with both ϕ_h and ϕ , cf. figure 1. It remains so even if we consider only up to the leading power in $1/Q$. A complete analysis of all the partonic channels of (a, b) in eq. (3.18) is beyond the scope of this paper. Rather, we focus on the channels sensitive to the Collins-type quark FEC. The leading contribution in SIDIS comes from the transition coefficient T_{qq} in eq. (3.18), which requires a transversely polarized proton target in the initial state and observation of the ϕ distribution in the final state. In this subsection, we first give a detailed analysis to the $(a, b) = (q, q)$ channel, for all possible spin configurations of both beam and target, and then provide a LO result. Other channels can be analyzed in a similar way.

3.3.1 Separation of leptonic and partonic subprocesses

In appendix A, we have expressed the hard coefficients C_{ab} , ΔC_{ab} , and T_{ab}^{ij} in terms of quark helicity structures. To further simplify the results in the Breit frame and especially to make the azimuthal modulations manifest, it is convenient to decompose the cross section into leptonic and hadronic parts, as indicated by the one-photon exchange approximation in eq. (3.1). This is done in a standard way [86] by separating the lepton-quark scattering amplitude in eq. (A.5) at the virtual photon $\gamma^*(q)$ which moves along the $-\hat{z}_B$ direction,

$$\mathcal{A}_{\lambda_e \lambda'_e, \lambda_1 \lambda_2}(\ell, k_1, \ell', k_2; X) = \frac{1}{Q^2} \sum_{\lambda_\gamma} (-1)^{\lambda_\gamma} \mathcal{L}_{\lambda_e \lambda'_e}^{\lambda_\gamma}(\ell, q) \mathcal{Q}_{\lambda_1 \lambda_2}^{\lambda_\gamma}(k_1, q, k_2; X), \quad (3.28)$$

where we have restored the helicity indices λ_e and λ'_e of the initials and final-state electrons, respectively, in $\mathcal{A}$, as compared to eq. (A.5). Here, $\lambda_\gamma = \pm 1$ or 0 refers to the helicity of γ^* . It is defined by rewriting its propagator as a sum over polarization vectors,

$$g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} = \sum_{\lambda_\gamma=0, \pm 1} (-1)^{\lambda_\gamma} \epsilon_{\lambda_\gamma}^\mu(q) \epsilon_{\lambda_\gamma}^{\nu*}(q), \quad (3.29)$$

where the $(-1)^{\lambda_\gamma}$ is due to the spacelike q , and the polarization vectors are chosen as

$$\epsilon_\pm^\mu(q) = \frac{1}{\sqrt{2}}(0, \mp 1, i, 0)_{\text{Cartesian}}, \quad \epsilon_0^\mu(q) = \frac{1}{Q}(q^\mu + 2xp^\mu) = \frac{1}{Q}\left(xp^+, \frac{Q^2}{2xp^+}, \mathbf{0}_T\right), \quad (3.30)$$

which satisfy $\epsilon_{\lambda_\gamma}(q) \cdot \epsilon_{\lambda'_\gamma}^*(q) = (-1)^{\lambda_\gamma} \delta_{\lambda_\gamma \lambda'_\gamma}$ and are specified in Cartesian and lightfront coordinates, respectively. In eq. (3.28), $\mathcal{L}_{\lambda_e \lambda'_e}^{\lambda_\gamma}$ is the LO helicity amplitude of eq. (3.1a),

$$\mathcal{L}_{\lambda_e \lambda'_e}^{\lambda_\gamma}(\ell, q) = \bar{u}_e(\ell + q, \lambda'_e) \gamma_\mu u_e(\ell, \lambda_e) \epsilon_{\lambda_\gamma}^{\mu*}(q), \quad (3.31)$$

with u_e and $\bar{u}_e$ being the electron spinors. The $\mathcal{Q}_{\lambda_1 \lambda_2}^{\lambda_\gamma}$ is the helicity amplitude of quark-photon scattering,

$$q(k_1, \lambda_1) + \gamma^*(q, \lambda_\gamma) \rightarrow q(k_2, \lambda_2) + X(k_X), \quad (3.32)$$

and can be formally defined with the QED current operator $J^\mu(x)$ as

$$\mathcal{Q}_{\lambda_1 \lambda_2}^{\lambda_\gamma}(k_1, q, k_2; X) = \langle q(k_2, \lambda_2), X(k_X); \text{out} | \epsilon_{\lambda_\gamma}(q) \cdot J(0) | q(k_1, \lambda_1); \text{in} \rangle. \quad (3.33)$$

The quark momenta k_1 and k_2 have been projected on shell and are related to p and p_h by

$$k_1 = \xi_1 p, \quad k_2 = p_h / \xi_2. \quad (3.34)$$

Thus the partonic version of the SIDIS variables in eqs. (3.3) and (3.4) is

$$\hat{x} = \frac{Q^2}{2k_1 \cdot q} = \frac{x}{\xi_1}, \quad \hat{z} = \frac{p \cdot k_2}{p \cdot q} = \frac{z}{\xi_2}. \quad (3.35)$$

k_2 is further described by its transverse component $k_{2T} = p_{hT} / \xi_2$ and azimuthal angle ϕ_h .

Using eq. (3.28), we can also decompose the hard factor in eq. (A.6) as

$$H_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}(\ell, k_1, \ell', k_2) = \sum_{\lambda_\gamma, \lambda'_\gamma} (-1)^{\lambda_\gamma + \lambda'_\gamma} L_{\lambda_\gamma \lambda'_\gamma}(\ell, q) W_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{\lambda_\gamma, \lambda'_\gamma}(k_1, q, k_2). \quad (3.36)$$

This takes the form of an unnormalized photon density matrix L multiplied by the photon quark hard scattering factor W . The former has averaged over the electron beam spin with the density matrix ρ^e in eq. (3.9),

$$L_{\lambda_\gamma \lambda'_\gamma}(\ell, q) = \frac{1}{Q^4} \sum_{\lambda_e, \bar{\lambda}_e, \lambda'_e} \rho_{\lambda_e \bar{\lambda}_e}^e(P_e) \mathcal{L}_{\lambda_e \lambda'_e}^{\lambda_\gamma}(\ell, q) \left[\mathcal{L}_{\bar{\lambda}_e \lambda'_e}^{\lambda'_\gamma}(\ell, q) \right]^*, \quad (3.37)$$

which introduces the electron polarization P_e and can be explicitly calculated to give

$$\begin{aligned} L_{\pm\pm} &= \frac{1}{Q^2(1-\varepsilon)} \left[1 \pm P_e \sqrt{1-\varepsilon^2} \right], \\ L_{00} &= \frac{1}{Q^2(1-\varepsilon)} (2\varepsilon), \\ L_{+-} &= L_{-+} = \frac{1}{Q^2(1-\varepsilon)} (-\varepsilon) = -\frac{1}{2} L_{00}, \\ L_{\pm 0} &= L_{0\pm} = \frac{1}{Q^2(1-\varepsilon)} \left[\pm \sqrt{\varepsilon(1+\varepsilon)} + P_e \sqrt{\varepsilon(1-\varepsilon)} \right], \end{aligned} \quad (3.38)$$

where we defined ε as the ratio of longitudinal to transverse photon flux,

$$\varepsilon = \frac{L_{00}}{L_{++} + L_{--}} = \frac{1-y}{1-y+y^2/2}, \quad (3.39)$$

which carries the whole dependence of eq. (3.36) on y , or the overall c.m. energy square s by eq. (3.3). The W in eq. (3.36) is defined as

$$W_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{\lambda_\gamma, \lambda'_\gamma}(k_1, q, k_2) = \sum_X \int d\Pi_X (2\pi)^4 \delta^{(4)}(q + k_1 - k_2 - k_X) \\ \times \mathcal{Q}_{\lambda_1 \lambda_2}^{\lambda_\gamma}(k_1, q, k_2; X) \mathcal{Q}_{\lambda'_1 \lambda'_2}^{\lambda'_\gamma*}(k_1, q, k_2; X), \quad (3.40)$$

which includes a sum over X , after subtracting the collinear singularities as discussed in section 3.2.3. Substituting the photon density matrix eq. (3.38) into eq. (3.36), we have

$$H_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}(\ell, k_1, \ell', k_2) = \frac{2}{Q^2(1-\varepsilon)} \left[\frac{1}{2} \left(W_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{+,+} + W_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{-,-} \right) + \varepsilon W_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{0,0} \right. \\ - \frac{\varepsilon}{2} \left(W_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{+,-} + W_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{-,+} \right) + \frac{P_e}{2} \sqrt{1-\varepsilon^2} \left(W_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{+,+} - W_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{-,-} \right) \\ - \frac{1}{2} \sqrt{\varepsilon(1+\varepsilon)} \left(W_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{+,0} + W_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{0,+} - W_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{-,0} - W_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{0,-} \right) \\ \left. - \frac{P_e}{2} \sqrt{\varepsilon(1-\varepsilon)} \left(W_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{+,0} + W_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{0,+} + W_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{-,0} + W_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{0,-} \right) \right]. \quad (3.41)$$

This form makes it straightforward to examine the symmetry properties.

First, by the rotational symmetry around the $\hat{z}_B$ axis, we can separate the ϕ_h dependence from the partonic tensor W ,

$$W_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{\lambda_\gamma, \lambda'_\gamma}(k_1, q, k_2) = W_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{\lambda_\gamma, \lambda'_\gamma}(Q^2, x/\xi_1, z/\xi_2, k_{2T}, \phi_h) \\ = e^{i[(\lambda_1 - \lambda'_1) - (\lambda_\gamma - \lambda'_\gamma)]\phi_h} \bar{W}_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{\lambda_\gamma, \lambda'_\gamma}(Q^2, x/\xi_1, z/\xi_2, k_{2T}). \quad (3.42)$$

Here, $\bar{W} \equiv W(\phi_h = 0)$ corresponds to the (inclusive) final state with $q(k_2)$ moving in the $\hat{x}_B$ - $\hat{z}_B$ plane. Up to the perturbative order we will work in the following, the $q \rightarrow q$ hard coefficients receive contributions only from the non-singlet (NS) channel, where quark line does not break in going from $q(k_1)$ to $q(k_2)$ in the cut diagram of figure 6. Then the chiral symmetry associated with the massless quark approximation implies that $\bar{W}_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{\lambda_\gamma, \lambda'_\gamma}$ is nonzero only when $\lambda_1 = \lambda_2$ and $\lambda'_1 = \lambda'_2$. This allows us to simplify the notations by defining

$$\mathcal{Q}_{\lambda_1 \lambda_2}^{\lambda_\gamma} \stackrel{\text{NS}}{=} \delta_{\lambda_1 \lambda_2} \mathcal{Q}_{\lambda_1}^{\lambda_\gamma}, \quad \bar{W}_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{\lambda_\gamma, \lambda'_\gamma} \stackrel{\text{NS}}{=} \delta_{\lambda_1 \lambda_2} \delta_{\lambda'_1 \lambda'_2} \bar{W}_{\lambda_1 \lambda_1}^{\lambda_\gamma, \lambda'_\gamma}, \quad H_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2} \stackrel{\text{NS}}{=} \delta_{\lambda_1 \lambda_2} \delta_{\lambda'_1 \lambda'_2} H_{\lambda_1 \lambda'_1}. \quad (3.43)$$

The NS channel also has a simple parity constraint,

$$\bar{W}_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2}^{\lambda_\gamma, \lambda'_\gamma}(Q^2, x/\xi_1, z/\xi_2, k_{2T}) = (-1)^{\lambda_\gamma + \lambda'_\gamma} \bar{W}_{-\lambda_1, -\lambda_2, -\lambda'_1, -\lambda'_2}^{-\lambda_\gamma, -\lambda'_\gamma}(Q^2, x/\xi_1, z/\xi_2, k_{2T}). \quad (3.44)$$

Together with the Hermiticity property, $\bar{W}_{\lambda \lambda'}^{\lambda_\gamma \lambda'_\gamma} = (\bar{W}_{\lambda'_\gamma \lambda_\gamma}^{\lambda'_\gamma \lambda_\gamma})^*$, these reduce $\bar{W}$ down to 18 independent real degrees of freedom, including 6 real components,

$$\bar{W}_{++}^{++}, \quad \bar{W}_{--}^{++}, \quad \bar{W}_{+-}^{+-}, \quad \bar{W}_{-+}^{+-}, \quad \bar{W}_{++}^{00}, \quad \bar{W}_{+-}^{00}, \quad (3.45)$$

and 6 complex ones,

$$\bar{W}_{+-}^{++}, \quad \bar{W}_{++}^{+-}, \quad \bar{W}_{++}^{+0}, \quad \bar{W}_{--}^{+0}, \quad \bar{W}_{+-}^{+0}, \quad \bar{W}_{-+}^{+0}. \quad (3.46)$$

Using these relations, we can write the hard coefficients C_{qq} , ΔC_{qq} , and T_{qq}^{ij} in eqs. (A.7) and (A.10) in terms of the $\bar{W}$ components in eqs. (3.45) and (3.46), while making the ϕ_h dependence explicit.

3.3.2 Hard coefficients and azimuthal modulations

The unpolarized hard coefficient C_{qq} is given by

$$\begin{aligned}
 C_{qq} &= \frac{1}{2} (H_{++} + H_{--}) \\
 &= \frac{2}{Q^2(1-\varepsilon)} \left[F_{UU,T} + \varepsilon F_{UU,L} + \sqrt{2\varepsilon(1+\varepsilon)} F_{UU}^{(1)} \cos \phi_h + \varepsilon F_{UU}^{(2)} \cos(2\phi_h) \right. \\
 &\quad \left. + P_e \sqrt{2\varepsilon(1-\varepsilon)} F_{LU} \sin \phi_h \right], \tag{3.47}
 \end{aligned}$$

which is organized by the azimuthal modulations and spin structures. We follow the SIDIS convention [100] to notate the coefficients as $F_{AB} = F_{AB}(Q^2, x/\xi_1, z/\xi_2, p_{hT}^2)$, using the subscripts A and B to separately indicate the electron and proton polarization states, with “ U ”, “ L ”, and “ T ” standing for “unpolarized”, “longitudinally polarized”, or “transversely polarized”, respectively. When F_{AB} receives contributions from both transverse and longitudinal γ^* polarizations, we use a third subscript $C = T$ or L in $F_{AB,C}$ to further distinguish them. Clearly, the $F_{AB,L}$ term will have an extra coefficient ε compared to $F_{AB,T}$. Furthermore, we use a superscript “ (n) ” to indicate the azimuthal modulation $\cos(n\phi_h)$ or $\sin(n\phi_h)$, which distinguishes F ’s with the same spin configuration. The coefficients in eq. (3.47) are given by the $\bar{W}$ components as

$$\begin{aligned}
 F_{UU,T} &= \frac{1}{2} (\bar{W}_{++}^{++} + \bar{W}_{--}^{++}), & F_{UU,L} &= \bar{W}_{++}^{00}, \\
 F_{UU}^{(1)} &= -\frac{1}{\sqrt{2}} \text{Re} (\bar{W}_{++}^{+0} + \bar{W}_{--}^{+0}), & F_{UU}^{(2)} &= -\text{Re} \bar{W}_{++}^{+-}, \\
 F_{LU} &= -\frac{1}{\sqrt{2}} \text{Im} (\bar{W}_{++}^{+0} + \bar{W}_{--}^{+0}). \tag{3.48}
 \end{aligned}$$

The polarized hard coefficient ΔC_{qq} is given by

$$\begin{aligned}
 \Delta C_{qq} &= \frac{1}{2} (H_{++} - H_{--}) \\
 &= \frac{2}{Q^2(1-\varepsilon)} \left\{ \sqrt{2\varepsilon(1+\varepsilon)} F_{UL}^{(1)} \sin \phi_h + \varepsilon F_{UL}^{(2)} \sin(2\phi_h) \right. \\
 &\quad \left. + P_e \left[\sqrt{1-\varepsilon^2} F_{LL} + \sqrt{2\varepsilon(1-\varepsilon)} F_{LL}^{(1)} \cos \phi_h \right] \right\}, \tag{3.49}
 \end{aligned}$$

where the coefficients are

$$\begin{aligned}
 F_{UL}^{(1)} &= -\frac{1}{\sqrt{2}} \text{Im} (\bar{W}_{++}^{+0} - \bar{W}_{--}^{+0}), & F_{UL}^{(2)} &= -\text{Im} \bar{W}_{++}^{+-}, \\
 F_{LL} &= \frac{1}{2} (\bar{W}_{++}^{++} - \bar{W}_{--}^{++}), & F_{LL}^{(1)} &= -\frac{1}{\sqrt{2}} \text{Re} (\bar{W}_{++}^{+0} - \bar{W}_{--}^{+0}). \tag{3.50}
 \end{aligned}$$

As shown in eq. (3.18), the hard coefficient T_{qq}^{ij} transfers the initial-state quark’s transverse spin $\mathbf{s}_T$ to that of the final-state quark, $S_{qT}^i \propto T_{qq}^{ij} s_T^j$. This transfer is most easily understood by referring to the scattering plane of $q(k_1) + \gamma^*(q) \rightarrow q(k_2)$ [96], where $\mathbf{s}_T$ has a component $s_{\parallel} = s_T \cos(\phi_S - \phi_h)$ on the plane and a component $s_{\perp} = s_T \sin(\phi_S - \phi_h)$ perpendicular to the plane, and similarly for $\mathbf{S}_{qT} = (S_{q\parallel}, S_{q\perp})$ of the final-state quark. In the simplest

case, let us consider the γ^* as unpolarized. Then by performing a parity transformation together with a rotation around this plane by π , one can easily see that the transverse spin components $s_\perp$ and $S_{q\perp}$ perpendicular to the plane stay unchanged, while the components $s_\parallel$ and $S_{q\parallel}$ in the plane are flipped. Hence, if parity is conserved, we should expect these spin components to transfer separately without mixing, i.e., $(S_{q\parallel}, S_{q\perp}) \propto (s_\parallel, s_\perp)$. Now noting that the components $(S_{q\parallel}, S_{q\perp})$ directly align with the components (S_{qT}^x, S_{qT}^y) in the hadron frame in eq. (3.5), we have $\mathbf{S}_{qT} \propto s_T (\cos(\phi_S - \phi_h), \sin(\phi_S - \phi_h))$. That is, the T_{qq}^{ij} is a rotation matrix of angle $-\phi_h$.

The interference of different γ^* helicities can induce additional $(\pm 1)\phi_h$ or $(\pm 2)\phi_h$ phase shift relative to that due to the basic rotation of $-\phi_h$. Therefore, we define the rotation basis for the spin transfer matrix T_{qq}^{ij} ,

$$t_n(\phi_h) = R_z(-(1+n)\phi_h) = \begin{pmatrix} \cos(1+n)\phi_h & \sin(1+n)\phi_h \\ -\sin(1+n)\phi_h & \cos(1+n)\phi_h \end{pmatrix}, \quad (3.51)$$

which is a rotation around the z axis by $-(1+n)\phi_h$. When parity-odd effects are involved, which arise due to the electron polarization P_e , the spin components in and perpendicular to the hard scattering plane will be transferred into each other, inducing a further rotation by $\pi/2$ on top of $t_n(\phi_h)$. Thus we also define the parity-odd rotation basis,

$$\tilde{t}_n(\phi) = R_z(\pi/2) R_z(-(1+n)\phi) = \begin{pmatrix} \sin(1+n)\phi & -\cos(1+n)\phi \\ \cos(1+n)\phi & \sin(1+n)\phi \end{pmatrix}. \quad (3.52)$$

Using these, we can write T_{qq}^{ij} as

$$\begin{aligned} (T_{qq}^{ij}) &\equiv \begin{pmatrix} T_{qq}^{xx} & T_{qq}^{xy} \\ T_{qq}^{yx} & T_{qq}^{yy} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} H_{+-} + H_{-+} & -i(H_{+-} - H_{-+}) \\ i(H_{+-} - H_{-+}) & H_{+-} + H_{-+} \end{pmatrix} \\ &= \frac{2}{Q^2(1-\varepsilon)} \left\{ (F_{UT,T} + \varepsilon F_{UT,L}) t_0(\phi_h) - \sqrt{2\varepsilon(1+\varepsilon)} [F_{UT}^{(1+)} t_1(\phi_h) + F_{UT}^{(1-)} t_{-1}(\phi_h)] \right. \\ &\quad \left. - \varepsilon [F_{UT}^{(2+)} t_2(\phi_h) + F_{UT}^{(2-)} t_{-2}(\phi_h)] \right. \\ &\quad \left. + P_e \left[\sqrt{1-\varepsilon^2} F_{LT} \tilde{t}_0(\phi_h) + \sqrt{2\varepsilon(1-\varepsilon)} (F_{LT}^{(1+)} \tilde{t}_1(\phi_h) + F_{LT}^{(1-)} \tilde{t}_{-1}(\phi_h)) \right] \right\}, \quad (3.53) \end{aligned}$$

where the coefficients are expressed in $\bar{W}$ as

$$\begin{aligned} F_{UT,T} &= \text{Re } \bar{W}_{+-}^{++}, & F_{UT,L} &= \text{Re } \bar{W}_{+-}^{00}, \\ F_{UT}^{(1+)} &= \frac{1}{\sqrt{2}} \text{Re } \bar{W}_{-+}^{+0}, & F_{UT}^{(1-)} &= \frac{1}{\sqrt{2}} \text{Re } \bar{W}_{+-}^{+0}, \\ F_{UT}^{(2+)} &= \frac{1}{2} \text{Re } \bar{W}_{-+}^{+-}, & F_{UT}^{(2-)} &= \frac{1}{2} \text{Re } \bar{W}_{+-}^{+-}, \\ F_{LT} &= -\text{Im } \bar{W}_{+-}^{++}, & F_{LT}^{(1+)} &= -\frac{1}{\sqrt{2}} \text{Im } \bar{W}_{-+}^{+0}, & F_{LT}^{(1-)} &= \frac{1}{\sqrt{2}} \text{Im } \bar{W}_{+-}^{+0}. \end{aligned} \quad (3.54)$$

These results show an interesting interplay between the spin transfer and azimuthal ϕ_h modulations. We have labeled coefficients of the rotation matrices $t_{\pm n}(\phi_h)$ or $\tilde{t}_{\pm n}(\phi_h)$ with superscripts “ $(n\pm)$ ” to indicate the helicity transfer unit originating from the γ^* interference.

Substituting eqs. (3.47) (3.49) and (3.53) back into eq. (3.18) exhibits all the azimuthal modulations. In particular, we note that the ϕ distribution of the energy flow around the measured hadron h is given by the transverse spin transfer T_{qq}^{ij} , with the two rotation modes in eqs. (3.51) and (3.52) generating, respectively, the modulations,

$$\begin{aligned} \frac{(\hat{p}_h \times \mathbf{n}_T)^i}{|\mathbf{n}_T|} t_n^{ij}(\phi_h) s_T^j &= -s_T \sin[\phi - \phi_S + (n+1)\phi_h], \\ \frac{(\hat{p}_h \times \mathbf{n}_T)^i}{|\mathbf{n}_T|} \tilde{t}_n^{ij}(\phi_h) s_T^j &= s_T \cos[\phi - \phi_S + (n+1)\phi_h]. \end{aligned} \quad (3.55)$$

Combining these with eq. (3.11) then gives the energy flow-weighted cross section,

$$\begin{aligned} &\left[\frac{y^2}{z(1-\varepsilon)} \frac{\alpha_e^2}{(8\pi^2 Q^2)^2} \right]^{-1} \frac{d\Sigma}{dx dQ^2 d\phi_S dz dp_{hT}^2 d\phi_h d\eta d\phi} \\ &= \mathcal{C}_{F_{UU,T}}[f, \mathcal{D}] + \varepsilon \mathcal{C}_{F_{UU,L}}[f, \mathcal{D}] \\ &\quad + \sqrt{2\varepsilon(1+\varepsilon)} \mathcal{C}_{F_{UU}^{(1)}}[f, \mathcal{D}] \cos \phi_h + \varepsilon \mathcal{C}_{F_{UU}^{(2)}}[f, \mathcal{D}] \cos(2\phi_h) \\ &\quad + P_e \sqrt{2\varepsilon(1-\varepsilon)} \mathcal{C}_{F_{LU}}[f, \mathcal{D}] \sin \phi_h \\ &\quad + P_N \left\{ \sqrt{2\varepsilon(1+\varepsilon)} \mathcal{C}_{F_{UL}^{(1)}}[g, \mathcal{D}] \sin \phi_h + \varepsilon \mathcal{C}_{F_{UL}^{(2)}}[g, \mathcal{D}] \sin(2\phi_h) \right\} \\ &\quad + P_e P_N \left\{ \sqrt{1-\varepsilon^2} \mathcal{C}_{F_{LL}}[g, \mathcal{D}] + \sqrt{2\varepsilon(1-\varepsilon)} \mathcal{C}_{F_{LL}^{(1)}}[g, \mathcal{D}] \cos \phi_h \right\} \\ &\quad + s_T \left\{ \left[-\mathcal{C}_{F_{UT,T}}[h, \mathcal{H}^\perp] - \varepsilon \mathcal{C}_{F_{UT,L}}[h, \mathcal{H}^\perp] \right] \sin(\phi - \phi_S + \phi_h) \right. \\ &\quad \quad + \sqrt{2\varepsilon(1+\varepsilon)} \left[\mathcal{C}_{F_{UT}^{(1+)}}[h, \mathcal{H}^\perp] \sin(\phi - \phi_S + 2\phi_h) + \mathcal{C}_{F_{UT}^{(1-)}}[h, \mathcal{H}^\perp] \sin(\phi - \phi_S) \right] \\ &\quad \quad \left. + \varepsilon \left[\mathcal{C}_{F_{UT}^{(2+)}}[h, \mathcal{H}^\perp] \sin(\phi - \phi_S + 3\phi_h) + \mathcal{C}_{F_{UT}^{(2-)}}[h, \mathcal{H}^\perp] \sin(\phi - \phi_S - \phi_h) \right] \right\} \\ &\quad + P_e s_T \left\{ \sqrt{1-\varepsilon^2} \mathcal{C}_{F_{LT}}[h, \mathcal{H}^\perp] \cos(\phi - \phi_S + \phi_h) \right. \\ &\quad \quad \left. + \sqrt{2\varepsilon(1-\varepsilon)} \left[\mathcal{C}_{F_{LT}^{(1+)}}[h, \mathcal{H}^\perp] \cos(\phi - \phi_S + 2\phi_h) + \mathcal{C}_{F_{LT}^{(1-)}}[h, \mathcal{H}^\perp] \cos(\phi - \phi_S) \right] \right\} \\ &\quad + (\text{other partonic channels}) + \mathcal{O}(\Lambda_F/Q, \Lambda_F/p_{hT}), \end{aligned} \quad (3.56)$$

where we introduced the notation for the convolution,

$$\begin{aligned} \mathcal{C}_F[f, \mathcal{D}] &= \mathcal{C}_F[f, \mathcal{D}](Q^2, x, z, p_{hT}^2; \Lambda_F) \\ &\equiv \int \frac{d\xi_1}{\xi_1} \int \frac{d\xi_2}{\xi_2^2} f_{q/p}(\xi_1) \mathcal{D}_{1,h/q}(\xi_2, \Lambda_F) F(Q^2, x/\xi_1, z/\xi_2, p_{hT}^2/\xi_2^2), \end{aligned} \quad (3.57)$$

with f being a PDF, $\mathcal{D}$ being an FEC, and F being a hard coefficient in eq. (3.48) (3.50) or (3.54).

It is the various ϕ modulations in eq. (3.56) that help to probe the Collins-type FEC. We re-emphasize the associated physics as follows. The matrix T_{qq} arising from the hard scattering in eq. (3.32) transfers the transverse spin from the initial-state quark to the

final-state quark, thus defining the transverse spin magnitude and direction of the latter. The fragmentation process of the final-state quark then conveys this spin information into an observational signal via the azimuthal energy deposition asymmetry by eq. (2.8). These modulations have been explicitly spelled out in eq. (3.56), appearing in simple trigonometric forms which allow their coefficients to be extracted by standard harmonic analysis. Clearly, due to chiral symmetry, nontrivial ϕ modulations and target transverse spin asymmetries (i.e., terms proportional to s_T) are correlated with each other, such that the ϕ dependence always appears in the form of $\phi - \phi_S$. Without measuring the ϕ distribution, all the terms involving s_T , Collins-type FEC, and transversity PDF will vanish.

While we have included all the polarization configurations in eq. (3.56), in practice, one may have only a subset of those terms. For example, we need only $s_T \neq 0$ to probe the Collins-type FEC, in which case all the terms involving P_e or P_N can be set to zero.

At LO, other partonic channels contributing in eq. (3.56) include $\bar{q} \rightarrow \bar{q}$, $q(\bar{q}) \rightarrow g$, $g \rightarrow q(\bar{q})$. As we will note in section 3.3.3, the $\bar{q} \rightarrow \bar{q}$ channel can be simply obtained from the $q \rightarrow q$ channel by a charge conjugation. The $q(\bar{q}) \rightarrow g$ channel will involve the gluon FEC, which has a linear polarization component that induces a $\cos(2\phi)$ or $\sin(2\phi)$ modulation. While this is an interesting signal by itself, it can be generated purely from perturbative origin [12, 13, 48, 49, 101], not intrinsically sensitive to chiral symmetry breaking; the modulations can also be readily distinguished from the quark case. Similarly, for the $g \rightarrow q(\bar{q})$ channel, only the unpolarized quark FEC can arise. Neither of these two channels allows the transversity PDF at the leading twist. Including them in a full analysis is left as a future development; we focus our attention on the $q \rightarrow q$ channel in this paper.

3.3.3 LO results for the $q \rightarrow q$ channel

Since we require a finite (and large) p_{hT} for the final-state hadron, the LO hard scattering has $X(k_X) = g(k_g)$ in eq. (3.32), given by

$$q(k_1, \lambda) + \gamma^*(q, \lambda_\gamma) \rightarrow q(k_2, \lambda) + g(k_g, \lambda_g). \quad (3.58)$$

The two diagrams are shown in figure 3. We denote its helicity amplitude as $\mathcal{Q}_{\lambda\lambda_g}^{\lambda_\gamma}$, making use of the chiral symmetry given in eq. (3.43). The partonic tensor defined in eq. (3.40) is given at this order by

$$W_{\lambda\lambda'}^{\lambda_\gamma\lambda'_\gamma}(k_1, q, k_2) = \sum_{\lambda_g=\pm 1} \int \frac{d^3\mathbf{k}_g}{(2\pi)^3 2|\mathbf{k}_g|} (2\pi)^4 \delta^{(4)}(k_1 + q - k_2 - k_g) \mathcal{Q}_{\lambda\lambda_g}^{\lambda_\gamma} \mathcal{Q}_{\lambda'\lambda_g}^{\lambda'_\gamma*}, \quad (3.59)$$

with the gluon spin summed over and phase space integrated out.

The partonic kinematics is described by the variables $\hat{x}$ and $\hat{z}$ defined in eq. (3.35), Q^2 , $k_{2T} = p_{hT}/\xi_2$, and ϕ_h . We also define the partonic c.m. energy squared,

$$\hat{s} = (k_1 + q)^2 = \frac{Q^2(1 - \hat{x})}{\hat{x}}. \quad (3.60)$$

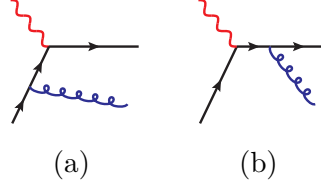


Figure 3. LO diagrams of the $q + \gamma^* \rightarrow q + g$ scattering amplitude.

After integrating over $\mathbf{k}_g$ with the delta function in eq. (3.59) and factoring out the ϕ_h phase as in eq. (3.42), we normalize $W_{\lambda\lambda'}^{\lambda_\gamma\lambda'_\gamma}$ by

$$W_{\lambda\lambda'}^{\lambda_\gamma\lambda'_\gamma}(\hat{x}, \hat{z}, Q^2, k_{2T}^2, \phi_h) = \left(8\pi^2 e_q^2 \alpha_s C_F\right) \hat{z} \delta(k_{2T}^2 - \hat{z}(1 - \hat{z})\hat{s}) \times e^{i[(\lambda - \lambda') - (\lambda_\gamma - \lambda'_\gamma)]\phi_h} \cdot \bar{w}_{\lambda\lambda'}^{\lambda_\gamma\lambda'_\gamma}(\hat{x}, \hat{z}), \quad (3.61)$$

where e_q is the electric charge of q , being $2/3$ and $-1/3$ for the up and down quarks, respectively, and $C_F = 4/3$ is a color factor in QCD. We see from the remaining delta function that k_{hT} is not independent of $\hat{z}$ and $\hat{x}$; it is given by

$$k_{2T}^2 = \hat{z}(1 - \hat{z})\hat{s} = Q^2 \frac{\hat{z}}{\hat{x}} (1 - \hat{z})(1 - \hat{x}). \quad (3.62)$$

This will not be the case beyond LO.

We present the explicit expressions of $\bar{w}_{\lambda\lambda'}^{\lambda_\gamma\lambda'_\gamma}$ in terms of the hard coefficients in eqs. (3.48), (3.50) and (3.54). Let us use a similar normalization to eq. (3.61),

$$F_X(Q^2, \hat{x}, \hat{z}, k_{2T}^2) = \left(8\pi^2 e_q^2 \alpha_s C_F\right) \hat{z} \delta(k_{2T}^2 - \hat{z}(1 - \hat{z})\hat{s}) \cdot f_X(\hat{x}, \hat{z}), \quad (3.63)$$

where X refers to the super- and/or sub-scripts of F . Then the hard coefficients for unpolarized target are

$$\begin{aligned} f_{UU,T} &= \frac{1}{2} (\bar{w}_{++}^{++} + \bar{w}_{--}^{++}) &= 2 \left[\frac{1 + \hat{x}^2 \hat{z}^2}{(1 - \hat{x})(1 - \hat{z})} + (1 - \hat{x})(1 - \hat{z}) \right], \\ f_{UU,L} &= \bar{w}_{++}^{00} &= 8\hat{x}\hat{z}, \\ f_{UU}^{(1)} &= -\frac{1}{\sqrt{2}} \text{Re} (\bar{w}_{++}^{+0} + \bar{w}_{--}^{+0}) &= -4 \sqrt{\frac{\hat{x}\hat{z}}{(1 - \hat{x})(1 - \hat{z})}} [(1 - \hat{x})(1 - \hat{z}) + \hat{x}\hat{z}], \\ f_{UU}^{(2)} &= -\text{Re} \bar{w}_{++}^{+-} &= 4\hat{x}\hat{z}, \\ f_{LU} &= -\frac{1}{\sqrt{2}} \text{Im} (\bar{w}_{++}^{+0} + \bar{w}_{--}^{+0}) &= 0. \end{aligned} \quad (3.64)$$

For polarized target, we have the hard coefficients,

$$\begin{aligned} f_{UL}^{(1)} &= -\frac{1}{\sqrt{2}} \text{Im} (\bar{w}_{++}^{+0} - \bar{w}_{--}^{+0}) &= 0, \\ f_{UL}^{(2)} &= -\text{Im} \bar{w}_{++}^{+-} &= 0, \\ f_{LL} &= \frac{1}{2} (\bar{w}_{++}^{++} - \bar{w}_{--}^{++}) &= 2 \left[\frac{1 + \hat{x}^2 \hat{z}^2}{(1 - \hat{x})(1 - \hat{z})} - (1 - \hat{x})(1 - \hat{z}) \right], \\ f_{LL}^{(1)} &= -\frac{1}{\sqrt{2}} \text{Re} (\bar{w}_{++}^{+0} - \bar{w}_{--}^{+0}) &= 4 \sqrt{\frac{\hat{x}\hat{z}}{(1 - \hat{x})(1 - \hat{z})}} [(1 - \hat{x})(1 - \hat{z}) - \hat{x}\hat{z}]. \end{aligned} \quad (3.65)$$

For transversely polarized target, the nonzero hard coefficients are

$$\begin{aligned}
 f_{UT,T} &= \bar{w}_{+-}^{++} &= 4, \\
 f_{UT}^{(1-)} &= \frac{1}{\sqrt{2}} \bar{w}_{+-}^{+0} &= 4 \sqrt{\frac{\hat{x}\hat{z}}{(1-\hat{x})(1-\hat{z})}}, \\
 f_{UT}^{(2-)} &= \frac{1}{2} \bar{w}_{+-}^{+-} &= -\frac{4\hat{x}\hat{z}}{(1-\hat{x})(1-\hat{z})},
 \end{aligned} \tag{3.66}$$

while all the other coefficients are 0,

$$f_{UT,L} = f_{UT}^{(1+)} = f_{UT}^{(2+)} = f_{LT} = f_{LT}^{(1+)} = f_{LT}^{(1-)} = 0. \tag{3.67}$$

Although we have not explicitly calculated the antiquark-to-antiquark ($\bar{q} \rightarrow \bar{q}$) channel, its properties can be inferred from our results for the quark channel. We summarize the key points as follows: (1) The hard coefficients are invariant under charge conjugation (C) and are therefore identical for the $\bar{q} \rightarrow \bar{q}$ and $q \rightarrow q$ channels. (2) The FECs are also related by C symmetry. For instance, the FEC for a $\bar{d}$ antiquark fragmenting into a π^0 is identical to that for a d quark fragmenting into a π^0 . (3) The most significant difference in the overall contribution stems from the PDFs. The cross section for the antiquark channel is weighted by the antiquark PDF, $f_{\bar{q}/p}(x), g_{\bar{q}/p}(x), h_{\bar{q}/p}(x)$, which is known to be different from the quark PDF, $f_{q/p}(x), g_{q/p}(x), h_{q/p}(x)$, especially at large x .

Eq. (3.63) along with eqs. (3.64)–(3.67) can be directly inserted into eq. (3.57) to give the convolution with PDF and FEC,

$$\begin{aligned}
 \mathcal{C}_{F_X}[f, \mathcal{D}] &= \frac{8\pi^2 e_q^2 \alpha_s C_F}{z} \int_x^1 \frac{d\xi_1}{\xi_1} \int_z^1 \frac{d\xi_2}{\xi_2} f_{q/p}(\xi_1) \mathcal{D}_{1,h/q}(\xi_2, \Lambda_F) f_X\left(\frac{x}{\xi_1}, \frac{z}{\xi_2}\right) \\
 &\times \delta\left(\frac{p_{hT}^2}{z^2} - Q^2 \frac{(\xi_1 - x)(\xi_2 - z)}{xz}\right).
 \end{aligned} \tag{3.68}$$

These can be readily integrated numerically, given an input of the PDF and FEC, to make predictions for the cross section in eq. (3.56).

4 FECs from energy-tagged SIDIS: inclusive with p_{hT}

It is advantageous in many cases to only record the longitudinal momentum fraction z in eq. (3.4) of the final-state hadron h , while integrating over its transverse momentum $\mathbf{p}_{hT}$. First, most of the events lie in the small p_{hT} region, where the factorization in eq. (3.18) does not apply. It needs to be replaced by a TMD factorization involving a TMD PDF of the target and TMD FEC of h [100, 102]. As discussed in section 2.2, we regard FEC as an alternative way to probe transverse fragmentation dynamics. Following that, it is the *collinear* factorization in eq. (3.4) that grants the FEC a unique advantage as compared to TMD distributions. Hence, rather than seeking a TMD factorization at small p_{hT} , we inclusively integrate over it. This still leads to a collinear factorization, with the hard scale associated with h being its minus momentum in the proton- γ^* c.m. frame,

$$p_{h,\text{c.m.}}^- = zQ \sqrt{\frac{1-x}{2x}} \gg m_h, \tag{4.1}$$

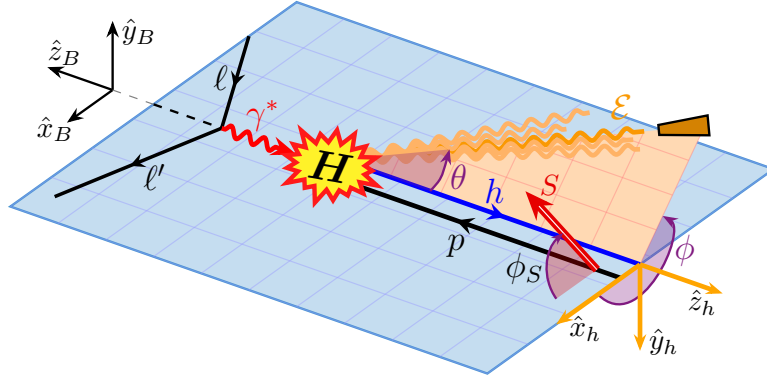


Figure 4. An illustration for the energy-tagged SIDIS with $\mathbf{p}_{hT}$ integrated. The kinematics shown here corresponds to the LO QCD process, where the recoiled quark moves along $-\hat{z}_B$ in the Breit frame.

which requires z not to be too small and x away from 1. Second, integrating over $\mathbf{p}_{hT}$ greatly reduces the differential cross section formula in eq. (3.56), making experimental extraction of the azimuthal modulation coefficients much simplified. The azimuthal angle ϕ still serves as a useful observable. It is defined on an event-by-event basis for each finite value of p_{hT} with respect to the hadron coordinate system in eq. (3.5). For $p_{hT} = 0$, we choose the latter as given by the backward-scattering limit of eq. (3.5) in the plane $\phi_h = \pi$, related to the Breit frame system by

$$\hat{z}_h = -\hat{z}_B, \quad \hat{x}_h = \hat{x}_B, \quad \hat{y}_h = -\hat{y}_B, \quad (4.2)$$

which is illustrated in figure 4. Its justification will be presented below in eq. (4.10).

In this section, we establish the framework for such an “inclusive” description. Roughly speaking, the cross section is obtained by integrating over p_{hT}^2 and ϕ_h in eq. (3.56), but there are some subtleties to be addressed. First, one immediately notices that for the $\mathbf{p}_{hT}$ integration not to affect the FECs, it is crucial for the latter to only depend on z and Λ_F , but not explicitly on the transverse momentum $\mathbf{p}_{hT}$ of h . This is the reason why we have reformulated FECs to be boost-invariant in section 2.3. Second, upon the integration of ϕ_h , all the modulations in eq. (3.56) that contain ϕ_h naively vanish. However, this is only true if the p_{hT}^2 integral is well behaved, especially as $p_{hT}^2 \rightarrow 0$. As we see from eqs. (3.64)–(3.66), the hard coefficients $f_{UU,T}$, f_{LL} , and $f_{UT}^{(2-)}$ do get singular at $\hat{z} = 1$, or $p_{hT}^2 = 0$. At this collinear kinematics, ϕ_h is undefined, and so the ϕ_h integral seems to become ambiguous. Addressing this issue is a main focus of this section. As it turns out, the singular p_{hT}^2 integration will produce a nonzero result to the ϕ_h integration, which contains collinear singularities that cancel with those from the virtual gluon loop corrections.

4.1 Factorization with $\vec{p}_{hT}$ integrated

To properly treat the collinear singularity as $p_{hT} \rightarrow 0$ in the $\mathbf{p}_T$ integral, we use dimensional regularization (DR) in $d = 4 - 2\epsilon$ dimensions to regulate both ultraviolet (UV) and IR divergences. This affects both the loop momenta and final-state phase space. First, we write

the phase space of the observed hadron h as

$$\frac{d^{d-1}\mathbf{p}_h}{(2\pi)^{d-1}2|\mathbf{p}_h|} = \frac{dz d^{d-2}\mathbf{p}_{hT}}{2z(2\pi)^{d-1}}. \quad (4.3)$$

This modifies the SIDIS cross section in eq. (3.6) to

$$\frac{d\sigma}{dx dQ^2 d\phi_S dz} = \frac{y^2}{z Q^2} \frac{\alpha_e^2}{4(2\pi)^{4-2\epsilon}} \int d^{d-2}\mathbf{p}_{hT} \overline{|\mathcal{M}|^2}, \quad (4.4)$$

where we only change the hadronic subprocess to d dimensions, while keeping the QED interaction and electron phase space in 4 dimensions; no ambiguity arises as we only work to the LO in QED. The $\overline{|\mathcal{M}|^2}$ in eq. (4.4) differs from eq. (3.8) in two ways: (1) the phase space Π_X of unobserved particles in X and the 4-momentum conservation delta function are in d dimensions, and (2) there is no requirement for h to carry large transverse momentum, and thus we allow the sum over X to include states with all particles collinear to the target.

The same modification applies to the energy flow-weighted cross section in eq. (3.11), which becomes

$$\frac{d\Sigma}{dx dQ^2 d\phi_S dz d\eta d\phi} = \frac{y^2}{z Q^2} \frac{\alpha_e^2}{4(2\pi)^{4-2\epsilon}} \int d^{d-2}\mathbf{p}_{hT} \overline{|\mathcal{M}_{\mathcal{E}}|^2}. \quad (4.5)$$

On the right-hand side the (η, ϕ) of the energy flow are fixed, which means that the energy flow keeps a fixed configuration with respect to the hadron h as its $\mathbf{p}_{hT}$ is integrated event by event. We restrict in the R_C region where Λ_F is small, so that the energy flow $\mathcal{E}$ resides in the nonperturbative FEC function and does not affect the perturbative calculation of the hard coefficients. Hence for simplicity, we keep the four-dimensional definition of the energy flow operator and (η, ϕ, Λ_F) . The $\overline{|\mathcal{M}_{\mathcal{E}}|^2}$ in eq. (4.5) differs from that in eq. (3.11) in the same way as noted above.

For the factorization of eq. (4.5) in the R_C region, the $\int d^{d-2}\mathbf{p}_{hT} \overline{|\mathcal{M}_{\mathcal{E}}|^2}$ contains an additional leading region compared to the $\overline{|\mathcal{M}_{\mathcal{E}}|^2}$ in eq. (3.11), which has $N = 1$ in figure 2(a), that is, with only two collinear subgraphs associated with the target p and observed hadron h , respectively. The approximations used for collinear and soft lines in section 3.2.1, however, stay unaffected, and lead to a factorization similar to eq. (3.18). The differences are:

- the hard coefficients that describe the partonic scattering in eq. (3.14) now contain $X = \text{vacuum}$, with $k_X = 0$; and
- the integration $\int d^{d-2}\mathbf{p}_{hT}$ propagates to the partonic level and becomes the integration over $\mathbf{k}_{2T} = \mathbf{p}_{hT}/\xi_2$ of the parton b in eq. (3.14) that fragments into h ,

$$\int d^{d-2}\mathbf{p}_{hT} = \xi_2^{2-2\epsilon} \int d^{d-2}\mathbf{k}_{2T}. \quad (4.6)$$

The factor ξ_2^2 in eq. (4.6) cancels the denominator in eq. (3.18), and the remaining $\xi_2^{-2\epsilon}$ is included in the definitions of FECs to properly normalize the latter and equip them with

probabilistic interpretations in d dimensions [37]. Also assigning the $(2\pi)^{2\epsilon}$ factor in eq. (4.5) to the hard coefficients C , ΔC , and T , we have the factorization formula,

$$\begin{aligned}
 & (2\pi)^{2\epsilon} \int d^{d-2} \mathbf{p}_{hT} |\overline{\mathcal{M}_{\mathcal{E}}}|^2 \\
 &= \sum_{a,b} \int \frac{d\xi_1}{\xi_1} \int d\xi_2 \left\{ \mathcal{D}_{1,h/b}(\xi_2, \Lambda_F, \mu) \left[f_{a/p}(\xi_1, \mu) [C_{ab}] \left(\frac{x}{\xi_1}, \frac{z}{\xi_2}, \frac{Q^2}{\mu^2} \right) \right. \right. \\
 & \quad \left. \left. + P_N g_{a/p}(\xi_1, \mu) [\Delta C_{ab}] \left(\frac{x}{\xi_1}, \frac{z}{\xi_2}, \frac{Q^2}{\mu^2} \right) \right] \right. \\
 & \quad \left. + \mathcal{H}_{1,h/b}^\perp(\xi_2, \Lambda_F, \mu) h_{a/p}(\xi_1, \mu) \sum_{i,j=1}^2 \frac{(\hat{p}_h \times \mathbf{n}_T)^i}{|\mathbf{n}_T|} [T_{ab}^{ij}] \left(\frac{x}{\xi_1}, \frac{z}{\xi_2}, \frac{Q^2}{\mu^2} \right) s_T^j \right\} \\
 &+ \mathcal{O}(\Lambda_F/Q, \Lambda_F/p_{h,\text{c.m.}}^-). \tag{4.7}
 \end{aligned}$$

The hard coefficients in eq. (4.7) are given by those in eq. (3.18) under an extra $\mathbf{k}_{2T}$ integration,⁴

$$\begin{aligned}
 & \{[C_{ab}], [\Delta C_{ab}], [T_{ab}^{ij}]\} \left(\frac{x}{\xi_1}, \frac{z}{\xi_2}, \frac{Q^2}{\mu^2} \right) \\
 &= (2\pi)^{2\epsilon} \int d^{d-2} \mathbf{k}_{2T} \left\{ C_{ab}, \Delta C_{ab}, T_{ab}^{ij} \right\} \left(\frac{x}{\xi_1}, \frac{z}{\xi_2}, \frac{\mathbf{k}_{2T}}{Q}, \frac{Q^2}{\mu^2} \right). \tag{4.8}
 \end{aligned}$$

Note that although the prefactor $\hat{p}_h \times \mathbf{n}_T$ of the spin transfer coefficient T depends on direction of $\mathbf{p}_h$, we are fixing the orientation of the energy flow *relative* to h , so that $(\hat{p}_h \times \mathbf{n}_T)^i/|\mathbf{n}_T| = (-\sin\phi, \cos\phi)^i$ is unaffected by the $\mathbf{k}_{2T}$ integration in eq. (4.7). These hard coefficients can be obtained similarly from the $\mathbf{k}_{2T}$ -integrated helicity matrix $[H]_{\lambda_1\lambda_2,\lambda'_1\lambda'_2}$ as in eqs. (A.7) and (A.10), which can be separated into the same leptonic tensor $L_{\lambda_\gamma\lambda'_\gamma}$ for the γ^* density matrix and $\mathbf{k}_{2T}$ -integrated hadronic tensor $[W]_{\lambda_1\lambda_2,\lambda'_1\lambda'_2}^{\lambda_\gamma\lambda'_\gamma}$ for the γ^* -parton scattering, yielding the same formulae as eqs. (3.36) and (3.41). The integrated W is defined as

$$\begin{aligned}
 [W]_{\lambda_1\lambda_2,\lambda'_1\lambda'_2}^{\lambda_\gamma\lambda'_\gamma} \left(\frac{x}{\xi_1}, \frac{z}{\xi_2}, \frac{Q^2}{\mu^2} \right) &= (2\pi)^{2\epsilon} \int d^{d-2} \mathbf{k}_{2T} \sum_X \int d\Pi_X (2\pi)^d \delta^{(d)}(q + k_1 - k_2 - k_X) \\
 &\times \mathcal{Q}_{\lambda_1\lambda_2}^{\lambda_\gamma}(k_1, q, k_2; X) \mathcal{Q}_{\lambda'_1\lambda'_2}^{\lambda'_\gamma*}(k_1, q, k_2; X). \tag{4.9}
 \end{aligned}$$

We will stick to the flavor NS channel, $(a, b) = (q, q)$, and use the shorthand notations $[W]_{\lambda\lambda'}^{\lambda_\gamma\lambda'_\gamma}$ and $\mathcal{Q}_\lambda^{\lambda_\gamma}$ for $[W]_{\lambda\lambda,\lambda'\lambda'}^{\lambda_\gamma\lambda'_\gamma}$ and $\mathcal{Q}_{\lambda\lambda}^{\lambda_\gamma}$, respectively, as in eq. (3.43). In the following, we distinguish two cases for the calculation of $[W]$, with X containing nothing, denoted as $X = 0$, or at least one particle, $X > 0$, and then formulate observables in $[C]$, $[\Delta C]$, and $[T]$.

4.2 Hard coefficients for $X = 0$

When X contains nothing, the partonic scattering is a simple $2 \rightarrow 1$ process,

$$q(k_1, \lambda) + \gamma^*(q, \lambda_\gamma) \rightarrow q(k_2, \lambda), \tag{4.10}$$

⁴We label $\mathbf{k}_{2T}$ -integrated quantities $[O]$ by enclosing them by square brackets to distinguish from the unintegrated ones O of the same labels.

which happens on top of the electron scattering plane, with $q(k_1)$ moving along $\hat{z}_B$ and γ^* and $q(k_2)$ along $-\hat{z}_B$.

The first question we need to answer is how the hadron coordinate system is defined for this kinematics, which is apparently a singularity of the definition in eq. (3.5). As remarked below eq. (A.8), the $\hat{x}_h$ and $\hat{y}_h$ are determined by the definition of the helicity state $|q(k_2, \lambda)\rangle$, which is given by a rotation from the state of the same energy and helicity moving along the $\hat{z}_B$ direction [103]. However, for k_2 moving along the $-\hat{z}_B$ direction, there is an ambiguity regarding its azimuthal angle ϕ_0 . We fix the convention by demanding that the parity relation in eq. (3.44) directly extrapolate to this case. Since a parity transformation relates a state along $\hat{n}(\theta, \phi)$ to one along $\hat{n}(\pi - \theta, \pi + \phi)$, for $\phi \in [0, \pi)$, we need $-\hat{z}_B = \hat{n}(\pi, \pi)$ to pair with $\hat{z}_B = \hat{n}(0, 0)$. Hence, we choose $\phi_0 = \pi$, such that the hadron system is obtained from the Breit system via a rotation around $\hat{y}_B$ by π followed by a rotation around $\hat{z}_B$ by π , which leads to eq. (4.2).

Setting $X = 0$ in eq. (4.9), we can use the delta function to integrate over $\mathbf{k}_{2T}$,

$$[W_{X=0}]_{\lambda\lambda'}^{\lambda_\gamma\lambda'_\gamma}(\hat{x}, \hat{z}; Q^2/\mu^2) = (2\pi)^4 \frac{2\delta(1-\hat{x})\delta(1-\hat{z})}{Q^2} \mathcal{Q}_\lambda^{\lambda_\gamma} \mathcal{Q}_{\lambda'}^{\lambda'_\gamma*}, \quad (4.11)$$

which constrains both $\hat{x}$ and $\hat{z}$ [defined in eq. (3.35)] to be 1. By angular momentum conservation along the $\hat{z}_B$ axis, the amplitude $\mathcal{Q}_\lambda^{\lambda_\gamma}$ is nonzero only when $2\lambda = \lambda_\gamma$, yielding two nonzero components $\mathcal{Q}_\pm^\pm$. Parity further constrains $\mathcal{Q}_+^+ = \mathcal{Q}_-^-$, and thus the four nonzero components of W are all equal and positive,

$$\begin{aligned} [W_{X=0}]_{++}^{++} &= [W_{X=0}]_{--}^{--} = [W_{X=0}]_{+-}^{+-} = [W_{X=0}]_{-+}^{-+} \\ &= (2\pi)^4 \frac{2\delta(1-\hat{x})\delta(1-\hat{z})}{Q^2} |\mathcal{Q}_+^+|^2 > 0. \end{aligned} \quad (4.12)$$

Applied to the integrated version of eq. (3.41), this gives the nonzero components of $[H]$,

$$\begin{aligned} [H_{X=0}]_{\pm\pm} &= \frac{2}{Q^2(1-\varepsilon)} \left(\frac{1 \pm P_e \sqrt{1-\varepsilon^2}}{2} [W_{X=0}]_{++}^{++} \right), \\ [H_{X=0}]_{+-} &= [H_{X=0}]_{-+} = \frac{2}{Q^2(1-\varepsilon)} \left(-\frac{\varepsilon}{2} [W_{X=0}]_{++}^{++} \right), \end{aligned} \quad (4.13)$$

which then determine the hard coefficients $[C]$, $[\Delta C]$, and $[T^{ij}]$ by eqs. (A.7) and (A.10),

$$[C_{qq, X=0}] = \frac{2}{Q^2(1-\varepsilon)} \left(\frac{1}{2} [W_{X=0}]_{++}^{++} \right), \quad (4.14a)$$

$$[\Delta C_{qq, X=0}] = \frac{2}{Q^2(1-\varepsilon)} \left(\frac{P_e \sqrt{1-\varepsilon^2}}{2} [W_{X=0}]_{++}^{++} \right), \quad (4.14b)$$

$$[T_{qq, X=0}^{xx}] = [T_{qq, X=0}^{yy}] = \frac{2}{Q^2(1-\varepsilon)} \left(-\frac{\varepsilon}{2} [W_{X=0}]_{++}^{++} \right), \quad (4.14c)$$

$$[T_{qq, X=0}^{xy}] = [T_{qq, X=0}^{yx}] = 0. \quad (4.14d)$$

The last two equations of eq. (4.14) imply that the quark transverse spin is transferred just as being “bent” on the electron scattering plane,

$$\mathbf{s}'_{T,q} \propto -\mathbf{s}_{T,q}. \quad (4.15)$$

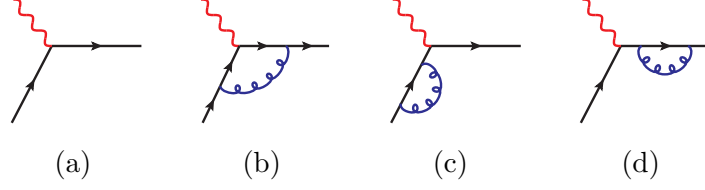


Figure 5. LO (a) and NLO (b)–(d) diagrams of the $q \rightarrow q$ scattering amplitude in eq. (4.10). Inclusion of the counterterm diagrams (not shown) for (b)–(d) is understood.

Notice that the coordinate systems describing the initial- and final-state quark spins, $\mathbf{s}_{T,q}$ and $\mathbf{s}'_{T,q}$, respectively, differ by flipping the y and z axes. Hence, eq. (4.15) means that the transverse spin component lying in the electron-scattering plane is flipped, while the perpendicular component stays unchanged.

The above discussion applies to all orders of the partonic process in eq. (4.10). With $X = 0$, the LO is given at tree level, and high-order corrections include only virtual gluon emissions. In the following, we present these results up to NLO for the $q \rightarrow q$ channel.

4.2.1 LO hard coefficients for $X = 0$

The LO amplitude for the scattering eq. (4.10) is shown in figure 5(a).⁵ Using the normalization of eq. (3.33), it can be explicitly computed as

$$\Delta_0 \mathcal{Q}_+^+ = \Delta_0 \mathcal{Q}_-^- = -\sqrt{2} i e_q Q. \quad (4.16)$$

We use the prefix notation Δ_n to label the order of α_s in perturbative expansions,

$$A = \Delta_0 A + \Delta_1 A + \Delta_2 A + \dots, \quad (4.17)$$

with $\Delta_n A \propto \alpha_s^n$. Eq. (4.16) gives the $[W]$ matrix in eq. (4.12),⁶

$$[\Delta_0 W]_{++}^{++} = [\Delta_0 W]_{--}^{--} = [\Delta_0 W]_{+-}^{+-} = [\Delta_0 W]_{-+}^{-+} = 4(2\pi)^4 e_q^2 \delta(1 - \hat{x}) \delta(1 - \hat{z}). \quad (4.18)$$

4.2.2 NLO hard coefficients for $X = 0$: virtual correction

The NLO corrections are shown in figure 5(b)–(d). Due to chiral and parity symmetries, they only contribute an overall factor to eq. (4.16). In the Feynman gauge, the vertex correction factor is given by the one-loop diagram in figure 5(b),

$$f_v(Q^2) = -\frac{\alpha_s C_F}{2\pi} \frac{e^{\epsilon(L+\gamma_E)}}{\Gamma(1-\epsilon)} \left[\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + 4 + \mathcal{O}(\epsilon) \right], \quad (4.19)$$

and its UV counterterm in the $\overline{\text{MS}}$ scheme,

$$\delta f_v = -\frac{\alpha_s C_F}{4\pi} \left[\frac{1}{\epsilon} + \ln(4\pi e^{-\gamma_E}) \right]. \quad (4.20)$$

⁵Note that here “LO” refers to the no-radiation case, while “NLO” has one virtual or real gluon radiation. This differs from the terminology for finite- p_{hT} observables in section 3.

⁶For simplicity, we suppress the subscript $X = 0$ as it is already encoded by the LO prefix Δ_0 .

Here, we have defined

$$\bar{\mu}^2 = 4\pi\mu^2 e^{-\gamma_E}, \quad L = \ln\left(\frac{\bar{\mu}^2}{Q^2}\right). \quad (4.21)$$

The external quark leg correction factor shown in figure 5(c) and (d) is given by the residue of the renormalized two-point function,

$$f_e = \frac{\alpha_s C_F}{4\pi} \left[\frac{1}{\epsilon} + \ln(4\pi e^{-\gamma_E}) \right] = -\delta f_v. \quad (4.22)$$

Combining these together, the vertex counterterm cancels the external quark leg correction, leaving just the factor $f_v(Q^2)$ from the bare vertex loop.

Adding the virtual correction to the tree-level amplitude in eq. (4.16), we have

$$\Delta_1 Q_+^+ = \Delta_1 Q_-^- = -\sqrt{2} i e_q Q f_v(Q^2). \quad (4.23)$$

This gives the NLO virtual correction to eq. (4.18),⁷

$$\begin{aligned} [\Delta_1^V W]_{++}^{++} &= [\Delta_1^V W]_{--}^{--} = [\Delta_1^V W]_{+-}^{+-} = [\Delta_1^V W]_{-+}^{-+} \\ &= 4(2\pi)^4 e_q^2 \delta(1-\hat{x}) \delta(1-\hat{z}) [2f_v(Q^2)]. \end{aligned} \quad (4.24)$$

4.3 Hard coefficients for $X > 0$: real correction

The LO contribution to the high- p_{hT} hadron production in section 3.3.3 now becomes the real correction to p_{hT} -integrated case at NLO. Most of the calculations are the same as section 3.3.3. We concentrate only on their differences.

First, the helicity amplitude $\mathcal{Q}_{\lambda\lambda'}^\gamma$ of the $2 \rightarrow 2$ scattering in eq. (3.58) stays the same, with only the coupling g modified to $g\mu^\epsilon$ in DR. Following eq. (4.9), we supplement eq. (3.61) with the factor $(4\pi^2\mu^2)^\epsilon$ and the $\mathbf{k}_{2T}$ integration,

$$\begin{aligned} [\Delta_1^R W]_{\lambda\lambda'}^{\lambda_\gamma\lambda'_\gamma}(\hat{x}, \hat{z}; Q^2/\mu^2) &= (4\pi^2\mu^2)^\epsilon \int d^{d-2} \mathbf{k}_{2T} \left(8\pi^2 e_q^2 \alpha_s C_F \right) \hat{z} \delta(k_{2T}^2 - \hat{z}(1-\hat{z})\hat{s}) \\ &\quad \times e^{i[(\lambda-\lambda')-(\lambda_\gamma-\lambda'_\gamma)]\phi_h} \cdot \bar{w}_{\lambda\lambda'}^{\lambda_\gamma\lambda'_\gamma}(\hat{x}, \hat{z}) \\ &= (2\pi)^4 \hat{z} e_q^2 \frac{\alpha_s C_F}{2\pi} \frac{e^{\epsilon(L+\gamma_E)}}{\Gamma(1-\epsilon)} \int_0^{2\pi} \frac{d\phi_h}{2\pi} \left[e^{i[(\lambda-\lambda')-(\lambda_\gamma-\lambda'_\gamma)]\phi_h} \frac{\hat{x}^\epsilon \hat{z}^{-\epsilon} \bar{w}_{\lambda\lambda'}^{\lambda_\gamma\lambda'_\gamma}(\hat{x}, \hat{z})}{(1-\hat{x})^\epsilon (1-\hat{z})^\epsilon} \right], \end{aligned} \quad (4.25)$$

where in the second step we used eq. (B.14) to separate the $\mathbf{k}_{2T}$ integration into k_{2T}^2 and ϕ_h , and used the delta function to integrate over k_{2T}^2 . The ϕ_h -independent factor $\bar{w}$ is the same as that in eq. (3.61) and has been presented in eqs. (3.64)–(3.66).

As noted at the beginning of this section, the ϕ_h integral is trivial if we take the integrations of k_{2T}^2 and ϕ_h as fully independent,

$$\int_0^{2\pi} \frac{d\phi_h}{2\pi} e^{i[(\lambda-\lambda')-(\lambda_\gamma-\lambda'_\gamma)]\phi_h} \stackrel{?}{=} \delta_{\lambda-\lambda', \lambda_\gamma-\lambda'_\gamma}. \quad (4.26)$$

⁷The prefix notation Δ_1^V indicating virtual corrections at NLO automatically implies the configuration $X = 0$, so we have suppressed the subscript “ $X = 0$ ”.

If this were indeed the case, the only surviving components in eq. (4.25) would be

$$\left[\Delta_1^R W\right]_{++}^{++}, \quad \left[\Delta_1^R W\right]_{--}^{++}, \quad \left[\Delta_1^R W\right]_{++}^{00}, \quad \left[\Delta_1^R W\right]_{+-}^{+0}, \quad (4.27)$$

plus their Hermitian-conjugate and parity counterparts. The $[W]_{+-}^{+-}$ would not receive any contribution from $X > 0$, and therefore, the IRC divergences of $[\Delta_1^Y W]_{+-}^{+-}$ in eq. (4.24) would remain uncanceled.

We rescue this paradox by noticing that the separation of k_{2T}^2 and ϕ_h is singular as $k_{2T}^2 \rightarrow 0$. Having used the delta function in eq. (4.25) to set k_{2T}^2 to eq. (3.62), we converted the singular kinematic point to $\hat{x} = 1$ and/or $\hat{z} = 1$. These correspond to three kinematic configurations:⁸

1. when $\hat{x} = 1$ and $\hat{z} \neq 1$, the gluon moves collinearly with the final-state quark in the $-\hat{z}_B$ direction;
2. when $\hat{x} \neq 1$ and $\hat{z} = 1$, the gluon is collinearly radiated by the initial-state quark in the $\hat{z}_B$ direction, with the final-state quark moving along $-\hat{z}_B$;
3. when $\hat{x} = \hat{z} = 1$, the final-state quark moves along the $-\hat{z}_B$ direction, with the gluon being soft.

In all these cases, the final-state quark moves in the $-\hat{z}_B$ direction, so that the kinematic singular point fixes ϕ_h to π , according to our discussion at the beginning of section 4.2. This means that if the $\hat{x}$ - $\hat{z}$ integrand in eq. (4.25) is singular at either $\hat{x} = 1$ or $\hat{z} = 1$, we should set $\phi_h = \pi$ for the singular piece and thus obtain a non-vanishing ϕ_h integral. This is the case for $[\Delta_1^R W]_{++}^{++}$ and $[\Delta_1^R W]_{+-}^{+-}$, exactly corresponding to the helicity structures for $X = 0$ in eq. (4.24). Their combinations would then cancel the soft divergences.

Below, we first consider the ϕ_h integrals for these singular coefficients, and then for remaining non-singular ones.

4.3.1 ϕ_h integrals for the singular hard coefficients

For $[\Delta_1^R W]_{++}^{++}$, the phase factor in eq. (4.25) reduces to unity, so its ϕ_h integral is trivial. From eqs. (3.64) and (3.65), we have

$$\bar{w}_{++}^{++}(\hat{x}, \hat{z}) = f_{UU,T} + f_{LL} = \frac{4(1 + \hat{x}^2 \hat{z}^2)}{(1 - \hat{x})(1 - \hat{z})}. \quad (4.28)$$

Plugging this into eq. (4.25) yields

$$\left[\Delta_1^R W\right]_{++}^{++} = 4(2\pi)^4 \hat{z} e_q^2 \frac{\alpha_s C_F}{2\pi} \frac{e^{\epsilon(L+\gamma_E)}}{\Gamma(1-\epsilon)} \left[\frac{(1 + \hat{x}^2 \hat{z}^2) \hat{x}^\epsilon \hat{z}^{-\epsilon}}{(1 - \hat{x})^{1+\epsilon} (1 - \hat{z})^{1+\epsilon}} \right]. \quad (4.29)$$

⁸The case with both the final-state quark and gluon being soft does not exist. Otherwise we would have $k_1^+ + q^+ = q^- = 0$, giving $Q^2 = 0$.

The IRC singularities at $\hat{x} = 1$ and $\hat{z} = 1$ are regulated with $\epsilon < 0$ in DR, and can be explicitly exposed by using the standard formula,

$$\begin{aligned}\frac{\hat{x}^\epsilon}{(1-\hat{x})^{1+\epsilon}} &= -\frac{\delta(1-\hat{x})}{\epsilon} + \frac{1}{(1-\hat{x})_+} + \epsilon \left[\frac{\ln \hat{x}}{1-\hat{x}} - \left(\frac{\ln(1-\hat{x})}{1-\hat{x}} \right)_+ \right] + \mathcal{O}(\epsilon^2), \\ \frac{\hat{z}^{-\epsilon}}{(1-\hat{z})^{1+\epsilon}} &= -\frac{\delta(1-\hat{z})}{\epsilon} + \frac{1}{(1-\hat{z})_+} - \epsilon \left[\frac{\ln \hat{z}}{1-\hat{z}} + \left(\frac{\ln(1-\hat{z})}{1-\hat{z}} \right)_+ \right] + \mathcal{O}(\epsilon^2).\end{aligned}\quad (4.30)$$

Then we rewrite eq. (4.29) as

$$\begin{aligned}[\Delta_1^R W]_{++}^{++} &= 4(2\pi)^4 \hat{z} e_q^2 \frac{\alpha_s C_F}{2\pi} \frac{e^{\epsilon(L+\gamma_E)}}{\Gamma(1-\epsilon)} \left\{ \frac{2\delta(1-\hat{x})\delta(1-\hat{z})}{\epsilon^2} + \frac{1+\hat{x}^2\hat{z}^2}{(1-\hat{x})_+(1-\hat{z})_+} \right. \\ &\quad + \delta(1-\hat{z}) \left[\frac{1}{\epsilon} \left(1+\hat{x} - \frac{2}{(1-\hat{x})_+} \right) - (1+\hat{x}^2) \left(\frac{\ln \hat{x}}{1-\hat{x}} - \left(\frac{\ln(1-\hat{x})}{1-\hat{x}} \right)_+ \right) \right] \\ &\quad \left. + \delta(1-\hat{x}) \left[\frac{1}{\epsilon} \left(1+\hat{z} - \frac{2}{(1-\hat{z})_+} \right) + (1+\hat{z}^2) \left(\frac{\ln \hat{z}}{1-\hat{z}} + \left(\frac{\ln(1-\hat{z})}{1-\hat{z}} \right)_+ \right) \right] + \mathcal{O}(\epsilon) \right\}.\end{aligned}\quad (4.31)$$

For $[\Delta_1^R W]_{+-}^{+-}$, we use eq. (3.66) to write its ϕ_h integral as

$$[\Delta_1^R W]_{+-}^{+-} = (2\pi)^4 \hat{z} e_q^2 \frac{\alpha_s C_F}{2\pi} \frac{e^{\epsilon(L+\gamma_E)}}{\Gamma(1-\epsilon)} \int_0^{2\pi} \frac{d\phi_h}{2\pi} \left[e^{-i\phi_h} \frac{(-8\hat{x}\hat{z}) \hat{x}^\epsilon \hat{z}^{-\epsilon}}{(1-\hat{x})^{1+\epsilon} (1-\hat{z})^{1+\epsilon}} \right]. \quad (4.32)$$

This contains the same IRC singularities and can be expanded in the same way as eq. (4.31). The difference, however, lies in the phase $e^{-i\phi_h}$ under the ϕ_h integral, which vanishes unless $\hat{x}$ or $\hat{z}$ is forced to 1 by the delta functions in eq. (4.30). Therefore, we retain only terms proportional to $\delta(1-\hat{x})$ or $\delta(1-\hat{z})$ and set $\phi_h = \pi$ in the integrand. This gives

$$\begin{aligned}[\Delta_1^R W]_{+-}^{+-} &= 8(2\pi)^4 \hat{z} e_q^2 \frac{\alpha_s C_F}{2\pi} \frac{e^{\epsilon(L+\gamma_E)}}{\Gamma(1-\epsilon)} \left\{ \frac{\delta(1-\hat{x})\delta(1-\hat{z})}{\epsilon^2} \right. \\ &\quad + \delta(1-\hat{z}) \left[\frac{1}{\epsilon} \left(1 - \frac{1}{(1-\hat{x})_+} \right) - \hat{x} \left(\frac{\ln \hat{x}}{1-\hat{x}} - \left(\frac{\ln(1-\hat{x})}{1-\hat{x}} \right)_+ \right) \right] \\ &\quad \left. + \delta(1-\hat{x}) \left[\frac{1}{\epsilon} \left(1 - \frac{1}{(1-\hat{z})_+} \right) + \hat{z} \left(\frac{\ln \hat{z}}{1-\hat{z}} + \left(\frac{\ln(1-\hat{z})}{1-\hat{z}} \right)_+ \right) \right] + \mathcal{O}(\epsilon) \right\}.\end{aligned}\quad (4.33)$$

In fact, we can argue to all orders that $[W]_{+-}^{+-}$ is the only nontrivial component (together with its conjugate $[W]_{-+}^{-+}$) under the ϕ_h integral. The reason is that one needs both a non-unity ϕ_h phase and a singularity at $\hat{x} = 1$ or $\hat{z} = 1$ in eq. (4.25). As we noted above, this singularity is of soft or collinear origin and happens at the edge of phase space with only one final-state collinear sector associated with the produced quark. It can therefore be factorized into a PDF for the initial-state quark, an FF for the final-state quark, and a hard coefficient that is exactly given by eq. (4.11) with no other particles in the final state. Chiral symmetry along with helicity conservation then fix the helicity structures to those in eq. (4.12), where only $[W]_{+-}^{+-}$ and $[W]_{-+}^{-+}$ have nontrivial phases.

4.3.2 ϕ_h integrals for non-singular hard coefficients

The other non-vanishing components are given by the last three terms in eq. (4.27). They are IRC safe, ϕ_h independent, and absent for $X = 0$, so we can remove their dimensional regulator and simply integrate over ϕ_h . This gives

$$\begin{aligned} \left\{ [\Delta_1 W]_{--}^{++}, [\Delta_1 W]_{++}^{00}, [\Delta_1 W]_{+-}^{+0} \right\} &= \left\{ [\Delta_1^R W]_{--}^{++}, [\Delta_1^R W]_{++}^{00}, [\Delta_1^R W]_{+-}^{+0} \right\} \\ &= 4(2\pi)^4 \hat{z} e_q^2 \frac{\alpha_s C_F}{2\pi} \left\{ (1-\hat{x})(1-\hat{z}), 2\hat{x}\hat{z}, \sqrt{\frac{2\hat{x}\hat{z}}{(1-\hat{x})(1-\hat{z})}} \right\}. \end{aligned} \quad (4.34)$$

4.4 Cancellation of IRC singularities

The factorization theorem in eq. (4.7) implies that (1) soft divergences cancel between real and virtual gluon emissions, and (2) collinear divergences are removed by subtractions associated with the PDFs and FECs. In this sub-section, we verify these two facts through our NLO calculation of the $q \rightarrow q$ channel, which also provides a consistency check of the perturbative result, particularly our treatment of the singular ϕ_h -dependent $\mathbf{k}_{2T}$ integral.

The three components in eq. (4.34) start appearing only at NLO, so eq. (4.34) is already their final expressions. Components related to them by Hermiticity and parity are

$$[W]_{--}^{++} = [W]_{++}^{--}, \quad [W]_{--}^{00} = [W]_{++}^{00}, \quad [W]_{+-}^{+0} = [W]_{-+}^{0+} = -[W]_{-+}^{-0} = -[W]_{+-}^{0-}. \quad (4.35)$$

The nontrivial components to be dealt with in the following are

$$[W]_{++}^{++} = [W]_{--}^{--}, \quad [W]_{+-}^{+-} = [W]_{-+}^{+-}. \quad (4.36)$$

All other components of $[W]$ vanish under the ϕ_h integration.

4.4.1 Combination of real and virtual corrections

The $[W]_{++}^{++}$ at NLO is given by the virtual contribution in eq. (4.24) and the real contribution in eq. (4.31). Combining them together gives

$$\begin{aligned} [\Delta_1 W]_{++}^{++} &= [\Delta_1^V W]_{++}^{++} + [\Delta_1^R W]_{++}^{++} \\ &= 4(2\pi)^4 \hat{z} e_q^2 \frac{\alpha_s C_F}{2\pi} \frac{e^{\epsilon(L+\gamma_E)}}{\Gamma(1-\epsilon)} \left\{ -8\delta(1-\hat{x})\delta(1-\hat{z}) + \frac{1+\hat{x}^2\hat{z}^2}{(1-\hat{x})_+(1-\hat{z})_+} \right. \\ &\quad \left. - \delta(1-\hat{z}) \left[\frac{P_{qq}(\hat{x})}{\epsilon} + (1+\hat{x}^2) \left(\frac{\ln \hat{x}}{1-\hat{x}} - \left(\frac{\ln(1-\hat{x})}{1-\hat{x}} \right)_+ \right) \right] \right. \\ &\quad \left. - \delta(1-\hat{x}) \left[\frac{K_{qq}(\hat{z})}{\epsilon} - (1+\hat{z}^2) \left(\frac{\ln \hat{z}}{1-\hat{z}} + \left(\frac{\ln(1-\hat{z})}{1-\hat{z}} \right)_+ \right) \right] + \mathcal{O}(\epsilon) \right\}, \end{aligned} \quad (4.37)$$

where the double pole $1/\epsilon^2$ is canceled, signifying the soft gluon cancellation. The coefficients of the remaining $1/\epsilon$ poles exactly organize into the (LO) evolution kernels of PDF and FF, $P_{qq}(\hat{x})$ and $K_{qq}(\hat{z})$, respectively [104],

$$P_{qq}(x) = K_{qq}(x) = -1 - x + \frac{2}{(1-x)_+} + \frac{3}{2}\delta(1-x). \quad (4.38)$$

Similarly, for $[W]_{+-}^{+-}$, we combine eqs. (4.24) and (4.33) and obtain the NLO contribution,

$$\begin{aligned}
 [\Delta_1 W]_{+-}^{+-} &= [\Delta_1^V W]_{+-}^{+-} + [\Delta_1^R W]_{+-}^{+-} \\
 &= 4(2\pi)^4 \hat{z} e_q^2 \frac{\alpha_s C_F}{2\pi} \frac{e^{\epsilon(L+\gamma_E)}}{\Gamma(1-\epsilon)} \left\{ -8\delta(1-\hat{x})\delta(1-\hat{z}) \right. \\
 &\quad - \delta(1-\hat{z}) \left[\frac{\delta_T P_{qq}(\hat{x})}{\epsilon} + 2\hat{x} \left(\frac{\ln \hat{x}}{1-\hat{x}} - \left(\frac{\ln(1-\hat{x})}{1-\hat{x}} \right)_+ \right) \right] \\
 &\quad \left. - \delta(1-\hat{x}) \left[\frac{\delta_T K_{qq}(\hat{z})}{\epsilon} - 2\hat{z} \left(\frac{\ln \hat{z}}{1-\hat{z}} + \left(\frac{\ln(1-\hat{z})}{1-\hat{z}} \right)_+ \right) \right] + \mathcal{O}(\epsilon) \right\}, \quad (4.39)
 \end{aligned}$$

where $\delta_T P_{qq}(\hat{x})$ and $\delta_T K_{qq}(\hat{z})$ are, respectively, the LO evolution kernels of the transversity PDF and FF [105],

$$\delta_T P_{qq}(x) = \delta_T K_{qq}(x) = -2 + \frac{2}{(1-x)_+} + \frac{3}{2}\delta(1-x). \quad (4.40)$$

The cancellation of the $1/\epsilon^2$ poles in eq. (4.39) is highly nontrivial: it shows the consistency of setting ϕ_h to π in evaluating the singular piece of eq. (4.32). For the virtual correction $[\Delta_1^V W]_{+-}^{+-}$, the framework set up in eq. (4.11) relates it to the amplitude of the backward $q \rightarrow q$ scattering, where we have chosen the azimuthal angle of the outgoing quark spinor as π when explicitly evaluating eqs. (4.16) and (4.23). Had we chosen it as $\phi_0 = \pi + \Delta\phi_0$ with $\Delta\phi_0 \neq 0$, the $X = 0$ contribution to $[W]_{+-}^{+-}$ would become, using eq. (4.11),

$$\begin{aligned}
 [W_{X=0}]_{+-}^{+-}(\phi_0) &\propto \langle q_+(\pi, \phi_0) | S | q_+, \gamma_+^* \rangle \langle q_-(\pi, \phi_0) | S | q_-, \gamma_-^* \rangle^* \\
 &= \langle q_+(\pi, \pi) | e^{iJ_3 \Delta\phi_0} S | q_+, \gamma_+^* \rangle \langle q_-(\pi, \pi) | e^{iJ_3 \Delta\phi_0} S | q_-, \gamma_-^* \rangle^* \\
 &= e^{-i\Delta\phi_0} \langle q_+(\pi, \pi) | S | q_+, \gamma_+^* \rangle \langle q_-(\pi, \pi) | S | q_-, \gamma_-^* \rangle^*, \quad (4.41)
 \end{aligned}$$

where we used the rotation operator in the second step to relate the quark state $|q_+(\pi, \phi_0)\rangle$ to $|q_+(\pi, \pi)\rangle$, and in the third step we used $J_3 |q_\pm(\pi, \pi)\rangle = \mp(1/2) |q_\pm(\pi, \pi)\rangle$ for it moves along the $-\hat{z}_B$ direction. This gives the relation between two different conventions,

$$[W_{X=0}]_{+-}^{+-}(\phi_0) = e^{-i\Delta\phi_0} [W_{X=0}]_{+-}^{+-}(\pi). \quad (4.42)$$

Obviously, the helicity diagonal components $[W_{X=0}]_{\pm\pm}^{\pm\pm}$ are unaffected by this convention. The phase in eq. (4.42) directly corresponds to the phase in eq. (4.32), where the backward scattering configuration is reached only as a limit in the integration and so setting ϕ_h to π when evaluating eq. (4.32) was not as obvious as in eqs. (4.16) and (4.23). Now we see that the soft cancellation demands a consistent choice: if we use $\phi_0 \neq \pi$ in eqs. (4.16) and (4.23), we also need to set ϕ_h to ϕ_0 in eq. (4.32); the net effect is an extra overall factor $e^{-i\Delta\phi_0}$ to both $[W_{X=0}]_{+-}^{+-}(\pi)$ and $[W_{X>0}]_{+-}^{+-}(\pi)$. Of course, our discussion here is only *a posteriori*. Recalling the derivation in appendix A of the hard coefficients in terms of helicity amplitudes, the choice of ϕ_0 is not free at all, but is dictated by the spinor projector in the definition of the Collins-type FEC or transversity FF.

4.4.2 Collinear subtraction

The $1/\epsilon$ poles in eqs. (4.37) and (4.39) are due to collinear subregions associated with the initial- or final-state quark. The corresponding subtraction terms can be obtained in a standard way by factorizing $[W]$ at parton level into the perturbative PDF and FF. At NLO, for the $q \rightarrow q$ NS channel, they are given as

$$[\Delta_1^{\text{col}} W]_{\lambda\lambda'}^{\lambda\gamma\lambda'\gamma}(\hat{x}, \hat{z}) = \int \frac{dx'}{x'} \int dz' [\Delta_0 W]_{\lambda\lambda'}^{\lambda\gamma\lambda'\gamma}\left(\frac{\hat{x}}{x'}, \frac{\hat{z}}{z'}\right) \times \left[\Delta_1 f_{q/q}^{\lambda\lambda'}(x') \delta(1-z') + \delta(1-\hat{x}/x') \Delta_1 d_{q/q}^{\lambda\lambda'}(z') \right], \quad (4.43)$$

where $[\Delta_0 W]$ is given in eq. (4.18), and the perturbative quark PDF $\Delta_1 f_{q/q}^{\lambda\lambda'}(x')$ and FF $\Delta_1 d_{q/q}^{\lambda\lambda'}(z')$ are defined in the helicity basis. At this order, they are equal to each other, given by the square of the $q \rightarrow qg$ splitting amplitudes [104],

$$\begin{aligned} \Delta_1 f_{q/q}^{++}(x) &= \Delta_1 f_{q/q}^{--}(x) = \Delta_1 d_{q/q}^{++}(x) = \Delta_1 d_{q/q}^{--}(x) = -\frac{S_\epsilon}{\epsilon} \frac{\alpha_s C_F}{2\pi} P_{qq}(x), \\ \Delta_1 f_{q/q}^{+-}(x) &= \Delta_1 f_{q/q}^{-+}(x) = \Delta_1 d_{q/q}^{+-}(x) = \Delta_1 d_{q/q}^{-+}(x) = -\frac{S_\epsilon}{\epsilon} \frac{\alpha_s C_F}{2\pi} \delta_T P_{qq}(x), \end{aligned} \quad (4.44)$$

where the factor S_ϵ is inserted to convert them to the $\overline{\text{MS}}$ scheme [37],

$$S_\epsilon = \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} = 1 + \epsilon \ln(4\pi e^{-\gamma_E}) + \mathcal{O}(\epsilon^2). \quad (4.45)$$

Using eq. (4.18), we get the subtraction terms for $[\Delta_1 W]_{++}^{++}$ and $[\Delta_1 W]_{+-}^{+-}$,

$$\begin{aligned} [\Delta_1^{\text{col}} W]_{++}^{++} &= 4(2\pi)^4 e_q^2 \hat{z} \frac{\alpha_s C_F}{2\pi} \left[-\frac{S_\epsilon}{\epsilon} \right] [P_{qq}(\hat{x}) \delta(1-\hat{z}) + \delta(1-\hat{x}) K_{qq}(\hat{z})], \\ [\Delta_1^{\text{col}} W]_{+-}^{+-} &= 4(2\pi)^4 e_q^2 \hat{z} \frac{\alpha_s C_F}{2\pi} \left[-\frac{S_\epsilon}{\epsilon} \right] [\delta_T P_{qq}(\hat{x}) \delta(1-\hat{z}) + \delta(1-\hat{x}) \delta_T K_{qq}(\hat{z})]. \end{aligned} \quad (4.46)$$

Other $[W]$ helicity structures in eq. (4.35) do not appear at LO and have no associated subtraction terms at NLO.

Subtracting eq. (4.46) from eqs. (4.37) and (4.39) removes the collinear $1/\epsilon$ poles and converts them to $\ln(\mu^2/Q^2)$. After that, we set ϵ to 0 to restore the finite results in four dimensions,

$$\begin{aligned} [\Delta_1^{\text{sub}} W]_{++}^{++} &= [\Delta_1 W]_{++}^{++} - [\Delta_1^{\text{col}} W]_{++}^{++} \\ &= 4(2\pi)^4 \hat{z} e_q^2 \frac{\alpha_s C_F}{2\pi} \left[-8\delta(1-\hat{x})\delta(1-\hat{z}) + \frac{1+\hat{x}^2\hat{z}^2}{(1-\hat{x})_+(1-\hat{z})_+} \right. \\ &\quad \left. - A(\hat{x})\delta(1-\hat{z}) - \delta(1-\hat{x})B(\hat{z}) \right], \end{aligned} \quad (4.47a)$$

$$\begin{aligned} [\Delta_1^{\text{sub}} W]_{+-}^{+-} &= [\Delta_1 W]_{+-}^{+-} - [\Delta_1^{\text{col}} W]_{+-}^{+-} \\ &= 4(2\pi)^4 \hat{z} e_q^2 \frac{\alpha_s C_F}{2\pi} \left[-8\delta(1-\hat{x})\delta(1-\hat{z}) \right. \\ &\quad \left. - A'(\hat{x})\delta(1-\hat{z}) - \delta(1-\hat{x})B'(\hat{z}) \right], \end{aligned} \quad (4.47b)$$

where we defined

$$\begin{aligned}
 A(\hat{x}) &= P_{qq}(\hat{x}) \ln \frac{\mu^2}{Q^2} + (1 + \hat{x}^2) \left[\frac{\ln \hat{x}}{1 - \hat{x}} - \left(\frac{\ln(1 - \hat{x})}{1 - \hat{x}} \right)_+ \right], \\
 B(\hat{z}) &= K_{qq}(\hat{z}) \ln \frac{\mu^2}{Q^2} - (1 + \hat{z}^2) \left[\frac{\ln \hat{z}}{1 - \hat{z}} + \left(\frac{\ln(1 - \hat{z})}{1 - \hat{z}} \right)_+ \right], \\
 A'(\hat{x}) &= \delta_T P_{qq}(\hat{x}) \ln \frac{\mu^2}{Q^2} + 2\hat{x} \left[\frac{\ln \hat{x}}{1 - \hat{x}} - \left(\frac{\ln(1 - \hat{x})}{1 - \hat{x}} \right)_+ \right], \\
 B'(\hat{z}) &= \delta_T K_{qq}(\hat{z}) \ln \frac{\mu^2}{Q^2} - 2\hat{z} \left[\frac{\ln \hat{z}}{1 - \hat{z}} + \left(\frac{\ln(1 - \hat{z})}{1 - \hat{z}} \right)_+ \right].
 \end{aligned} \tag{4.48}$$

We have used the prefix Δ_1^{sub} in eq. (4.47) to stand for the collinear-subtracted NLO contribution, which is to be added to the LO result in eq. (4.18) to give the final hard coefficients at NLO.

The parity relation in eq. (4.36) is preserved by the collinear subtraction terms and therefore continues to hold for the subtracted components,

$$[\Delta_1^{\text{sub}} W]_{++}^{++} = [\Delta_1^{\text{sub}} W]_{--}^{--}, \quad [\Delta_1^{\text{sub}} W]_{+-}^{+-} = [\Delta_1^{\text{sub}} W]_{-+}^{-+}. \tag{4.49}$$

4.5 Observables

Up to NLO, all the $[W]$ components have been found to be real. Combining eqs. (4.35) and (4.36) with eq. (3.41), we obtain the nonzero $[H]$ components,

$$\begin{aligned}
 [H]_{\pm\pm} &= \frac{2}{Q^2(1 - \varepsilon)} \left([F_{UU,T}] + \varepsilon [F_{UU,L}] \pm P_e \sqrt{1 - \varepsilon^2} [F_{LL}] \right), \\
 [H]_{+-} &= [H]_{-+} = -\frac{2}{Q^2(1 - \varepsilon)} \left(\varepsilon [F_{UT}^{(2-)}] + \sqrt{2\varepsilon(1 + \varepsilon)} [F_{UT}^{(1-)}] \right),
 \end{aligned} \tag{4.50}$$

where we used the same F notations as eqs. (3.48) (3.50) and (3.54) to make correspondence of the ε coefficients,

$$\begin{aligned}
 [F_{UU,T}] &= \frac{1}{2} ([W]_{++}^{++} + [W]_{--}^{++}), \quad [F_{UU,L}] = [W]_{++}^{00}, \\
 [F_{LL}] &= \frac{1}{2} ([W]_{++}^{++} - [W]_{--}^{++}), \\
 [F_{UT}^{(2-)}] &= \frac{1}{2} [W]_{+-}^{+-}, \quad [F_{UT}^{(1-)}] = \frac{1}{\sqrt{2}} [W]_{+-}^{+0}.
 \end{aligned} \tag{4.51}$$

The bracketed versions here, however, are no longer associated with any ϕ_h modulations. Using eqs. (A.7) and (A.10), we obtain the NS hard coefficients at NLO,

$$[C_{qq}] = \frac{2}{Q^2(1 - \varepsilon)} \left([F_{UU,T}] + \varepsilon [F_{UU,L}] \right), \tag{4.52a}$$

$$[\Delta C_{qq}] = \frac{2}{Q^2(1 - \varepsilon)} \left(P_e \sqrt{1 - \varepsilon^2} [F_{LL}] \right), \tag{4.52b}$$

$$[T_{qq}^{ij}] = \frac{2}{Q^2(1 - \varepsilon)} (-\delta^{ij}) \left(\varepsilon [F_{UT}^{(2-)}] + \sqrt{2\varepsilon(1 + \varepsilon)} [F_{UT}^{(1-)}] \right). \tag{4.52c}$$

Defining the normalization for $[F_X]$,

$$[F_X] = 2(2\pi)^4 e_q^2 [f_X], \quad (4.53)$$

we can combine the LO and NLO calculations, in eqs. (4.18) (4.34) and (4.47), to get

$$\begin{aligned} \{[f_{UU,T}], [f_{LL}]\} = & \left[1 - \frac{4\alpha_s C_F}{\pi} \right] e_q^2 \delta(1-\hat{x}) \delta(1-\hat{z}) + e_q^2 \frac{\alpha_s C_F}{2\pi} \hat{z} \left[\frac{1 + \hat{x}^2 \hat{z}^2}{(1-\hat{x})_+(1-\hat{z})_+} \right. \\ & \left. \pm (1-\hat{x})(1-\hat{z}) - A(\hat{x}) \delta(1-\hat{z}) - \delta(1-\hat{x}) B(\hat{z}) \right], \end{aligned} \quad (4.54a)$$

$$\begin{aligned} [f_{UT}^{(2-)}] = & \left[1 - \frac{4\alpha_s C_F}{\pi} \right] e_q^2 \delta(1-\hat{x}) \delta(1-\hat{z}) \\ & + e_q^2 \frac{\alpha_s C_F}{2\pi} \hat{z} \left[A'(\hat{x}) \delta(1-\hat{z}) + \delta(1-\hat{x}) B'(\hat{z}) \right], \end{aligned} \quad (4.54b)$$

$$\{[f_{UU,L}], [f_{UT}^{(1-)}]\} = e_q^2 \frac{\alpha_s C_F}{\pi} \hat{z} \left\{ 2\hat{x}\hat{z}, \sqrt{\frac{\hat{x}\hat{z}}{(1-\hat{x})(1-\hat{z})}} \right\}. \quad (4.54c)$$

Substituting eq. (4.52) into eq. (4.7) and then into eq. (4.5), we have the $\mathbf{p}_{hT}$ -integrated differential cross section,

$$\begin{aligned} \frac{d\Sigma}{dx dQ^2 d\phi_S dz d\eta d\phi} = & \frac{y^2}{z Q^4} \frac{\alpha_e^2}{(1-\varepsilon)} \left\{ \mathcal{C}'_{f_{UU,T}}[f, \mathcal{D}] + \varepsilon \mathcal{C}'_{f_{UU,L}}[f, \mathcal{D}] \right. \\ & + s_T \left(\varepsilon \mathcal{C}'_{f_{UT}^{(2-)}}[h, \mathcal{H}^\perp] + \sqrt{2\varepsilon(1+\varepsilon)} \mathcal{C}'_{f_{UT}^{(1-)}}[h, \mathcal{H}^\perp] \right) \sin(\phi - \phi_S) \\ & \left. + P_N P_e \sqrt{1-\varepsilon^2} \mathcal{C}'_{f_{LL}}[g, \mathcal{D}] \right\} \\ & + \mathcal{O}(\Lambda_F/Q, \Lambda_F/p_{h,\text{c.m.}}), \end{aligned} \quad (4.55)$$

where we defined the notation $\mathcal{C}'$ for the $\mathbf{p}_{hT}$ -integrated convolution,

$$\begin{aligned} \mathcal{C}'_K[f, \mathcal{D}] = & \mathcal{C}'_K[f, \mathcal{D}](Q^2, x, z; \Lambda_F) \\ \equiv & \int \frac{d\xi_1}{\xi_1} \int d\xi_2 f_{q/p}(\xi_1) \mathcal{D}_{1,h/q}(\xi_2, \Lambda_F) K(x/\xi_1, z/\xi_2; Q^2/\mu^2). \end{aligned} \quad (4.56)$$

Compared with eq. (3.56) for a finite p_{hT} , we see that eq. (4.5) achieves a great simplification. Unpolarized scattering cross section gives a handle for the unpolarized FEC,

$$\frac{d\Sigma^{\text{unp.}}}{dx dQ^2 d\phi_S dz d\eta d\phi} = \frac{y^2}{z Q^4} \frac{\alpha_e^2}{(1-\varepsilon)} \mathcal{C}'_{f_{UU,T}+\varepsilon f_{UU,L}}[f, \mathcal{D}], \quad (4.57)$$

which can also be probed via the double helicity asymmetry,

$$A_{LL} = \frac{d\Sigma(P_N, P_e) - d\Sigma(P_N, -P_e)}{d\Sigma(P_N, P_e) + d\Sigma(P_N, -P_e)} = P_N P_e \frac{\sqrt{1-\varepsilon^2} \mathcal{C}'_{f_{LL}}[g, \mathcal{D}]}{\mathcal{C}'_{f_{UU,T}+\varepsilon f_{UU,L}}[f, \mathcal{D}]}, \quad (4.58)$$

along with the polarized PDF $g_{q/p}$. As for the extraction of the Collins-type FEC, one needs to use the single transverse spin asymmetry of the target along with the ϕ modulation,

$$A_{UT} = \frac{d\Sigma(\phi_S) - d\Sigma(\phi_S + \pi)}{d\Sigma(\phi_S) + d\Sigma(\phi_S + \pi)} = s_T \sin(\phi - \phi_S) \frac{\mathcal{C}'_{\varepsilon f_{UT}^{(2-)} + \sqrt{2\varepsilon(1+\varepsilon)} f_{UT}^{(1-)}}[h, \mathcal{H}^\perp]}{\mathcal{C}'_{f_{UU,T}+\varepsilon f_{UU,L}}[f, \mathcal{D}]}, \quad (4.59)$$

where the transverse spin magnitude s_T is held fixed while its direction is flipped in constructing the asymmetry. In contrast to eq. (3.56), there is only one such ϕ modulation in this case. This is because the spin transfer matrix T is an identity, cf. eq. (4.52). As noted below eq. (4.14) for the case of $|X\rangle = |0\rangle$, one may heuristically interpret this as that by integrating over $\mathbf{p}_{hT}$, the quark can be thought of as being scattered in the backward direction, and its transverse spin is bent with respect to the electron scattering plane.⁹

Before closing this section, we would make one final remark. Unlike eq. (3.56), which is valid to all orders, the discussion in this section is primarily confined to NLO. What shall we expect beyond NLO (for the NS channel)? First of all, eqs. (4.35) and (4.36) contain all the non-vanishing $[W]$ components, which have all shown up at NLO and entered the cross section in eq. (4.55). The only qualitatively new feature that can arise is the imaginary part of $[W]_{+-}^{+0}$. That is, the last identity in eq. (4.35) should be changed to

$$[W]_{+-}^{+0} = -[W]_{-+}^{-0} = ([W]_{-+}^{0+})^* = -([W]_{+-}^{0-})^*, \quad (4.60)$$

while all the other $[W]$ components stay real. This will induce an imaginary part to $[H]_{\pm\mp}$, which become

$$\begin{aligned} [H]_{\pm\mp} &= -\frac{2}{Q^2(1-\varepsilon)} \left\{ \frac{\varepsilon}{2} [W]_{+-}^{+-} + \sqrt{\varepsilon(1+\varepsilon)} \operatorname{Re} [W]_{+-}^{+0} \pm i P_e \sqrt{\varepsilon(1-\varepsilon)} \operatorname{Im} [W]_{+-}^{+0} \right\} \\ &= -\frac{2}{Q^2(1-\varepsilon)} \left\{ \varepsilon [F_{UT}^{(2-)}] + \sqrt{2\varepsilon(1+\varepsilon)} [F_{UT}^{(1-)}] \pm i P_e \sqrt{2\varepsilon(1-\varepsilon)} [F_{LT}^{(1-)}] \right\}, \end{aligned} \quad (4.61)$$

instead of eq. (4.50). The spin transfer matrix T will then generate an off-diagonal element,

$$[T]_{qq} = \begin{pmatrix} \operatorname{Re} [H]_{+-} & \operatorname{Im} [H]_{+-} \\ -\operatorname{Im} [H]_{+-} & \operatorname{Re} [H]_{+-} \end{pmatrix} \quad (4.62)$$

which will contribute an additional modulation $\cos(\phi - \phi_S)$ to eq. (4.55). It requires a longitudinal beam polarization and consequently leads to a double spin asymmetry A_{LT} , following eq. (4.59). This imaginary part, $\operatorname{Im} [W]_{+-}^{+0}$, can arise at NNLO from a threshold loop, so is expected to be suppressed by $\mathcal{O}(\alpha_s^2)$. There are no other observables expected to arise beyond NNLO; higher orders only induce quantitative corrections.

5 Summary

Parton fragmentation dynamics in the transverse direction encodes a large amount of the nonperturbative hadronization information, and has been extensively studied in the literature with various types of observables. In this paper, we give a detailed study of the FEC in the SIDIS. As a hybrid of nonperturbative parton FF and IRC-safe energy flow operator, the FEC provides a powerful observable to probe the transverse fragmentation dynamics.

We have introduced a modification of the original definition in ref. [33] to equip the FEC with a boost-invariant property, which makes the global extraction of FECs from different experiments more consistent and allows us to study two types of SIDIS observables

⁹However, one should keep in mind that this picture does not correspond to actual event analysis in experiments, where ϕ is measured with respect to the hadron production plane which varies from event to event. Only ϕ_S is measured with respect to the fixed electron scattering plane in the Breit frame.

in this paper: one with a large transverse momentum of the observed hadron and the other with the transverse momentum inclusively integrated over. After arguing for an all-order factorizations of the FEC in both cases, we performed a leading- or next-to-leading-order calculation and confirmed the consistency. Especially, we have given a detailed discussion of an azimuthal singularity in the perturbative calculation in the second type of observables, which is characteristic of the transverse spin transfer phenomenon in SIDIS and is presented here for the first time.

A great advantage of FEC lies in its collinear factorization which shares a parallel structure as the normal SIDIS. Theoretically, this allows one to directly copy the hard coefficient calculations from SIDIS. Experimentally, this provides an observable that can be directly obtained via a straightforward analysis or re-analysis of the SIDIS experiments that have been or are being analyzed. One only needs to record the energy flow next to the measured hadron, without any extra measurements. The same SIDIS kinematics applies, whether one integrates over the hadron transverse momentum or not. The extraction of FEC can also be done in a parallel way to the FF.

Without measuring the spin of the final state hadron, the azimuthal asymmetry of the energy flow around the hadron provides an extra handle for the transverse spin of the fragmenting quark. This asymmetry, encoded in the Collins-type FEC, is a particular focus of this paper. It is an analogy of the transverse-momentum-dependent Collins function, both being sensitive to the chiral symmetry breaking effect in QCD, but the FEC is more easily measured and reveals complementary dynamic information. We have given a thorough discussion of the azimuthal modulations that are associated with the measurement of such FEC. Our formalism also applies to lepton-lepton and hadron-hadron collisions in a similar way. The universality of FECs allows them to be extracted in a global analysis. We will present our phenomenology study in a separate paper.

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A Hard coefficients in terms of helicity amplitudes

Here, we present the definitions of the hard coefficients C_{qq} , ΔC_{qq} , and T_{qq}^{ij} in eq. (3.18) for the quark-to-quark channel, $(a, b) = (q, q)$.

Denote $\mathcal{A}_{\beta\alpha}(\ell, k_1, \ell, k_2; k_X)$ as the amplitude of the partonic scattering,

$$e(\ell) + q(k_1) \rightarrow e(\ell') + q(k_2) + X(k_X), \quad (\text{A.1})$$

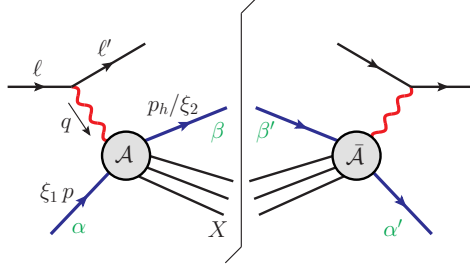


Figure 6. Hard scattering of the factorization formulae in eqs. (3.12) and (3.18).

amputated on the external quark legs, whose momenta k_1 and k_2 have been projected on shell as in eq. (3.34). (Recall that we use massless hadron approximation.) Then the hard factor resulted from the factorization of region R_C in figure 2(a) is given by

$$H_{\beta\alpha,\alpha'\beta'} = H_{\beta\alpha,\alpha'\beta'}(\ell, k_1, \ell', k_2) = \int d\Pi_X (2\pi)^4 \delta^{(4)}(q + k_1 - k_2 - k_X) \mathcal{A}_{\beta\alpha} \bar{\mathcal{A}}_{\alpha'\beta'}, \quad (\text{A.2})$$

which inclusively sums over X and its phase space Π_X , and where $\bar{\mathcal{A}} = \gamma^0 \mathcal{A}^\dagger \gamma^0$. This is shown in figure 6. $H_{\beta\alpha,\alpha'\beta'}$ also includes the subtraction of collinear singularities associated with the initial- and final-state quarks; however, this will not affect the discussion that follows.

The hard coefficients C_{qq} , ΔC_{qq} , and T_{qq}^{ij} are obtained from the projectors in eq. (3.17), given by $H_{\beta\alpha,\alpha'\beta'}$ traced with certain spinor matrices,

$$C_{qq}(x/\xi_1, z/\xi_2, Q^2, \mathbf{p}_{hT}) \equiv (\gamma \cdot k_2)_{\beta'\beta} H_{\beta\alpha,\alpha'\beta'} \left(\frac{\gamma \cdot k_1}{2} \right)_{\alpha\alpha'}, \quad (\text{A.3a})$$

$$\Delta C_{qq}(x/\xi_1, z/\xi_2, Q^2, \mathbf{p}_{hT}) \equiv (\gamma \cdot k_2)_{\beta'\beta} H_{\beta\alpha,\alpha'\beta'} \left(\frac{\gamma_5 \gamma \cdot k_1}{2} \right)_{\alpha\alpha'}, \quad (\text{A.3b})$$

$$T_{qq}^{ij}(x/\xi_1, z/\xi_2, Q^2, \mathbf{p}_{hT}) \equiv (\gamma \cdot k_2 \gamma^i \gamma_5)_{\beta'\beta} H_{\beta\alpha,\alpha'\beta'} \left(\frac{\gamma \cdot k_1 \gamma^j \gamma_5}{2} \right)_{\alpha\alpha'}, \quad (\text{A.3c})$$

where repeated spinor indices are summed over. They all take real values by Hermiticity. As remarked below eq. (3.19), the transverse indices i and j are defined in different coordinate systems, which complicates actual calculations. This can be avoided by expressing the hard coefficients in terms of helicity amplitudes, and then C_{qq} , ΔC_{qq} , and T^{ij} will be manifested as different helicity structures.

We first note the following identities for any on-shell spinors $u(k, \lambda)$ and $\bar{u}(k, \lambda)$ of a lightlike momentum k and helicity λ ,

$$\begin{aligned} \gamma \cdot k &= u(k, +) \bar{u}(k, +) + u(k, -) \bar{u}(k, -), \\ \gamma_5 \gamma \cdot k &= u(k, +) \bar{u}(k, +) - u(k, -) \bar{u}(k, -). \end{aligned} \quad (\text{A.4})$$

This motivates us to define the helicity amplitude for eq. (A.1),

$$\mathcal{A}_{\lambda_1 \lambda_2}(\ell, k_1, \ell, k_2; k_X) \equiv \bar{u}_\beta(k_2, \lambda_2) \mathcal{A}_{\beta\alpha}(\ell, k_1, \ell, k_2; k_X) u_\alpha(k_1, \lambda_1) \quad (\text{A.5})$$

and the corresponding hard factor in the helicity space,

$$\begin{aligned} H_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2} &\equiv \int d\Pi_X (2\pi)^4 \delta^{(4)}(q + k_1 - k_2 - k_X) \mathcal{A}_{\lambda_1 \lambda_2} \mathcal{A}_{\lambda'_1 \lambda'_2}^* \\ &= u_{\beta'}(k_2, \lambda'_2) \bar{u}_{\beta}(k_2, \lambda_2) H_{\beta\alpha, \alpha'\beta'} u_{\alpha}(k_1, \lambda_1) \bar{u}_{\alpha'}(k_1, \lambda'_1). \end{aligned} \quad (\text{A.6})$$

By eq. (A.4), we can then rewrite C_{qq} and ΔC_{qq} as

$$C_{qq} = \frac{1}{2} (H_{++,++} + H_{--,--} + H_{+,-,+} + H_{-+,-}), \quad (\text{A.7a})$$

$$\Delta C_{qq} = \frac{1}{2} (H_{++,++} - H_{--,--} + H_{+,-,+} - H_{-+,-}). \quad (\text{A.7b})$$

Clearly, the C_{qq} sums over both initial- and final-state spins, while ΔC_{qq} only sums over the final-state spin and incorporates the helicity difference of the initial-state quark.

The Dirac projectors in T_{qq} are chirally odd, and therefore mix spinors of different helicities. Their exact expressions, like eq. (A.4), depend on the relative phase between $u(k, +)$ and $\bar{u}(k, -)$. We adopt the convention of the γ matrices as given in ref. [106] and take as part of the definition for $u(k, +)$ and $\bar{u}(k, -)$ the identity,

$$u(k, +) \bar{u}(k, -) = \frac{1}{2} \gamma \cdot k (\gamma^x + i\gamma^y) \gamma_5, \quad u(k, -) \bar{u}(k, +) = \frac{1}{2} \gamma \cdot k (\gamma^x - i\gamma^y) \gamma_5, \quad (\text{A.8})$$

the two of which are related by Hermitian conjugation. Eq. (A.8) can be readily checked if k moves along the z direction. For k moving along an arbitrary direction described by (θ, ϕ) , its spinors $u(k, \pm)$ are defined by rotating both sides of eq. (A.8) first by θ around the y axis and then by ϕ around the z axis, in exactly the same way as for single-particle states [103]. This also shows how the x and y indices on the right-hand sides of eq. (A.8) are defined,

$$\hat{y}_k = \frac{\hat{z} \times \mathbf{k}}{|\hat{z} \times \mathbf{k}|}, \quad \hat{x}_k = \hat{y}_k \times \frac{\mathbf{k}}{|\mathbf{k}|}, \quad (\text{A.9})$$

where $\hat{z}$ is the unit vector along the lab-frame z axis, and $\hat{x}_k$ and $\hat{y}_k$ are the unit vectors defining the x and y indices associated with the momentum k . See also ref. [92] for a detailed review of this. Using eq. (A.8), we can then rewrite the four components of eq. (A.3c) as

$$T_{qq}^{xx} = \frac{1}{2} (H_{++,--} + H_{--,++} + H_{+,-,+} + H_{-+,-}), \quad (\text{A.10a})$$

$$T_{qq}^{yy} = \frac{1}{2} (H_{++,--} + H_{--,++} - H_{+,-,+} - H_{-+,-}), \quad (\text{A.10b})$$

$$T_{qq}^{xy} = \frac{-i}{2} (H_{++,--} - H_{--,++} + H_{+,-,+} - H_{-+,-}), \quad (\text{A.10c})$$

$$T_{qq}^{yx} = \frac{i}{2} (H_{++,--} - H_{--,++} - H_{+,-,+} + H_{-+,-}). \quad (\text{A.10d})$$

By the simple Hermiticity property encoded in eq. (A.6),

$$H_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2} = H_{\lambda'_1 \lambda'_2, \lambda_1 \lambda_2}^*, \quad (\text{A.11})$$

we easily see that the expressions in eqs. (A.7) and (A.10) are all real functions, in agreement with the remark below eq. (A.3).

B Angular integral in d dimensions

The discussion here follows appendix B.3.2 of Schwartz [107] and is an analytic continuation from integer-dimensional space.

A vector $\mathbf{r}_d$ in d dimensions can be written as

$$\mathbf{r}_d = r \mathbf{n}_d = r (n_d^1, n_d^2, \dots, n_d^d), \quad (\text{B.1})$$

where $r = \sqrt{\mathbf{r}_d \cdot \mathbf{r}_d} \geq 0$ and $\mathbf{n}_d$ is a unit vector in d dimensions. Since $|\mathbf{n}_d|^2 = 1$, it can be parametrized by $(d-1)$ angular variables $(\phi_1, \dots, \phi_{d-1})$ that span the hypersphere S_{d-1} . The first few nontrivial examples are

$$\begin{aligned} \mathbf{n}_2 &= (\cos \phi_1, \sin \phi_1), \\ \mathbf{n}_3 &= (\sin \phi_2 \cos \phi_1, \sin \phi_2 \sin \phi_1, \cos \phi_2), \\ \mathbf{n}_4 &= (\sin \phi_3 \sin \phi_2 \cos \phi_1, \sin \phi_3 \sin \phi_2 \sin \phi_1, \sin \phi_3 \cos \phi_2, \cos \phi_3), \quad \dots \end{aligned} \quad (\text{B.2})$$

from which one can easily deduce the iterative formula,

$$\mathbf{n}_d = (\sin \phi_{d-1}(\mathbf{n}_{d-1}), \cos \phi_{d-1}), \quad \text{with } d \geq 2 \text{ and } \mathbf{n}_2 = (\cos \phi_1, \sin \phi_1). \quad (\text{B.3})$$

For $d = 2$, the angle ϕ_1 should go from 0 to 2π to cover the whole circle, such that the two components n_2^1 and n_2^2 can have all possible signs: $(n_2^1, n_2^2) = (+, +), (-, -), (+, -)$, and $(-, +)$. As one builds $\mathbf{n}_d$ upon the iteration in eq. (B.3), since the internal $\mathbf{n}_{d-1}$ has covered the whole hypersphere S_{d-2} , one only needs to require $\cos \phi_{d-1}$ to cover $[-1, 1]$ while keeping $\sin \phi_{d-1} \geq 0$, i.e., $\phi_{d-1} \in [0, \pi]$ (otherwise having $\phi_{d-1} \in [0, 2\pi)$ gives double-counted directions). That is, we have

$$\phi_1 \in [0, 2\pi), \quad \phi_i \in [0, \pi], \quad \text{for } i = 2, \dots, d-1. \quad (\text{B.4})$$

From eq. (B.3), one can get a general explicit expression for $\mathbf{n}_d = (n_d^1, n_d^2, \dots, n_d^d)$,

$$n_d^1 = \cos \phi_1 \prod_{k=2}^{d-1} \sin \phi_k, \quad n_d^2 = \sin \phi_1 \prod_{k=2}^{d-1} \sin \phi_k, \quad n_d^j = \cos \phi_{j-1} \prod_{k=j}^{d-1} \sin \phi_k, \quad j = 3, \dots, d, \quad (\text{B.5})$$

with the convention that $\prod_{k=a}^b c_k = 1$ for any quantity c if $a > b$.¹⁰

The integration over $\mathbf{r}_d$ can be written as an integration over r and $(d-1)$ -dimensional integration over the angles,¹¹

$$\begin{aligned} \int d^d \mathbf{r}_d &= \int dr r^{d-1} \int d\Omega_{d-1} \\ &= \int dr r^{d-1} \int_0^{2\pi} d\phi_1 \int_0^\pi d\phi_2 \cdots \int_0^\pi d\phi_{d-1} J_{d-1}(\phi_1, \dots, \phi_{d-1}). \end{aligned} \quad (\text{B.6})$$

¹⁰Note that the convention for ϕ_1 is different from all the other ϕ_j . If we defined $\mathbf{n}_2 = (\sin \phi_1, \cos \phi_1)$, then eq. (B.5) would be simpler,

$$n_d^j = \cos \phi_{j-1} \prod_{k=j}^{d-1} \sin \phi_k, \quad j = 1, \dots, d,$$

with the understanding that $\phi_0 = 0$.

¹¹Note that the notation for Ω_d differs from Schwartz's [107] in that here Ω_d refers to the solid angle in S_d whereas in [107] the Ω_d refers to the solid angle in S_{d-1} .

Here J_{d-1} is the Jacobian associated with the $(d-1)$ -dimensional angular integral. It can be obtained by an explicit calculation,

$$J_{d-1} = \frac{1}{r^{d-1}} \text{abs} \begin{vmatrix} \partial \mathbf{r}_d / \partial r \\ \partial \mathbf{r}_d / \partial \phi_{d-1} \\ \partial \mathbf{r}_d / \partial \phi_{d-2} \\ \dots \\ \partial \mathbf{r}_d / \partial \phi_1 \end{vmatrix} = \text{abs} \begin{vmatrix} \sin \phi_{d-1}(\mathbf{n}_{d-1}) & \cos \phi_{d-1} \\ \cos \phi_{d-1}(\mathbf{n}_{d-1}) & -\sin \phi_{d-1} \\ \sin \phi_{d-1} \partial \mathbf{n}_{d-1} / \partial \phi_{d-2} & 0 \\ \dots & \dots \\ \sin \phi_{d-1} \partial \mathbf{n}_{d-1} / \partial \phi_1 & 0 \end{vmatrix}, \quad (\text{B.7})$$

where we have singled out the last column in the determinant. Factorizing all the coefficients of $\mathbf{n}_{d-1}$ in the other columns gives

$$J_{d-1} = \left| \sin^{d-1} \phi_{d-1} \cos \phi_{d-1} \right| \text{abs} \begin{vmatrix} \mathbf{n}_{d-1} & \cot \phi_{d-1} \\ \mathbf{n}_{d-1} & -\tan \phi_{d-1} \\ \partial \mathbf{n}_{d-1} / \partial \phi_{d-2} & 0 \\ \dots & \dots \\ \partial \mathbf{n}_{d-1} / \partial \phi_1 & 0 \end{vmatrix}, \quad (\text{B.8})$$

which allows us to subtract the second row from the first to obtain a recursion relation,

$$J_{d-1} = \left| \frac{\sin^{d-1} \phi_{d-1} \cos \phi_{d-1}}{\sin \phi_{d-1} \cos \phi_{d-1}} \right| \text{abs} \begin{vmatrix} \mathbf{n}_{d-1} \\ \partial \mathbf{n}_{d-1} / \partial \phi_{d-2} \\ \dots \\ \partial \mathbf{n}_{d-1} / \partial \phi_1 \end{vmatrix} = \sin^{d-2} \phi_{d-1} J_{d-2}, \quad (\text{B.9})$$

where

$$J_{d-1} = J_{d-1}(\phi_1, \dots, \phi_{d-1}), \quad J_{d-2} = J_{d-2}(\phi_1, \dots, \phi_{d-2}). \quad (\text{B.10})$$

The initial value of this recursion can be easily calculated to be $J_1 = 1$. Then, we obtain the Jacobian

$$J_{d-1}(\phi_1, \dots, \phi_{d-1}) = \sin^{d-2} \phi_{d-1} \sin^{d-3} \phi_{d-2} \dots \sin \phi_2, \quad (\text{B.11})$$

and hence the angular integration in S_{d-1} ,

$$\int d\Omega_{d-1} = \int_0^{2\pi} d\phi_1 \int_0^\pi d\phi_2 \dots \int_0^\pi d\phi_{d-1} \sin^{d-2} \phi_{d-1} \sin^{d-3} \phi_{d-2} \dots \sin \phi_2. \quad (\text{B.12})$$

For the purpose of studying azimuthal modulation in dimensional regularization, we keep $\phi_1 \equiv \phi$ and integrate over all other angles. This gives

$$\int d\Omega_{d-1} = \frac{\pi^{\frac{d-2}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_0^{2\pi} d\phi. \quad (\text{B.13})$$

Applying this to the transverse momentum $\mathbf{p}_T$ integration, we note that $\mathbf{p}_T$ has $(d-2) = (2-2\epsilon)$ dimensions in dimensional regularization, and contains $(d-3) = (1-2\epsilon)$ angular variables. Therefore, we have

$$\begin{aligned} \int d^{d-2} \mathbf{p}_T &= \int_0^\infty dp_T p_T^{d-3} \int d\Omega_{d-3} = \int_0^\infty dp_T p_T^{d-3} \left[\frac{\pi^{\frac{d-4}{2}}}{\Gamma\left(\frac{d-2}{2}\right)} \int_0^{2\pi} d\phi \right] \\ &= \frac{\pi^{-\epsilon}}{2\Gamma(1-\epsilon)} \int_0^\infty dp_T^2 (p_T^2)^{-\epsilon} \int_0^{2\pi} d\phi. \end{aligned} \quad (\text{B.14})$$

This yields the formula used in eq. (4.25).

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Code Availability Statement. This article has no associated code or the code will not be deposited.

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