AN ASYMPTOTIC PRESERVING DISCONTINUOUS GALERKIN METHOD FOR A LINEAR BOLTZMANN SEMICONDUCTOR MODEL*

VICTOR P. DECARIA†, CORY D. HAUCK‡, AND STEFAN R. SCHNAKE§

Abstract. A key property of the linear Boltzmann semiconductor model is that as the collision frequency tends to infinity, the phase space density $f = f(x, v, t)$ converges to an isotropic function $M(v)\rho(x, t)$, called the drift-diffusion limit, where M is a Maxwellian and the physical density ρ satisfies a second-order parabolic PDE known as the drift-diffusion equation. Numerical approximations that mirror this property are said to be asymptotic preserving. In this paper we build a discontinuous Galerkin method to the semiconductor model, and we show this scheme is both uniformly stable in ε, where $1/\varepsilon$ is the scale of the collision frequency, and asymptotic preserving. In particular, we discuss what properties the discrete Maxwellian must satisfy in order for the schemes to converge in ε to an accurate h-approximation of the drift-diffusion limit. Discrete versions of the drift-diffusion equation and error estimates in several norms with respect to ε and the spacial resolution are also included.

Key words. drift-diffusion, asymptotic preserving, discontinuous Galerkin, semiconductor models

AMS subject classifications. 65M08, 65M12, 65M15, 65M60

1. Introduction.

Kinetic equations are an established tool for modeling charged-particle transport in semiconductors, particularly in non-equilibrium settings [18, 26, 29]. However, numerical simulations of such equations are known to be challenging, due to the size of the space on which they are defined (in general three position, three momentum variables, plus time) and the multiscale nature of the equations. With regards to the latter, it is well-known that for large collision frequencies and long-time scales, the kinetic solution is well-approximated by a drift-diffusion equation which depends on space and time only. Under reasonable conditions, this limit was established rigorously for the case of an applied electric field in [28]. The case of a self-consistent field was later treated in [1, 27].

Because of the drift-diffusion approximation, solving a kinetic model of charge transport in collisional regimes may be unnecessarily expensive; to ameliorate this cost, methods which leverage the drift-diffusion approximation, either via domain decomposition [20] or acceleration [10, 22] are sometimes used. At a minimum, it is important that a discretization of the kinetic equation recover a stable and con-

*This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of Advanced Scientific Computing Research, as part of their Applied Mathematics Research Program. The work was performed at the Oak Ridge National Laboratory, which is managed by UT-Battelle, LLC under Contract No. De-AC05-00OR22725. The United States Government retains and the publisher, by accepting the article for publication, acknowledges that the United States Government retains a non-exclusive, paid-up, irrevocable, world-wide license to publish or reproduce the published form of this manuscript, or allow others to do so, for the United States Government purposes. The Department of Energy will provide public access to these results of federally sponsored research in accordance with the DOE Public Access Plan (http://energy.gov/downloads/doe-public-access-plan).

†Mathematics in Computation Section, Computer Science and Mathematics Division, Oak Ridge National Laboratory, Oak Ridge, TN 37831, USA(vpdecaria@gmail.com).

‡Mathematics in Computation Section, Computer Science and Mathematics Division, Oak Ridge National Laboratory, Oak Ridge, TN 37831, USA and Mathematics Department, University of Tennessee, Knoxville, TN 37996, USA (hauckc@ornl.gov).

§Mathematics in Computation Section, Computer Science and Mathematics Division, Oak Ridge National Laboratory, Oak Ridge, TN 37831, USA (schnakesr@ornl.gov).

sistent discretization of the drift-diffusion limit as the collision frequency becomes infinitely large; this is the so-called asymptotic preserving (AP) property [15, 16]. While standard finite volume or finite-difference methods that that rely on upwinding to discretize advection terms are not asymptotic preserving, there are specialized spatial discretizations [31] and operator splitting techniques [16, 17, 21] that are.

A different approach for capturing the numerical drift-diffusion limit is to use discontinuous Galerkin (DG) methods. These methods have been developed both for kinetic semiconductor equations [6–9, 25] and for the drift-diffusion equations [5, 24]. While not yet rigorously established in the literature, it is reasonable to assume that DG methods will recover the numerical drift-diffusion limit. Such a conjecture rests on a similar body of work for kinetic equations of radiation transport. In that setting, collisional dynamics over long time scales lead to a standard diffusion equation [3, 14, 23]. The asymptotic preserving properties of DG methods for transport equations were first established in [23] for one-dimensional (slab) geometries and later extended to the general multi-dimensional setting in [2]. In [13], the work in [2] was re-established using a rigorous functional analysis framework. The work presented here follows in the spirit of that framework.

In the current paper, we rigorously prove the numerical drift-diffusion limit for a DG method applied to a linear kinetic semiconductor equation. In particular, the collision operator approximates very complicated material interactions with a simple relaxation model and the electric field is not self-consistent, but rather assumed to be given. The DG method relies on a reformulation of the kinetic equation in terms of a weighted distribution function. While such a reformulation is likely not necessary, partially due to the numerical results in [22], it does make some stability results easier to prove. Such results are challenging because, unlike the radiation transport case, the advection operators and collision operator of the kinetic semiconductor equation are stable in L^2 spaces with two different weightings. Even at the analytical level, this mismatch poses significant challenges [27]. Even so, we expect that the analysis presented here can be leveraged for "more standard" implementations.

Beyond linearity, there are several other assumptions made in the analysis. Some of these are technical, but others are quite important. Among these, the most important is a zero-inflow boundary condition which precludes the development of a boundary layer. We also assume that the initial data is well-prepared in the sense that it is consistent with the state of local thermal equilibrium. Removing these three assumptions—linearity, zero inflow, and well-prepared initial data—will be important steps in future work. In addition, uniform error estimates independent of the collision frequency, along the lines of [32] for the radiation transport case, should be considered. However, the analysis here is already fairly involved and requires more work than the radiation transport case. The main novelty of this work is the rigorous analysis of the numerical diffusion limit of a kinetic equation. In contrast to [13] whose work and novelty closely resembles and inspires the work here, the kinetic equation in the current setting is time-dependent, involves advection in both the physical and velocity variables, is defined over an unbounded velocity domain, and has a collision operator with a kernel (the local thermal equilibrium) that is not contained in a standard finite element space. Additionally, this work provides several lemmas concerning stability and control of projecting discontinuous Galerkin finite element functions onto a continuous Galekrin finite element space. These technical results will aid in the current and future numerical analysis of AP discontinuous Galerkin schemes.

The remainder of the paper is organized as follows. In Section 2, we introduce the relevant equations, preliminary notation, assumptions used to construct a discrete

Maxwellian, and the numerical method for solving the kinetic semiconductor model given in (2.1) below. We characterize the collision frequency by an asymptotic parameter $\varepsilon > 0$ which is inversely proportional to the mean-free-path between collisions, and in Section 3, we develop stability and pre-compactness estimates that allow us to take the ε-limit to 0. Additionally, we give several technical results which will aid in the general analysis of discrete drift-diffusion limits. In Section 4, we show the numerical density ρ_h^ε of the kinetic model converges to the solution of a discretized version ρ_h^0 of the drift diffusion system, given in (2.5) below. In Section 5, we show error estimates for $\|\rho_h^\varepsilon - \rho_h^0\|$ in ε and h as well as error estimates for $\|\rho_h^0 - \rho^0\|$ in h. This allows us to build estimates for $\|\rho_h^\varepsilon - \rho_0\|$ in ε and h.

2. Background, Preliminaries, and Assumptions. Given $\varepsilon > 0$, a Lipschitz spatial domain $\Omega_x \subset \mathbb{R}^3$, and data f_0 prescribed on Ω_x, let $f_\varepsilon(x, v, t)$ be the solution of the following kinetic semiconductor model

$$\varepsilon \frac{\partial f^\varepsilon}{\partial t} + v \cdot \nabla_x f^\varepsilon + E(x,t) \cdot \nabla_v f^\varepsilon - \frac{1}{\varepsilon} Q(f^\varepsilon) = 0, \qquad (x,v) \in \Omega_x \times \mathbb{R}^3,\ t > 0; \tag{2.1a}$$

$$f^\varepsilon(x,v,t) = f_-(x,v,t), \qquad (x,v) \in \partial\Omega_-,\ t > 0; \tag{2.1b}$$

$$f^\varepsilon(x,v,0) = f_0(x,v), \qquad (x,v) \in \Omega_x \times \mathbb{R}^3, \tag{2.1c}$$

where $E \in W^{1,\infty}([0,T]; L^\infty(\Omega_x))$ is a given electric field, f_- is the inflow data, and

$$\partial\Omega_- = \{(x,v) \in \partial\Omega_x \times \mathbb{R}^3 : v \cdot n_x(x) < 0\}, \tag{2.2}$$

with $n_x(x_0)$ being the normal to Ω_x at the point x_0, is the inflow component of the boundary. Additionally the collision operator Q is defined by

$$Q(f^\varepsilon) = \omega(M\rho^\varepsilon - f^\varepsilon) \tag{2.3}$$

where $\rho^\varepsilon(x,t) = \int_{\mathbb{R}^3} f^\varepsilon(x,v,t)\,\mathrm{d}v$ is the number density and $M(v) = (2\pi\theta)^{\frac{-3}{2}} e^{-\frac{|v|^2}{2\theta}}$ with $\theta > 0$ the (lattice) temperature and $\omega \in L^\infty(\Omega_x)$ with $0 < \omega_{\min} \leq \omega$ on Ω_x the (scaled) collision frequency. The moment $J^\varepsilon = \varepsilon^{-1} \int_{\mathbb{R}^3} v f^\varepsilon(x,v,t)\,\mathrm{d}v$ is the current density.

If the inflow and initial data are isotropic, that is, $f_-(x,v,t) = m(x)M(v)$ and $f_0(x,v) = \rho_0(x,t)M(v)$, then as $\varepsilon \to 0$, f^ε converges to $M(v)\rho^0(x)$, where ρ^0 solves the drift-diffusion equation [28]:

$$\frac{\partial \rho^0}{\partial t} + \nabla_x \cdot \left(\frac{1}{\omega}(-\theta \nabla_x \rho^0 + E\rho^0) \right) = 0, \qquad x \in \Omega_x,\ t > 0; \tag{2.4a}$$

$$\rho^0(x,t) = m(x), \qquad x \in \partial\Omega_x,\ t > 0; \tag{2.4b}$$

$$\rho^0(x,0) = \rho_0(x), \qquad x \in \Omega_x. \tag{2.4c}$$

If $J^0 = \frac{1}{\omega}(-\theta\nabla_x \rho^0 + E\rho^0)$, the pair $\{\rho^0, J^0\}$ solves the equivalent first-order system:

$$\frac{\partial \rho^0}{\partial t} + \nabla_x \cdot J^0 = 0, \qquad x \in \Omega_x,\ t > 0; \tag{2.5a}$$

$$\omega J^0 + \theta \nabla_x \rho^0 - E\rho^0 = 0, \qquad x \in \Omega_x,\ t > 0; \tag{2.5b}$$

$$\rho^0(x,t) = m(x), \qquad x \in \partial\Omega_x,\ t > 0; \tag{2.5c}$$

$$\rho^0(x,0) = \rho_0(x), \qquad x \in \Omega_x. \tag{2.5d}$$

2.1. Notation. Given a measureable open set $D \subset \mathbb{R}^3$, let $L^2(D)$ and $W^{k,p}(D)$ be the standard Lebesgue and Sobolev spaces of functions on D and let $H^k(D) := W^{k,2}(D)$. When D is a volume (a three-dimensional manifold) in $\mathbb{R}^3$, we use $(\cdot,\cdot)_D$ to denote the standard L^2 inner product with respect to the Lebesgue measure $\mathrm{d}x$. If D is a surface (a two-dimensional manifold) in $\mathbb{R}^3$, we use $\langle\cdot,\cdot\rangle_D$ to denote the L^2 inner product with respect to the Lebesgue measure on the surface. These inner products can be extended to vector valued functions in a natural way by use of the Euclidean inner product.

To discretize (2.1), we first restrict the domain in v. Given $L > 0$, let $\Omega_v = [-L, L]^3$ and define $\Omega = \Omega_x \times \Omega_v$. Given a mesh parameters $h_x > 0$, let $\mathcal{T}_{x,h} := \mathcal{T}_{x,h_x}$ be a mesh on Ω_x constructed from open polyhedral cells K of maximum diameter h_x, and let $\mathcal{E}^I_{x,h}$ be the interior skeleton of $\mathcal{T}_{x,h}$, i.e., the set of edges $e \subset \partial K \not\subset \partial\Omega_x$. Similarly, given $h_v > 0$, let $\mathcal{T}_{v,h} := \mathcal{T}_{v,h_v}$ and $\mathcal{E}^I_{v,h}$ be a mesh and interior skeleton for Ω_v respectively. We assume that $\mathcal{T}_{x,h}$ is quasi-uniform and shape regular. The conditions of $\mathcal{T}_{v,h}$ are given in Subsection 2.3.

Given an edge $e \in \mathcal{E}^I_{x,h} = \partial K^+ \cap \partial K^-$ for some $K^+, K^- \in \mathcal{T}_{x,h}$, let $z \in L^2(\Omega_x)$ and $\tau \in [L^2(\Omega_x)]^3$ be scalar and vector-valued functions, respectively, each with well-defined traces on K^+ and K^-. For such functions, we define the average and jump methods

$$\{\!\{z\}\!\} = \frac{1}{2}\left(z\big|_{K^+} + z\big|_{K^-}\right), \qquad [\![z]\!] = z\big|_{K^+} n_x^+ + z\big|_{K^-} n_x^-, \tag{2.6a}$$

$$\{\!\{\tau\}\!\} = \frac{1}{2}(\tau|_{K^+} + \tau|_{K^-}), \qquad [\![\tau]\!] = \tau|_{K^+} \cdot n_x^+ + \tau|_{K^-} \cdot n_x^-, \tag{2.6b}$$

where $n_x^\pm$ are the unit normal vectors pointing outward from $K^\pm$, respectively. These definitions can be modified to average and jumps in the v-direction in a natural way with unit normal vector n_v.

For ease of presentation we will use $a \lesssim b$ to denote $a \le Cb$ where $C > 0$ is a constant independent of h_x and ε. The constant additionally depends on the data ω and E (see Assumption 2.8), the final time T, Ω_x, L, h_x-independent mesh parameters of $\mathcal{T}_{x,h}$, and the discrete Maxwellian discussed in Subsection 2.3 which depends on h_v, L, and θ.

Given integers $k_x \ge 0$ and $k_v \ge 0$, let

$$V_{z,h} = \{q \in L^2(\Omega_x) : q\big|_T \in \mathbb{Q}_{k_z}(T)\ \forall K \in \mathcal{T}_{z,h}\} \tag{2.7}$$

for $z = v, x$ where $\mathbb{Q}_k(T)$ is the set of all polynomials on K with k being the maximum degree in any variable, and let $V_h = V_{x,h} \otimes V_{v,h}$ be the tensor DG discrete space. For purposes of this paper, we assume $k_x \ge 1$[1] and $k_v \ge 1$.[2] For any function $z_h \in V_h$, let $\nabla_x z_h \subset V_h$ and $\nabla_v z_h \subset V_h$ denote piece-wise gradients defined on K for all $T \in \mathcal{T}_h$.[3]

For the discretization in x, some addition notation is needed. Let $S^0_{x,h} = V^0_{x,h} \cap H^1_0(\Omega_x)$ be the continuous finite element analogue to the DG space with zero trace. Let $\mathcal{S}_h$ be an L^2-orthogonal projection operator from $L^2(\Omega_x)$ onto $S^0_{x,h}$. Moreover, let $(S^0_{x,h})^*$ denote the topological dual of $S^0_{x,h}$ and let $\rightharpoonup$ represent convergence in the

[1]The assumption on k_x guarantees that locking will not occur as the discrete limiting density in ε will be continuous in x; see Theorem 4.1.

[2]The assumption on k_v is to enable the construction of the discrete Maxwellian; see Subsection 2.3.

[3]The discrete gradient ignores the jumps in z_h across the boundary, but agrees with the standard definition of gradient for continuous functions.

weak topology. Additionally, define the discrete dual-norm $H_h^{-1}(\Omega_x)$ by

$$\|z_h\|_{H_h^{-1}(\Omega_x)} = \sup_{q_h \in S^0_{x,h}\setminus\{0\}} \frac{(z_h, q_h)_{\Omega_x}}{\|\nabla_x q_h\|_{L^2(\Omega_x)}}, \tag{2.8}$$

and a discrete H^1 norm on V_h by

$$\|q_h\|^2_{H_h^1(\Omega_x)} = \|\nabla_x q_h\|^2_{L^2(\Omega_x)} + \frac{1}{h_x}\|[\![q_h]\!]\|^2_{L^2(\mathcal{E}^I_{x,h})} + \frac{1}{h_x}\|q_h\|^2_{L^2(\partial\Omega_x)}. \tag{2.9}$$

A discrete Poincaré-Friedrichs inequality [4, Theorem 10.6.12] yields

$$\|q_h\|_{L^2(\Omega_x)} \lesssim \|q_h\|_{H_h^1(\Omega_x)}. \tag{2.10}$$

Given a Banach space X, $1 \le p \le \infty$, and the final time T, we let $L^p_T(X) := L^p([0,T];X)$, $C^0([0,T];X)$, and $H^1([0,T];X)$ be the standard $L^p/C^0/H^1$ spaces of Banach-space valued functions with Bochner integration.

Finally we will often write $\frac{\partial}{\partial t}$ as ∂_t in order to keep the spacing consistent in longer estimates. Both will be used interchangeably.

2.2. Alternate form of the PDE. It is easy to show that the collision operator Q, defined in (2.3), is semi-coercive in the weighted norm $\|M^{-\frac{1}{2}}(\cdot)\|_{L^2(\Omega)}$. Indeed, testing Q by $M^{-1}f^\varepsilon$ gives

$$-(M^{-1}f^\varepsilon, Q(f^\varepsilon))_\Omega = \|\omega^{\frac{1}{2}}M^{-\frac{1}{2}}(f^\varepsilon - M\rho^\varepsilon)\|^2_{L^2(\Omega)}. \tag{2.11}$$

This structure is critical to achieving the drift-diffusion limit. However since standard discretizations of (2.1) do not allow test functions with an M^{-1} weight, we instead rewrite (2.1) in terms of the weighted distribution $g^\varepsilon = M^{-\frac{1}{2}}f^\varepsilon$:

$$\varepsilon\frac{\partial g^\varepsilon}{\partial t} + v\cdot\nabla_x g^\varepsilon + E(x,t)\cdot\nabla_v g^\varepsilon - \frac{\omega}{\varepsilon}\left(M^{\frac{1}{2}}\rho^\varepsilon - g^\varepsilon\right) = \frac{1}{2\theta}E(x,t)\cdot v g^\varepsilon; \tag{2.12a}$$

$$g^\varepsilon(x,v,t) = f_-(x,v,t)/M^{\frac{1}{2}}(v); \tag{2.12b}$$

$$g^\varepsilon(x,v,0) = f_0(x,v)/M^{\frac{1}{2}}(v), \tag{2.12c}$$

where, in terms of g^ε, $\rho^\varepsilon = (M^{\frac{1}{2}}, g^\varepsilon)_{\mathbb{R}^3}$ and (x,v,t) is defined in (2.4). Since $\|g^\varepsilon\|_{L^2(\Omega)} = \|M^{-\frac{1}{2}}f^\varepsilon\|_{L^2(\Omega)}$, the weighted collision operator $M^{-\frac{1}{2}}Q(M^{\frac{1}{2}}g^\varepsilon) = \omega\left(M^{\frac{1}{2}}\rho^\varepsilon - g^\varepsilon\right)$ will be L^2-coercive and symmetric as a function of g^ε. We refer to the function $M^{\frac{1}{2}}g^\varepsilon$ as the weighted equilibrium. The cost of this additional structure is the electric field term on the right-hand side of (2.12a).

2.3. Construction of Discrete Maxwellian. In order to recover the proper drift-diffusion limit, we need to construct a suitable discrete Maxwellian on the bounded domain Ω_v. This is done via an approximation of the square root of the one-dimensional Maxwellian. Assume that $\mathcal{T}_{v,h}$ is a tensor product mesh, i.e., $\mathcal{T}_{v,h} = \mathcal{T}^1_{v,h}\otimes\cdots\otimes\mathcal{T}^3_{v,h}$, and let $M^{\frac{1}{2}}_{h,i}$ be a continuous, strictly positive, piecewise-polynomial approximation of the one-dimensional root-Maxwellian over $\mathcal{T}^i_{v,h}$:

$$M^{\frac{1}{2}}_{h,i}(v_i) \approx M^{\frac{1}{2}}_i(v_i) := \left(\frac{1}{\sqrt{2\pi\theta}}e^{\frac{-v_i^2}{2\theta}}\right)^{1/2}, \qquad i = 1,\dots,3, \tag{2.13}$$

with the following properties:

Assumption 2.1. *For each $i = 1,\dots,3$, the function $M^{\frac{1}{2}}_{h,i}$ satisfies:*

a. $(M_{h,i}^{\frac{1}{2}}, M_{h,i}^{\frac{1}{2}})_{[-L,L]} = 1,$

b. $M_{h,i}^{\frac{1}{2}}(L) = M_{h,i}^{\frac{1}{2}}(-L),$

c. $(\partial_v M_{h,i}^{\frac{1}{2}}, M_{h,i}^{\frac{1}{2}})_{[-L,L]} = 0.$

Define the discrete temperature vector $\theta_* \in \mathbb{R}^3$, which depends on h_v and L, and its element-wise reciprocal $\theta_*^{-1} \in \mathbb{R}^3$ by

$$(\theta_*)_i = \theta_{*,i} := \tfrac{1}{4}(\partial_v M_{h,i}^{\frac{1}{2}}, \partial_v M_{h,i}^{\frac{1}{2}})_{[-L,L]}^{-1}, \qquad \text{and} \qquad (\theta_*^{-1})_i = \tfrac{1}{\theta_{*,i}}. \tag{2.14}$$

Definition 2.2. *The discrete root-Maxwellian $M_h^{\frac{1}{2}} \in V_{v,h} \cap C^0(\overline{\Omega_v})$ is*

$$M_h^{\frac{1}{2}}(v) = \prod_{i=1}^{3} M_{h,i}^{\frac{1}{2}}(v_i), \tag{2.15}$$

and the discrete velocity v_h is

$$v_h = -2(M_h^{\frac{1}{2}})^{-1}(\theta_* \odot \nabla_v M_h^{\frac{1}{2}}) = -2\theta_* \odot \nabla_v \log(M_h^{\frac{1}{2}}) \tag{2.16}$$

where $\odot$ is the element-wise Hadamard product of two vectors.

Since $M_h^{\frac{1}{2}} > 0$ on $\overline{\Omega_v}$, it follows that $v_h \in [L^\infty(\Omega_v)]^3$. Moreover $v_h M_h^{\frac{1}{2}} \in [V_{v,h}]^3$, even though $v_h \notin [V_{v,h}]^3$.

Remark 2.3. *In defining the discrete root-Maxwellian:*

1. *The continuity requirement on $M_{h,i}^{\frac{1}{2}}$ is the reason for assumption $k_v \geq 1$ in Subsection 2.1.*
2. *Assumption 2.1.c is not independent, but rather is implied by Assumption 2.1.b.* ∎
3. *Since each of the 1D discrete root-Maxwellian's can have a different H^1 semi-norm, we must treat the discrete temperature θ_* as a vector rather than a scalar. If each of the $\theta_{*,i}$'s are equal, then θ_* can be treated as a scalar and the Hadamard product in (2.16) and throughout the rest of the paper becomes a scalar multiplication.*

Remark 2.4.

1. *To create a discrete Maxwellian that satisfies Assumption 2.1, take $M_{h,i}^{\frac{1}{2}}$ to be the Lagrange piecewise linear nodal interpolant of $M_i^{\frac{1}{2}}$ and scale it to have an L^2 norm of 1. With mild restrictions on L and h_v based on θ, we show in Lemma B.1, given in the appendix, that $M_{h,i}^{\frac{1}{2}}$ is an $\mathcal{O}(h_v^2)$ approximation to $M_i^{\frac{1}{2}}$ in the L^2-norm and an $\mathcal{O}(h_v)$ approximation in the H^1-norm.*[4]
2. *It is readily seen from Lemma B.1 that $\theta_{*,i}$ is an $\mathcal{O}(h_v)$ approximation to θ for $i = 1, 2, 3$. If $k_v > 1$, then higher order interpolations of the root-Maxwellian can also be used to improve the asymptotic accuracy of θ_* provided positivity of the interpolant is preserved.*

2.4. The Numerical Method. We now give our numerical method for (2.12).

[4] We are neglecting the errors due to the finite velocity domain. See Lemma B.1 for the full estimate.

Problem 2.5. *Find $g_h^\varepsilon \in H^1([0,T];V_h)$ such that*

$$\varepsilon\left(\frac{\partial g_h^\varepsilon}{\partial t}, z_h\right)_\Omega + \mathcal{A}(g_h^\varepsilon, z_h) + \mathcal{B}(g_h^\varepsilon, z_h) + \mathcal{D}(g_h^\varepsilon, z_h) - \frac{1}{\varepsilon}\mathcal{Q}(g_h^\varepsilon, z_h) = \mathcal{C}(g_h^\varepsilon, z_h) + \mathcal{R}(z_h), \tag{2.17a}$$

$$g_h^\varepsilon(0) := g_{0,h} \tag{2.17b}$$

for all $z_h \in V_h$ and a.e. $0 < t \leq T$ where

$$\mathcal{A}(w_h, z_h) = -(v_h w_h, \nabla_x z_h)_\Omega + \left\langle v_h \{\!\{w_h\}\!\} + \frac{|v_h \cdot n_x|}{2}[\![w_h]\!], [\![z_h]\!]\right\rangle_{\mathcal{E}^I_{x,h}\times\Omega_v} \tag{2.18a}$$

$$+ \langle v_h w_h, n_x z_h\rangle_{\partial\Omega_+}, \tag{2.18b}$$

$$\mathcal{B}(w_h, z_h) = -(E w_h, \nabla_v z_h)_\Omega + \left\langle E\{\!\{w_h\}\!\} + \frac{|E \cdot n_v|}{2}[\![w_h]\!], [\![z_h]\!]\right\rangle_{\Omega_x\times\mathcal{E}^I_{v,h}}, \tag{2.18c}$$

$$\mathcal{D}(w_h, z_h) = \left\langle E M_h^{\frac{1}{2}} P(w_h), n_v z_h\right\rangle_{\Omega_x\times\partial\Omega_v}, \qquad P(w_h) = (M_h^{\frac{1}{2}}, w_h)_{\Omega_v}, \tag{2.18d}$$

$$\mathcal{Q}(w_h, z_h) = \left(\omega(M_h^{\frac{1}{2}} P(w_h) - w_h), z_h\right)_\Omega, \tag{2.18e}$$

$$\mathcal{C}(w_h, z_h) = \frac{1}{2}(E \cdot (\theta_*^{-1} \odot v_h) w_h, z_h)_\Omega, \tag{2.18f}$$

$$\mathcal{R}(z_h) = -\langle v_h g_{-,h}, n_x z_h\rangle_{\partial\Omega_-}, \tag{2.18g}$$

and where $g_{-,h} \in V_h$ and $g_{0,h} \in V_h$ are the discrete inflow and initial data respectively.

Definition 2.6. *The discrete number density ρ_h^ε and current density J_h^ε are*

$$\rho_h^\varepsilon = P(g_h^\varepsilon) = (M_h^{\frac{1}{2}}, g_h^\varepsilon)_{\Omega_v}, \tag{2.19}$$

$$J_h^\varepsilon = \frac{1}{\varepsilon}P(v_h g_h^\varepsilon) = \frac{1}{\varepsilon}(M_h^{\frac{1}{2}} v_h, g_h^\varepsilon)_{\Omega_v}. \tag{2.20}$$

Remark 2.7. *In Problem 2.5,*

1. *The bilinear form $\mathcal{A}$ and functional $\mathcal{R}$ are the result of upwind discretizations of the operator $v \cdot \nabla_x$ with v replaced by v_h.*
2. *The bilinear forms $\mathcal{B}$ and $\mathcal{D}$ are constructed using standard upwind fluxes for the Vlasov operator $E \cdot \nabla_v$ but, due to the velocity boundary domain restriction, we weakly impose the boundary condition $g_h^\varepsilon = M_h^{\frac{1}{2}}\rho_h^\varepsilon$ on $\Omega_x \times \partial\Omega_v$. This provides two benefits. First, the density g_h^ε will not lose or gain mass out of the velocity boundary. Second, this boundary condition keeps the restriction of v to the bounded domain Ω_v from polluting the discrete drift-diffusion limit.*
3. *The bilinear form $\mathcal{Q}$ is a standard discretization of the collision operator.*
4. *The bilinear form $\mathcal{C}$ is a standard discretization of the term $\frac{1}{2\theta}E \cdot vg$, with v replaced by v_h and θ replaced by θ_*.*

2.5. Assumptions. We collect any other assumptions used in the analysis of Problem 2.5.

Assumption 2.8. *The collision frequency ω and the electric field E in (2.1) are specified as $\omega \in L^\infty(\Omega_x)$ with $0 < \omega_{\min} \leq \omega$ on Ω_x and $E \in W^{1,\infty}([0,T]; L^\infty(\Omega_x))$.*

Assumption 2.9. *The inflow data in (2.1) is isotropic, that is, $f_0(x,v) = \rho_0(x)M(v)$. Additionally we require $g_{0,h}$ in Problem 2.5 to be discretely* *isotropic, that is,* $g_{0,h} = \rho_{0,h} M_h^{\frac{1}{2}}$ *where* $\rho_{0,h} \in V_{x,h}$ *is defined by* $(\rho_{0,h}, q_h)_{\Omega_x} = (\rho_0, q_h)_{\Omega_x} \quad \forall q_h \in V_{x,h}$.

Assumption 2.9 is a common assumption made in the study of the drift-diffusion limit on the continuous PDE (2.1); see [27, 28]. It follows from Assumption 2.9, Assumption 2.1.a, and Assumption 2.1.c that

$$\rho_h^\varepsilon|_{t=0} = (M_h^{\frac{1}{2}}, g_{0,h})_{\Omega_v} = (M_h^{\frac{1}{2}}, M_h^{\frac{1}{2}})_{\Omega_v} \rho_{0,h} = \rho_{0,h}, \tag{2.21}$$

$$J_h^\varepsilon|_{t=0} = \frac{1}{\varepsilon}(v_h M_h^{\frac{1}{2}}, g_{0,h})_{\Omega_v} = \frac{1}{\varepsilon}(v_h M_h^{\frac{1}{2}}, M_h^{\frac{1}{2}})_{\Omega_v} \rho_{0,h} = 0. \tag{2.22}$$

Assumption 2.10. *The continuous incoming data* f_- *in* (2.1b)*, the discrete incoming data* $g_{-,h}$ *in Problem 2.5 and, consequently, the functional* $\mathcal{R}$ *in Problem 2.5 are identically zero.*

Assumptions like Assumption 2.10 are commonly made in the analysis of diffusion limits to avoid handling complications due to boundary layers.

Assumption 2.11. *The inverse Laplacian* $S : L^2(\Omega_x) \to H_0^1(\Omega_x)$ *defined by*

$$(\nabla Sq, \nabla w)_{\Omega_x} = (q, w)_{\Omega_x} \quad \forall w \in H_0^1(\Omega), \tag{2.23}$$

is a bounded linear operator from $L^2(\Omega_x) \to H^2(\Omega_x) \cap H_0^1(\Omega_x)$*, that is,*

$$\|Sq\|_{H^2(\Omega_x)} \lesssim \|q\|_{L^2(\Omega_x)} \quad \forall q \in L^2(\Omega_x). \tag{2.24}$$

Assumption 2.11 is needed for the proof of stability of the L^2 projection $\mathcal{S}_h$ in the $H_h^1(\Omega_x)$-norm (see Lemma 3.5). If Ω_x is convex, then Assumption 2.11 is automatically satisfied [12, Section 3.2].

Assumption 2.12. *There is a constant* $C > 0$ *such that* $\frac{\varepsilon}{h_x} < C$*, that is,* h_x *cannot go to zero faster than* ε*.*

Assumption 2.12 is to improve the readability of the paper. With it, $1 + \frac{\varepsilon}{h_x} \lesssim 1$ and $1 + \sqrt{\frac{\varepsilon}{h_x}} \lesssim 1$.

3. A Priori Estimates.

In this section we develop space-time stability estimates for g_h^ε, the number density ρ_h^ε, and the current density J_h^ε in Problem 2.5.

3.1. Preliminary Estimates and Identities.

In this subsection we list an inverse and trace inequality, derived from the standard estimates in [30], technical interpolation and projections estimates, and a useful integration by parts identity.

Lemma 3.1 (Trace Inequality). *For any* $z_h \in V_h$ *and* $q_h \in V_{x,h}$ *we have*

$$\|[\![z_h]\!]\|^2_{L^2(\mathcal{E}^I_{x,h} \times \Omega_v)} + \|z_h\|^2_{L^2(\partial\Omega_x \times \Omega_v)} \le C_T h_x^{-1} \|z_h\|^2_{L^2(\Omega)}, \tag{3.1a}$$

$$\|[\![q_h]\!]\|^2_{L^2(\mathcal{E}^I_{x,h})} + \|q_h\|^2_{L^2(\partial\Omega_x)} \le C_T h_x^{-1} \|q_h\|^2_{L^2(\Omega_x)}, \tag{3.1b}$$

$$\|[\![z_h]\!]\|^2_{L^2(\Omega_x \times \mathcal{E}^I_{v,h})} + \|z_h\|^2_{L^2(\Omega_x \times \partial\Omega_v)} \le C_T h_v^{-1} \|z_h\|^2_{L^2(\Omega)}. \tag{3.1c}$$

Here $C_T > 0$ *is an* ε*,* h_x*, and* h_v*-independent constant.* C_T *depends on the polynomial degree of* V_h *and other* h_x *and* h_v*-independent mesh parameters of* $\mathcal{T}_{x,h}$ *and* $\mathcal{T}_{v,h}$*.*

Lemma 3.2 (Inverse Inequality). *For any* $q_h \in V_{x,h}$ *we have*

$$\|\nabla q_h\|_{L^2(\Omega_x)} \le C h_x^{-1} \|q_h\|_{L^2(\Omega_x)} \tag{3.2}$$

where $C > 0$ *is some* ε *and* h_x*-independent constant that depends on the the polynomial degree of* $V_{x,h}$*.*

LEMMA 3.3 (Integration by Parts (IBP)). *For any $q_h \in V_{x,h}$ and $\tau_h \in [V_{x,h}]^3$,*

$$(q_h, \nabla_x \cdot \tau_h)_{\Omega_x} = -(\nabla_x q_h, \tau_h)_{\Omega_x} + \langle \{\!\{q_h\}\!\}, [\![\tau_h]\!]\rangle_{\mathcal{E}^I_{x,h}} + \langle [\![q_h]\!], \{\!\{\tau_h\}\!\}\rangle_{\mathcal{E}^I_{x,h}} + \langle q_h n_x, \tau_h\rangle_{\partial\Omega_x}. \tag{3.3}$$

3.2. Technical Estimates for Drift-Diffusion Analysis. In this subsection, we present several technical results which are useful both in the analysis in the drift-diffusion limit to Problem 2.5, and for the future analysis of similar problems.

The first result is an error estimate of an interpolant from $V_{x,h} \to S^0_{x,h}$. The result allows us to control a DG function's distance to $S^0_{x,h}$, in the H^1_h-norm, by the function's interior jumps and boundary data. The interpolant onto the conforming finite element space, I_h, is the KP interpolant from [19, Theorem 2.2].

LEMMA 3.4 (Conforming Interpolant). *There is an interpolant $I_h : V_{x,h} \to S^0_{x,h}$ such that for any $q_h \in V_{x,h}$ we have*

$$\|q_h - I_h q_h\|^2_{H^1_h(\Omega_x)} \lesssim \frac{1}{h_x}\|q_h\|^2_{L^2(\partial\Omega_x)} + \frac{1}{h_x}\|[\![q_h]\!]\|^2_{L^2(\mathcal{E}^I_{x,h})}. \tag{3.4}$$

Additionally, we give an H^1_h stability estimate for the L^2 projection from $V_{x,h}$ to $S^0_{x,h}$. This interpolant, $\mathcal{S}_h$ is vital to the analysis of evolution problems and this estimate is needed to extend the results of the interpolant I_h to $\mathcal{S}_h$; see the proof of Lemma 3.9. We move the proof of Lemma 3.5 to the appendix.

LEMMA 3.5. *The projection $\mathcal{S}_h$ is stable on $V_{x,h}$ with respect to the $H^1_h(\Omega_x)$-norm:*

$$\|\mathcal{S}_h q_h\|_{H^1_h(\Omega_x)} \lesssim \|q_h\|_{H^1_h(\Omega_x)} \quad \forall q_h \in V_{x,h}. \tag{3.5}$$

3.3. Initial Estimates. We first focus on estimates for g^ε_h that will lead to estimates for ρ^ε_h and J^ε_h.

LEMMA 3.6. *The bilinear forms defined in Problem 2.5 satisfy the bounds*

$$-\mathcal{Q}(g^\varepsilon_h, g^\varepsilon_h) \geq \omega_{\min} \left\| g^\varepsilon_h - M_h^{\frac{1}{2}} \rho^\varepsilon_h \right\|^2_{L^2(\Omega)}, \tag{3.6}$$

$$\mathcal{C}(g^\varepsilon_h, g^\varepsilon_h) \leq \frac{1}{2}\left(2C_1 + \frac{\omega_{\min}}{\varepsilon}\right) \left\| g^\varepsilon_h - M_h^{\frac{1}{2}} \rho^\varepsilon_h \right\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2\omega_{\min}} C_3^2 \|g^\varepsilon_h\|^2_{L^2(\Omega)}, \tag{3.7}$$

$$\mathcal{A}(g^\varepsilon_h, g^\varepsilon_h) = \left\langle \frac{|v_h \cdot n_x|}{2}[\![g^\varepsilon_h]\!], [\![g^\varepsilon_h]\!] \right\rangle_{\partial\Omega_x \times \Omega_v} + \left\langle \frac{|v_h \cdot n_x|}{2}[\![g^\varepsilon_h]\!], [\![g^\varepsilon_h]\!] \right\rangle_{\mathcal{E}^I_{x,h} \times \Omega_v}, \tag{3.8}$$

(3.9)
$$\mathcal{B}(g^\varepsilon_h, g^\varepsilon_h) + \mathcal{D}(g^\varepsilon_h, g^\varepsilon_h) \geq \left\langle \frac{|E \cdot n_v|}{2}[\![g^\varepsilon_h]\!], [\![g^\varepsilon_h]\!] \right\rangle_{\Omega_x \times \mathcal{E}^I_{v,h}} - \frac{C_2}{2h_v} \left\| g^\varepsilon_h - M_h^{\frac{1}{2}} \rho^\varepsilon_h \right\|^2_{L^2(\Omega)}$$

where

$$C_1 := \frac{1}{2}\|E \cdot (\theta_*^{-1} \odot v_h)\|_{L^\infty(\Omega)}, \qquad C_2 := C_T \|E\|_{L^\infty_T(L^\infty(\Omega_x))}, \tag{3.10a}$$

$$C_3 := \frac{3}{2\theta_*^{\min}} \|E\|_{L^\infty_T(L^\infty(\Omega_x))}, \qquad \theta_*^{\min} := \min\{\theta_{*,1}, \theta_{*,2}, \theta_{*,3}\}. \tag{3.10b}$$

are constants independent of ε, h_x, and h_v; and $C_T > 0$ is a constant from the trace inequality (3.1b),

Proof. For (3.6), the definition of ρ^ε_h, along with Assumption 2.1.a, implies that

$$(M_h^{\frac{1}{2}} \rho^\varepsilon_h, M_h^{\frac{1}{2}} \rho^\varepsilon_h)_{\Omega_v} = |\rho^\varepsilon_h|^2 = (M_h^{\frac{1}{2}} \rho^\varepsilon_h, g^\varepsilon_h)_{\Omega_v}. \tag{3.11}$$

Therefore by expansion we see

$$\|g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon\|^2_{L^2(\Omega_v)} = (g_h^\varepsilon, g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon)_{\Omega_v}. \tag{3.12}$$

Using (3.12) gives

$$-\mathcal{Q}(g_h^\varepsilon, g_h^\varepsilon) = \left(\omega, \|g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h\|^2_{L^2(\Omega_v)}\right)_{\Omega_x} \geq \omega_{\min}\left\|g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon\right\|^2_{L^2(\Omega)}, \tag{3.13}$$

which is (3.6). For (3.7), by Assumption 2.1.c,

$$\mathcal{C}(M_h^{\frac{1}{2}}\rho_h^\varepsilon, M_h^{\frac{1}{2}}\rho_h^\varepsilon) = -(E\rho_h^\varepsilon, \rho_h^\varepsilon)_{\Omega_v}\cdot(\nabla_v M_h^{\frac{1}{2}}, M_h^{\frac{1}{2}})_{\Omega_v} = 0. \tag{3.14}$$

Thus by (3.14),

$$\mathcal{C}(g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon, g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon) = \mathcal{C}(g_h^\varepsilon, g_h^\varepsilon) - 2\mathcal{C}(g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon, M_h^{\frac{1}{2}}\rho_h^\varepsilon). \tag{3.15}$$

Rewriting (3.15) yields

$$\mathcal{C}(g_h^\varepsilon, g_h^\varepsilon) = \mathcal{C}(g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon, g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon) + 2\mathcal{C}(g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon, M_h^{\frac{1}{2}}\rho_h^\varepsilon) := I_1 + I_2. \tag{3.16}$$

Applying Hölder's inequality to (3.16) gives

$$|I_1| \leq \tfrac{1}{2}\|E\cdot(\theta_*^{-1}\odot v_h)\|_{L^\infty(\Omega)}\|g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon\|^2_{L^2(\Omega)}. \tag{3.17}$$

Using Hölder's inequality, (2.16), and (2.14), we bound I_2 as

$$\begin{aligned} |I_2| &\leq \|E\cdot(\theta_*^{-1}\odot v_h)M_h^{\frac{1}{2}}\rho_h^\varepsilon\|_{L^2(\Omega)}\|g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon\|_{L^2(\Omega)} \\ &\leq \frac{3}{2\theta_*^{\min}}\|E\|_{L^\infty(\Omega_x)}\|\rho_h^\varepsilon\|_{L^2(\Omega_x)}\|g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon\|_{L^2(\Omega)}. \end{aligned} \tag{3.18}$$

Meanwhile, integrating (3.11) over Ω_x gives (with Cauchy-Schwartz)

$$\|\rho_h^\varepsilon\|_{L^2(\Omega_x)} = \|M_h^{\frac{1}{2}}\rho_h^\varepsilon\|_{L^2(\Omega)} \leq \|g_h^\varepsilon\|_{L^2(\Omega)}. \tag{3.19}$$

Substituting (3.17)-(3.19) into (3.16) and invoking Young's inequality gives

$$\begin{aligned} \mathcal{C}(g_h^\varepsilon, g_h^\varepsilon) &\leq C_1\|g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon\|^2_{L^2(\Omega)} + C_3\|g_h^\varepsilon\|_{L^2(\Omega)}\|g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon\|_{L^2(\Omega)} \\ &\leq \left(C_1 + \frac{\omega_{\min}}{2\varepsilon}\right)\|g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2\omega_{\min}}C_3^2\|g_h^\varepsilon\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.20}$$

which yields (3.7). For (3.8), it follows from Lemma 3.3 (setting $q_h = g_h^\varepsilon$ and $\tau_h = v_h g_h^\varepsilon$) that

$$(v_h g_h^\varepsilon, \nabla_x g_h^\varepsilon)_\Omega = \langle v_h\{\!\{g_h^\varepsilon\}\!\}, [\![g_h^\varepsilon]\!]\rangle_{\mathcal{E}_{x,h}^I\times\Omega_v} + \frac{1}{2}\langle v_h g_h^\varepsilon, n_x g_h^\varepsilon\rangle_{\partial\Omega_x\times\Omega_v}. \tag{3.21}$$

Direct substitution of this formula into the definition of $\mathcal{A}$ gives

$$\mathcal{A}(g_h^\varepsilon, g_h^\varepsilon) = \left\langle\frac{|v_h\cdot n_x|}{2}[\![g_h^\varepsilon]\!], [\![g_h^\varepsilon]\!]\right\rangle_{\mathcal{E}_{x,h}^I\times\Omega_v} + \left\langle\frac{|v_h\cdot n_x|}{2}g_h^\varepsilon, g_h^\varepsilon\right\rangle_{\partial\Omega_x\times\Omega_v}, \tag{3.22}$$

which is (3.8). For (3.9), a formula similar to (3.21) gives

$$\mathcal{B}(g_h^\varepsilon, g_h^\varepsilon) = \left\langle \frac{|E\cdot n_v|}{2}[\![g_h^\varepsilon]\!], [\![g_h^\varepsilon]\!] \right\rangle_{\Omega_x\times\mathcal{E}_{v,h}^I} - \frac{1}{2}\left\langle E g_h^\varepsilon, n_v g_h^\varepsilon\right\rangle_{\Omega_x\times\partial\Omega_v}. \tag{3.23}$$

Meanwhile, invoking the divergence theorem and Assumption 2.1.c yields

$$\left\langle E\cdot n_v, (M_h^{\frac{1}{2}})^2\right\rangle_{\partial\Omega_v} = 2E\cdot(\nabla_v M_h^{\frac{1}{2}}, M_h^{\frac{1}{2}})_{\Omega_v} = 0. \tag{3.24}$$

Therefore applying the polarization identity $ab = \frac{1}{2}(a^2+b^2-(a-b)^2)$ with $a := g_h^\varepsilon$ and $b := M_h^{\frac{1}{2}}\rho_h$ and (3.24) gives the following bound on $\mathcal{D}$:

$$\begin{aligned}\mathcal{D}(g_h^\varepsilon, g_h^\varepsilon) &= \frac{1}{2}\left\langle E\cdot n_v, (g_h^\varepsilon)^2\right\rangle_{\Omega_x\times\partial\Omega_v} - \frac{1}{2}\left\langle E\cdot n_v, (g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon)^2\right\rangle_{\Omega_x\times\partial\Omega_v}\\ &\geq \frac{1}{2}\left\langle E g_h^\varepsilon, n_v g_h^\varepsilon\right\rangle_{\Omega_x\times\partial\Omega_v} - \frac{\|E\|_{L_T^\infty(L^\infty(\Omega_x))}}{2}\|g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon\|^2_{L^2(\Omega_x\times\partial\Omega_v)}.\end{aligned} \tag{3.25}$$

Applying the discrete trace estimate (3.1c) to (3.25) gives

$$\mathcal{D}(g_h^\varepsilon, g_h^\varepsilon) \geq \frac{1}{2}\left\langle E g_h^\varepsilon, n_v g_h^\varepsilon\right\rangle_{\Omega_x\times\partial\Omega_v} - \frac{C_T\|E\|_{L_T^\infty(L^\infty(\Omega_x))}}{2h_v}\|g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon\|^2_{L^2(\Omega)}, \tag{3.26}$$

with $C_T > 0$ a constant independent of ε, h_x, and h_v. Adding (3.23) and (3.26) yields yields (3.9). The proof is complete. ∎

Using Lemma 3.6, we derive a space-time estimates for g_h^ε and $M_h^{\frac{1}{2}}\rho_h^\varepsilon - g_h^\varepsilon$. Let

$$\varepsilon_{h_v} := \frac{\omega_{\min} h_v}{4C_1 h_v + 2C_2}. \tag{3.27}$$

LEMMA 3.7. *Given $h_v > 0$, $h_x > 0$, and $\varepsilon \leq \varepsilon_{h_v}$,*

$$\begin{aligned}\|g_h^\varepsilon(T)\|^2_{L^2(\Omega)} &+ \frac{\omega_{\min}}{2\varepsilon^2}\|g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon\|^2_{L_T^2(L^2(\Omega))} + \int_0^T\left[\frac{1}{\varepsilon}\left\langle |v_h\cdot n_x|[\![g_h^\varepsilon]\!], [\![g_h^\varepsilon]\!]\right\rangle_{\mathcal{E}_{x,h}^I\times\Omega_v}\right.\\ &\left.+\frac{1}{\varepsilon}\left\langle |v_h\cdot n_x| g_h^\varepsilon, g_h^\varepsilon\right\rangle_{\partial\Omega_x\times\Omega_v}\right]\mathrm{d}t \leq \|g_{0,h}\|^2_{L^2(\Omega)}\exp\left(\frac{C_3^2}{\omega_{\min}}T\right).\end{aligned} \tag{3.28}$$

Proof. Setting $z_h = \frac{2}{\varepsilon}g_h^\varepsilon$ in (2.17a) yields the energy equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\|g_h^\varepsilon\|^2_{L^2(\Omega)} + \frac{2}{\varepsilon}\mathcal{A}(g_h^\varepsilon, g_h^\varepsilon) + \frac{2}{\varepsilon}(\mathcal{B}(g_h^\varepsilon, g_h^\varepsilon) + \mathcal{D}(g_h^\varepsilon, g_h^\varepsilon)) - \frac{2}{\varepsilon^2}\mathcal{Q}(g_h^\varepsilon, g_h^\varepsilon) = \frac{2}{\varepsilon}\mathcal{C}(g_h^\varepsilon, g_h^\varepsilon). \tag{3.29}$$

Substituting the estimates in Lemma 3.6 yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|g_h^\varepsilon\|^2_{L^2(\Omega)} &+ \frac{1}{\varepsilon}\left\langle |v_h\cdot n_x|[\![g_h^\varepsilon]\!], [\![g_h^\varepsilon]\!]\right\rangle_{\mathcal{E}_{x,h}^I\times\Omega_v} + \frac{1}{\varepsilon}\left\langle |v_h\cdot n_x| g_h^\varepsilon, g_h^\varepsilon\right\rangle_{\partial\Omega_x\times\Omega_v}\\ &+ \frac{1}{\varepsilon}\left\langle |E\cdot n_v|[\![g_h^\varepsilon]\!], [\![g_h^\varepsilon]\!]\right\rangle_{\Omega_x\times\mathcal{E}_{v,h}^I} - \frac{C_2}{\varepsilon h_v}\|M_h^{\frac{1}{2}}\rho_h^\varepsilon - g_h^\varepsilon\|^2_{L^2(\Omega)}\\ &+ \frac{2\omega_{\min}}{\varepsilon^2}\|M_h^{\frac{1}{2}}\rho_h^\varepsilon - g_h^\varepsilon\|^2_{L^2(\Omega)}\\ &\leq \frac{1}{\omega_{\min}}C_3^2\|g_h^\varepsilon\|^2_{L^2(\Omega)} + \left(\frac{2C_1}{\varepsilon} + \frac{\omega_{\min}}{\varepsilon^2}\right)\|g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon\|^2_{L^2(\Omega)}\end{aligned} \tag{3.30}$$

Dropping $\frac{1}{\varepsilon}\left\langle |E\cdot n_v| [\![g_h^\varepsilon]\!], [\![g_h^\varepsilon]\!]\right\rangle_{\Omega_x\times\mathcal{E}_{v,h}^I} \geq 0$ and collecting like terms gives

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|g_h^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon}\left\langle |v_h\cdot n_x| [\![g_h^\varepsilon]\!], [\![g_h^\varepsilon]\!]\right\rangle_{\mathcal{E}_{x,h}^I\times\Omega_v} + \frac{1}{\varepsilon}\left\langle |v_h\cdot n_x| g_h^\varepsilon, g_h^\varepsilon\right\rangle_{\partial\Omega_x\times\Omega_v} \\ + \frac{1}{\varepsilon^2}\left(\omega_{\min} - 2\varepsilon C_1 - \varepsilon\frac{C_2}{h_v}\right)\|M_h^{\frac{1}{2}}\rho_h^\varepsilon - g_h^\varepsilon\|_{L^2(\Omega)}^2 \leq \frac{C_3^2}{\omega_{\min}}\|g_h^\varepsilon\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.31}$$

Since $\varepsilon_{h_v} = \frac{\omega_{\min} h_v}{4C_1 h_v + 2C_2}$ (see (3.27)), it follows that for any $\varepsilon \leq \varepsilon_{h_v}$, $\omega_{\min} - 2\varepsilon C_1 - \varepsilon\frac{C_2}{h_v} \geq \frac{\omega_{\min}}{2}$. Therefore

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|g_h^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon}\left\langle |v_h\cdot n_x| [\![g_h^\varepsilon]\!], [\![g_h^\varepsilon]\!]\right\rangle_{\mathcal{E}_{x,h}^I\times\Omega_v} + \frac{1}{\varepsilon}\left\langle |v_h\cdot n_x| g_h^\varepsilon, g_h^\varepsilon\right\rangle_{\partial\Omega_x\times\Omega_v} \\ + \frac{\omega_{\min}}{2\varepsilon^2}\|M_h^{\frac{1}{2}}\rho_h^\varepsilon - g_h^\varepsilon\|_{L^2(\Omega)}^2 \leq \frac{C_3^2}{\omega_{\min}}\|g_h^\varepsilon\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.32}$$

Applying Grönwall's inequality to (3.32) yields (3.28). The proof is complete. ◻

With Lemma 3.7 in hand, we can obtain stability estimates for ρ_h^ε and J_h^ε as well as some projection estimates which will be useful in the next section. We first list a technical lemma whose proof is provided in the appendix.

Lemma 3.8. *Let*

$$\gamma_I(x) := \left(\tfrac{|v_h\cdot n_x(x)|}{2}M_h^{\frac{1}{2}}, M_h^{\frac{1}{2}}\right)_{\Omega_v} \quad \text{and} \quad \gamma_B(x) := \left(v_h M_h^{\frac{1}{2}}, M_h^{\frac{1}{2}}\right)_{\{v: v_h(v)\cdot n_x(x) > 0\}} \tag{3.33}$$

for $x \in \mathcal{E}_{x,h}^I$ and $x \in \partial\Omega_x$ respectively. Then there exists $\gamma_* = \gamma_*(h_v) > 0$ such that $\gamma_I > \gamma_*$ on $\mathcal{E}_{x,h}^I$ and $\gamma_B \cdot n > \gamma_*$ on $\partial\Omega_x$ for all h_x and ε.

Lemma 3.9. *Recall the definitions of ρ_h^ε and J_h^ε from (2.19) and (2.20), respectively. For all $h_x > 0$ and every $\varepsilon \leq \varepsilon_{h_v}$, where ε_{h_v} is defined in (3.27), the following space-time stability estimates hold:*

$$\|\rho_h^\varepsilon\|_{L_T^\infty(L^2(\Omega_x))} \leq \|g_h^\varepsilon\|_{L_T^\infty(L^2(\Omega))} \lesssim \|g_{0,h}\|_{L^2(\Omega)}, \tag{3.34}$$

$$\tfrac{1}{\varepsilon}\|M_h^{\frac{1}{2}}\rho_h^\varepsilon - g_h^\varepsilon\|_{L_T^2(L^2(\Omega))} \lesssim \|g_{0,h}\|_{L^2(\Omega)}, \tag{3.35}$$

$$\|J_h^\varepsilon\|_{L_T^2(L^2(\Omega_x))} \lesssim \|g_{0,h}\|_{L^2(\Omega)}. \tag{3.36}$$

Moreover, given γ_* is defined in Lemma 3.8, we have

$$\tfrac{\sqrt{\gamma_*}}{\sqrt{\varepsilon}}\|[\![\rho_h^\varepsilon]\!]\|_{L_T^2(L^2(\mathcal{E}_{x,h}^I))} + \tfrac{\sqrt{\gamma_*}}{\sqrt{\varepsilon}}\|\rho_h^\varepsilon\|_{L_T^2(L^2(\partial\Omega_x))} \lesssim \left(\sqrt{\tfrac{\varepsilon}{h_x}} + 1\right)\|g_{0,h}\|_{L^2(\Omega)} \tag{3.37}$$

$$\sqrt{\tfrac{h_x}{\varepsilon}}\|\rho_h^\varepsilon - \mathcal{S}_h\rho_h^\varepsilon\|_{L_T^2(L^2(\Omega_x))} + \sqrt{\tfrac{h_x}{\varepsilon}}\|\rho_h^\varepsilon - \mathcal{S}_h\rho_h^\varepsilon\|_{L_T^2(H_h^1(\Omega_x))} \lesssim \left(\sqrt{\tfrac{\varepsilon}{h_x}} + 1\right)\|g_{0,h}\|_{L^2(\Omega)} \tag{3.38}$$

Proof. Estimates (3.34) and (3.35) follow from (3.28). For (3.36), the definition of J_h^ε and Assumption 2.1.c give

$$J_h^\varepsilon = \tfrac{1}{\varepsilon}(v_h M_h^{\frac{1}{2}}, g_h^\varepsilon)_{\Omega_v} = -\tfrac{2}{\varepsilon}(\theta_*\odot\nabla_v M_h^{\frac{1}{2}}, g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon)_{\Omega_v}. \tag{3.39}$$

Together (3.35) and (3.39) imply that

$$\|J_h^\varepsilon\|_{L_T^2(L^2(\Omega_x))} \lesssim \varepsilon^{-1}\|M_h^{\frac{1}{2}}\|_{H^1(\Omega_v)}\|M_h^{\frac{1}{2}}\rho_h^\varepsilon - g_h^\varepsilon\|_{L_T^2(L^2(\Omega))} \lesssim \|g_{0,h}\|_{L^2(\Omega)}. \tag{3.40}$$

We now focus on (3.37). We will only prove the bound on the first term of (3.37) as the bound on the second term is similar. Using the definition of γ_I, adding and subtracting g_h^ε, and using the trace inequality (3.1a) we obtain

$$
\begin{aligned}
\sqrt{\gamma_*}\|[\![\rho_h^\varepsilon]\!]\|_{L^2(\mathcal{E}_{x,h}^I)} &\le \|\sqrt{\gamma_I}[\![\rho_h^\varepsilon]\!]\|_{L^2(\mathcal{E}_{x,h}^I)} = \Big\|\sqrt{\tfrac{|v_h\cdot n_x|}{2}}[\![M_h^{\frac{1}{2}}\rho_h^\varepsilon]\!]\Big\|_{L^2(\mathcal{E}_{x,h}^I\times\Omega_v)} \\
&\le \Big\|\sqrt{\tfrac{|v_h\cdot n_x|}{2}}[\![M_h^{\frac{1}{2}}\rho_h^\varepsilon - g_h^\varepsilon]\!]\Big\|_{L^2(\mathcal{E}_{x,h}^I\times\Omega_v)} + \Big\|\sqrt{\tfrac{|v_h\cdot n_x|}{2}}[\![g_h^\varepsilon]\!]\Big\|_{L^2(\mathcal{E}_{x,h}^I\times\Omega_v)} \\
&\lesssim \tfrac{1}{\sqrt{h_x}}\|M_h^{\frac{1}{2}}\rho_h^\varepsilon - g_h^\varepsilon\|_{L^2(\Omega)} + \Big\|\sqrt{\tfrac{|v_h\cdot n_x|}{2}}[\![g_h^\varepsilon]\!]\Big\|_{L^2(\mathcal{E}_{x,h}^I\times\Omega_v)}.
\end{aligned}
\tag{3.41}
$$

Integrating (3.41) from 0 to T and using both (3.28) and (3.35) yields

$$
\sqrt{\gamma_*}\|[\![\rho_h^\varepsilon]\!]\|_{L_T^2(L^2(\mathcal{E}_{x,h}^I))} \lesssim \left(\tfrac{\varepsilon}{\sqrt{h_x}} + \sqrt{\varepsilon}\right)\|g_{0,h}\|_{L^2(\Omega)}.
\tag{3.42}
$$

We can divide (3.42) by $\sqrt{\varepsilon}$ and arrive at (3.37).

We now focus on (3.38). Recall I_h from Lemma 3.4. From (3.4) and (3.37) we have

$$
\begin{aligned}
\sqrt{\tfrac{h_x}{\varepsilon}}\|(\rho_h^\varepsilon - I_h\rho_h^\varepsilon)\|_{L_T^2(H_h^1(\Omega_x))} &\lesssim \tfrac{1}{\sqrt{\varepsilon}}\|[\![\rho_h^\varepsilon]\!]\|_{L_T^2(L^2(\mathcal{E}_{x,h}^I))} + \tfrac{1}{\sqrt{\varepsilon}}\|\rho_h^\varepsilon\|_{L_T^2(L^2(\partial\Omega_x))} \\
&\lesssim \left(\sqrt{\tfrac{\varepsilon}{h_x}} + 1\right)\|g_{0,h}\|_{L^2(\Omega)}.
\end{aligned}
\tag{3.43}
$$

We now extend (3.43) to $\rho_h^\varepsilon - \mathcal{S}_h\rho_h^\varepsilon$ using the stability of $\mathcal{S}_h$. By Lemma 3.5 we can show $\|\rho_h^\varepsilon - \mathcal{S}_h\rho_h^\varepsilon\|_{H_h^1(\Omega_x)} \lesssim \|\rho_h^\varepsilon - I_h\rho_h^\varepsilon\|_{H_h^1(\Omega_x)}$. Applying this result to (3.43) yields

$$
\sqrt{\tfrac{h_x}{\varepsilon}}\|(\rho_h^\varepsilon - \mathcal{S}_h\rho_h^\varepsilon)\|_{L_T^2(H_h^1(\Omega_x))} \lesssim \left(\sqrt{\tfrac{\varepsilon}{h_x}} + 1\right)\|g_{0,h}\|_{L^2(\Omega)}.
\tag{3.44}
$$

We can extend (3.44) to the $L_T^2(L^2(\Omega_x))$ norm by (2.10) and arrive at (3.38). □

3.4. Time Derivative Estimates. In this subsection, we construct temporal estimates for $\partial_t\rho_h^\varepsilon$ and $\partial_t J_h^\varepsilon$ by determining the evolution equations for ρ_h^ε and J_h^ε. The evolution equations (see Lemmas 3.10 and 3.12) are formed by choosing a particular type of test function in Problem 2.5. By adding and subtracting the discrete weighted equilibrium $M_h^{\frac{1}{2}}\rho_h^\varepsilon$, we can write the evolution equations (3.45) and (3.58) into the terms that will build the discretization of (2.5) and the remainder terms Θ_i where Θ_i is uniformly bounded in ε when integrated over time.

We begin with the evolution equation for ρ_h^ε:

Lemma 3.10. *For any $\varepsilon > 0$, ρ_h^ε and J_h^ε satisfy*

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}\rho_h^\varepsilon, q_h\right)_{\Omega_x} &- (J_h^\varepsilon, \nabla_x q_h)_{\Omega_x} + \langle\{\!\{J_h^\varepsilon\}\!\}, [\![q_h]\!]\rangle_{\mathcal{E}_{x,h}^I} + \frac{1}{\varepsilon}\langle\gamma_I[\![\rho_h^\varepsilon]\!], [\![q_h]\!]\rangle_{\mathcal{E}_{x,h}^I} \\
&+ \frac{1}{\varepsilon}\langle\gamma_B\rho_h^\varepsilon, n_x q_h\rangle_{\partial\Omega_x} = \Theta_1(\tilde{g}_h^\varepsilon, q_h) + \Theta_2(\tilde{g}_h^\varepsilon, q_h),
\end{aligned}
\tag{3.45}
$$

for all $q_h \in V_{x,h}$, where $\tilde{g}_h^\varepsilon = g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon$ and

$$
\Theta_1(\tilde{g}_h^\varepsilon, q_h) = -\frac{1}{\varepsilon}\left\langle\frac{|v_h\cdot n_x|}{2}[\![g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon]\!], [\![M_h^{\frac{1}{2}}q_h]\!]\right\rangle_{\mathcal{E}_{x,h}^I\times\Omega_v},
\tag{3.46}
$$

$$
\Theta_2(\tilde{g}_h^\varepsilon, q_h) = -\frac{1}{\varepsilon}\left\langle v_h(g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon), n_x M_h^{\frac{1}{2}}q_h\right\rangle_{\partial\Omega_+}.
\tag{3.47}
$$

Additionally, we have the following bounds:

$$|\Theta_1(\tilde{g}_h^\varepsilon, q_h)| \lesssim \frac{1}{\varepsilon\sqrt{h_x}} \|g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon\|_{L^2(\Omega)} \|[\![q_h]\!]\|_{L^2(\mathcal{E}_{x,h}^I)}, \tag{3.48}$$

$$|\Theta_2(\tilde{g}_h^\varepsilon, q_h)| \lesssim \frac{1}{\varepsilon\sqrt{h_x}} \|g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon\|_{L^2(\Omega)} \|q_h\|_{L^2(\partial\Omega_x)}. \tag{3.49}$$

Proof. To show (3.45), we let $q_h \in V_{x,h}$ and choose $z_h = M_h^{\frac{1}{2}} q_h \in V_h$ into (2.17a) and evaluate term by term. First, the time derivative term reduces to

$$\varepsilon \left(\frac{\partial}{\partial t} g_h^\varepsilon, M_h^{\frac{1}{2}} q_h \right)_\Omega = \varepsilon \left(\frac{\partial}{\partial t} (g_h^\varepsilon, M_h^{\frac{1}{2}})_{\Omega_v}, q_h \right)_{\Omega_x} = \varepsilon \left(\frac{\partial}{\partial t} \rho_h^\varepsilon, q_h \right)_{\Omega_x}. \tag{3.50}$$

Next, using the definition of J_h^ε in (2.20), we compute

$$\begin{aligned} \mathcal{A}(g_h^\varepsilon, M_h^{\frac{1}{2}} q_h) = &-\varepsilon (J_h^\varepsilon, \nabla q_h)_{\Omega_x} + \varepsilon \langle \{\!\{J_h^\varepsilon\}\!\}, [\![q_h]\!] \rangle_{\mathcal{E}_{x,h}^I} \\ &+ \left\langle \tfrac{1}{2} |v_h \cdot n_x| [\![g_h^\varepsilon]\!], [\![M_h^{\frac{1}{2}} q_h]\!] \right\rangle_{\mathcal{E}_{x,h}^I \times \Omega_v} + \left\langle v_h g_h^\varepsilon, n_x M_h^{\frac{1}{2}} q_h \right\rangle_{\partial\Omega_+}. \end{aligned} \tag{3.51}$$

Adding and subtracting $M_h^{\frac{1}{2}} \rho_h^\varepsilon$ from the last two terms of (3.51) and using the definitions of Θ_1 and Θ_2 gives

$$\begin{aligned} \mathcal{A}(g_h^\varepsilon, M_h^{\frac{1}{2}} q_h) = &-\varepsilon (J_h^\varepsilon, \nabla q_h)_{\Omega_x} + \varepsilon \langle \{\!\{J_h^\varepsilon\}\!\}, [\![q_h]\!] \rangle_{\mathcal{E}_{x,h}^I} + \varepsilon \langle \gamma_I [\![\rho_h^\varepsilon]\!], [\![q_h]\!] \rangle_{\mathcal{E}_{x,h}^I} \\ &+ \langle \gamma_B \rho_h^\varepsilon, n_x q_h \rangle_{\partial\Omega_x} - \varepsilon \Theta_1(\tilde{g}_h^\varepsilon, q_h) - \varepsilon \Theta_2(\tilde{g}_h^\varepsilon, q_h). \end{aligned} \tag{3.52}$$

After division by ε, (3.50) and (3.52) recover (3.45). Thus it remains to show that

$$\mathcal{B}(g_h^\varepsilon, M_h^{\frac{1}{2}} q_h) + \mathcal{D}(g_h^\varepsilon, M_h^{\frac{1}{2}} q_h) + \mathcal{Q}(g_h^\varepsilon, M_h^{\frac{1}{2}} q_h) = \mathcal{C}(g_h^\varepsilon, M_h^{\frac{1}{2}} q_h). \tag{3.53}$$

For $\mathcal{B}$, any edge integral in (2.18c) vanishes because $M_h^{\frac{1}{2}} q_h$ is continuous in v. Thus by the definition of the discrete velocity v_h in (2.16),

$$\mathcal{B}(g_h^\varepsilon, M_h^{\frac{1}{2}} q_h) = -(E g_h^\varepsilon, q_h \nabla_v M_h^{\frac{1}{2}})_\Omega = \frac{1}{2\theta_*} (E \cdot v_h g_h^\varepsilon, M_h^{\frac{1}{2}} q_h)_\Omega = \mathcal{C}(g_h^\varepsilon, M_h^{\frac{1}{2}} q_h). \tag{3.54}$$

For $\mathcal{D}$, Assumption 2.1.b implies that $\mathcal{D}(g_h^\varepsilon, M_h^{\frac{1}{2}} q_h) = 0$. For $\mathcal{Q}$, because $M_h^{\frac{1}{2}} q_h$ is isotropic, $\mathcal{Q}(g_h^\varepsilon, M_h^{\frac{1}{2}} q_h) = 0$ as well. Thus (3.53) holds and consequently so does (3.45). The bounds on Θ_1 and Θ_2 are obtained by applications of Hölder's inequality and the trace inequality (3.1a) and are omitted for brevity. The proof is complete. ∎

Using (3.45), we derive an ε-independent bound for $\partial_t \rho_h^\varepsilon$ in various norms.

LEMMA 3.11. *For any $\varepsilon \leq \varepsilon_{h_v}$, where ε_{h_v} is defined in (3.27), and all $h_x > 0$,*

$$\|\partial_t \rho_h^\varepsilon\|_{L^2(H_h^{-1}(\Omega_x))} \lesssim \|g_{0,h}\|_{L^2(\Omega_x)}, \tag{3.55}$$

$$\|\partial_t \mathcal{S}_h \rho_h^\varepsilon\|_{L_T^2(L^2(\Omega_x))} \lesssim \frac{1}{h_x} \|g_{0,h}\|_{L^2(\Omega_x)}, \tag{3.56}$$

where the $H_h^{-1}(\Omega_x)$ norm is defined in (2.8).

Proof. We first focus on (3.55). Choose $q_h \in S^0_{x,h}$ in (3.45), with $q_h \neq 0$. Due to the continuity and boundary data of q_h, we have

$$(\partial_t \rho_h^\varepsilon, q_h) = (J_h^\varepsilon, \nabla_x q_h)_{\Omega_x} \lesssim \|\nabla_x q_h\|_{L^2(\Omega_x)} \|J_h^\varepsilon\|_{L^2(\Omega_x)}. \tag{3.57}$$

Dividing (3.57) by $\|\nabla_x q_h\|_{L^2(\Omega_x)}$, taking the supremum over all q_h, integrating over $t \in [0,T]$, and finally applying (3.36) yields (3.55).

To show (3.56), we choose $q_h = \partial_t \mathcal{S}_h \rho_h^\varepsilon$ in (3.57). Using the inverse inequality (3.2) and the identity $(\partial_t \rho_h^\varepsilon, \partial_t \mathcal{S}_h \rho_h^\varepsilon)_{\Omega_x} = (\partial_t \mathcal{S}_h \rho_h^\varepsilon, \partial_t \mathcal{S}_h \rho_h^\varepsilon)_{\Omega_x}$ we see that the desired estimate holds. The proof is complete. ∎

We next turn to the evolution equation for J_h^ε.

LEMMA 3.12. *For any $\varepsilon > 0$, ρ_h^ε and J_h^ε satisfy*

$$\begin{aligned} \varepsilon^2 \left(\frac{\partial}{\partial t} J_h^\varepsilon, \tau_h \right)_{\Omega_x} &+ (\omega J_h^\varepsilon, \tau_h)_{\Omega_x} + (\theta_* \odot \nabla_x \rho_h^\varepsilon, \tau_h)_{\Omega_x} - \langle \theta_* \odot [\![\rho_h^\varepsilon]\!], \{\!\{\tau_h\}\!\} \rangle_{\mathcal{E}^I_{x,h}} \\ &- (E\rho_h^\varepsilon, \tau_h)_{\Omega_x} = \varepsilon \Theta_3(\tilde{g}_h^\varepsilon, \tau_h) + \sqrt{\varepsilon}\Theta_4(\rho_h^\varepsilon, \tau_h) + \sqrt{\varepsilon}\Theta_5(\rho_h^\varepsilon, \tau_h), \end{aligned} \tag{3.58}$$

for all $\tau_h \in [V_{x,h}]^3$. Here Θ_3 is a remainder that includes several terms that depend on $\tilde{g}_h^\varepsilon = g_h^\varepsilon - M_h^{\frac{1}{2}} \rho_h^\varepsilon$; it satisfies the bound

$$|\Theta_3(\tilde{g}_h^\varepsilon, \tau_h)| \lesssim \frac{1}{\varepsilon h_x} \|g_h^\varepsilon - M_h^{\frac{1}{2}} \rho_h^\varepsilon\|_{L^2(\Omega)} \|\tau_h\|_{L^2(\Omega_x)}. \tag{3.59}$$

The terms

$$\Theta_4(\rho_h^\varepsilon, \tau_h) = -\frac{1}{\sqrt{\varepsilon}} \left\langle \tfrac{|v_h \cdot n_x|}{2} M_h^{\frac{1}{2}} [\![\rho_h^\varepsilon]\!], [\![M_h^{\frac{1}{2}} v_h \cdot \tau_h]\!] \right\rangle_{\mathcal{E}^I_{x,h} \times \Omega_v}, \tag{3.60}$$

$$\Theta_5(\rho_h^\varepsilon, \tau_h) = \frac{1}{\sqrt{\varepsilon}} \left\langle v_h \cdot n_x M_h^{\frac{1}{2}} \rho_h^\varepsilon, M_h^{\frac{1}{2}} v_h \cdot \tau_h \right\rangle_{\partial\Omega_-}, \tag{3.61}$$

are also remainder terms satisfying the bounds

$$|\Theta_4(\rho_h^\varepsilon, \tau_h)| \lesssim \frac{1}{\sqrt{\varepsilon h_x}} \|[\![\rho_h^\varepsilon]\!]\|_{L^2(\mathcal{E}^I_{x,h})} \|\tau_h\|_{L^2(\Omega_x)} \lesssim \frac{1}{\sqrt{\varepsilon} h_x} \|\rho_h^\varepsilon\|_{L^2(\Omega_x)} \|\tau_h\|_{L^2(\Omega_x)}, \tag{3.62}$$

$$|\Theta_5(\rho_h^\varepsilon, \tau_h)| \lesssim \frac{1}{\sqrt{\varepsilon h_x}} \|\rho_h^\varepsilon\|_{L^2(\partial\Omega_x)} \|\tau_h\|_{L^2(\Omega_x)} \lesssim \frac{1}{\sqrt{\varepsilon} h_x} \|\rho_h^\varepsilon\|_{L^2(\Omega_x)} \|\tau_h\|_{L^2(\Omega_x)}. \tag{3.63}$$

Proof. Let $v_h^i = v_h \cdot e_i$, where e_i is the standard unit basis vector. It follows (2.16) that $v_h^i M_h^{\frac{1}{2}} \in V_{v,h}$. To derive (3.58), we let $z_h = v_h^i M_h^{\frac{1}{2}} \tau_h \in V_h$ in (2.17a), where $\tau_h \in V_{x,h}$ is arbitrary, and evaluate the result term by term. For brevity, we identify each term that belongs to Θ_3 and show that it satisfies the bound in (3.59).

For the time derivative, the definition of J_h^ε in (2.20) gives

$$\varepsilon(\partial_t g_h^\varepsilon, v_h^i M_h^{\frac{1}{2}} \tau_h)_\Omega = \varepsilon(\partial_t (g_h^\varepsilon, v_h M_h^{\frac{1}{2}})_{\Omega_v}, \tau_h e_i)_{\Omega_x} = \varepsilon^2 (\partial_t J_h^\varepsilon, \tau_h e_i)_{\Omega_x}. \tag{3.64}$$

To evaluate $\mathcal{A}$, we add and subtract $M_h^{\frac{1}{2}} \rho_h^\varepsilon$ from the first argument and write

$$\mathcal{A}(g_h^\varepsilon, v_h^i M_h^{\frac{1}{2}} \tau_h) = \mathcal{A}(M_h^{\frac{1}{2}} \rho_h^\varepsilon, v_h^i M_h^{\frac{1}{2}} \tau_h) - \varepsilon \tfrac{\mathcal{A}(g_h^\varepsilon - M_h^{\frac{1}{2}} \rho_h^\varepsilon, v_h^i M_h^{\frac{1}{2}} \tau_h)}{\varepsilon} =: I_1 - \varepsilon\{I_2\}. \tag{3.65}$$

The term I_2 belongs to Θ_3. Since I_2 contains $g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon$, following a similar treatment to Θ_1 in Lemma 3.10, we can show I_2 satisfies (3.59). For I_1, the definition of $\mathcal{A}$ in (2.18b) implies that $I_1 = I_3 - \sqrt{\varepsilon}\{I_4\} - \sqrt{\varepsilon}\{I_5\}$, where

$$\begin{aligned} I_3 &= -(\rho_h^\varepsilon, \nabla_x \tau_h)_{\Omega_x} \cdot (v_h M_h^{\frac{1}{2}}, v_h^i M_h^{\frac{1}{2}})_{\Omega_v} + \langle \{\!\{\rho_h^\varepsilon\}\!\}, [\![\tau_h]\!]\rangle_{\mathcal{E}_{x,h}^I} \cdot (v_h M_h^{\frac{1}{2}}, v_h^i M_h^{\frac{1}{2}})_{\Omega_v} \\ &\quad + \langle \rho_h, n_x \tau_h\rangle_{\partial\Omega_x} \cdot (v_h M_h^{\frac{1}{2}}, v_h^i M_h^{\frac{1}{2}})_{\Omega_v}, \\ I_4 &= -\frac{1}{\sqrt{\varepsilon}} \left\langle \tfrac{|v_h\cdot n_x|}{2} M_h^{\frac{1}{2}} [\![\rho_h^\varepsilon]\!], [\![M_h^{\frac{1}{2}} v_h \cdot e_i \tau_h]\!] \right\rangle_{\mathcal{E}_{x,h}^I \times \Omega_v}, \\ I_5 &= \frac{1}{\sqrt{\varepsilon}} \left\langle v_h \cdot n_x M_h^{\frac{1}{2}} \rho_h^\varepsilon, M_h^{\frac{1}{2}} v_h \cdot e_i \tau_h \right\rangle_{\partial\Omega_-}. \end{aligned}$$

The definitions of $M_h^{\frac{1}{2}}$ and v_h in Definition 2.2, combined with Assumptions 2.1.c and (2.14), imply that

$$(v_h M_h^{\frac{1}{2}}, v_h^i M_h^{\frac{1}{2}})_{\Omega_v} = (-2\theta_* \odot \nabla_v M_h^{\frac{1}{2}}, -2\theta_{*,i}\partial_{v_i} M_h^{\frac{1}{2}})_{\Omega_v} = \theta_{*,i} e_i = \theta_* \odot e_i. \tag{3.66}$$

Substituting (3.66) into the definition of I_3 and then applying the discrete IBP formula from (3.3) gives

$$I_3 = (\theta_* \odot \nabla_x \rho_h^\varepsilon, \tau_h e_i) - \langle \theta_* \odot [\![\rho_h^\varepsilon]\!], \{\!\{\tau_h e_i\}\!\}\rangle_{\mathcal{E}_{x,h}^I}. \tag{3.67}$$

Meanwhile I_4 is the only component of Θ_4 and can be bounded using the trace inequality (3.1b) to obtain

$$|I_4| \lesssim \frac{1}{\sqrt{h_x \varepsilon}} \|v_h\|_{L^\infty(\Omega_v)}^2 \|M_h^{\frac{1}{2}}\|_{L^\infty(\Omega_v)}^2 \|[\![\rho_h^\varepsilon]\!]\|_{L^2(\mathcal{E}_{x,h}^I)} \|\tau_h\|_{L^2(\Omega_x)}. \tag{3.68}$$

Likewise, I_5 is the only component of Θ_5 and can be bounded in a similar fashion.

To evaluate $\mathcal{B}$, we add and subtract $M_h^{\frac{1}{2}}\rho_h^\varepsilon$ from the first argument and write

$$\begin{aligned} \mathcal{B}(g_h^\varepsilon, v_h^i M_h^{\frac{1}{2}}\tau_h) &= \mathcal{B}(M_h^{\frac{1}{2}}\rho_h^\varepsilon, v_h^i M_h^{\frac{1}{2}}\tau_h) - \varepsilon\left\{\tfrac{1}{\varepsilon}\mathcal{B}(g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon, v_h^i M_h^{\frac{1}{2}}\tau_h)\right\} \\ &= I_5 + \varepsilon\{I_6\}. \end{aligned} \tag{3.69}$$

Here I_6 is a remainder term in Θ_3 and satisfies the bound in (3.59) due to the trace estimate (3.1c) and inverse estimate (3.2). Meanwhile, the upwind penalty term in $\mathcal{B}(M_h^{\frac{1}{2}}\rho_h^\varepsilon, v_h^i M_h^{\frac{1}{2}}\tau_h)$ vanishes because the first argument is continuous in v; this leaves

$$I_5 = -(E\rho_h^\varepsilon, \tau_h)_{\Omega_x} \cdot \left[\left(M_h^{\frac{1}{2}}, \nabla_v(v_h^i M_h^{\frac{1}{2}})\right)_{\Omega_v} - \left\langle M_h^{\frac{1}{2}}, [\![v_h^i M_h^{\frac{1}{2}}]\!]\right\rangle_{\mathcal{E}_{v,h}^I}\right]. \tag{3.70}$$

From the integration-by-parts identity (3.3) (applied to functions in $V_{v,h}$) and continuity of $M_h^{\frac{1}{2}}$,

$$\left(M_h^{\frac{1}{2}}, \nabla_v(v_h^i M_h^{\frac{1}{2}})\right)_{\Omega_v} - \left\langle M_h^{\frac{1}{2}}, [\![v_h^i M_h^{\frac{1}{2}}]\!]\right\rangle_{\mathcal{E}_{v,h}^I} = -\left(\nabla_v M_h^{\frac{1}{2}}, v_h^i M_h^{\frac{1}{2}}\right)_{\Omega_v} + \left\langle M_h^{\frac{1}{2}}, n_v v_h^i M_h^{\frac{1}{2}}\right\rangle_{\partial\Omega_v} \tag{3.71}$$

The definition of v_h in (2.16), along with (3.66), implies that for the first term above,

$$-\left(\nabla_v M_h^{\frac{1}{2}}, v_h^i M_h^{\frac{1}{2}}\right)_{\Omega_v} = \tfrac{1}{2}\theta_*^{-1} \odot \left(v_h M_h^{\frac{1}{2}}, v_h^i M_h^{\frac{1}{2}}\right)_{\Omega_v} = \tfrac{1}{2} e_i. \tag{3.72}$$

Substituting (3.71) with (3.72) into (3.70) and recalling (2.18d) gives

$$I_5 = -\frac{1}{2}(E\rho_h^\varepsilon, \tau_h e_i)_{\Omega_x} - \mathcal{D}(g_h^\varepsilon, v_h^i M_h^{\frac{1}{2}} \tau_h). \tag{3.73}$$

To evaluate $\mathcal{Q}$, we use Assumption 2.1.c:

$$\frac{1}{\varepsilon}\mathcal{Q}(g_h^\varepsilon, v_h^i M_h^{\frac{1}{2}} \tau_h) = -\frac{1}{\varepsilon}(\omega(M_h^{\frac{1}{2}}\rho_h^\varepsilon - g_h^\varepsilon), v_h^i M_h^{\frac{1}{2}}\tau_h)_\Omega = (\omega J_h^\varepsilon, \tau_h e_i)_{\Omega_x}. \tag{3.74}$$

Lastly, to evaluate $\mathcal{C}$, we add and subtract $M_h^{\frac{1}{2}}\rho_h^\varepsilon$ from the first argument and write

$$\begin{aligned}\mathcal{C}(g_h^\varepsilon, v_h^i M_h^{\frac{1}{2}}\tau_h) &= \tfrac{1}{2}(E\cdot(\theta_*^{-1}\odot v_h)M_h^{\frac{1}{2}}\rho_h^\varepsilon, v_h^i M_h^{\frac{1}{2}}\tau_h)_\Omega \\ &\quad + \varepsilon\{\tfrac{1}{2\varepsilon}(E\cdot(\theta_h^{-1}\odot v_h)(g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon), v_h^i M_h^{\frac{1}{2}}\tau_h)_\Omega\} = I_7 + \varepsilon\{I_8\}.\end{aligned} \tag{3.75}$$

where I_8 is a remainder term of Θ_3 that satisfies the bound in (3.59) and

$$I_7 = \frac{1}{2}\left(E\rho_h^\varepsilon, \tau_h\right)_{\Omega_x}\cdot\left(\theta_*^{-1}\odot(v_h M_h^{\frac{1}{2}}, v_h^i M_h^{\frac{1}{2}})_{\Omega_v}\right) = \frac{1}{2}(E\rho_h^\varepsilon, \tau_h e_i)_{\Omega_x}, \tag{3.76}$$

because of (3.66). Hence we have shown (3.58) for all $\tau_h e_i$ where $\tau_h \in V_{x,h}$, and therefore (3.58) holds for all $\tau_h \in [V_{x,h}]^3$. The proof is complete. ∎

We now use (3.58) to build a space-time bound on $\partial_t J_h^\varepsilon$.

Lemma 3.13. *Assume $\varepsilon \le \varepsilon_{h_v} \lesssim h_x \le 1$, where ε_{h_v} is defined in (3.27). Then*

$$\|J_h^\varepsilon\|_{L_T^\infty(L^2(\Omega_x))} \lesssim \frac{1}{h_x}\|g_{0,h}\|_{L^2(\Omega)}, \tag{3.77}$$

$$\varepsilon^{3/2}\,\|\partial_t J_h^\varepsilon\|_{L_T^2(L^2(\Omega_x))} \lesssim \frac{1}{\sqrt{h_x}}\|g_{0,h}\|_{L^2(\Omega)}. \tag{3.78}$$

Proof. We will first prove (3.77) which is an estimate needed to obtain (3.78). Setting $\tau_h = J_h^\varepsilon$ in (3.58) gives

$$\begin{aligned}\frac{\varepsilon^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|J_h^\varepsilon\|_{L^2(\Omega_x)}^2 + \omega_{\min}\|J_h^\varepsilon\|_{L^2(\Omega_x)}^2 &\le (-\theta_*\odot\nabla_x\rho_h^\varepsilon, J_h^\varepsilon)_{\Omega_x} + \langle\theta_*\odot[\![\rho_h^\varepsilon]\!], \{\!\{J_h^\varepsilon\}\!\}\rangle_{\mathcal{E}_{x,h}^I} \\ &\quad + (E\rho_h^\varepsilon, J_h^\varepsilon)_{\Omega_x} + \varepsilon\Theta_3(\tilde{g}_h^\varepsilon, J_h^\varepsilon) + \sqrt{\varepsilon}\Theta_4(\rho_h^\varepsilon, J_h^\varepsilon) + \sqrt{\varepsilon}\Theta_5(\rho_h^\varepsilon, J_h^\varepsilon).\end{aligned} \tag{3.79}$$

The first three terms on the RHS of (3.79) can be bounded with Cauchy-Schwarz and the inverse inequality (3.2) for the first, a trace inequality (3.1b) for the second, and the L^∞ bound on E for the third. After absorbing ε-independent constants,

$$(-\theta_*\odot\nabla_x\rho_h^\varepsilon, J_h^\varepsilon)_{\Omega_x} + \langle\theta_*\odot[\![\rho_h^\varepsilon]\!], \{\!\{J_h^\varepsilon\}\!\}\rangle_{\mathcal{E}_{x,h}^I} + (E\rho_h^\varepsilon, J_h^\varepsilon)_{\Omega_x} \lesssim \frac{1}{h_x}\|\rho_h^\varepsilon\|_{L^2(\Omega_x)}\|J_h^\varepsilon\|_{L^2(\Omega_x)} \tag{3.80}$$

Meanwhile, the bounds on Θ_3, Θ_4, Θ_5 from Lemma 3.10 and Lemma 3.12 imply that

$$\begin{aligned}&\varepsilon\Theta_3(\tilde{g}_h^\varepsilon, J_h^\varepsilon) + \sqrt{\varepsilon}\Theta_4(\rho_h^\varepsilon, J_h^\varepsilon) + \sqrt{\varepsilon}\Theta_5(\rho_h^\varepsilon, J_h^\varepsilon) \\ &\qquad \lesssim {h_x}^{-1}\left(\|\rho_h^\varepsilon\|_{L^2(\Omega_x)} + \|g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon\|_{L^2(\Omega)}\right)\|J_h^\varepsilon\|_{L^2(\Omega_x)}.\end{aligned} \tag{3.81}$$

Substituting these bounds into (3.79) and dividing by $\varepsilon^2\|J_h\|_{L^2(\Omega_x)}$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|J_h^\varepsilon\|_{L^2(\Omega_x)} + \frac{\omega_{\min}}{\varepsilon^2}\|J_h^\varepsilon\|_{L^2(\Omega_x)} \lesssim \frac{1}{\varepsilon^2 h_x}\left(\|\rho_h^\varepsilon\|_{L^2(\Omega_x)} + \|g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon\|_{L^2(\Omega)}\right). \tag{3.82}$$

According to (2.22), $J_h^\varepsilon|_{t=0} = 0$. Thus Grönwall's Lemma applied to (3.82), along with the L^∞ in time bound in (3.34), recovers (3.77).

We now prove (3.78). As the proof below is quite technical, we first briefly summarize the process. The idea is to pass the time derivative from J_h^ε to ρ_h^ε for the terms that are not sufficiently small in ε to bound with the usual techniques. However, since $\partial_t \rho_h^\varepsilon$ is not uniformly bounded w.r.t ε in $L^2_T(L^2(\Omega_x))$, we will first add and subtract its projection $\mathcal{S}_h \rho_h^\varepsilon$, whose time derivative is uniformly bounded in ε by Lemma 3.11, before passing the time derivative over. Having an explicit bound on the size of $\rho_h^\varepsilon - \mathcal{S}_h\rho_h^\varepsilon$ by (3.38), we can obtain the $\varepsilon^{3/2}$ scale in (3.78).

Setting $\tau_h = \varepsilon \partial_t J_h^\varepsilon$ in (3.58) gives

$$\begin{aligned} &\varepsilon^3 \|\partial_t J_h^\varepsilon\|^2_{L^2(\Omega_x)} + \tfrac{\varepsilon}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\omega} J_h^\varepsilon\|^2_{L^2(\Omega_x)} \\ &\quad = \varepsilon^{3/2}\Theta_4(\rho_h^\varepsilon, \partial_t J_h^\varepsilon) + \varepsilon^{3/2}\Theta_5(\rho_h^\varepsilon, \partial_t J_h^\varepsilon) - \varepsilon(\theta_* \odot \nabla_x \rho_h^\varepsilon, \partial_t J_h^\varepsilon)_{\Omega_x} \\ &\quad + \varepsilon\, \langle \theta_* \odot [\![\rho_h^\varepsilon]\!], \{\!\{\partial_t J_h^\varepsilon\}\!\}\rangle_{\mathcal{E}^I_{x,h}} + \varepsilon(E\rho_h^\varepsilon, \partial_t J_h^\varepsilon)_{\Omega_x} + \varepsilon^2\Theta_3(\tilde{g}_h^\varepsilon, \partial_t J_h^\varepsilon). \end{aligned} \tag{3.83}$$

We then integrate (3.83) over $t \in [0,T]$, use the zero initial condition in (2.22), and drop the positive term $\|\sqrt{\omega}J_h^\varepsilon\|^2_{L^2(\Omega_x)}|_{t=T}$. This gives

$$\begin{aligned} \varepsilon^3 \|\partial_t J_h^\varepsilon\|^2_{L^2_T(L^2(\Omega_x))} \le \int_0^T \Big[&- \varepsilon(\theta_* \odot \nabla_x \rho_h^\varepsilon, \partial_t J_h^\varepsilon)_{\Omega_x} + \varepsilon\, \langle \theta_* \odot [\![\rho_h^\varepsilon]\!], \{\!\{\partial_t J_h^\varepsilon\}\!\}\rangle_{\mathcal{E}^I_{x,h}} \\ +\varepsilon(E\rho_h^\varepsilon, \partial_t J_h^\varepsilon)_{\Omega_x} + \varepsilon^2\Theta_3(\tilde{g}_h^\varepsilon, \partial_t J_h^\varepsilon) &+ \varepsilon^{3/2}\Theta_4(\rho_h^\varepsilon, \partial_t J_h^\varepsilon) + \varepsilon^{3/2}\Theta_5(\rho_h^\varepsilon, \partial_t J_h^\varepsilon)\Big]\,\mathrm{d}t. \end{aligned} \tag{3.84}$$

We add and subtract $\mathcal{S}_h\rho_h^\varepsilon$ (recall $\mathcal{S}_h$ is an L^2 projection) to several terms of (3.84):

$$\begin{aligned} \varepsilon^3 \|\partial_t J_h^\varepsilon\|^2_{L^2_T(L^2(\Omega_x))} \le \int_0^T \Big[&\Big\{ - \varepsilon(\theta_* \odot \nabla_x(\rho_h^\varepsilon - \mathcal{S}_h\rho_h^\varepsilon), \partial_t J_h^\varepsilon)_{\Omega_x} \\ &+ \varepsilon\, \langle \theta_* \odot [\![\rho_h^\varepsilon - \mathcal{S}_h\rho_h^\varepsilon]\!], \{\!\{\partial_t J_h^\varepsilon\}\!\}\rangle_{\mathcal{E}^I_{x,h}} + \varepsilon(E(\rho_h^\varepsilon - \mathcal{S}_h\rho_h^\varepsilon), \partial_t J_h^\varepsilon)_{\Omega_x}\Big\} \\ &+ \Big\{ - \varepsilon(\theta_* \odot \nabla_x \mathcal{S}_h\rho_h^\varepsilon, \partial_t J_h^\varepsilon)_{\Omega_x} + \varepsilon\, \langle \theta_* \odot [\![\mathcal{S}_h\rho_h^\varepsilon]\!], \{\!\{\partial_t J_h^\varepsilon\}\!\}\rangle_{\mathcal{E}^I_{x,h}}\Big\} \\ &+ \Big\{\varepsilon(E\mathcal{S}_h\rho_h^\varepsilon, \partial_t J_h^\varepsilon)_{\Omega_x}\Big\} \\ &+ \Big\{\varepsilon^2\Theta_3(\tilde{g}_h^\varepsilon, \partial_t J_h^\varepsilon) + \varepsilon^{3/2}\Theta_4(\rho_h^\varepsilon, \partial_t J_h^\varepsilon) + \varepsilon^{3/2}\Theta_5(\rho_h^\varepsilon, \partial_t J_h^\varepsilon)\Big\}\Big]\,\mathrm{d}t \\ \le \int_0^T &\Big[\{I_1\} + \{I_2\} + \{I_3\} + \{I_4\}\Big]\,\mathrm{d}t. \end{aligned} \tag{3.85}$$

We will bound I_1, I_2, I_3, and I_4 independently. For I_1, after applying Cauchy-Schwarz, the trace inequality (3.1b), the projection bound (3.38), Assumption 2.12, and Young's inequality, we have, for any $\nu > 0$,

$$\begin{aligned} \int_0^T I_1\,\mathrm{d}t &\lesssim \varepsilon\left(\|\rho_h^\varepsilon - \mathcal{S}_h\rho_h^\varepsilon\|_{L^2_T(L^2(\Omega_x))} + \|\rho_h^\varepsilon - \mathcal{S}_h\rho_h^\varepsilon\|_{L^2(H^1_h(\Omega_x))}\right)\|\partial_t J_h^\varepsilon\|_{L^2_T(L^2(\Omega_x))} \\ &\lesssim \frac{\varepsilon^{3/2}}{\sqrt{h_x}}\left(\sqrt{\frac{\varepsilon}{h_x}} + 1\right)\|g_{0,h}\|_{L^2(\Omega)}\|\partial_t J_h^\varepsilon\|_{L^2_T(L^2(\Omega_x))} \\ &\lesssim \frac{1}{h_x\nu}\|g_{0,h}\|^2_{L^2(\Omega)} + \nu\varepsilon^3\|\partial_t J_h^\varepsilon\|^2_{L^2_T(L^2(\Omega_x))}. \end{aligned} \tag{3.86}$$

For I_2, we integrate by parts in time to obtain

$$\begin{aligned}\int_0^T I_2\,\mathrm{d}t &= \left\{\varepsilon\int_0^T (\theta_*\odot\nabla_x\partial_t\mathcal{S}_h\rho_h^\varepsilon, J_h^\varepsilon)_{\Omega_x} - \langle\theta_*\odot[\![\partial_t\mathcal{S}_h\rho_h^\varepsilon]\!], \{\!\{J_h^\varepsilon\}\!\}\rangle_{\mathcal{E}^I_{x,h}}\,\mathrm{d}t\right\}\\ &\quad+\left\{-\varepsilon\,(\theta_*\odot\nabla_x\mathcal{S}_h\rho_h^\varepsilon, J_h^\varepsilon)_{\Omega_x}\Big|_0^T + \varepsilon\,\langle\theta_*\odot[\![\mathcal{S}_h\rho_h^\varepsilon]\!], \{\!\{J_h^\varepsilon\}\!\}\rangle_{\mathcal{E}^I_{x,h}}\Big|_0^T\right\}\\ &= \{K_1\}+\{K_2\}.\end{aligned}\tag{3.87}$$

We first bound K_1. Using Cauchy-Schwarz, the inverse inequality (3.2), and the trace inequality (3.1b), as well as the bound on $\partial_t\mathcal{S}_h\rho_h^\varepsilon$ in (3.56) and the bound on J_h^ε in (3.36), we have

$$K_1 \lesssim \frac{\varepsilon}{h_x}\|\partial_t\mathcal{S}_h\rho_h^\varepsilon\|_{L^2_T(L^2(\Omega_x))}\|J_h^\varepsilon\|_{L^2_T(L^2(\Omega_x))} \lesssim \frac{\varepsilon}{h_x^2}\|g_{0,h}\|^2_{L^2(\Omega)}.\tag{3.88}$$

For K_2, terms evaluated at $t=0$ vanish due to Assumption 2.9. Following a similar treatment as for K_1, but instead using the L^∞_T estimates (3.36) and (3.34), we obtain

$$K_2 \lesssim \frac{\varepsilon}{h_x}\|\mathcal{S}_h\rho_h^\varepsilon\|_{L^\infty(L^2(\Omega_x))}\|J_h^\varepsilon\|_{L^\infty(L^2(\Omega_x))} \lesssim \frac{\varepsilon}{h_x^2}\|g_{0,h}\|^2_{L^2(\Omega)}.\tag{3.89}$$

For I_3, the treatment is similar to that of I_2. Integrating by parts in time and applying bounds similar to those used for K_1 and K_2, we find

$$\begin{aligned}\int_0^T I_3\,\mathrm{d}t &= -\varepsilon\int_0^T (\partial_t(E\mathcal{S}_h\rho_h^\varepsilon), J_h^\varepsilon)_{\Omega_x}\,\mathrm{d}t + \varepsilon\,(E\mathcal{S}_h\rho_h^\varepsilon, J_h^\varepsilon)_{\Omega_x}\Big|_0^T\\ &= -\varepsilon\int_0^T\Big[(\mathcal{S}_h\rho_h^\varepsilon\partial_t E, J_h^\varepsilon)_{\Omega_x} + (E\partial_t\mathcal{S}_h\rho_h^\varepsilon, J_h^\varepsilon)_{\Omega_x}\Big]\,\mathrm{d}t + \varepsilon\,(E\mathcal{S}_h\rho_h^\varepsilon, J_h^\varepsilon)_{\Omega_x}\Big|_0^T\\ &\lesssim \varepsilon\|\rho_h^\varepsilon\|_{L^2_T(L^2(\Omega_x))}\|J_h^\varepsilon\|_{L^2_T(L^2(\Omega_x))} + \varepsilon\|\partial_t\mathcal{S}_h\rho_h^\varepsilon\|_{L^2_T(L^2(\Omega_x))}\|J_h^\varepsilon\|_{L^2_T(L^2(\Omega_x))}\\ &\quad+\varepsilon\|\rho_h^\varepsilon\|_{L^\infty(L^2(\Omega_x))}\|J_h^\varepsilon\|_{L^\infty(L^2(\Omega_x))} \lesssim \frac{\varepsilon}{h_x}\|g_{0,h}\|^2_{L^2(\Omega)}.\end{aligned}\tag{3.90}$$

We now focus on each term of I_4. To bound Θ_3, we use (3.59), (3.35), and Young's inequality:

$$\begin{aligned}\int_0^T \varepsilon^2\Theta_3(\tilde{g}_h^\varepsilon, \partial_t J_h^\varepsilon)\,\mathrm{d}t &\lesssim \frac{\sqrt{\varepsilon}}{h_x}\left(\frac{1}{\varepsilon}\|g_h^\varepsilon - M_h^{\frac{1}{2}}\rho_h^\varepsilon\|_{L^2_T(L^2(\Omega))}\right)\left(\varepsilon^{3/2}\|\partial_t J_h^\varepsilon\|_{L^2_T(L^2(\Omega_x))}\right)\\ &\lesssim \frac{\varepsilon}{\nu h_x^2}\|g_{0,h}\|^2_{L^2(\Omega)} + \nu\varepsilon^3\|\partial_t J_h^\varepsilon\|^2_{L^2_T(L^2(\Omega_x))}\end{aligned}\tag{3.91}$$

for any $\nu>0$. We treat Θ_4 in a similar manner. Using (3.62) and (3.37) with Assumption 2.12, we have

$$\begin{aligned}\int_0^T \varepsilon^{3/2}\Theta_4(\rho_h^\varepsilon, \partial_t J_h^\varepsilon)\,\mathrm{d}t &\lesssim \frac{1}{\sqrt{h_x}}\left(\frac{1}{\varepsilon^{1/2}}\|[\![\rho_h^\varepsilon]\!]\|_{L^2_T(L^2(\mathcal{E}^I_{x,h}))}\right)\left(\varepsilon^{3/2}\|\partial_t J_h^\varepsilon\|_{L^2_T(L^2(\Omega_x))}\right)\\ &\lesssim \frac{1}{\nu h_x}\|g_{0,h}\|^2_{L^2(\Omega)} + \nu\varepsilon^3\|\partial_t J_h^\varepsilon\|^2_{L^2_T(L^2(\Omega_x))}\end{aligned}\tag{3.92}$$

for any $\nu>0$. We treat Θ_5 similar to Θ_4; cf. (3.92):

$$\int_0^T \varepsilon^{3/2}\Theta_5(\rho_h^\varepsilon, \partial_t J_h^\varepsilon)\,\mathrm{d}t \lesssim \frac{1}{\nu h_x}\|g_{0,h}\|^2_{L^2(\Omega)} + \nu\varepsilon^3\|\partial_t J_h^\varepsilon\|^2_{L^2_T(L^2(\Omega_x))}\tag{3.93}$$

for any $\nu > 0$. Combining (3.91)-(3.93) yields the following bound for I_4:

$$\int_0^T I_4 \,\mathrm{d}t \lesssim \frac{1}{\nu}\left(\frac{1}{h_x} + \frac{\varepsilon}{h_x^2}\right)\|g_{0,h}\|^2_{L^2(\Omega)} + \nu\varepsilon^3\|\partial_t J_h^\varepsilon\|^2_{L^2_T(L^2(\Omega_x))}. \tag{3.94}$$

for any $\nu > 0$. Combining (3.85)-(3.89) and (3.94) we obtain

$$\varepsilon^3\,\|\partial_t J_h^\varepsilon\|^2_{L^2_T(L^2(\Omega_x))} \lesssim \frac{1}{\nu}\left(\frac{1}{h_x} + \frac{\varepsilon}{h_x^2}\right)\|g_{0,h}\|^2_{L^2(\Omega)} + \frac{\varepsilon}{h_x^2}\|g_{0,h}\|^2_{L^2(\Omega)} + \nu\varepsilon^3\|\partial_t J_h^\varepsilon\|^2_{L^2_T(L^2(\Omega_x))}. \tag{3.95}$$

Choosing ν, independent of ε and h_x, sufficiently small to move $\varepsilon^3\|\partial_t J_h^\varepsilon\|^2_{L^2_T(L^2(\Omega_x))}$ from the right-hand side of (3.95) and applying Assumption 2.12 to the first two terms on the right-hand side of (3.95) gives us (3.78). The proof is complete. ◻

4. The Drift Diffusion Limit. The bounds in Section 3 facilitate the limit of ρ_h^ε and J_h^ε as $\varepsilon \to 0$. In this section, we show these limits satisfy (4.2), a discrete version of the drift-diffusion equations (2.5). Recall the definition of ε_{h_v} from (3.27).

THEOREM 4.1. *Let $h_x, h_v > 0$ be fixed. Then for all $\varepsilon \le \varepsilon_{h_v}$, we have $\rho_h^\varepsilon \in L^2_T(L^2(\Omega_x))$, $\frac{\partial}{\partial t}\rho_h^\varepsilon \in L^2_T(H_h^{-1}(\Omega_x))$ and $J_h^\varepsilon \in L^2_T(L^2(\Omega_x))$ with bound*

$$\|\rho_h^\varepsilon\|_{L^2_T(L^2(\Omega_x))} + \|\partial_t\rho_h^\varepsilon\|_{L^2(H_h^{-1}(\Omega_x))} + \|J_h^\varepsilon\|_{L^2_T(L^2(\Omega_x))} \lesssim \|g_{0,h}\|_{L^2(\Omega)}. \tag{4.1}$$

Moreover, there exist functions $\rho_h^0 \in H^1([0,T]; S^0_{x,h})$, $J_h^0 \in L^2([0,T];[V_{x,h}]^3)$, and subsequences of $\{\rho_h^\varepsilon\}_\varepsilon$ and $\{J_h^\varepsilon\}_\varepsilon$, not relabeled, such that $\rho_h^\varepsilon \rightharpoonup \rho_h^0$ in $L^2_T(L^2(\Omega_x))$, $\partial_t\rho_h^\varepsilon \rightharpoonup \partial_t\rho_h^0$, and $J_h^\varepsilon \rightharpoonup J_h^0$ in $L^2_T([L^2(\Omega_x)]^3)$.

Additionally, ρ_h^0 and J_h^0 satisfy the following drift-diffusion system:

$$\left(\frac{\partial}{\partial t}\rho_h^0, q_h\right)_{\Omega_x} - (J_h^0, \nabla_x q_h)_{\Omega_x} = 0 \qquad \forall q_h \in S^0_{x,h}, \tag{4.2a}$$

$$(\omega J_h^0, \tau_h)_{\Omega_x} + (\theta_* \odot \nabla_x\rho_h^0, \tau_h)_{\Omega_x} - (E\rho_h^0, \tau_h)_{\Omega_x} = 0 \qquad \forall \tau_h \in [V_{x,h}]^3, \tag{4.2b}$$

$$(\rho_h^0(0), q_h)_{\Omega_x} = (\rho_{0,h}, q_h)_{\Omega_x} \qquad \forall q_h \in S^0_{x,h}. \tag{4.2c}$$

Proof. The proof proceeds in two steps.
Step 1: Existence of Limits. It follows from (3.34), (3.36), and (3.55) that $\|\rho_h^\varepsilon\|_{L^2_T(L^2(\Omega_x))}$, $\|J_h^\varepsilon\|_{L^2_T(L^2(\Omega_x))}$, and $\|\partial_t\rho_h^\varepsilon\|_{L^2_T(H^{-1}_{h,\beta}(\Omega_x))}$, respectively, (4.1) holds and each term is uniformly bounded in ε. Since each of these spaces are Hilbert spaces, we can extract a subsequence ρ_h^ε and J_h^ε, not relabeled, and limiting functions $\rho_h^0 \in L^2_T(V_{x,h})$ and $J_h^0 \in L^2_T([V_{x,h}]^3)$ such that $\rho_h^\varepsilon \rightharpoonup \rho_h^0$ in $L^2_T(L_2(\Omega_x))$ and $J_h^\varepsilon \rightharpoonup J_h^0$ in $L^2_T([L^2(\Omega_x)]^3)$. We now show $\rho_h^0 \in L^2_T(V^0_{x,h})$. By (3.37),

$$\|[\![\rho_h^\varepsilon]\!]\|^2_{L^2_T(L^2(\mathcal{E}^I_{x,h}))} + \|\rho_h^\varepsilon\|^2_{L^2_T(L^2(\partial\Omega_x))} \lesssim \varepsilon\|g_{0,h}\|^2_{L^2(\Omega)}. \tag{4.3}$$

Since V_h is finite dimensional, then $\rho_h^\varepsilon \rightharpoonup \rho_h^\varepsilon$ in $L^2_T(L^2(\partial\Omega_x))$. Because the norm is weakly lower semi-continuous, (4.3) implies that $\rho_h^0(t)$ is continuous in x and 0 on $\partial\Omega_x$ for a.e. time t; therefore $\rho_h^0 \in L^2_T(S^0_{x,h})$.

By (3.55), $\partial_t\rho_h^0$ is uniformly bounded in ε in $L^2_T((S^0_{x,h})^*)$ where $(S^0_{x,h})^*$ is the dual space of $S^0_{x,h}$. Hence there is a subsequence of $\partial_t\rho_h^\varepsilon$ such that $\partial_t\rho_h^\varepsilon \rightharpoonup \zeta$ for some $\zeta \in L^2_T((S^0_{x,h})^*)$. Since $S^0_{x,h}$ is finite-dimensional and thus a Hilbert space with respect to the L^2 inner product on Ω_x, we can apply the Riesz representation theorem

to show there is $\zeta_h \in L^2_T(S^0_{x,h})$ such that $\zeta(t; q_h) = (\zeta_h(t), q_h(t))_{\Omega_x}$ for all $q_h \in S^0_{x,h}$ and a.e. t where $\zeta(t;\cdot) \in (S^0_{x,h})^*$. By a standard density argument (see [11, Chapter 7, Problem 5]), we have $\zeta_h = \partial_t \rho^0_h$. Therefore $\rho^0_h \in H^1([0,T]; S^0_{x,h}) \hookrightarrow C^0([0,T]; S^0_{x,h})$.
Step 2: *The Limiting System.* We first recover (4.2a). We choose $q_h \in L^2_T(V^0_{x,h})$ in (3.45) and integrate in time to obtain

$$\int_0^T \left[(\partial_t \rho^\varepsilon_h, q_h)_{\Omega_x} - (J^\varepsilon_h, \nabla_x q_h)_{\Omega_x}\right] \mathrm{d}t = 0. \tag{4.4}$$

Note that the right hand side of (3.45) and the interior flux terms vanish since $q_h \in S^0_{x,h}$ Therefore we can pass the weak limit as $\varepsilon \to 0$ in (4.4) and arrive at (4.2a).

For (4.2b), choose $\tau_h \in L^2_T([V_{x,h}]^3)$ in (3.58) and integrate in time to obtain

$$\begin{aligned}\int_0^T &(\omega J^\varepsilon_h, \tau_h)_{\Omega_x} + (\theta_* \odot \nabla_x \rho^\varepsilon_h, \tau_h)_{\Omega_x} - \langle \theta_* \odot [\![\rho^\varepsilon_h]\!], \{\!\{\tau_h\}\!\}\rangle_{\mathcal{E}^I_{x,h}} - (E\rho^\varepsilon_h, \tau_h)_{\Omega_x}\,\mathrm{d}t \\ &= \int_0^T \varepsilon\Theta_3(\tilde{g}^\varepsilon_h, \tau_h) + \sqrt{\varepsilon}\Theta_4(\rho^\varepsilon_h, \tau_h) + \sqrt{\varepsilon}\Theta_5(\rho^\varepsilon_h, \tau_h) - \varepsilon^2 \left(\partial_t J^\varepsilon_h, \tau_h\right)_{\Omega_x} \mathrm{d}t.\end{aligned} \tag{4.5}$$

Since $\varepsilon^{3/2}\|\partial_t J^\varepsilon_h\|_{L^2_T(L^2(\Omega_x))}$ is bounded in ε by (3.78), the time derivative term in (4.5) vanishes as $\varepsilon \to 0$. Additionally since $\int_0^T \Theta_3(\tilde{g}^\varepsilon_h, \tau_h)\,\mathrm{d}t$, $\int_0^T \Theta_4(\rho^\varepsilon_h, \tau_h)\,\mathrm{d}t$, and $\int_0^T \Theta_5(\rho^\varepsilon_h, \tau_h)\,\mathrm{d}t$ are bounded in ε by (3.59), (3.62), and (3.63) respectively, then the entire right hand side of (4.5) vanishes as ε vanishes. Therefore taking the limit as $\varepsilon \to 0$ of (4.5) yields

$$(\omega J^0_h, \tau_h)_{\Omega_x} + (\theta_* \odot \nabla_x \rho^0_h, \tau_h)_{\Omega_x} - \left\langle \theta_* \odot [\![\rho^0_h]\!], \{\!\{\tau_h\}\!\}\right\rangle_{\mathcal{E}^I_{x,h}} - (E\rho^0_h, \tau_h)_{\Omega_x} = 0 \tag{4.6}$$

for all $\tau_h \in [V_{x,h}]^3$ and a.e. $0 < t \leq T$. Since $\rho^0_h \in L^2_T(S^0_{x,h})$, we drop the interior flux terms in (4.6) and recover (4.2b). The proof of the projected initial condition (4.2c) follows the standard arguments shown in [11, Page 379] and is not included. The proof is complete. ∎

Now we show that the whole sequence $\{\rho^\varepsilon_h\}_\varepsilon$ and $\{J^\varepsilon_h\}_\varepsilon$ must converge to ρ^0_h and J^0_h respectively. We do this by showing uniqueness of solutions to the drift-diffusion system (4.2) with the following lemma.

Lemma 4.2. *Suppose ρ_h and J_h satisfy (4.2), then for any $h_x > 0$:*

$$\|\rho_h\|^2_{L^\infty_T(L^2(\Omega_x))} + \tfrac{\omega_{\min}}{\theta^{\max}_*}\|J_h\|^2_{L^2_T(L^2(\Omega_x))} \leq \exp(C_4 T)\|\rho_{0,h}\|^2_{L^2(\Omega_x)}. \tag{4.7}$$

where $C_4 := \frac{\|\theta_*^{-1}\odot E\|_{L^\infty([0,T]\times\Omega_x)}}{\omega_{\min}}$ and $\theta^{\max}_*$ is the element-wise maximum of θ_*. Moreover, the solution pair $\{\rho_h, J_h\}$ to (4.2) is unique.

Proof. We first focus on (4.7). Choose $q_h = \rho_h$ and $\tau_h = \theta_*^{-1} \odot J_h$ in (4.2). Adding both equations in (4.2) gives us

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho_h\|^2_{L^2(\Omega_x)} + (\omega J_h, \theta_*^{-1} \odot J_h)_{\Omega_x} = (\theta_*^{-1} \odot E\rho_h, J_h)_{\Omega_x}. \tag{4.8}$$

Bounding the right hand side of (4.8) by Hölder's and Young's inequality, bounding θ_*^{-1} from below, and multiplying by 2 we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho_h\|^2_{L^2(\Omega_x)} + \frac{\omega_{\min}}{\theta^{\max}_*}\|J_h\|^2_{L^2(\Omega_x)} \leq \frac{\|\theta_*^{-1} \odot E\|_{L^\infty([0,T]\times\Omega_x)}}{\omega_{\min}}\|\rho_h\|^2_{L^2(\Omega_x)}. \tag{4.9}$$

Applying Grönwall's to (4.9) gives us (4.7). Uniqueness of the solution pair follows from applying (4.7) to the case where $\rho_{0,h} = 0$. The proof is complete. ☐

Due to the uniqueness result from Lemma 4.2, we attach the additional corollary.

COROLLARY 4.3. *The full sequences* $\{\rho_h^\varepsilon\}_\varepsilon$ *and* $\{J_h^\varepsilon\}_\varepsilon$ *weakly converge to* ρ_h^0 *and* J_h^0 *respectively in the topologies given in Theorem 4.1.*

5. Error Estimates. In this section we develop error estimates for ρ_h^ε against the true drift-diffusion limit ρ^0 which solves (2.4). The error estimates are created by comparing both against the discrete drift-diffusion ρ_h^0 which solves (4.2). This is summarized in the following theorem whose proof we delay until the end of the section.

THEOREM 5.1. *Suppose* $\rho^0 \in L_T^2(H^s(\Omega))$ *and* $J^0 \in L_T^2([L^2(\Omega_x)]^3)$ *satisfy* (2.5) *for some* $s \geq 2$, $\omega \in W^{r,\infty}(\Omega_x)$, *and* $E \in L_T^\infty(W^{r,\infty}(\Omega_x))$ *for some* $r \geq s-1$. *Define*

$$C_{\omega,r} = \|\omega\|_{W^{r,\infty}(\Omega_x)}\|E\|_{L_T^\infty(W^{r,\infty}(\Omega_x))} \quad \text{and} \quad \bar{\theta} = [\theta,\theta,\theta] \in \mathbb{R}^3. \tag{5.1}$$

Then for any $\varepsilon \leq \varepsilon_{h_v}$ *where* ε_{h_v} *is defined in* (3.27) *we have the following error estimate:*

$$\begin{aligned}\|\rho_h^\varepsilon - \rho^0\|_{L_T^2(L^2(\Omega_x))} \lesssim \sqrt{\frac{\varepsilon}{h_x}}\|g_{0,h}\|_{L^2(\Omega)} + (h_x^{\min\{k+1,s\}-1} + |\theta_* - \bar{\theta}|)\|\rho^0\|_{L_T^2(H^s(\Omega_x))} \\ + C_{\omega,r}h_x^{\min\{k+1,s-1\}}\|\rho^0\|_{L_T^2(H^s(\Omega_x))}.\end{aligned} \tag{5.2}$$

5.1. Error Estimates in ε. Here we build estimates comparing ρ_h^ε to ρ_h^0. Define $e_\rho^\varepsilon = \rho_h^\varepsilon - \rho_h^0$ and $e_J^\varepsilon = J_h^\varepsilon - J_h^0$. Subtracting (4.2) from the system (3.45) and (3.58) gives us the following error equations for all test functions $\tau_h \in [V_{x,h}]^3$ and $q_h \in S_{x,h}^0$:

$$\begin{aligned}&\left(\partial_t e_\rho^\varepsilon, q_h\right)_{\Omega_x} - (e_J^\varepsilon, \nabla_x q_h)_{\Omega_x} = 0,\\ &(\omega e_J^\varepsilon, \tau_h)_{\Omega_x} + (\theta_* \odot \nabla_x e_\rho^\varepsilon, \tau_h)_{\Omega_x} - \left\langle \theta_* \odot [\![e_\rho^\varepsilon]\!], \{\!\{\tau_h\}\!\}\right\rangle_{\mathcal{E}_{x,h}^I} - (E e_\rho^\varepsilon, \tau_h)_{\Omega_x}\\ &\qquad = \varepsilon\Theta_3(\tilde{g}_h^\varepsilon, \tau_h) + \sqrt{\varepsilon}\Theta_4(\rho_h^\varepsilon, \tau_h) + \sqrt{\varepsilon}\Theta_5(\rho_h^\varepsilon, \tau_h) + \sqrt{\varepsilon}\left(\varepsilon^{3/2}\partial_t J_h^\varepsilon, \tau_h\right)_{\Omega_x}.\end{aligned} \tag{5.3}$$

In order to bound the error of e_ρ^ε, we decompose the ρ error as $e_\rho^\varepsilon = \eta_\rho^\varepsilon - \xi_\rho^\varepsilon := (\rho_h^\varepsilon - \mathcal{S}_h\rho_h^\varepsilon) - (\rho_h^0 - \mathcal{S}_h\rho_h^\varepsilon)$. Thus $\eta_\rho^\varepsilon \in V_{x,h}$ and $\xi_\rho^\varepsilon \in S_{x,h}^0$.

LEMMA 5.2. *For any* $h_x > 0$, $h_v > 0$, *and* $\varepsilon \leq \varepsilon_{h_v}$ *where* ε_{h_v} *is defined in Lemma 3.7,* e_ρ^ε *and* e_J^ε *satisfy the following error bound:*

$$\|\xi_\rho^\varepsilon(T)\|_{L^2(\Omega)}^2 + \frac{\omega_{\min}}{2}\|e_J^\varepsilon\|_{L_T^2(L^2(\Omega_x))}^2 \lesssim \frac{\varepsilon}{h_x}\|g_{0,h}\|_{L^2(\Omega)}^2. \tag{5.4}$$

Proof. Choose $q_h = -\theta_*\xi_\rho^\varepsilon$ and $\tau_h = e_{\theta_*^{-1},J}^\varepsilon := \theta_*^{-1} \odot e_J^\varepsilon$ in (5.3). Adding the two equations in (5.3) we arrive at

$$\begin{aligned}\left(\partial_t \xi_\rho^\varepsilon, \xi_\rho^\varepsilon\right)_{\Omega_x} + (\omega e_J^\varepsilon, e_{\theta_*^{-1},J}^\varepsilon)_{\Omega_x} &= -(\nabla_x \eta_\rho^\varepsilon, e_J^\varepsilon)_{\Omega_x} + \left\langle \{\!\{e_J^\varepsilon\}\!\}, [\![\eta_\rho^\varepsilon]\!]\right\rangle_{\mathcal{E}_{x,h}^I} + (E\eta_\rho^\varepsilon, e_{\theta_*^{-1},J}^\varepsilon)_{\Omega_x}\\ &- (E\xi_\rho^\varepsilon, e_{\theta_*^{-1},J}^\varepsilon)_{\Omega_x} - \varepsilon\Theta_3(\tilde{g}_h^\varepsilon, e_{\theta_*^{-1},J}^\varepsilon) + \sqrt{\varepsilon}\Theta_4(\rho_h^\varepsilon, e_{\theta_*^{-1},J}^\varepsilon)\\ &+ \sqrt{\varepsilon}\Theta_5(\rho_h^\varepsilon, e_{\theta_*^{-1},J}^\varepsilon) + \sqrt{\varepsilon}\left(\varepsilon^{3/2}\partial_t J_h^\varepsilon, e_{\theta_*^{-1},J}^\varepsilon\right)_{\Omega_x} + \left(\partial_t \eta_\rho^\varepsilon, \xi_\rho^\varepsilon\right)_{\Omega_x}.\end{aligned} \tag{5.5}$$

We seek to bound each of the terms on the right hand side of (5.5). As the entries θ_* do not depend on ε or h_x, we will absorb them into the generic constant in $\lesssim$. Let

$$I_1 := -(\nabla_x \eta_\rho^\varepsilon, e_J^\varepsilon)_{\Omega_x} + \langle \{\!\{e_J^\varepsilon\}\!\}, [\![\eta_\rho^\varepsilon]\!]\rangle_{\mathcal{E}_{x,h}^I} + (E\eta_\rho^\varepsilon, e^\varepsilon_{\theta_*^{-1},J})_{\Omega_x}. \tag{5.6}$$

By use of inverse inequalities, trace inequalities, and Young's inequality, we can bound I_1 for any $\nu > 0$ by

$$|I_1| \lesssim \frac{1}{\nu}\|\eta_\rho^\varepsilon\|^2_{L^2(\Omega_x)} + \frac{1}{\nu}\|\eta_\rho^\varepsilon\|^2_{H_h^1(\Omega_x)} + \nu\|e_J^\varepsilon\|^2_{L^2(\Omega_x)}. \tag{5.7}$$

We can bound $(E\xi_\rho^\varepsilon, e^\varepsilon_{\theta_*^{-1},J})_{\Omega_x}$ with Young's inequality to obtain

$$|(E\xi_\rho^\varepsilon, e^\varepsilon_{\theta_*^{-1},J})_{\Omega_x}| \lesssim \frac{1}{\nu}\|\xi_\rho^\varepsilon\|^2_{L^2(\Omega_x)} + \nu\|e_J^\varepsilon\|^2_{L^2(\Omega_x)} \tag{5.8}$$

for any $\nu > 0$. We use (3.59) and Young's inequality to obtain

$$|\varepsilon\Theta_3(\tilde{g}_h^\varepsilon, e^\varepsilon_{\theta_*^{-1},J})| \lesssim \frac{1}{\nu h_x^2}\|M_h^{\frac{1}{2}}\rho_h^\varepsilon - g_h^\varepsilon\|^2_{L^2(\Omega)} + \nu\|e_J^\varepsilon\|^2_{L^2(\Omega_x)} \tag{5.9}$$

for any $\nu > 0$. We bound Θ_4 and Θ_5 by (3.62), (3.63), and Young's inequality:

$$|\sqrt{\varepsilon}\Theta_4(\rho_h^\varepsilon, e^\varepsilon_{\theta_*^{-1},J}) + \sqrt{\varepsilon}\Theta_5(\rho_h^\varepsilon, e^\varepsilon_{\theta_*^{-1},J})| \lesssim \frac{1}{\nu h_x}\|[\![\rho_h^\varepsilon]\!]\|^2_{L^2(\mathcal{E}_{x,h}^I)} + \frac{1}{\nu h_x}\|\rho_h^\varepsilon\|^2_{L^2(\partial\Omega_x)} + \nu\|e_J^\varepsilon\|^2_{L^2(\Omega_x)} \tag{5.10}$$

for any $\nu > 0$. We bound the $\partial_t J_h^\varepsilon$ term as

$$\left|\sqrt{\varepsilon}\left(\varepsilon^{3/2}\partial_t J_h^\varepsilon, e^\varepsilon_{\theta_*^{-1},J}\right)_{\Omega_x}\right| \lesssim \frac{\varepsilon}{\nu}\|\varepsilon^3\partial_t J_h^\varepsilon\|^2_{L^2(\Omega_x)} + \nu\|e_J^\varepsilon\|^2_{L^2(\Omega_x)} \tag{5.11}$$

for any $\nu > 0$. The term $\left(\partial_t\eta_\rho^\varepsilon, \xi_\rho^\varepsilon\right)_{\Omega_x}$ is zero since by definition η_ρ^ε is orthogonal to ξ_ρ^ε in $L^2(\Omega_x)$. Injecting (5.7) through (5.11) into (5.5) gives us

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\|\xi_\rho^\varepsilon\|^2_{L^2(\Omega_x)} + \|e_J^\varepsilon\|^2_{L^2(\Omega)} &\lesssim \tfrac{1}{\nu}\|\xi_\rho^\varepsilon\|^2_{L^2(\Omega_x)} + \tfrac{1}{\nu}\|\eta_\rho^\varepsilon\|^2_{L^2(\Omega_x)} + \tfrac{1}{\nu}\|\eta_\rho^\varepsilon\|^2_{H_h^1(\Omega_x)} \\ &\quad + \tfrac{1}{\nu h_x}\left(1 + \tfrac{1}{h_x}\right)\|M_h^{\frac{1}{2}}\rho_h^\varepsilon - g_h^\varepsilon\|^2_{L^2(\Omega)} + \tfrac{\varepsilon}{\nu}\|\varepsilon^3\partial_t J_h^\varepsilon\|^2_{L^2(\Omega_x)} \\ &\quad + \tfrac{1}{\nu h_x}\|[\![\rho_h^\varepsilon]\!]\|^2_{L^2(\mathcal{E}_{x,h}^I)} + \tfrac{1}{\nu h_x}\|\rho_h^\varepsilon\|^2_{L^2(\partial\Omega_x)} + \nu\|e_J^\varepsilon\|^2_{L^2(\Omega)}\end{aligned} \tag{5.12}$$

for all $\nu > 0$. Choosing ν, independent of ε and h_x, sufficiently small we can move the last two terms on the right of (5.12) over to the left. Since $\xi_\rho^\varepsilon(0) = 0$, we can then apply Grönwall's to obtain (assuming $h_x \lesssim 1$)

$$\begin{aligned}\|\xi_\rho^\varepsilon\|^2_{L_T^\infty(L^2(\Omega_x))} + \|e_J^\varepsilon\|^2_{L_T^2(L^2(\Omega_x))} &\lesssim \|\eta_\rho^\varepsilon\|^2_{L_T^2(L^2(\Omega_x))} + \|\eta_\rho^\varepsilon\|^2_{L_T^2(H_h^1(\Omega_x))} \\ &\quad + \tfrac{1}{\nu h_x^2}\|M_h^{\frac{1}{2}}\rho_h^\varepsilon - g_h^\varepsilon\|^2_{L_T^2(L^2(\Omega))} + \varepsilon\|\varepsilon^3\partial_t J_h^\varepsilon\|^2_{L_T^2(L^2(\Omega_x))} \\ &\quad + \tfrac{1}{h_x}\|[\![\rho_h^\varepsilon]\!]\|^2_{L_T^2(L^2(\mathcal{E}_{x,h}^I))} + \tfrac{1}{h_x}\|\rho_h^\varepsilon\|^2_{L_T^2(L^2(\partial\Omega_x))}.\end{aligned} \tag{5.13}$$

Note that the appropriate norms of $\eta_\rho^\varepsilon = \rho_h^\varepsilon - \mathcal{S}_h\rho_h^\varepsilon$, ρ_h^ε, and $M_h^{\frac{1}{2}}\rho_h^\varepsilon - g_h^\varepsilon$ can all be bounded from Lemma 3.9 while $\partial_t J_h^\varepsilon$ can be bounded in L^2 by Lemma 3.13. Applying these bounds to (5.13), recalling Assumption 2.12, and noticing $\varepsilon^2 \lesssim \varepsilon$ gives us (5.4). The proof is complete. ∎

We can now show the ε-error estimate.

THEOREM 5.3. *Let $\{\rho_h^0, J_h^0\}$ satisfy (4.2). Then for any $\varepsilon \le \varepsilon_{h_v}$, where ε_{h_v} is defined in (3.27), we have the following error estimate:*

$$\|\rho_h^\varepsilon - \rho_h^0\|_{L_T^2(L^2(\Omega_x))} + \|J_h^\varepsilon - J_h^0\|_{L_T^2(L^2(\Omega_x))} \lesssim \sqrt{\tfrac{\varepsilon}{h_x}}\|g_{0,h}\|_{L^2(\Omega)}. \tag{5.14}$$

Proof. The $L_T^2 L^2$ bound of η_ρ^ε and ξ_ρ^ε are provided by (3.38) and (5.4) respectively. The J_h^ε error estimate follows from (5.4). The proof is complete. □

5.2. Error Estimate in h. We now focus on the error estimates of the limiting drift diffusion system (4.2). We note that DG discretizations of (2.5) have been studied for the one-dimensional case in [24]; however, their discretization defines the auxiliary variable in the system as a scalar multiple of $\nabla_x \rho_h^0$ rather than J_h^ε. Additionally, their error estimates rely on ρ_h^0 being discontinuous in space and the fluxes for ρ_h^0 and J_h^0 to be alternating so that the Gauss-Radau projection can be utilized. Since both of these properties do not hold for (4.2), we include our own error estimates in h.

LEMMA 5.4. *Suppose $\rho^0 \in L_T^2(H^s(\Omega))$ and $J^0 \in L_T^2([L^2(\Omega_x)]^3)$ satisfy (2.5) for some $s \ge 2$, $\omega \in W^{r,\infty}(\Omega_x)$, and $E \in L_T^\infty(W^{r,\infty}(\Omega_x))$ for some $r \ge 1$. Define $\mu = \min\{r, s-1\}$ and recall $C_{\omega,r}$ and $\bar\theta$ from Theorem 5.1. Then*

$$\begin{aligned}|\rho_h^0 - \rho^0\|_{L_T^2(L^2(\Omega_x))} &+ \|J_h^0 - J^0\|_{L_T^2(L^2(\Omega_x))} \\ &\lesssim (h_x^{\min\{k_x+1,s\}-1} + |\theta_* - \bar\theta| + C_{\omega,r} h_x^{\min\{k_x+1,\mu\}})\|\rho^0\|_{L_T^2(H^s(\Omega_x))}.\end{aligned} \tag{5.15}$$

Proof. We decompose $e_{h,\rho} := \rho_h^0 - \rho_0$ and $e_{h,J} := J_h^0 - J^0$ as

$$e_{h,\rho} = \xi_{h,\rho} - \eta_{h,\rho} := (\rho_h^0 - \mathcal{S}_h \rho^0) - (\rho^0 - \mathcal{S}_h \rho^0), \tag{5.16a}$$

$$e_{h,J} = \xi_{h,J} - \eta_{h,J} := (J_h^0 - \mathcal{P}_h J^0) - (J^0 - \mathcal{P}_h J^0), \tag{5.16b}$$

where $\mathcal{P}_h$ is the L^2-orthogonal projection onto V_h. Because ρ^0, J^0 also solve (4.2) but with θ_* replaced by $\bar\theta$, the differences $e_{h,\rho}$ and $e_{h,J}$ satisfy

$$\begin{aligned}(\partial_t e_{h,\rho}, q_h)_{\Omega_x} - (e_{h,J}, \nabla_x q_h)_{\Omega_x} &= 0, \\ (\omega e_{h,J}, \tau_h)_{\Omega_x} + (\theta_* \odot \nabla_x e_{h,\rho}, \tau_h)_{\Omega_x} - (E e_{h,\rho}, \tau_h)_{\Omega_x} &= ((\bar\theta - \theta_*) \odot \nabla_x \rho^0, \tau_h)_{\Omega_x}.\end{aligned} \tag{5.17}$$

for all $\tau_h \in [V_{x,h}]^3$ and $q_h \in S_{x,h}^0$. Choosing $q_h = \xi_{h,\rho}$ and $\tau_h = \theta_*^{-1} \odot \xi_{h,J}$ in (5.17) gives

$$\begin{aligned}(\partial_t \xi_{h,\rho}, \xi_{h,\rho})_{\Omega_x} - (\xi_{h,J}, \nabla_x \xi_{h,\rho})_{\Omega_x} &= (\partial_t \eta_{h,\rho}, \xi_{h,\rho})_{\Omega_x} - (\eta_{h,J}, \nabla_x \xi_{h,\rho})_{\Omega_x}, \\ (\omega \xi_{h,J}, \theta_*^{-1} \odot \xi_{h,J})_{\Omega_x} + (\nabla_x \xi_{h,\rho}, \xi_{h,J})_{\Omega_x} &- (E \xi_{h,\rho}, \theta_*^{-1} \odot \xi_{h,J})_{\Omega_x} \\ &= (\omega \eta_{h,J}, \theta_*^{-1} \odot \xi_{h,J})_{\Omega_x} + (\nabla_x \eta_{h,\rho}, \xi_{h,J})_{\Omega_x} \\ &\quad - (E \eta_{h,\rho}, \theta_*^{-1} \odot \xi_{h,J})_{\Omega_x} + ((\bar\theta - \theta_*) \odot \nabla_x \rho^0, \theta_*^{-1} \odot \xi_{h,J})_{\Omega_x}.\end{aligned} \tag{5.18}$$

As in the proof of Lemma 5.2, we will absorb θ_*^{-1} into the constant with the exception $\theta_* - \bar\theta$ By properties of the L^2-projection. $(\partial_t \eta_{h,\rho}, \xi_{h,\rho})_{\Omega_x} = 0$ and $(\eta_{h,J}, \nabla_x \xi_{h,\rho})_{\Omega_x} = 0$. Adding the equations in (5.18) gives

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\|\xi_{h,\rho}\|_{L^2(\Omega_x)}^2 + \|\xi_{h,J}\|_{L^2(\Omega_x)}^2 &\lesssim (E \xi_{h,\rho}, \theta_*^{-1} \odot \xi_{h,J})_{\Omega_x} \\ &+ (\omega \eta_{h,J}, \theta_*^{-1} \odot \xi_{h,J})_{\Omega_x} + (\nabla_x \eta_{h,\rho}, \xi_{h,J})_{\Omega_x} \\ &- (E \eta_{h,\rho}, \theta_*^{-1} \odot \xi_{h,J})_{\Omega_x} + ((\bar\theta - \theta_*) \odot \nabla_x \rho^0, \theta_*^{-1} \odot \xi_{h,J})_{\Omega_x}.\end{aligned} \tag{5.19}$$

We now bound the terms on the right hand side of (5.17) using a combination of Cauchy-Schwarz, inverse (3.2), and Young's inequalities; this yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_{h,\rho}\|^2_{L^2(\Omega_x)} + \|\xi_{h,J}\|^2_{L^2(\Omega_x)} \lesssim \|\xi_{h,\rho}\|^2_{L^2(\Omega_x)} + \frac{1}{h_x^2}\|\eta_{h,\rho}\|^2_{L^2(\Omega_x)} + \|\eta_{h,J}\|^2_{L^2(\Omega_x)} + |\bar\theta - \theta_*|^2\|\nabla_x\rho^0\|^2_{L^2(\Omega_x)} \tag{5.20}$$

Since $\omega, E(t) \in W^{r,\infty}(\Omega)$, then $J^0(t) \in H^\mu(\Omega_x)$ where $\mu = \min\{r, s-1\}$. From standard finite element interpolation theory (c.f. [4]) we have the projection estimates

$$\begin{aligned}\|\eta_{h,\rho}\|_{L^2_T(L^2(\Omega_x))} &\lesssim h_x^{\min\{k_x+1,s\}}\|\rho^0\|_{L^2_T(H^s(\Omega_x))},\\ \|\eta_{h,J}\|_{L^2_T(L^2(\Omega_x))} &\lesssim C_{\omega,r}h_x^{\min\{k_x+1,\mu\}}\|\rho^0\|_{L^2_T(H^s(\Omega))}.\end{aligned} \tag{5.21}$$

Integrating (5.20) from 0 to T, applying the bounds (5.21), and invoking Grönwall's lemma we have

$$\begin{aligned}\|\xi_{h,\rho}\|^2_{L^\infty_T(L^2(\Omega_x))} + \|\xi_{h,J}\|^2_{L^2_T(L^2(\Omega_x))} &\lesssim (h_x^{2\min\{k_x+1,s\}-2} + |\theta_* - \bar\theta|^2)\|\rho^0\|^2_{L^2_T(H^s(\Omega_x))}\\ &\quad + C^2_{\omega,r}h_x^{2\min\{k_x+1,\mu\}}\|\rho^0\|^2_{L^2_T(H^s(\Omega_x))}.\end{aligned} \tag{5.22}$$

Thus by (5.22) and a triangle inequality we have (5.15). The proof is complete. ◻

We can now prove Theorem 5.1 as a consequence of Theorem 5.3 and Lemma 5.4.

Proof of Theorem 5.1. Using a triangle inequality and applying Theorem 5.3 and Lemma 5.4 immediately implies the result. The proof is complete. ◻

REMARK 5.5. *Consider an H^2 solution ρ^0 with $k_x = 1$. Then the error in (5.2) is $\mathcal{O}(\sqrt{\varepsilon/h_x} + h_x)$ which is optimal if we set $h_x = \varepsilon^{1/3}$.*

6. Conclusion. We have developed a stable discontinuous Galerkin method for the a linear Boltzmann semiconductor problem, rigorously showed that it is asymptotically preserving, and explicitly showed its limiting discrete drift-diffusion systems as $\varepsilon \to 0$. Future work includes extending the results presented in this paper to a self-consistent electric field E, non-homogeneous inflow boundary data f_-, and non-isotropic initial data.

Appendix A. Technical Lemmas.

A.1. Proof of Lemma 3.5. We will prove Lemma 3.5 by first showing an equivalent result. Given $\gamma > 0$, Define the DG discrete Laplacian energy on $V_{x,h}$ via the symmetric bilinear form

$$\begin{aligned}(q_h, z_h)_E :=& (\nabla q_h, \nabla z_h)_{\Omega_x} - \langle\{\!\{\nabla q_h\}\!\}, [\![z_h]\!]\rangle_{\mathcal{E}^I_{x,h}} - \langle[\![q_h]\!], \{\!\{\nabla z_h\}\!\}\rangle_{\mathcal{E}^I_{x,h}} - \langle\nabla q_h, z_h n_x\rangle_{\partial\Omega_x}\\ &- \langle q_h n_x, \nabla z_h\rangle_{\partial\Omega_x} + \frac{\gamma}{h_x}\langle[\![q_h]\!], [\![z_h]\!]\rangle_{\mathcal{E}^I_{x,h}} + \frac{\gamma}{h_x}\langle q_h, z_h\rangle_{\partial\Omega_x}.\end{aligned} \tag{A.1}$$

Standard DG elliptic theory shows that there exists a $\gamma_* > 0$, independent of h, such that $(\cdot,\cdot)_E$ is an inner product on $V_{x,h}$ for all $\gamma > \gamma_*$ [30]. We fix some $\gamma > \gamma_*$ and therefore $(\cdot,\cdot)_E$ induces a norm $\|\cdot\|_E$ on $V_{x,h}$. Moreover, by use of the trace inequality, Lemma 3.1, we have

$$\|q_h\|_{H^1_h(\Omega_x)} \lesssim \|q_h\|_E \lesssim \|q_h\|_{H^1_h(\Omega_x)} \tag{A.2}$$

for all $q_h \in V_{x,h}$. From (A.2), the following lemma immediately implies Lemma 3.5.

LEMMA A.1. *The L^2-projection $\mathcal{S}_h$ is stable on $V_{x,h}$ with respect to $\|\cdot\|_E$, that is, there is a constant $C > 0$, independent of h_x, such that*

$$\|\mathcal{S}_h q_h\|_E \leq C\|q_h\|_E \quad \forall q_h \in V_{x,h}. \tag{A.3}$$

Proof. Let $\{\psi_i\}_{i=1}^M \subset V_{x,h}$ be an orthonormal eigenbasis in $L^2(\Omega_x)$ with associated eigenvalues $\lambda_i > 0$, in increasing order, for the following eigenvalue problem: find $\psi \in V_{x,h}$ and $\lambda \in \mathbb{R}$ such that

$$(\psi, q_h)_E = \lambda(\psi, q_h)_{\Omega_x} \quad \forall q_h \in V_{x,h}. \tag{A.4}$$

Similarly, let $\{\varphi_j\}_{j=1}^N \subset S^0_{x,h}$ be an orthonormal eigenbasis in $L^2(\Omega_x)$ with associated eigenvalues $\mu_j > 0$, in increasing order, for the following eigenvalue problem on $S^0_{x,h}$: find $\varphi \in S^0_{x,h}$ and $\mu \in \mathbb{R}$ such that

$$(\varphi, w_h)_E = \mu(\varphi, w_h)_{\Omega_x} \quad \forall w_h \in S^0_{x,h}. \tag{A.5}$$

We also define the solution operators $T_h : V_{x,h} \to V_{x,h}$, and $T^0_h : S^0_{x,h} \to S^0_{x,h}$ by

$$(T_h q_h, z_h)_E = (q_h, z_h)_{\Omega_x} \quad \forall z_h \in V_{x,h}, \tag{A.6}$$

$$(T^0_h w_h, s_h)_E = (w_h, s_h)_{\Omega_x} \quad \forall s_h \in S^0_{x,h}. \tag{A.7}$$

We note that while written using the inner product $(\cdot\,,\cdot)_E$, (A.7) is the standard continuous Galerkin finite element method for the Poisson problem and (A.5) is its respective eigenvalue problem. Note ψ_i and φ_j are eigenvectors of T_h and T^0_h with associated eigenvalues λ_i^{-1} and μ_j^{-1} respectively. We also recall the inverse Laplacian $S : L^2(\Omega_x) \to H^2(\Omega_x) \cap H^1_0(\Omega_x)$ which is given in Assumption 2.11. Standard continuous and discontinuous Galerkin theory [4, 30] and Assumption 2.11 yield the following estimates:

$$\|T_h q_h - S q_h\|_E \lesssim h_x\|S q_h\|_{H^2(\Omega)} \lesssim h_x\|q_h\|_{L^2(\Omega_x)} \quad \forall q_h \in V_{x,h}, \tag{A.8}$$

$$\|T^0_h w_h - S w_h\|_E \lesssim h_x\|S w_h\|_{H^2(\Omega)} \lesssim h_x\|w_h\|_{L^2(\Omega_x)} \quad \forall w_h \in S^0_{x,h}. \tag{A.9}$$

Therefore by (A.8)-(A.9) we have

$$\|T_h w_h - T^0_h w_h\|_E \lesssim h_x\|w_h\|_{L^2(\Omega_x)} \quad \forall w_h \in S^0_{x,h}. \tag{A.10}$$

Given $q_h \in V_{x,h}$, let $\alpha \in \mathbb{R}^M$ be the coefficients of q_h w.r.t the basis $\{\psi_i\}$ given by $\alpha_i = (q_h, \psi_i)_{\Omega_x}$. Similarly, given $w_h \in S^0_{x,h}$, let $\xi \in \mathbb{R}^N$ be the coefficients of w_h w.r.t the basis $\{\varphi_j\}$ given by $\xi_j = (w_h, \varphi_j)_{\Omega_x}$. Due to the eigenbasis decomposition we have

$$\|q_h\|^2_{L^2(\Omega_x)} = \textstyle\sum_{i=1}^M \alpha_i^2 = |\alpha|^2, \qquad \|w_h\|^2_{L^2(\Omega_x)} = \textstyle\sum_{j=1}^N \xi_j^2 = |\xi|^2, \tag{A.11a}$$

$$\|q_h\|^2_E = \textstyle\sum_{i=1}^M \lambda_i\alpha_i^2 =: |\alpha|^2_E, \qquad \|w_h\|^2_E = \textstyle\sum_{i=1}^M \mu_j\xi_j^2 =: |\xi|^2_E, \tag{A.11b}$$

$$T_h q_h = \textstyle\sum_{i=1}^M \frac{\alpha_i}{\lambda_i}\psi_i, \qquad T^0_h w_h = \textstyle\sum_{j=1}^N \frac{\xi_j}{\mu_j}\varphi_j. \tag{A.11c}$$

Additionally, we define the operator $A^0_h : S^0_{x,h} \to S^0_{x,h}$ by $A^0_h w_h = \sum_{j=1}^N \xi_j\mu_j^{1/2}\varphi_j$. The fact that $\{\varphi_j\}$ is an orthonormal set in $L^2(\Omega_x)$ and (A.11b) yield

$$\|A^0_h w_h\|_{L^2(\Omega_x)} = \|w_h\|_E. \tag{A.12}$$

Also, φ_j is an eigenvector of A_h^0 with associated eigenvalue $\mu_j^{1/2}$.

Note that (A.3) is equivalent to uniformly bounding

$$\sup_{q_h \in V_{x,h}\setminus\{0\}} \frac{\|\mathcal{S}_h q_h\|_E}{\|q_h\|_E} = \sup_{q_h \in V_{x,h}\setminus\{0\}} \sup_{w_h \in S^0_{x,h}\setminus\{0\}} \frac{(\mathcal{S}_h q_h, w_h)_E}{\|q_h\|_E \|w_h\|_E} \tag{A.13}$$

in h_x; thus we seek to bound $(\mathcal{S}_h q_h, w_h)_E$. Using (A.5), (A.11b), and (A.12) we have

$$\begin{aligned}(\mathcal{S}_h q_h, w_h)_E &= \sum_{ij} \alpha_i (\mathcal{S}_h \psi_i, \varphi_j)_E \xi_j = \sum_{ij} \alpha_i \lambda_i^{1/2} (\lambda_i^{-1/2} \psi_i, A_h^0 \varphi_j)_{\Omega_x} \xi_j \mu_j^{1/2} \\ &\le |\alpha|_E \|\overline{C}\|_2 |\xi|_E = \|\overline{C}\|_2 \|q_h\|_E \|w_h\|_E.\end{aligned} \tag{A.14}$$

Here $\overline{C} \in \mathbb{R}^{M\times N}$ with $\overline{C}_{ij} = (\lambda_i^{-1/2}\psi_i, A_h^0\varphi_j)_{\Omega_x}$. To bound $\|\overline{C}\|_2$, we use the decomposition of w_h along with (A.11c) and (A.6) to compute show $\xi^T \bar{C}^T \bar{C} \xi = (T_h A_h^0 w_h, T_h A_h^0 w_h)_E$. Hence.

$$\|\overline{C}\|_2 = \max_{w_h \in S^0_{x,h}\setminus\{0\}} \frac{\|T_h A_h^0 w_h\|_E^2}{\|w_h\|^2_{L^2(\Omega_x)}}. \tag{A.15}$$

Therefore the desired estimate (A.3) holds provided we can show

$$\|T_h A_h^0 w_h\|_E \lesssim \|w_h\|_{L^2(\Omega_x)} \quad \forall w_h \in S^0_{x,h}. \tag{A.16}$$

To show (A.16), we will add and subtract $T_h^0 A_h^0 w_h$ where T_h^0 is given in (A.7) and apply the triangle inequality to obtain

$$\|T_h A_h^0 w_h\|_E \le \|(T_h - T_h^0) A_h^0 w_h\|_E + \|T_h^0 A_h^0 w_h\|_E. \tag{A.17}$$

To bound $\|(T_h - T_h^0)A_h^0 w_h\|_E$, we use (A.10), (A.12), (A.2), and (3.2):

$$\|(T_h - T_h^0) A_h^0 w_h\|_E \lesssim h_x \|A_h^0 w_h\|_{L^2(\Omega_x)} = h_x \|w_h\|_E \lesssim h_x \|w_h\|_{H^1_h(\Omega)} \lesssim \|w_h\|_{L^2(\Omega_x)}.$$

Direct computation of $T_h^0 A_h^0 w_h$ using the eigenvalue decomposition of w_h gives us $T_h^0 A_h^0 w_h = \sum_{j=1}^N \xi_j \mu_j^{-1/2} \varphi_j$. Using this calculation and (A.11a) we can show $\|T_h^0 A_h^0 w_h\|_E^2 = \|w_h\|^2_{L^2(\Omega_x)}$. Therefore (A.16) holds. The proof is complete. ∎

A.2. Proof of Lemma 3.8. Lemma 3.8 is a result of the following lemma:

LEMMA A.2. *There exists $\gamma_* > 0$ independent of ε and h_x such that*

$$\inf_{\substack{\xi \in \mathbb{R}^d \\ \|\xi\|_2 = 1}} \left(v_h M_h^{\frac{1}{2}}, \xi M_h^{\frac{1}{2}} \right)_{\{v : v_h(v)\cdot \xi > 0\}} > \gamma_*, \tag{A.18}$$

$$\inf_{\substack{\xi \in \mathbb{R}^d \\ \|\xi\|_2 = 1}} \left(\frac{|v_h \cdot \xi|}{2} M_h^{\frac{1}{2}}, M_h^{\frac{1}{2}} \right)_{\Omega_v} > \gamma_*. \tag{A.19}$$

Proof. We first focus on (A.18). We show the function $\gamma : \mathbb{R}^d \to \mathbb{R}$ defined by $\gamma(\xi) = \left(v_h M_h^{\frac{1}{2}}, \xi M_h^{\frac{1}{2}} \right)_{\{v : v_h(v)\cdot \xi > 0\}}$ is lower semi-continuous. Let $\xi_n \to \xi$. Define $\gamma_n : \Omega_v \to \mathbb{R}$ by $\gamma_n(v) = v_h \cdot \xi_n M_h^{\frac{1}{2}} M_h^{\frac{1}{2}} \chi_{\{v_h(v)\cdot\xi_n > 0\}}$ where χ_A is the indicator function for the set A. By Fatou's Lemma we have

$$\liminf_{n\to\infty} \gamma(\xi_n) = \liminf_{n\to\infty} \int_{\Omega_v} \gamma_n(v)\, \mathrm{d}v \ge \int_{\Omega_v} \liminf_{n\to\infty} \gamma_n(v)\, \mathrm{d}v. \tag{A.20}$$

We claim

$$\lim_{n\to\infty} \gamma_n(v) = v_h \cdot \xi M_h^{\frac{1}{2}} M_h^{\frac{1}{2}} \chi_{\{v_h(v)\cdot\xi>0\}} \tag{A.21}$$

for all a.e. $v \in \Omega_v$. Let $v \in \Omega_v$ with $v_h(v)\cdot\xi < 0$. Then eventually we have $v_h(v)\cdot\xi_k < 0$ for all k sufficiently large. Thus the indicator function evaluates to zero and $\gamma_k(v) = 0$; thus (A.21) holds. Let $v \in \Omega_v$ with $v_h(v)\cdot\xi < 0$. Then similarly $v_h(v)\cdot\xi_k > 0$ for all k sufficiently large. Hence the indication function evaluates to 1 and we can pass the limit to show (A.21) holds. Since the set $\{v : v_h(v)\cdot\xi = 0\}$ is a set of measure zero, (A.21) holds for all a.e. $v \in \Omega_v$. Using (A.21) we continue (A.20) to obtain

$$\liminf_{n\to\infty} \gamma(\xi_n) \geq \int_{\Omega_v} \liminf_{n\to\infty} \gamma_n(v)\,\mathrm{d}v = \int_{\Omega_v} v_h \cdot \xi M_h^{\frac{1}{2}} M_h^{\frac{1}{2}} \chi_{\{v_h(v)\cdot\xi>0\}}\,\mathrm{d}v = \gamma(\xi).$$

Therefore γ is lower semi-continuous. Since $\gamma > 0$ on the compact unit sphere, it obtains a positive minimum. Thus the first equality of (A.18) holds. For (A.19), we note that the function $\xi \to \big(\frac{|v_h\cdot\xi|}{2} M_h^{\frac{1}{2}}, M_h^{\frac{1}{2}}\big)_{\Omega_v}$ is Lipschitz continuous. Since it is also positive on the compact unit sphere, it obtains a positive minimum. The proof is complete. ∎

Appendix B. Maxwellian Approximation.

Lemma B.1 gives precise bounds for the discrete 1D root Maxwellian constructed in Remark 2.4.

Lemma B.1. *Let $L > 0$ with $L \geq \sqrt{\theta}$, and suppose $\Omega_v = [-L, L]$. Furthermore, assume $h_v^2 \leq \frac{4}{\sqrt{3}}\theta$. Let $Q_{h_v} : C^0(\overline{\Omega_v}) \to S_{v,h_v}$ with $k_v = 1$ be the piecewise linear nodal Lagrange interpolant. Define $\widetilde{Q}_{h_v} := \frac{Q_{h_v} u}{\|Q_{h_v} u\|_{L^2(\Omega_v)}}$. For $i = 1, \ldots, 3$, let $M_{h,i}^{\frac{1}{2}} = \widetilde{Q}_h(M_i^{\frac{1}{2}})$ where $M_i^{\frac{1}{2}}(v_i)$ is defined in (2.13). Then $M_{h,i}^{\frac{1}{2}}$ is positive, continuous, and satisfies Assumption 2.1.a, Assumption 2.1.b, and Assumption 2.1.c. Moreover, we have the following approximation results:*

$$\|M_i^{\frac{1}{2}} - \widetilde{Q}_{h_v} M_i^{\frac{1}{2}}\|_{L^2(\Omega_v)} \leq \frac{5}{2}\left(1 - \operatorname{erf}(\tfrac{L}{\sqrt{2\theta}})^{1/2}\right) + 5h_v^2 \frac{\sqrt{3}}{8\theta}, \tag{B.1}$$

$$\|\partial_v(M_i^{\frac{1}{2}} - \widetilde{Q}_{h_v} M_i^{\frac{1}{2}})\|_{L^2(\Omega_v)} \leq \frac{5}{2}\left(1 - \operatorname{erf}(\tfrac{L}{\sqrt{2\theta}})^{1/2}\right) + \frac{5}{2}h_v^2 \frac{\sqrt{3}}{16\theta^{3/2}} + \frac{5}{2}\frac{\sqrt{3}}{\sqrt{2}}\frac{1}{4\theta}h_v. \tag{B.2}$$

Proof. For ease of notation set $\mathcal{M}(v) = M_i^{\frac{1}{2}}(v)$. Direction calculation and $L \geq \sqrt{\theta}$ yields

$$\tfrac{16}{25} \leq \operatorname{erf}(\tfrac{1}{\sqrt{2}}) \leq \|\mathcal{M}\|^2_{L^2(\Omega_v)} = \operatorname{erf}(\tfrac{L}{\sqrt{2\theta}}) \leq 1. \tag{B.3a}$$

$$\|\partial_v \mathcal{M}\|^2_{L^2(\Omega_v)} = \tfrac{1}{4\theta}\operatorname{erf}(\tfrac{L}{\sqrt{2\theta}}) - \tfrac{L}{2\sqrt{2\pi\theta^3}}\exp\left(\tfrac{-L^2}{2\theta}\right) \leq \tfrac{1}{4\theta}\operatorname{erf}(\tfrac{L}{\sqrt{2\theta}}). \tag{B.3b}$$

$$\|\partial_v^2 \mathcal{M}\|^2_{L^2(\Omega_v)} = \tfrac{3}{16\theta^2}\operatorname{erf}(\tfrac{L}{\sqrt{2\theta}}) - \tfrac{L(\theta - L^2)}{8\sqrt{2\pi\theta^7}}\exp\left(\tfrac{-L^2}{2\theta}\right) \leq \tfrac{3}{16\theta^2}\operatorname{erf}(\tfrac{L}{\sqrt{2\theta}}) \tag{B.3c}$$

We can give precise bounds on the interpolation error $\|\mathcal{M} - \widetilde{Q}_{h_v}\mathcal{M}\|$ from the proof in [4, Theorem (0.4.5)], (B.3c), and (B.3a):

$$\begin{aligned}\|\mathcal{M} - Q_{h_v}\mathcal{M}\|_{L^2(\Omega_v)} &+ \tfrac{1}{\sqrt{2}}h_v\|\partial_v(\mathcal{M} - Q_{h_v}\mathcal{M})\|_{L^2(\Omega_x)} \\ &\leq \tfrac{1}{2}h_v^2\|\partial_v^2\mathcal{M}\|_{L^2(\Omega_v)} \leq \tfrac{\sqrt{3}}{8\theta}h_v^2\|\mathcal{M}\|_{L^2(\Omega_v)} \leq \tfrac{\sqrt{3}}{8\theta}h_v^2.\end{aligned} \tag{B.4}$$

Using the reverse triangle inequality, (B.4), and (B.3a), we obtain

$$\|Q_h\mathcal{M}\|_{L^2(\Omega_v)} \geq (1 - \tfrac{\sqrt{3}h_v^2}{8\theta})\|\mathcal{M}\|_{L^2(\Omega_v)} \geq \tfrac{4}{5}(1 - h_v^2\tfrac{\sqrt{3}}{8\theta}) \geq \tfrac{2}{5} \tag{B.5}$$

where the last inequality follows from the assumed h_v restriction.

We now show (B.1). Define $\alpha_{\mathcal{M}} = \|Q_{h_v}\mathcal{M}\|_{L^2(\Omega_v)}$. Using in definition of $\widetilde{Q}_{h_v}$ we obtain

$$\|\mathcal{M} - \widetilde{Q}_{h_v\mathcal{M}}\|_{L^2(\Omega_v)} = \frac{1}{\alpha_{\mathcal{M}}}\|\alpha_{\mathcal{M}}\mathcal{M} - Q_{h_v}\mathcal{M}\|_{L^2(\Omega_v)}. \tag{B.6}$$

Adding and subtracting key quantities and several uses of the standard and reverse triangle inequalities both yield

$$\begin{aligned} \|\alpha_{\mathcal{M}}\mathcal{M} - Q_{h_v}\mathcal{M}\|_{L^2(\Omega_v)} &\leq |1-\alpha_{\mathcal{M}}|\|\mathcal{M}\|_{L^2(\Omega_v)} + \|\mathcal{M} - Q_{h_v}\mathcal{M}\|_{L^2(\Omega_v)} \\ &\leq |1 - \|\mathcal{M}\|_{L^2(\Omega_v)}|\|\mathcal{M}\|_{L^2(\Omega_v)} + \|\mathcal{M} - Q_{h_v}\mathcal{M}\|_{L^2(\Omega_v)}\|\mathcal{M}\|_{L^2(\Omega_v)} \\ &\quad + \|\mathcal{M} - Q_{h_v}\mathcal{M}\|_{L^2(\Omega_v)}. \end{aligned} \tag{B.7}$$

The terms on the right hand side of (B.7) can be bounded using (B.3) and (B.4). These estimates along with (B.6) and (B.5) yield (B.1). For (B.2), a similar H^1 estimate to (B.7) can be formed, namely:

$$\begin{aligned} &\|\partial_v(\alpha_{\mathcal{M}}\mathcal{M} - Q_{h_v}\mathcal{M})\|_{L^2(\Omega_v)} \leq |1 - \|\mathcal{M}\|_{L^2(\Omega_v)}|\|\partial_v\mathcal{M}\|_{L^2(\Omega_v)} \\ &\qquad + \|\mathcal{M} - Q_{h_v}\mathcal{M}\|_{L^2(\Omega_v)}\|\partial_v\mathcal{M}\|_{L^2(\Omega_v)} + \|\partial_v(\mathcal{M} - Q_{h_v}\mathcal{M})\|_{L^2(\Omega_v)}. \end{aligned} \tag{B.8}$$

Estimate (B.8) along with the estimates above yield (B.2). The proof is complete. □

REFERENCES

[1] N. B. Abdallah and M. L. Tayeb. Diffusion approximation for the one dimensional boltzmann-poisson system. Discrete & Continuous Dynamical Systems-B, 4(4):1129, 2004.

[2] M. L. Adams. Discontinuous finite element transport solutions in thick diffusive problems. Nuclear science and engineering, 137(3):298–333, 2001.

[3] A. Bensoussan, P.-L. Lions, and G. C. Papanicolaou. Boundary layers and homogenizatlon of transport processes. Publications of the Research Institute for Mathematical Sciences, 15(1):53–157, 1979.

[4] S. C. Brenner, L. R. Scott, and L. R. Scott. The mathematical theory of finite element methods, volume 3. Springer, 2008.

[5] L. Chen and H. Bagci. Steady-state simulation of semiconductor devices using discontinuous galerkin methods. IEEE Access, 8:16203–16215, 2020.

[6] Y. Cheng, I. M. Gamba, A. Majorana, and C.-W. Shu. Discontinuous galerkin solver for the semiconductor boltzmann equation. In Simulation of Semiconductor Processes and Devices 2007, pages 257–260. Springer, 2007.

[7] Y. Cheng, I. M. Gamba, A. Majorana, and C.-W. Shu. Discontinuous galerkin solver for boltzmann-poisson transients. Journal of Computational Electronics, 7(3):119–123, 2008.

[8] Y. Cheng, I. M. Gamba, A. Majorana, and C.-W. Shu. A discontinuous galerkin solver for boltzmann–poisson systems in nano devices. Computer Methods in Applied Mechanics and Engineering, 198(37-40):3130–3150, 2009.

[9] Y. Cheng, I. M. Gamba, A. Majorana, and C.-W. Shu. A brief survey of the discontinuous galerkin method for the boltzmann-poisson equations. SeMA Journal, 54(1):47–64, 2011.

[10] V. P. DeCaria, C. D. Hauck, and M. T. P. Laiu. Analysis of a new implicit solver for a semiconductor model. SIAM Journal on Scientific Computing, 43(3):B733–B758, 2021.

[11] L. Evans. Partial Differential Equations. Graduate studies in mathematics. American Mathematical Society, 2010.

[12] P. Grisvard. Elliptic problems in nonsmooth domains. SIAM, 2011.

[13] J.-L. Guermond and G. Kanschat. Asymptotic analysis of upwind discontinuous galerkin approximation of the radiative transport equation in the diffusive limit. SIAM Journal on Numerical Analysis, 48(1):53–78, 2010.

[14] G. Habetler and B. Matkowsky. Uniform asymptotic expansions in transport theory with small mean free paths, and the diffusion approximation. Journal of Mathematical Physics, 16(4):846–854, 1975.

[15] S. Jin. Efficient asymptotic-preserving (ap) schemes for some multiscale kinetic equations. SIAM Journal on Scientific Computing, 21(2):441–454, 1999.

[16] S. Jin. Asymptotic preserving (ap) schemes for multiscale kinetic and hyperbolic equations: a review. Lecture notes for summer school on methods and models of kinetic theory (M&MKT), Porto Ercole (Grosseto, Italy), pages 177–216, 2010.

[17] S. Jin and L. Pareschi. Discretization of the multiscale semiconductor boltzmann equation by diffusive relaxation schemes. Journal of Computational Physics, 161(1):312–330, 2000.

[18] A. Jüngel. Transport equations for semiconductors, volume 773. Springer, 2009.

[19] O. A. Karakashian and F. Pascal. A posteriori error estimates for a discontinuous galerkin approximation of second-order elliptic problems. SIAM Journal on Numerical Analysis, 41(6):2374–2399, 2003.

[20] A. Klar. Asymptotic-induced domain decomposition methods for kinetic and drift diffusion semiconductor equations. SIAM Journal on Scientific Computing, 19(6):2032–2050, 1998.

[21] A. Klar. A numerical method for kinetic semiconductor equations in the drift-diffusion limit. SIAM Journal on Scientific Computing, 20(5):1696–1712, 1999.

[22] M. P. Laiu, Z. Chen, and C. D. Hauck. A fast implicit solver for semiconductor models in one space dimension. Journal of Computational Physics, 417:109567, 2020.

[23] E. W. Larsen and J. B. Keller. Asymptotic solution of neutron transport problems for small mean free paths. Journal of Mathematical Physics, 15(1):75–81, 1974.

[24] Y. Liu and C.-W. Shu. Analysis of the local discontinuous galerkin method for the drift-diffusion model of semiconductor devices. Science China Mathematics, 59(1):115–140, 2016.

[25] A. Majorana, G. Nastasi, and V. Romano. Simulation of bipolar charge transport in graphene by using a discontinuous galerkin method. Commun. Comput. Phys., 26(1):114–134, 2019.

[26] P. A. Markowich, C. A. Ringhofer, and C. Schmeiser. Semiconductor equations. Springer Science & Business Media, 2012.

[27] N. Masmoudi and M. L. Tayeb. Diffusion limit of a semiconductor boltzmann–poisson system. SIAM journal on mathematical analysis, 38(6):1788–1807, 2007.

[28] F. Poupaud. Diffusion approximation of the linear semiconductor boltzmann equation: analysis of boundary layers. Asymptotic analysis, 4(4):293–317, 1991.

[29] F. Poupaud. Mathematical theory of kinetic equations for transport modelling in semiconductors. In Advances in Kinetic Theory and Computing: Selected Papers, pages 141–168. World Scientific, 1994.

[30] B. Rivière. Discontinuous Galerkin methods for solving elliptic and parabolic equations: theory and implementation. SIAM, 2008.

[31] C. Schmeiser and A. Zwirchmayr. Convergence of moment methods for linear kinetic equations. SIAM journal on numerical analysis, 36(1):74–88, 1998.

[32] Q. Sheng and C. Hauck. Uniform convergence of an upwind discontinuous galerkin method for solving scaled discrete-ordinate radiative transfer equations with isotropic scattering. Mathematics of Computation, 90(332):2645–2669, 2021.