Superluminal matter waves

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(Received 30 April 2024; accepted 4 October 2024; published 29 October 2024)

The Dirac equation has resided among the greatest successes of modern physics since its emergence as the first quantum mechanical theory fully compatible with special relativity. This compatibility ensures that the expectation value of the velocity is less than the vacuum speed of light. Here, we show that the Dirac equation admits free-particle solutions where the peak amplitude of the wave function can travel at any velocity, including those exceeding the vacuum speed of light, despite having a subluminal velocity expectation value. The solutions are constructed by superposing basis functions with correlations in momentum space. These arbitrary velocity wave functions feature a near-constant profile and may impact quantum mechanical processes that are sensitive to the local value of the probability density as opposed to expectation values.

DOI: 10.1103/PhysRevResearch.6.043075

I. INTRODUCTION

The discovery of the Dirac equation stands as a foundational achievement of modern physics [1]. As the first quantum mechanical theory fully compatible with special relativity, the Dirac equation resolved several inconsistencies that beset the, otherwise widely successful, Schrödinger equation [2]. The equation and its solutions preserve many of the quantum mechanical concepts developed within the context of the Schrödinger equation, such as probability currents, expectation values, and operators. Unlike the Schrödinger equation, however, the Dirac equation was formulated from the outset to exhibit a Lorentz-covariant and Hamiltonian structure consistent with special relativity. In doing so, the Dirac equation precluded unphysical phenomena fully allowed by the Schrödinger equation, namely, expectation values for velocity that exceed the vacuum speed of light.

At a fundamental level, the Dirac equation is a wave equation. Within first quantization, the solutions, or wave functions ψ, describe the quantum mechanical state of a charged lepton, while their adjoint product $\psi^\dagger\psi$ provides the probability density of finding the lepton within a particular region of space-time or momentum-energy. In the absence of fields (and vacuum nonlinearities), the simplest solution is a plane wave modulating a constant spinor [3]. Physically occurring, localized wave functions are typically formed by superposing these solutions with amplitudes and phases that are uncorrelated in space-time or momentum-energy. Superpositions featuring correlations in space-time or momentum-energy allow for wave functions with more complex and potentially advantageous structures.

Insight into the fundamental properties of matter and light and the potential for applications has driven a growing interest in structured solutions for quantum mechanical and electromagnetic waves, with ideas from one often being adapted to the other [4–17]. One of the most iconic examples is the concept of orbital angular momentum from quantum mechanics being adapted to electromagnetism [5,6]. Within quantum mechanics, this "wave-function engineering" has been applied to the formation of self-accelerating solutions, which appear to move under the influence of a potential in the absence of any potential [12,15]; transverse shaping of electron wave functions for added flexibility in scanning electron microscopes [16]; and the formation of free-electron crystals for enhancing x-ray radiation [17]. In each of these, the physical processes of interest are sensitive to the local properties of the wave function as opposed to expectation values.

Here, we introduce a structured free-particle solution to the Dirac equation where the peak amplitude of the wave function can travel at any velocity, including those exceeding the vacuum speed of light, despite having a subluminal velocity expectation value (Fig. 1). Motivated by similar solutions found in electromagnetism (e.g., space-time wave packets [18–20] and flying focus pulses [21–27]), the solutions feature a near-constant profile and are constructed by superposing basis functions with correlations in momentum-energy. The solutions can be generated using the Kapitza-Dirac effect to apply a time-dependent focusing phase to an initial wave function [11,28–36]. While previous studies have described superluminal solutions to the Dirac equation in the context of tunneling and wave-packet spreading [37–43], the solutions introduced here do not require a potential or external fields and have a well-defined, controllable velocity. Furthermore, the formalism presented here can be used to construct solutions with more exotic properties, such as a modified

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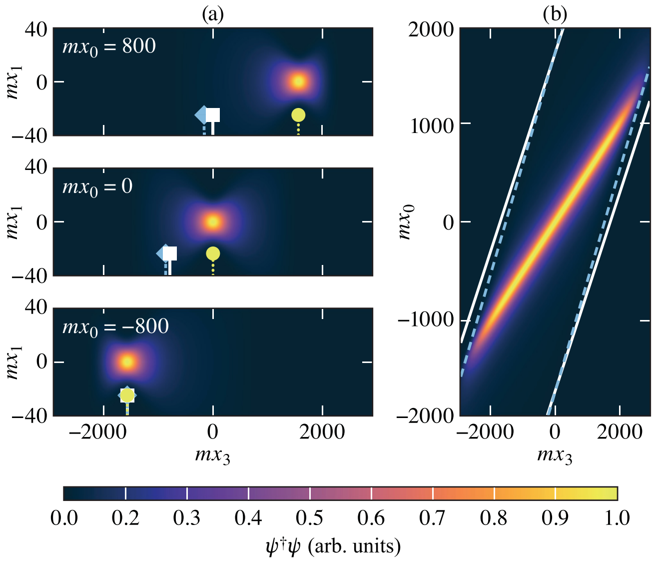

FIG. 1. Evolution of the probability density $\psi^\dagger\psi$ for a lepton with a velocity expectation value $\langle v_3\rangle = 0.9$ and a group velocity $v_a = 2$: (a) snapshots of the spatial profile at $x_2 = 0$ and (b) the on-axis profile (i.e., at $x_1 = x_2 = 0$). The peak of the probability density travels superluminally while maintaining a near-constant profile. The yellow (circle, dotted), blue (diamond, dashed), and white (square, solid) lines indicate signals traveling at the specified group velocity $v_a = 2$, the velocity expectation value $\langle v_3\rangle = 0.9$, and the vacuum speed of light $v = 1$, respectively. In this example, $\bar{\mathcal{P}} = 2m$, $w = 0.1m$, $\xi_0 = 250m^{-1}$, and $\zeta_0 = 1400m^{-1}$ [see Eqs. (23)–(25)].

effective mass in the absence of fields. These arbitrary velocity solutions may impact quantum mechanical phenomena that are sensitive to the local properties of the probability density as opposed to expectation values [44].

II. ARBITRARY VELOCITY WAVEFUNCTIONS

The wave function of a spin one-half charged particle evolves according to the Dirac equation. In natural units ($\hbar = c = 1$), the Schrödinger form of the Dirac equation is given by

$$i\partial_t\psi = H\psi, \tag{1}$$

where $H = -i\boldsymbol{\alpha}\cdot\nabla + \beta m$ is the Hamiltonian,

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

the σ_j with $j = 1, 2, 3$ are the Pauli matrices, I is the identity matrix, and m is the mass of the particle. Here and throughout, bold font denotes three-vectors, $\boldsymbol{a}^2 \equiv \boldsymbol{a}\cdot\boldsymbol{a}$, the shorthand notations $p \equiv p^\mu$ and $x \equiv x^\mu$ are used for the momentum and position four-vectors, $a\cdot b \equiv a^\mu b_\mu = a_0 b_0 - \boldsymbol{a}\cdot\boldsymbol{b}$, and relativistic notation for subscripts and superscripts is not used.

The Dirac equation admits four plane-wave solutions, corresponding to positive and negative energy and spin states. A general wave function can be expressed as a superposition of these solutions. For simplicity and definiteness, a superposition of solutions with positive energy and spin in the rest frame will be considered, such that

$$\psi(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{1}{\sqrt{2p_0}}u(p)f(p)e^{-ip\cdot x}\delta(p_0 - E_{\boldsymbol{p}})d^4p, \tag{2}$$

where

$$u(p) = \sqrt{E_{\boldsymbol{p}} + m}\begin{pmatrix} \chi_+ \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_{\boldsymbol{p}}+m}\chi_+ \end{pmatrix} \tag{3}$$

is the bispinor normalized so that $u^\dagger\beta u = 2m$, $\dagger$ indicates a Hermitian conjugate, $\chi_+ = \binom{1}{0}$, $E_{\boldsymbol{p}} = (m^2 + \boldsymbol{p}^2)^{1/2} > 0$, and the Dirac delta function δ ensures the "on-shell" condition. The complex scalar function $f(p)$ determines the relative phase and amplitude of each plane wave that composes the wave function. Here, $f(p)$ is expressed as an integral over an auxiliary parameter q:

$$f(p) = \int N(p, q)dq. \tag{4}$$

The utility of this parameter will become clear below. The functions $f(p)$ and $N(p, q)$ are constrained by the normalization condition $\int \psi^\dagger\psi d^3x = \int |f(p)|^2 d^3p = 1$, where $\psi^\dagger\psi$ is the probability density and, for the remainder, $p_0 = E_{\boldsymbol{p}}$ is implied. Note that the wave functions described here only require positive-energy states, which is crucial to avoid pair production and stay within a single-particle framework.

The evolution of the wave function can be characterized by three velocities: the phase velocities of the plane-wave solutions, the eigenvalues (or expectation values) of the velocity operator, and the group velocity. The phase velocities, $\boldsymbol{v}_p = E_{\boldsymbol{p}}\boldsymbol{p}/\boldsymbol{p}^2$, are always superluminal ($|\boldsymbol{v}_p| > 1$). The velocity operator $\boldsymbol{v}$ is found by evaluating the commutator of the Hamiltonian and position vector: $\boldsymbol{v} = i[H, \boldsymbol{x}] = \boldsymbol{\alpha}$. Using Eq. (2), one finds $\langle\boldsymbol{v}\rangle = \int \psi^\dagger\boldsymbol{v}\psi d^3x = \int(\boldsymbol{p}/E_{\boldsymbol{p}})|f(p)|^2 d^3p = \langle\boldsymbol{p}/E_{\boldsymbol{p}}\rangle$, which, as expected, is always subluminal ($|\boldsymbol{p}/E_{\boldsymbol{p}}| < 1$).

The group velocity $\boldsymbol{v}_g$ can take any value in any direction. For a typical wave function, the components of the four momenta must satisfy the on-shell condition but are otherwise independent. As a result, $\partial p_j/\partial p_k = \delta_{jk}$ and

$$v_{g,j} = \frac{\partial E_{\boldsymbol{p}}}{\partial p_j} = \frac{p_j}{E_{\boldsymbol{p}}}, \tag{5}$$

which is consistent with the eigenvalues of the velocity operator $\langle\boldsymbol{v}\rangle = \langle\boldsymbol{p}/E_{\boldsymbol{p}}\rangle$. However, if the momenta have some correlation, i.e., they are interdependent, then $\partial p_j/\partial p_k \neq \delta_{jk}$ and

$$v_{g,j} = \frac{\partial E_{\boldsymbol{p}}}{\partial p_j} = \frac{p_j}{E_{\boldsymbol{p}}} + \frac{1}{2E_{\boldsymbol{p}}}\frac{\partial}{\partial p_j}(\boldsymbol{p}^2 - p_j^2). \tag{6}$$

In this case, the correlated momenta introduce an additional contribution (the second term on the right-hand side) that decouples the group velocity from $\boldsymbol{p}/E_{\boldsymbol{p}}$.

As an example, consider the case of an arbitrary, constant group velocity $\boldsymbol{v}_a$. Direct integration of $\partial E_{\boldsymbol{p}}/\partial p_j$ yields

$$E_{\boldsymbol{p}} = \boldsymbol{v}_a\cdot\boldsymbol{p} + \kappa, \tag{7}$$

where κ is a constant. Setting $(m^2 + \boldsymbol{p}^2)^{1/2} = \boldsymbol{v}_a\cdot\boldsymbol{p} + \kappa$ ensures that the on-shell condition is satisfied and determines

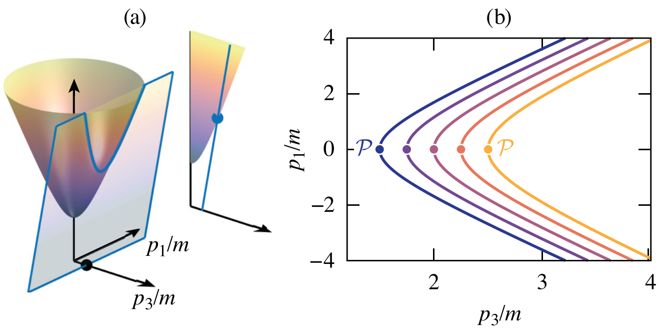

FIG. 2. (a) Satisfying the on-shell condition for an arbitrary group velocity v_a is equivalent to finding the intersection between a four-dimensional hyperboloid, $m^2 = E_{\boldsymbol{p}}^2 - \boldsymbol{p}^2$, and the hyperplane $E_{\boldsymbol{p}} = \boldsymbol{v}_a \cdot \boldsymbol{p} + \kappa$. Here, the dimensionality has been reduced for visualization purposes by setting $p_2 = 0$. (b) Projections of the intersection in the p_3-p_1 plane for different values of $\mathcal{P}$. The values of $\mathcal{P}$ define the vertices of the projection, i.e., the values of p_3 where $p_1 = p_2 = 0$ [Eq. (16)]. The parameters are the same as in Fig. 1.

the interdependence of the momenta (Fig. 2). Specifically,

$$\boldsymbol{p}^2 - (\boldsymbol{v}_a \cdot \boldsymbol{p})^2 - 2\kappa(\boldsymbol{v}_a \cdot \boldsymbol{p}) + m^2 - \kappa^2 = 0. \tag{8}$$

Without loss of generality, the specified group velocity $\boldsymbol{v}_a$ will be aligned along the $j = 3$ axis, i.e., $\boldsymbol{v}_a = v_a \boldsymbol{e}_3$, where $\boldsymbol{e}_j$ denotes a unit vector. Substitution of Eqs. (7) and (8) into Eq. (6) verifies that indeed $v_{g,3} = v_a$. The condition on the momenta [Eq. (8)] is a quadratic equation whose solution provides p_3 in terms of $\boldsymbol{p}_\perp \equiv p_1\boldsymbol{e}_1 + p_2\boldsymbol{e}_2$. Upon solving this equation, one finds

$$p_3 = p_c \equiv \kappa\gamma_a^2 v_a \pm \left[\left(\kappa\gamma_a^2\right)^2 - \gamma_a^2\left(m^2 + \boldsymbol{p}_\perp^2\right)\right]^{1/2}, \tag{9}$$

where $\gamma_a \equiv 1/\sqrt{1 - v_a^2}$. Note that γ_a only appears in even powers. Thus, a superluminal group velocity ($|v_a| > 1$) does not produce a complex valued p_c. In the special case $v_a = -1$, one solution diverges and is discarded, while the second, physical solution is given by $p_c = (\kappa^2 - m^2 - \boldsymbol{p}_\perp^2)/2\kappa$.

The interdependence of the momenta, as described by Eq. (9), can be built into the wave function by making use of the auxiliary parameter q. Expressing $q = q(\kappa)$ as a function of κ (to be determined below) and setting $N(p, q) = \mathcal{N}(p, q)\delta(p_3 - p_c)$ yields

$$f(p) = \int \mathcal{N}(p, q)\delta(p_3 - p_c)dq, \tag{10}$$

where the dependence of p_c and q on κ is understood. Together, Eqs. (2) and (10) describe a superposition of solutions with correlated momenta parametrized by κ. Each solution has the same arbitrary group velocity $v_{g,3} = v_a$.

To elucidate the physical meaning of κ and facilitate further calculation, it is helpful to substitute Eq. (10) into Eq. (2) and define the function

$$\Phi(x, q) = \int \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} u(p)\mathcal{N}(p, q)e^{-ip\cdot x}\delta(p_3 - p_c)d^3p, \tag{11}$$

such that $\psi(x) = (2\pi)^{-3/2}\int \Phi(x, q)dq$. Several properties of the arbitrary group velocity solutions can be analyzed by considering the phase in Eq. (11): $p \cdot x = p_0x_0 - p_3x_3 - \boldsymbol{p}_\perp \cdot \boldsymbol{x}_\perp$. Making use of the delta function, the phase contribution $\phi \equiv p_0x_0 - p_3x_3$ becomes

$$\phi = (p_3v_a + \kappa)x_0 - p_3x_3 = \kappa x_0 + p_3(v_ax_0 - x_3). \tag{12}$$

The first term is proportional to κ but does not depend on $\boldsymbol{p}$. As a result, the factor $e^{-i\kappa x_0}$ can be extracted from the integral. This factor shows that κ quantifies the temporal phase advance of the wave function. The second term in Eq. (12) reveals that the integrand depends on x_0 and x_3 only in the combination $v_ax_0 - x_3$. Thus, the integrand advects at the group velocity v_a.

An intuitive, and often helpful, picture of a wave function is that of an "envelope" modulated by a "carrier" wave. The envelope describes the bulk motion of the probability density and propagates at the group velocity, while the carrier describes the motion of the phase fronts, which propagate at the phase velocity. The function $\Phi(x, q)$ can be expressed in this same framework. Adding and subtracting a longitudinal (along x_3) momentum offset $\mathcal{P}$ to Eq. (12) provides

$$\phi = \mathcal{P}\left[\left(v_a + \frac{\kappa}{\mathcal{P}}\right)x_0 - x_3\right] + (p_3 - \mathcal{P})(v_ax_0 - x_3). \tag{13}$$

With this offset, the first and second terms in ϕ describe the phase of the carrier wave, which can be extracted from the integral as before, and the advection of the integral (i.e., the envelope) at the group velocity. Choosing

$$\kappa = \mathcal{E} - v_a\mathcal{P}, \tag{14}$$

where $\mathcal{E} = (m^2 + \mathcal{P}^2)^{1/2}$, yields

$$\phi = \mathcal{P}(\beta_px_0 - x_3) + (p_3 - \mathcal{P})(v_ax_0 - x_3) \tag{15}$$

and ensures that the phase fronts travel at the velocity $\beta_p \equiv \mathcal{E}/\mathcal{P}$. Note that this is equivalent to enveloping the wave function about the longitudinal momentum $\mathcal{P}$ and energy $\mathcal{E}$.

With $\kappa = \mathcal{E} - v_a\mathcal{P}$, the condition on the interdependence of the momenta [Eq. (9)] simplifies to

$$p_c - \mathcal{P} = \gamma_a^2(v_a\mathcal{E} - \mathcal{P}) - \varpi\left[\gamma_a^4(v_a\mathcal{E} - \mathcal{P})^2 - \gamma_a^2\boldsymbol{p}_\perp^2\right]^{1/2}, \tag{16}$$

where the choice of sign $\varpi \equiv \mathrm{sgn}[\gamma_a^2(v_a\mathcal{E} - \mathcal{P})]$ ensures that the envelope varies slowly compared to the phase fronts. Equation (16) demonstrates two important points: first, that $\mathcal{P}$ corresponds to the value of p_3 when $\boldsymbol{p}_\perp = \boldsymbol{0}$, and second that arbitrary group velocity solutions are precluded in one dimension. In one dimension, the wave function is composed solely of plane waves with $\boldsymbol{p}_\perp = \boldsymbol{0}$, such that p_c is always equal to $\mathcal{P}$. This eliminates the second term in Eq. (15), which is responsible for the movement of the envelope at v_a. Figure 2(b) displays a family of longitudinal momenta p_3 [Eq. (16)] parametrized by $\mathcal{P}$ as a function of perpendicular momentum $|\boldsymbol{p}_\perp|$ for a superluminal group velocity $v_a = 2$.

Lorentz transformations of the arbitrary group velocity solutions [Eqs. (11) and (16)] provide additional insight into their interpretation. Upon performing a longitudinal Lorentz transformation to a frame moving at a velocity v_L with respect to the nominal frame, Eq. (15) becomes

$$\phi = \frac{\mathcal{P}}{\gamma_L(1 + \beta_p'v_L)}(\beta_p'x_0' - x_3') + \gamma_L(p_3 - \mathcal{P})[(v_a - v_L)x_0' - (1 - v_av_L)x_3'], \tag{17}$$

where the prime $'$ denotes quantities in the moving frame and $\beta_p' = (\beta_p - v_L)/(1 - \beta_p v_L)$. One may expect that $v_L = v_a$ would be a natural choice for the frame velocity. However, care must be taken to ensure that the phase velocity is superluminal (i.e., the on-shell condition is satisfied). For $|v_a| < 1$, one can indeed set $v_L = v_a$ to find

$$\phi_< = \frac{\mathcal{P}}{\gamma_a(1+\beta_p' v_a)}(\beta_p' x_0' - x_3') - \frac{(p_3 - \mathcal{P})}{\gamma_a}x_3', \tag{18}$$

but for $|v_a| > 1$ the on-shell condition requires $v_L = v_a^{-1}$, which yields

$$\phi_> = \frac{v_a\mathcal{P}}{\gamma_a(v_a+\beta_p')}(\beta_p' x_0' - x_3') - \frac{(p_3 - \mathcal{P})}{v_a\gamma_a}x_0'. \tag{19}$$

Recall that the second terms in Eqs. (18) and (19) determine the space-time dependence of the envelope. Thus, the subluminal and superluminal solutions represent Lorentz transformations of solutions with time-independent and longitudinal-coordinate-independent envelopes, respectively.

The phase [Eq. (15)] and momentum condition [Eq. (16)] appearing in Eq. (11) depend directly on $\mathcal{P}$. Thus, a convenient choice for the auxiliary parameter is $q(\kappa) = \mathcal{P}(\kappa) = \gamma_a^2 v_a\kappa \pm (\gamma_a^4\kappa^2 - \gamma_a^2 m^2)^{1/2}$ [see Eq. (14)]. Using this choice and applying the delta functions in Eq. (11) provides

$$\Phi(x,\mathcal{P}) = e^{-i\mathcal{P}\eta}\int d\boldsymbol{p}_\perp \tilde{\mathcal{N}}(\boldsymbol{p}_\perp,\mathcal{P}) \times \left(\frac{\chi_+}{\dfrac{\sigma_3 p_c + \boldsymbol{\sigma}_\perp\cdot\boldsymbol{p}_\perp}{v_a(p_c-\mathcal{P})+\mathcal{E}+m}\chi_+}\right) e^{-i(p_c-\mathcal{P})\xi + i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp}, \tag{20}$$

where $\eta = \beta_p x_0 - x_3$, $\xi = v_a x_0 - x_3$,

$$\tilde{\mathcal{N}}(\boldsymbol{p}_\perp,\mathcal{P}) = \sqrt{\frac{v_a(p_c-\mathcal{P})+\mathcal{E}+m}{2[v_a(p_c-\mathcal{P})+\mathcal{E}]}}\int \mathcal{N}(p,\mathcal{P})\delta(p_3-p_c)dp_3,$$

and the $\mathcal{P}$ dependence of β_p, $\mathcal{E}$, and p_c is understood. As before, the integral (envelope) depends solely on x_0 and x_3 in the combination $\xi = v_a x_0 - x_3$ and advects at v_a.

Approximate analytical solutions for $\Phi(x,\mathcal{P})$ and $\psi(x)$ can be found for a particle moving predominantly in the $\boldsymbol{e}_3$ direction when $|v_a\beta_p - 1|\mathcal{P}^2 \gg w^2$, where w is the transverse momentum spread. This condition describes a "paraxial" limit in which Eq. (16) simplifies to

$$p_c \approx \mathcal{P} + \frac{\boldsymbol{p}_\perp^2}{2\mathcal{P}(v_a\beta_p - 1)}. \tag{21}$$

In the special case $v_a = -1$, Eq. (21) is an exact equality. With the quadratic dependence of p_c on $\boldsymbol{p}_\perp$, a natural choice of basis functions for $\tilde{\mathcal{N}}(\boldsymbol{p}_\perp,\mathcal{P})$ are Laguerre-Gaussian modes,

$$\tilde{\mathcal{N}}(\boldsymbol{p}_\perp,\mathcal{P}) = \sum_{n,\ell}\mathcal{A}_{n\ell}(\mathcal{P})\rho^{|\ell|}L_n^{|\ell|}(\rho^2)\exp\left(-\tfrac{1}{2}\rho^2\right)e^{i\ell\theta_p}, \tag{22}$$

where $\rho = w^{-1}(\boldsymbol{p}_\perp^2)^{1/2}$, $L_n^{|\ell|}$ is a generalized Laguerre polynomial, and $\theta_p = \text{atan}(p_2/p_1)$ is the momentum-space azimuth. Upon substituting Eqs. (21) and (22) into Eq. (20), dropping terms higher order than $\boldsymbol{p}_\perp^2$ in the spinor, and integrating, one finds

$$\Phi(x,\mathcal{P}) \approx 2\pi e^{-i\mathcal{P}\eta}\sum_{n,\ell} i^{2n+\ell}\mathcal{A}_{n\ell}(\mathcal{P})\begin{pmatrix}U_1(x)\\0\\U_3(x)\\U_4(x)\end{pmatrix}. \tag{23}$$

Each component of the spinor can be expressed in terms of U_1 as follows:

$$\begin{aligned} U_1(x) &= \frac{w}{R}\left(\frac{r}{R}\right)^{|\ell|}L_n^{|\ell|}\left(\frac{r^2}{R^2}\right)\exp\left[-\left(1 - i\frac{\xi}{\xi_0}\right)\frac{r^2}{2R^2} + i\Lambda(x)\right],\\ U_3(x) &= \frac{\mathcal{P}}{\mathcal{E}+m}\left[1 - \frac{(\mathcal{E} - v_a\mathcal{P}+m)}{2w^2\xi_0(\mathcal{E}+m)\mathcal{P}}\nabla_\perp^2\right]U_1(x),\\ U_4(x) &= -\frac{i}{(\mathcal{E}+m)}r^{|\ell|}e^{i\theta}\partial_r r^{-|\ell|}U_1(x), \end{aligned} \tag{24}$$

where $r = (\boldsymbol{x}_\perp^2)^{1/2}$, $\theta = \text{atan}(x_2/x_1)$ is the configuration-space azimuth, $R = w^{-1}[1 + (\xi/\xi_0)^2]^{1/2}$, $\Lambda(x) = \ell\theta - (2n + |\ell| + 1)\text{atan}(\frac{\xi}{\xi_0})$, and $\xi_0 = \mathcal{P}(v_a\beta_p - 1)w^{-2}$. Equations (23) and (24) show that the envelope advects at the group velocity v_a while maintaining a constant profile characterized by the duration ξ_0/v_a.

The $\mathcal{A}_{n\ell}(\mathcal{P})$ quantify the distribution of $\mathcal{P}$ values for every n and ℓ mode that compose the wave function. Said differently, the wave function comprises solutions that fall along a curve defined by Eq. (21) and parametrized by $\mathcal{P}$ [Fig. 2(b)]: $\psi(x) = (2\pi)^{-3/2}\int \Phi(x,\mathcal{P})d\mathcal{P}$. If the $\mathcal{A}_{n\ell}(\mathcal{P})$ are narrowly peaked about some $\mathcal{P} = \bar{\mathcal{P}}$, an approximate expression for the wave function can be obtained. Expressing $\Phi(x,\mathcal{P}) = \sum_{n,\ell}\Phi_{n\ell}(x,\mathcal{P})$ yields

$$\psi(x) \approx \sum_{n,\ell}\frac{\Phi_{n\ell}(x,\bar{\mathcal{P}})}{\mathcal{A}_{n\ell}(\bar{\mathcal{P}})}\int \mathcal{A}_{n\ell}(\bar{\mathcal{P}}+\Delta)e^{-i\Delta\zeta}d\Delta, \tag{25}$$

where $\Delta = \mathcal{P} - \bar{\mathcal{P}}$, $\zeta = v_n x_0 - x_3$, and $v_n = \bar{\mathcal{P}}/(m^2 + \bar{\mathcal{P}}^2)^{1/2}$. Figure 1 illustrates the evolution of the probability density for a superluminal velocity $v_a = 2$, $n = \ell = 0$, and $\int \mathcal{A}_{00}(\bar{\mathcal{P}}+\Delta)e^{-i\Delta\zeta}d\Delta \propto \exp[-(\zeta/\zeta_0)^8]$. The overall profile is causal and travels at the velocity $\langle v_3\rangle \approx v_n = 0.9$, while the peak amplitude travels at $v_a = 2$. The superluminal peak maintains a near-constant profile over a distance $L = 2\zeta_0/|v_a - \langle v_3\rangle|$.

III. DISCUSSION AND CONCLUSION

To generate these solutions using the Kapitza-Dirac effect, one can draw an analogy with equivalent solutions for electromagnetic waves. As shown in Ref. [26], the equivalent electromagnetic solutions can be realized by applying a time-dependent focusing phase. Thus, the motion of the peak probability density at v_a can be understood as a moving or "flying" focus. When a charged lepton passes through the standing wave of counterpropagating light waves, it acquires a phase $\phi_f = -A(\zeta)\sin^2(kx_1)$, where $A(\zeta)$ depends on the ponderomotive potential experienced by each time slice of the wave function ζ as it passes through the standing wave and k is the wavenumber of the light waves [32]. Locally, this phase appears parabolic $\phi_f \approx -A(\zeta)k^2x_1^2$ and acts as a lens with a time-dependent focal length $f = p_0/2k^2A(\zeta)$. If the lepton passes through an additional, orthogonal standing wave, the phase can be made cylindrically symmetric $\phi_f \approx -A(\zeta)k^2(x_1^2 + x_2^2)$. For a velocity v_a, the time dependence of the ponderomotive potential should be set so that $\partial f/\partial x_0 = |v_a/(v_a - \langle v_3 \rangle)|$ [26].

Solutions to the Dirac equation can feature a peak probability density that moves at any velocity, including those exceeding the speed of light, while maintaining a near-constant profile. This motion is independent of the velocity expectation value. The solutions are on shell and are constructed by superposing basis functions with correlations in the longitudinal and transverse momenta. Future work will investigate the formation of these solutions using the Kapitza-Dirac effect by applying analytical and numerical methods [36,45–47]; pursue additional structures by replacing Eq. (7) with other dependencies such as $E_{\boldsymbol{p}} = p_3^2/\mathcal{M} + \kappa$, which exhibits nonrelativistic dispersion with an effective mass $\mathcal{M}$ in vacuum; and explore the benefit of these wave functions in Cherenkov radiation and nonlinear Compton scattering [48]. As an example, in the head-on collision of a high-energy electron traveling in the positive x_3 direction with an ultrashort laser pulse, a $v_a = -1$ would allow the peak probability amplitude to cotravel with the peak intensity of the pulse over an extended distance.

ACKNOWLEDGMENTS

The authors would like to thank R. R. Almeida and D. H. Froula for insightful discussions. The work of M.F. is supported by the European Union's Horizon Europe research and innovation program under the Marie Skłodowska-Curie Grant Agreement No. 101105246-STEFF. The work of J.P.P., D.R., and A.D. is supported by the Office of Fusion Energy Sciences under Award No. DE-SC0021057, the Department of Energy National Nuclear Security Administration under Award No. DE-NA0004144, the University of Rochester, and the New York State Energy Research and Development Authority.

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