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Topical Review

AdS$_3$ Einstein gravity and boundary description: pedagogical review

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Abstract

We review the various aspects of the 3D Einstein gravity theory with a negative cosmological constant and its boundary description. We also explore its connections to conformal field theories (CFTs), modular symmetry, and holography. It is worth noting that this particular theory is topological in nature, which means that all the physical degrees of freedom are located on the boundary. Additionally, we can derive the boundary description on a torus, which takes the form of a 2D Schwarzian theory. This observation suggests that the relevant degrees of freedom for the theory can be described using this 2D theory. Because of the renormalizability of the 3D gravity theory, one can probe the quantum regime. This suggests that it is possible to investigate quantum phenomena. Unlike the conventional CFTs, when considering the

AdS_3 background, the boundary theory loses modular symmetry. This represents a departure from the usual behavior of CFT and is quite intriguing. The Weyl transformation induces anomaly in CFTs, and we indicate that applying this transformation to the 2D Schwarzian theory leads to similar results. Summing over all geometries with the asymptotic AdS_3 boundary condition is equivalent to summing over a modular group. The partition function is one-loop exact and therefore an analytical expression from the summation. This theory holds potential applications in Quantum Information and is a recurring theme in the study of holography, where gravitational theories are connected with CFTs.

Keywords: gauge formulation, 2D Schwarzian theory, AdS_3, modular symmetry, and holographic entanglement entropy

1. Introduction

Black hole entropy [1] is a concept that assigns a measure of entropy to a black hole. This idea emerged from efforts to satisfy the laws of thermodynamics [2] with the physics of black holes. The black hole's entropy depends on its mass, electric charge, and angular momentum. These parameters are crucial in defining a black hole's properties. The entropy of a black hole is related to the area of its event horizon, indicating that entropy should be a monotonic function of the event horizon [3]. Black hole entropy has profound implications in theoretical physics. One of the notable consequences is the *Holographic Principle* [4, 5]. The Holographic Principle is a concept in theoretical physics that suggests that the information and degrees of freedom of Quantum Gravity can be encoded or represented on a lower-dimensional boundary surrounding it [4, 5].

The realization of the holographic principle is motivated by String Theory, which describes the dynamics of one-dimensional objects (strings). The particular example is a duality in theoretical physics that connects two seemingly different theories: the gravitational theory in a $(d+1)$-dimensional Anti-de Sitter (AdS_{d+1}) space and a d-dimensional conformal field theory (CFT_d). This duality establishes a connection between gravity and field theories called *AdS/CFT* correspondence. String Theory is a candidate for a theory of Quantum Gravity. It has the potential to naturally include the graviton, which is the hypothetical quantum particle related to gravity. Additionally, it is considered ultraviolet (UV) complete, making it promising evidence for the AdS/CFT correspondence conjecture. Einstein gravity in four dimensions is not renormalizable, which is why the AdS/CFT correspondence is valuable. However, the issue of renormalizability can be overcome in the context of quantum field theory (QFT), which is another reason why the AdS/CFT correspondence is important. As a result, researchers can use the AdS/CFT correspondence to examine gravity within the framework of QFT, providing a distinct perspective on quantum gravity. Recently, Quantum Information goes into the AdS/CFT correspondence beginning from the Ryu–Takayanagi (RT) conjecture [6, 7], which connects the entanglement entropy (EE) of CFT_d to the minimum area of surfaces (codimension two surfaces on a given time slice) in AdS_{d+1} space [6, 7]. This connection suggests that *Quantum Entanglement* plays a role in generating spacetime, supporting the idea of *Emergent Spacetime*. For a spherical entangling surface, a conformal transformation maps EE to thermal entropy [8]. Consistent results were obtained between the holographic method (minimum surface) and field theory. The *replica trick* is a mathematical technique used to calculate EE in field theory [9]. When applied to bulk gravity theory, the result also gives the co-dimensional two surface [10–12]. Introducing the Hayward term [13] to the gravity theory is an equivalent approach to avoiding the use of the replica trick at the classical level [14]. Rényi entropy also has a similar holographic formulation through the cosmic brane [15] and the Hayward term [16]. Hence, the minimum surface approach provides various pieces of evidence at the classical level. The concept of excitation EE is introduced to illustrate the difference between the EE of general excitation and their arbitrary descendants [17]. This observable is a finite quantity independent of the cutoff and can be computed by the holographic formula [17]. The holographic formula is a tool that simplifies the study of many-body physics by reducing the complexity of computations related to EE. Because we hope to have the gauge invariant measure, the bipartition becomes subtle if a holographic system has the gauge symmetry. However, from the operator algebra point of view, we can define gauge invariant measures with consistent results in the p-form gauge theory [18, 19]. In $\mathrm{AdS}_3/\mathrm{CFT}_2$ case, the RT conjecture can have the simplified interpretation of the relationship between the length of the geodesic line and the EE by the differential entropy [20, 21] or kinematic space [22, 23].

Because it is hard to have the complete Lagrangian description for the bulk gravitational theory and CFT simultaneously, we only have a few examples to establish concrete AdS/CFT correspondence by comparing results between the bulk and boundary theories. One of the most well-known examples of the AdS/CFT correspondence involves String Theory in $\mathrm{AdS}_5 \times S^5$, where S^5 is a five-dimensional sphere manifold, is dual to $\mathcal{N} = 4$ Super Yang–Mills (SYM) theory in four dimensions. In String Theory, there is a clear correspondence between perturbation computations and integrability techniques in SYM theory. However, we do not have a direct way to derive SYM theory from String Theory. In the realm of 3D Einstein gravity, there are no propagating gravitational degrees of freedom in the bulk. Instead, the physical degrees of freedom reside on the boundary. This allows for direct derivation of the boundary theory. Moreover, the system is simple enough to compute the stress tensor n-point correlators with ease [24]. The most general solutions to the AdS_3 Einstein gravity with Brown–Henneaux boundary conditions can also be classified [25–27]. The doubled Chern–Simons theory with SL(2) gauge groups reformulates the 3D Einstein gravity theory and resolves the issue of renormalizability with a dimensionless coupling constant [28]. Because the functional measure of the metric formulation is only over the non-singular vielbeins, the metric formulation differs from the gauge formulation non-perturbatively [29]. However, the Chern–Simons description [30] provides a simple way to treat the quantum correction of 3D Einstein gravity. When considering a negative cosmological constant in 3D Einstein gravity, the conformal symmetry corresponds to the bulk gauge symmetry (or SO(2, 2) gauge group). According to the theory, the boundary is expected to be a CFT_2 [31]. However, research has found that pure 3D Einstein gravity does not possess a physical CFT description [32, 33]. The summation over the asymptotic AdS_3 boundary condition results in a negative density of states [32, 33]. This negative density is associated with a non-unitary CFT. The error approximating the discrete spectrum to a continuous spectrum that is not sensitive to the spin [34, 35] and the generalization of the usual Cardy formula [36] help the CFT computation to support the same conclusion [37, 38]. The boundary theory of AdS_3 Einstein gravity has been a subject of study. It was initially proposed as Liouville theory [39, 40], but this proposal was not acceptable due to issues with the non-normalizable vacuum. Recent work has shown that the correct boundary on the torus manifold (asymptotic boundary for the Euclidean case) has SL(2) gauge symmetry with the Schwarzian form [41], leading to what is called *2D Schwarzian theory*. This theory is dual to chiral scalar fields [42–44], and the classical limit is the Liouville theory [45].

The partition function of 2D Schwarzian theory on the torus manifold is *one-loop exact* [41, 46]. This suggests that an analytical solution for the partition function can be obtained without considering higher-loop contributions. It can be generalized to an n-sheet covering and related to EE with the bulk dual, known as the Wilson line [47–49]. Even though 2D Schwarzian theory lacks a covariant Lagrangian description, the *Weyl transformation* acting on the torus case generates the *Liouville theory* [50]. This allows for simplifications using Liouville theory or Weyl anomaly, similar to CFT_2 [50, 51]. The 2D Schwarzian theory is a simple quantum system for studying the bulk/boundary correspondence. When considering the AdS/CFT correspondence on a wormhole manifold, introducing an ensemble to the boundary theory is a common approach [52]. However, this ensemble averaging does not provide a smooth description of N to the Hamiltonian H_N describing a black hole state [53], which implies differences between the boundary dual of Einstein gravity and String Theory in this context [53, 54].

1.1. Outline

The outline of this review is as follows. This section introduces the gauge formulation of 3D Einstein Gravity in section 2. It covers the basic concepts and equations related to this topic.

We explore the boundary description of AdS_3 Einstein Gravity in section 3. It includes discussions on the 2D Schwarizna theory, the equivalence between chiral scalar fields and the boundary theory, and the examination of the Weyl transformation. Additionally, we will calculate the partition function by summing over all manifolds with the asymptotic AdS_3 boundary condition.

We introduce the concept of holographic EE in section 4. We plan to demonstrate the RT conjecture using the AdS_3 geometry or an exact solution. Furthermore, we will apply the gauge formulation to find the dual of EE using the bulk Wilson line. We discuss potential future research directions or areas of study related to the topics in section 5.

2. 3D Einstein gravity

We first introduce the second-order and first-order formulations of 3D Einstein Gravity and then rewrite it to get the gauge formulation, SL(2) doubled Chern–Simons [29]. Because the coupling constant in the gauge formulation is dimensionless, it is easier to treat the quantum fluctuation. The path integration is over the singular and non-singular vielbeins. We can find the difference at the non-perturbative level.

2.1. Second-order formulation

The Lagrangian description of metric formulation (or second-order formulation) for 3D Einstein Gravity with the Lorentz signature $(-1,1,1)$ is

$$S_{3DS} = \frac{1}{16\pi G_3}\int \mathrm{d}^3x\,\sqrt{|\det - g_{\mu\nu}|}\,(R-2\Lambda)\,, \tag{1}$$

where G_3 is the 3D gravitational constant, and Λ is the cosmological constant. The $g_{\mu\nu}$ represents the metric that is used to define the spacetime interval. This interval can be determined by the formula

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{2}$$

We denote the bulk spacetime indices by $\mu, \nu, \ldots$. The Ricci curvature tensor $R_{\mu\nu}$ is defined by

$$R_{\mu\nu} \equiv \partial_\delta \Gamma^\delta_{\nu\mu} - \partial_\nu \Gamma^\delta_{\delta\mu} + \Gamma^\delta_{\delta\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\delta_{\nu\lambda}\Gamma^\lambda_{\delta\mu}, \tag{3}$$

where Christoffel symbol $\Gamma^\mu_{\nu\delta}$ is

$$\Gamma^\mu_{\nu\delta} \equiv \frac{1}{2} g^{\mu\lambda}\left(\partial_\delta g_{\lambda\nu} + \partial_\nu g_{\lambda\delta} - \partial_\lambda g_{\nu\delta}\right), \tag{4}$$

The Ricci scalar is a mathematical object defined using the metric tensor and the Ricci tensor. It is given by the expression

$$R \equiv g^{\mu\nu} R_{\mu\nu}, \tag{5}$$

where $g^{\mu\nu}$ is the inverse of the metric tensor and $R_{\mu\nu}$ is the Ricci tensor.

2.2. First-order formulation

The action (1) can be rewritten as the first-order formulation from the vielbein

$$e_\mu \equiv e_\mu{}^a J_a \tag{6}$$

and spin connection

$$\omega_\mu \equiv \omega_\mu{}^a J_a \tag{7}$$

$$S_{3DF} = \frac{1}{16\pi G_3}\int \mathrm{d}^3x\, e\left(R_{\mu\nu}{}^{ab} e^\mu{}_a e^\nu{}_b - 2\Lambda\right), \tag{8}$$

where

$$R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} + \left(\omega_\mu{}^{ac}\omega_\nu{}^{bd} - \omega_\nu{}^{ac}\omega_\mu{}^{bd}\right)\eta_{cd};$$
$$e_d \equiv \sqrt{\det\left(-e_{\mu,a} e^a_\nu\right)}. \tag{9}$$

The $\omega_\mu{}^{bc}$ is defined by the spin connection

$$\omega_\mu{}^a = \frac{1}{2}\epsilon^a{}_{bc}\omega_\mu{}^{bc}. \tag{10}$$

The J_a are SL(2) generators:

$$J_0 \equiv \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix};\ J_1 \equiv \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix};\ J_2 \equiv \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \tag{11}$$

which satisfies the commutation and the trace relations:

$$[J^a, J^b] = \epsilon^{abc};\ \mathrm{Tr}\left(J^a J^b\right) = \frac{1}{2}\eta^{ab}. \tag{12}$$

The Lie algebra indices are raised or lowered by

$$\eta \equiv \mathrm{diag}\left(-1, 1, 1\right). \tag{13}$$

The metric now can be rewritten in terms of vielbein as

$$g_{\mu\nu} = 2\mathrm{Tr}\left(e_\mu e_\nu\right). \tag{14}$$

We will reformulate 3D Einstein Gravity via the gauge fields (gauge formulation) [28].

2.3. Gauge formulation

We reformulate 3D Einstein Gravity with a negative cosmological constant from the Chern–Simons theory [28]

$$S_{3DG} = S_{\mathrm{CS}}(A) - S_{\mathrm{CS}}(\bar{A}), \tag{15}$$

where

$$S_{\mathrm{CS}} = \frac{k}{4\pi}\int \mathrm{d}^3x\,\mathrm{Tr}\left(\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \frac{2}{3}\epsilon^{\mu\nu\rho}A_\mu A_\nu A_\rho\right), \tag{16}$$

The constant k is defined as [28]

$$k \equiv \frac{l}{4G_3}, \tag{17}$$

where

$$\frac{1}{l^2} \equiv -\Lambda. \tag{18}$$

The gauge fields are defined by the vielbein and spin connection [28]:

$$A_\mu = J_a\left(\omega_\mu{}^a + \frac{1}{l}e_\mu{}^a\right), \qquad \bar{A}_\mu = J_a\left(\omega_\mu{}^a - \frac{1}{l}e_\mu{}^a\right). \tag{19}$$

We substitute the vielbein and spin connection to the gauge fields, and then we obtain that:

$$\begin{aligned}
\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho &= \epsilon^{\mu\nu\rho}J_aJ_b\left(\omega_\mu{}^a\partial_\nu\omega_\rho{}^b + \frac{1}{l}\omega_\mu{}^a\partial_\nu e_\rho{}^b + \frac{1}{l}e_\mu{}^a\partial_\nu\omega_\rho{}^b + \frac{1}{l^2}e_\mu{}^a\partial_\nu e_\rho{}^b\right);\\
\epsilon^{\mu\nu\rho}\bar{A}_\mu\partial_\nu\bar{A}_\rho &= \epsilon^{\mu\nu\rho}J_aJ_b\left(\omega_\mu{}^a\partial_\nu\omega_\rho{}^b - \frac{1}{l}\omega_\mu{}^a\partial_\nu e_\rho{}^b - \frac{1}{l}e_\mu{}^a\partial_\nu\omega_\rho{}^b + \frac{1}{l^2}e_\mu{}^a\partial_\nu e_\rho{}^b\right);\\
\frac{2}{3}\epsilon^{\mu\nu\rho}A_\mu A_\nu A_\rho &= \frac{2}{3}\epsilon^{\mu\nu\rho}J_aJ_bJ_c\left(\omega_\mu{}^a\omega_\nu{}^b\omega_\rho{}^c + \frac{1}{l}\omega_\mu{}^a\omega_\nu{}^b e_\rho{}^c + \frac{1}{l}\omega_\mu{}^a e_\nu{}^b\omega_\rho{}^c + \frac{1}{l}e_\mu{}^a\omega_\nu{}^b\omega_\rho{}^c\right.\\
&\quad\left. + \frac{1}{l^2}\omega_\mu{}^a e_\nu{}^b e_\rho{}^c + \frac{1}{l^2}e_\mu{}^a\omega_\nu{}^b e_\rho{}^c + \frac{1}{l^3}e_\mu{}^a e_\nu{}^b e_\rho{}^c\right);\\
\frac{2}{3}\epsilon^{\mu\nu\rho}\bar{A}_\mu\bar{A}_\nu\bar{A}_\rho &= \frac{2}{3}\epsilon^{\mu\nu\rho}J_aJ_bJ_c\left(\omega_\mu{}^a\omega_\nu{}^b\omega_\rho{}^c - \frac{1}{l}\omega_\mu{}^a\omega_\nu{}^b e_\rho{}^c - \frac{1}{l}\omega_\mu{}^a e_\nu{}^b\omega_\rho{}^c - \frac{1}{l}e_\mu{}^a\omega_\nu{}^b\omega_\rho{}^c\right.\\
&\quad\left. + \frac{1}{l^2}\omega_\mu{}^a e_\nu{}^b e_\rho{}^c + \frac{1}{l^2}e_\mu{}^a\omega_\nu{}^b e_\rho{}^c - \frac{1}{l^3}e_\mu{}^a e_\nu{}^b e_\rho{}^c\right).
\end{aligned} \tag{20}$$

From equation (20), we obtain the action:

$$S_{3DG}=S_{\mathrm{CS}}\left(A\right)-S_{\mathrm{CS}}\left(\bar{A}\right)=\frac{k}{4\pi}\int\mathrm{d}^3x\ \epsilon^{\mu\nu\rho}\mathrm{Tr}\bigg[J_aJ_b\left(\frac{1}{l}\omega_\mu{}^a\partial_\nu e_\rho{}^b+\frac{1}{l}e_\mu{}^a\partial_\nu\omega_\rho{}^b\right)$$
$$+J_aJ_b\left(\frac{1}{l}\omega_\mu{}^a\partial_\nu e_\rho{}^b+\frac{1}{l}e_\mu{}^a\partial_\nu\omega_\rho{}^b\right)+\frac{2}{3}J_aJ_bJ_c\left(\frac{1}{l}\omega_\mu{}^a\omega_\nu{}^b e_\rho{}^c+\frac{1}{l}\omega_\mu{}^a e_\nu{}^b\omega_\rho{}^c\right.$$
$$\left.+e_\mu{}^a\omega_\nu{}^b\omega_\rho{}^c+\frac{1}{l^3}e_\mu{}^a e_\nu{}^b e_\rho{}^c\right)+\frac{2}{3}J_aJ_bJ_c\left(\frac{1}{l}\omega_\mu{}^a\omega_\nu{}^b e_\rho{}^c+\frac{1}{l}\omega_\mu{}^a e_\nu{}^b\omega_\rho{}^c\right.$$
$$\left.+\frac{1}{l}e_\mu{}^a\omega_\nu{}^b\omega_\rho{}^c+\frac{1}{l^3}e_\mu{}^a e_\nu{}^b e_\rho{}^c\right)\bigg].\tag{21}$$

Now we compute the first two terms of the last equality in equation (21):

$$\frac{k}{4\pi}\int\mathrm{d}^3x\ \mathrm{Tr}\left[\epsilon^{\mu\nu\rho}J_aJ_b\left(\frac{1}{l}\omega_\mu{}^a\partial_\nu e_\rho{}^b+\frac{1}{l}e_\mu{}^a\partial_\nu\omega_\rho{}^b\right)\right]$$
$$=\frac{k}{4\pi}\int\mathrm{d}^3x\ \left[\frac{1}{2l}\epsilon^{\mu\nu\rho}\left(\partial_\mu\omega_\nu{}^a-\partial_\nu\omega_\mu{}^a\right)e_{\rho,a}\right];$$
$$-\frac{k}{4\pi}\int\mathrm{d}^3x\ \mathrm{Tr}\left[\epsilon^{\mu\nu\rho}J_aJ_b\left(\frac{1}{l}\omega_\mu{}^a\partial_\nu e_\rho{}^b+\frac{1}{l}e_\mu{}^a\partial_\nu\omega_\rho{}^b\right)\right]$$
$$=\frac{k}{4\pi}\int\mathrm{d}^3x\ \left[\frac{1}{2l}\epsilon^{\mu\nu\rho}\left(\partial_\mu\omega_\nu{}^a-\partial_\nu\omega_\mu{}^a\right)e_{\rho,a}\right].\tag{22}$$

The above equalities are up to a total derivative term. Hence we obtain that:

$$\frac{k}{4\pi}\int\mathrm{d}^3x\ \mathrm{Tr}\left[\epsilon^{\mu\nu\rho}J_aJ_b\left(\frac{1}{l}\omega_\mu{}^a\partial_\nu e_\rho{}^b+\frac{1}{l}e_\mu{}^a\partial_\nu\omega_\rho{}^b\right)\right]$$
$$-\frac{k}{4\pi}\int\mathrm{d}^3x\ \mathrm{Tr}\left[\epsilon^{\mu\nu\rho}J_aJ_b\left(\frac{1}{l}\omega_\mu{}^a\partial_\nu e_\rho{}^b+\frac{1}{l}e_\mu{}^a\partial_\nu\omega_\rho{}^b\right)\right]$$
$$=\frac{k}{4\pi l}\int\mathrm{d}^3x\ \left[\epsilon^{\mu\nu\rho}\left(\partial_\mu\omega_\nu{}^a-\partial_\nu\omega_\mu{}^a\right)e_{\rho,a}\right].\tag{23}$$

Due to the following identity

$$\epsilon^{\mu\nu\rho}\epsilon_{abc}=e_d\left(e^\mu{}_{[a}e^\nu{}_b e^\rho{}_{c]}\right),\tag{24}$$

we obtain that:

$$\begin{aligned}\epsilon^{\mu\nu\rho}\partial_\mu\omega_\nu{}^a e_{\rho,a}&=\frac{1}{2}\epsilon^{\mu\nu\rho}\epsilon^{abc}\left(\partial_\mu\omega_{\nu,bc}\right)e_{\rho,a}\\&=\frac{1}{2}e_d\left(e^\mu{}_{[a}e^\nu{}_b e^\rho{}_{c]}\right)\left(\partial_\mu\omega_\nu{}^{bc}\right)e_\rho{}^a\\&=e_d\left(e^\mu{}_a e^\nu{}_b e^\rho{}_c\right)\left(\partial_\mu\omega_\nu{}^{bc}\right)e_\rho{}^a+e_d\left(e^\mu{}_c e^\nu{}_a e^\rho{}_b\right)\left(\partial_\mu\omega_\nu{}^{bc}\right)e_\rho{}^a\\&\quad+e_d\left(e^\mu{}_b e^\nu{}_c e^\rho{}_a\right)\left(\partial_\mu\omega_\nu{}^{bc}\right)e_\rho{}^a\\&=e_d\left(e^\nu{}_b e^\mu{}_c\right)\partial_\mu\omega_\nu{}^{bc}+e_d\left(e^\mu{}_c e^\nu{}_b\right)\partial_\mu\omega_\nu{}^{bc}+3e_d\left(e^\mu{}_b e^\nu{}_c\right)\partial_\mu\omega_\nu{}^{bc}\\&=e_d\left(e^\mu{}_b e^\nu{}_c\right)\partial_\mu\omega_\nu{}^{bc},\end{aligned}\tag{25}$$

in which we used

$$e^\mu{}_a e_\mu{}^b=\eta^b{}_a\tag{26}$$

in the fourth equality. Hence we obtain

$$\frac{k}{4\pi}\int d^3x\left[\frac{1}{2l}\epsilon^{\mu\nu\rho}\left(\partial_\mu\omega_\nu{}^a-\partial_\nu\omega_\mu{}^a\right)e_{\rho,a}\right]=\frac{k}{4\pi l}\int d^3x\, e_d\left(e^\mu{}_b e^\nu{}_c\right)\left(\partial_\mu\omega_\nu^{bc}-\partial_\nu\omega_\mu^{bc}\right). \quad (27)$$

We regulate the remaining terms of equation (21) as

$$\begin{aligned}&\frac{k}{4\pi l}\int d^3x\,\epsilon^{\mu\nu\rho}\mathrm{Tr}\bigg[\frac{2}{3}J_aJ_bJ_c\left(\omega_\mu{}^a\omega_\nu{}^b e_\rho{}^c+\omega_\mu{}^a e_\nu{}^b\omega_\rho{}^c+e_\mu{}^a\omega_\nu{}^b\omega_\rho{}^c+\frac{1}{l^2}e_\mu{}^a e_\nu{}^b e_\rho{}^c\right)\\&\quad+\frac{2}{3}J_aJ_bJ_c\left(\omega_\mu{}^a\omega_\nu{}^b e_\rho{}^c+\omega_\mu{}^a e_\nu{}^b\omega_\rho{}^c+e_\mu{}^a\omega_\nu{}^b\omega_\rho{}^c+\frac{1}{l^2}e_\mu{}^a e_\nu{}^b e_\rho{}^c\right)\bigg]\\&\qquad=\frac{k}{4\pi l}\int d^3x\,\epsilon^{\mu\nu\rho}\mathrm{Tr}\bigg[\frac{4}{3}J_aJ_bJ_c\left(\omega_\mu{}^a\omega_\nu{}^b e_\rho{}^c+\omega_\mu{}^a e_\nu{}^b\omega_\rho{}^c+e_\mu{}^a\omega_\nu{}^b\omega_\rho{}^c+\frac{1}{l^2}e_\mu{}^a e_\nu{}^b e_\rho{}^c\right)\bigg].\end{aligned} \quad (28)$$

Now we rewrite the first term of equation (28):

$$\begin{aligned}&\frac{k}{4\pi l}\int d^3x\,\mathrm{Tr}\left(\frac{2}{3}\epsilon^{\mu\nu\rho}J_aJ_bJ_c\omega_\mu{}^a\omega_\nu{}^b e_\rho{}^c-\frac{2}{3}\epsilon^{\mu\nu\rho}J_aJ_cJ_b\omega_\mu{}^a\omega_\nu{}^b e_\rho{}^c\right)\\&=\frac{k}{4\pi l}\int d^3x\,\mathrm{Tr}\left(\frac{2}{3}\epsilon^{\mu\nu\rho}J_a[J_b,J_c]\omega_\mu{}^a\omega_\nu{}^b e_\rho{}^c\right)\\&=\frac{k}{4\pi l}\int d^3x\,\mathrm{Tr}\left(\frac{2}{3}\epsilon^{\mu\nu\rho}\epsilon_{bcd}J_aJ^d\omega_\mu{}^a\omega_\nu{}^b e_\rho{}^c\right)\\&=\frac{k}{4\pi l}\int d^3x\,\left(\frac{1}{3}\epsilon^{\mu\nu\rho}\epsilon_{bca}\omega_\mu{}^a\omega_\nu{}^b e_\rho{}^c\right),\end{aligned} \quad (29)$$

then we can get

$$\begin{aligned}&\frac{k}{4\pi l}\int d^3x\,\epsilon_{\mu\nu\rho}\mathrm{Tr}\left[\frac{4}{3}J_aJ_bJ_c\left(\omega_\mu{}^a\omega_\nu{}^b e_\rho{}^c+\omega_\mu{}^a e_\nu{}^b\omega_\rho{}^c+e_\mu{}^a\omega_\nu{}^b\omega_\rho{}^c\right)\right]\\&=\frac{k}{4\pi l}\int d^3x\,\epsilon^{\mu\nu\rho}\epsilon_{bca}\omega_\mu{}^a\omega_\nu{}^b e_\rho{}^c.\end{aligned} \quad (30)$$

By the following equalities:

$$\begin{aligned}\epsilon^{\mu\nu\rho}\epsilon_{bca}\omega_\mu{}^a\omega_\nu{}^b e_\rho{}^c&=3e_de^\mu{}_ae^\nu{}_b\omega_\mu{}^a\omega_\nu{}^b-3e_de^\mu{}_be^\nu{}_a\omega_\mu{}^a\omega_\nu{}^b+e_de^\nu{}_ae^\mu{}_b\omega_\mu{}^a\omega_\nu{}^b\\&\quad-e_de^\nu{}_be^\mu{}_a\omega_\mu{}^a\omega_\nu{}^b+e_de^\mu{}_be^\nu{}_a\omega_\mu{}^a\omega_\nu{}^b-e_de^\mu{}_ae^\nu{}_b\omega_\mu{}^a\omega_\nu{}^b\\&=-\frac{1}{4}e_de^\mu{}_be^\nu{}_a\epsilon^a{}_{cd}\epsilon^b{}_{ef}\omega_\mu{}^{cd}\omega_\nu{}^{ef}+\frac{1}{4}e_de^\mu{}_ae^\nu{}_b\epsilon^a{}_{cd}\epsilon^b{}_{ef}\omega_\mu{}^{cd}\omega_\nu{}^{ef}\\&=-e_de^\mu{}_be^\nu{}_af^{ab}{}_{cd\,ef}\omega_\mu{}^{cd}\omega_\nu{}^{ef},\end{aligned} \quad (31)$$

where

$$f^{ab}{}_{cd\,ef}=\frac{1}{4}\left(\epsilon^a{}_{cd}\epsilon^b{}_{ef}-\epsilon^b{}_{cd}\epsilon^a{}_{ef}\right), \quad (32)$$

we can obtain

$$\frac{k}{4\pi l}\int d^3x\,\epsilon^{\mu\nu\rho}\epsilon_{bca}\omega_\mu{}^a\omega_\nu{}^b e_\rho{}^c=-\frac{k}{4\pi l}\int d^3x\,e_de^\mu{}_ae^\nu{}_b\left(\omega_\mu{}^{ac}\omega_\nu{}^{db}-\omega_\nu{}^{ac}\omega_\mu{}^{db}\right)\eta_{cd}. \quad (33)$$

The last term of equation (28) can be rewritten as the cosmological constant term:

$$\begin{aligned}\frac{k}{4\pi l}\int d^3x\,\mathrm{Tr}\left(\frac{4}{3l^2}\epsilon^{\mu\nu\rho}J_aJ_bJ_ce_\mu{}^ae_\nu{}^be_\rho{}^c\right) &= \frac{k}{4\pi l}\int d^3x\,\mathrm{Tr}\left(\frac{2}{3l^2}\epsilon^{\mu\nu\rho}\epsilon_{abd}J^dJ_ce_\mu{}^ae_\nu{}^be_\rho{}^c\right)\\ &= \frac{k}{4\pi l}\int d^3x\,\left(\frac{1}{3l^2}\epsilon^{\mu\nu\rho}\epsilon_{abc}e_\mu{}^ae_\nu{}^be_\rho{}^c\right)\\ &= \frac{k}{4\pi l}\int d^3x\,e_d\frac{2}{l^2},\end{aligned} \tag{34}$$

in which we used:

$$\begin{aligned}\mathrm{Tr}\left(\epsilon^{\mu\nu\rho}J_aJ_bJ_ce_\mu{}^ae_\nu{}^be_\rho{}^c\right) &= -\mathrm{Tr}\left(\epsilon^{\mu\nu\rho}J_bJ_aJ_ce_\mu{}^ae_\nu{}^be_\rho{}^c\right)\\ &= \frac{1}{2}\epsilon^{\mu\nu\rho}\mathrm{Tr}\left([J_a,J_b]J_ce_\mu{}^ae_\nu{}^be_\rho{}^c\right)\\ &= \frac{1}{2}\epsilon^{\mu\nu\rho}\epsilon_{abd}\mathrm{Tr}\left(J^dJ_ce_\mu{}^ae_\nu{}^be_\rho{}^c\right)\end{aligned} \tag{35}$$

in the first equality. By combining equations (27), (33) and (34), we show the equivalence between the gauge formulation and the first-order formulation [28]

$$S_{3DG} = S_{3DF}, \tag{36}$$

up to a total derivative term. Because G_3 is not a dimensionless constant but k is, the gauge formulation is more convenient to treat the quantum fluctuation [28]. The path integration is only over the non-singular vielbein in the metric formulation

$$\int \mathcal{D}g_{\mu\nu}, \tag{37}$$

but the measure of the gauge formulation

$$\int \mathcal{D}A\mathcal{D}\bar{A} \tag{38}$$

is also over the singular one. Therefore, the metric and gauge formulations are distinct from the non-perturbative effect, as stated in [29].

3. Boundary description

Because the gauge formulation has the SL(2)×SL(2) gauge symmetry [29], the boundary theory has the conformal symmetry. Therefore, we first review the conformal symmetry. We then derive the boundary description of AdS_3 Einstein gravity theory, 2D Schwarzian theory [41], which is also dual to chiral scalar fields [42], and discuss the loss of modular symmetry [41]. We cannot apply the CFT_2 result of the Weyl anomaly directly to the 2D Schwarzian theory due to covariance loss in the Lagrangian [41]. Nevertheless, we perform a direct calculation on the torus manifold to demonstrate the Liouville theory resulting from the Weyl transformation [50]. Finally, we present the analytical expression for the partition function when summing over all manifolds with the asymptotic AdS_3 boundary condition [32].

3.1. CFT

We first introduce the conformal transformation and then the conformal algebra in CFT_d. We then discuss the correspondence between the gauge symmetry of the gauge formulation and the conformal symmetry. Because the gauge symmetry is preserved even with quantum correction, the boundary description of gauge formulation is from CFT_2.

3.1.1. Conformal transformation. A conformal transformation is an invertible coordinate transformation

$$x^\mu \to \tilde{x}^\mu \tag{39}$$

with a transformation of a metric field

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \Omega(x)\, g_{\mu\nu}(x). \tag{40}$$

When $d \neq 2$, a conformal transformation of the field ϕ is

$$\delta\phi(z) = \epsilon\partial\phi(z) + \Delta(\partial\epsilon)\,\phi(z), \tag{41}$$

where

$$\partial \equiv \frac{\partial}{\partial z}. \tag{42}$$

The ϵ is a holomorphic transformation parameter, and Δ is a conformal dimension of the holomorphic part. When $d=2$, a conformal transformation of the field ϕ needs to include the anti-holomorphic part

$$\delta\phi(z,\bar{z}) = \epsilon\partial\phi(z,\bar{z}) + \Delta(\partial\epsilon)\,\phi(z,\bar{z}) + \bar{\epsilon}\bar{\partial}\phi(z,\bar{z}) + \bar{\Delta}\left(\bar{\partial}\bar{\epsilon}\right)\phi(z,\bar{z}), \tag{43}$$

where

$$\bar{\partial} \equiv \frac{\partial}{\partial \bar{z}}. \tag{44}$$

The $\bar{\epsilon}$ is an anti-holomorphic transformation parameter, and $\bar{\Delta}$ is a conformal dimension of the anti-holomorphic part.

3.1.2. Conformal algebra. The conformal transformations are given by the following generators:

- Translation $P_\mu = -i\partial_\mu$;
- Lorentz rotation $M_{\mu\nu} = -i(x_\mu\partial_\nu - x_\nu\partial_\mu)$;
- Dilaton $D = -ix^\mu\partial_\mu$;
- Special conformal transformation $K_\mu = -i\left(2x_\mu(x^\nu\partial_\nu) - x^2\partial_\mu\right)$.

The conformal algebra is in the following:

$$\begin{aligned}
[M_{\mu\nu},P_\rho] &= -i\left(\eta_{\nu\rho}P_\mu-\eta_{\mu\rho}P_\nu\right);\\
[M_{\mu\nu},K_\rho] &= -i\left(\eta_{\nu\rho}K_\mu-\eta_{\mu\rho}K_\nu\right);\\
[M_{\mu\nu},M_{\rho\sigma}] &= -i\left(\eta_{\nu\rho}M_{\mu\sigma}-\eta_{\mu\rho}M_{\nu\sigma}+\eta_{\nu\sigma}M_{\mu\rho}-\eta_{\mu\sigma}M_{\rho\nu}\right);\\
[D,P_\mu] &= iP_\mu;\\
[D,K_\mu] &= -iK_\mu;\\
[K_\mu,P_\nu] &= 2i\left(\eta_{\mu\nu}D+M_{\mu\nu}\right),
\end{aligned} \tag{45}$$

where

$$\eta_{\mu\nu}=\mathrm{diag}\left(-1,1,\ldots,1\right). \tag{46}$$

The group associated with the above algebra is isomorphic to SO(2, d). Therefore, the number of generators are

$$C_2^{d+2}=\frac{(d+2)(d+1)}{2} \tag{47}$$

in CFT$_d$. When $d=2$, the conformal symmetry group SO(2, 2) is isomorphic to the gauge symmetry group SL(2)×SL(2) of the gauge formulation. Due to the non-breaking of gauge symmetry by quantum effects, the gauge formulation's boundary description is CFT$_2$.

3.2. AdS$_3$ solution

The AdS$_{d+1}$ solution can be immersed in a $(d+2)$-dimensional flat spacetime

$$\mathrm{d}s^2_{d+1}=-\mathrm{d}X_1^2-\mathrm{d}X_2^2+\sum_{j=3}^{d+2}\mathrm{d}X_j^2, \tag{48}$$

in which the embedding coordinates satisfies

$$-X_1^2-X_2^2+\sum_{j=3}^{d+2}X_j^2=\frac{d(d-1)}{2\Lambda}. \tag{49}$$

The AdS$_{d+1}$ solution can be parametrized in the global coordinate:

$$\begin{aligned}
X_1 &= \sqrt{\frac{-d(d-1)}{2\Lambda}}\cosh(\rho)\cos(\tau);\\
X_2 &= \sqrt{\frac{-d(d-1)}{2\Lambda}}\cosh(\rho)\sin(\tau);\\
X_j &= \sqrt{\frac{-d(d-1)}{2\Lambda}}\sinh(\rho)\hat{x}_j,
\end{aligned} \tag{50}$$

where

$$\sum_{j=3}^{d+2}\hat{x}_j^2=1, \tag{51}$$

and Poincaré coordinate:

$$X_1 = -\frac{d(d-1)}{4r\Lambda}\left[1+\frac{4r^2\Lambda^2}{d^2(d-1)^2}\left(-\frac{d(d-1)}{2\Lambda}+\left(\sum_{j=3}^{d+1}x_j^2\right)-t^2\right)\right];$$

$$X_2 = \sqrt{-\frac{2\Lambda}{d(d-1)}}rt;$$

$$X_j = \sqrt{-\frac{2\Lambda}{d(d-1)}}rx_j;$$

$$X_{d+2} = -\frac{d(d-1)}{4r\Lambda}\left[1-\frac{4r^2\Lambda^2}{d^2(d-1)^2}\left(-\frac{d(d-1)}{2\Lambda}-\left(\sum_{j=3}^{d+1}x_j^2\right)+t^2\right)\right]. \tag{52}$$

In the global coordinate, the range of parameters is:

$$-\infty < \tau < \infty;\ \rho > 0. \tag{53}$$

The range of parameters in the Poincaré coordinate is:

$$-\infty < t < \infty;\ r > 0. \tag{54}$$

We first introduce the properties of the AdS_3 metric in the Poincaré and global coordinates. We then demonstrate the AdS_3 metric in the global coordinate from the gauge formulation.

3.2.1. Poincaré coordinate. We introduce the parameter

$$z \equiv -\frac{1}{\Lambda r}, \tag{55}$$

and then the AdS_3 metric becomes

$$\mathrm{d}s_{3P}^2 = -\frac{1}{\Lambda z^2}\left(\mathrm{d}z^2+\mathrm{d}x_3^2-\mathrm{d}t^2\right). \tag{56}$$

Due to $z > 0$, the Poincaré coordinate only covers the upper half region. The asymptotic boundary of AdS_3 is at $z \to 0$.

3.2.2. Global coordinate. We introduce the parameters:

$$r \equiv \sqrt{-\frac{1}{\Lambda}}\sinh(\rho);\ t \equiv \sqrt{-\frac{1}{\Lambda}}\tau, \tag{57}$$

and then the AdS_3 metric becomes

$$\mathrm{d}s_{3G}^2 = -\left(1-\Lambda r^2\right)\mathrm{d}t^2+\frac{1}{1-\Lambda r^2}\mathrm{d}r^2+r^2\mathrm{d}\theta^2, \tag{58}$$

where $0 < \theta \leqslant 2\pi$. This coordinate covers the entire AdS_3 spacetime, and is referred to as the global coordinate. The AdS_3 boundary is located at $r \to \infty$ and takes the form of a cylinder.

The cylinder manifold is isomorphic to a sphere manifold by removing the top and bottom points. After doing the Wick rotation with the Euclidean time

$$\psi \equiv it \tag{59}$$

and the identification

$$z_t \equiv \theta + i\psi \sim z_t + 2\pi\tau, \tag{60}$$

where τ is a complex structure, the boundary becomes a torus manifold.

3.2.3. *Gauge formulation.*

The equations of motion for A_μ and $\bar{A}_\mu$ in the gauge formulation (S_{3DG}) are:

$$\begin{aligned} 0 = F_{\mu\nu} &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]; \\ 0 = \bar{F}_{\mu\nu} &\equiv \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu + [\bar{A}_\mu, \bar{A}_\nu]. \end{aligned} \tag{61}$$

We can write the solutions in terms of the SL(2) transformations, g and $\bar{g}$:

$$A = g^{-1}ag + g^{-1}\mathrm{d}g;\ \bar{A} = \bar{g}^{-1}a\bar{g} + \bar{g}^{-1}\mathrm{d}\bar{g}. \tag{62}$$

The field strength $\tilde{F}_{\mu\nu}$ associated to a is zero:

$$\tilde{F}_{\mu\nu} \equiv \partial_\mu a_\nu - \partial_\nu a_\mu + [a_\mu, a_\nu] = 0. \tag{63}$$

We choose the solution of the gauge fields with $F_{\mu\nu} = \bar{F}_{\mu\nu} = 0$:

$$\begin{aligned} lA &= \sqrt{-\Lambda r^2 + 1} J_0 \mathrm{d}x^+ + \sqrt{-\Lambda} r J_1 \mathrm{d}x^+ + \frac{\mathrm{d}r}{\sqrt{-\Lambda r^2 + 1}} J_2; \\ l\bar{A} &= -\sqrt{-\Lambda r^2 + 1} J_0 \mathrm{d}x^- + \sqrt{-\Lambda} r J_1 \mathrm{d}x^- - \frac{\mathrm{d}r}{\sqrt{-\Lambda r^2 + 1}} J_2, \end{aligned} \tag{64}$$

where $x^\pm \equiv t \pm \theta$, for obtaining the AdS$_3$ solution in the global coordinate. The vielbein can be determined by A and $\bar{A}$ or equation (19):

$$e = \frac{l}{2}(A - \bar{A}) \equiv e_+ \mathrm{d}x^+ + e_- \mathrm{d}x^- + e_r \mathrm{d}x^r. \tag{65}$$

We can obtain the AdS$_3$ metric in the global coordinate by substituting the components of the vielbein to equation (14).

3.3. *Boundary term in gauge formulation*

To derive a non-trivial boundary description, we introduce a boundary term (at $r \to \infty$) to the gauge formulation. The Lagrangian description is

$$
\begin{aligned}
S_{\mathrm{G}} = & \frac{k}{2\pi}\int \mathrm{d}^3x\,\mathrm{Tr}\left(A_tF_{r\theta} - \frac{1}{2}\left(A_r\partial_tA_\theta - A_\theta\partial_tA_r\right)\right)\\
& - \frac{k}{2\pi}\int \mathrm{d}^3x\,\mathrm{Tr}\left(\bar{A}_t\bar{F}_{r\theta} - \frac{1}{2}\left(\bar{A}_r\partial_t\bar{A}_\theta - \bar{A}_\theta\partial_t\bar{A}_r\right)\right)\\
& - \frac{k}{4\pi}\int \mathrm{d}t\mathrm{d}\theta\,\mathrm{Tr}\left(\frac{E_t^+}{E_\theta^+}A_\theta^2\right) + \frac{k}{4\pi}\int \mathrm{d}t\mathrm{d}\theta\,\mathrm{Tr}\left(\frac{E_t^-}{E_\theta^-}\bar{A}_\theta^2\right)
\end{aligned}
\tag{66}
$$

when the boundary zweibein E is:

$$
E_t^+ = -E_t^- = E_\theta^+ = E_\theta^- = 1. \tag{67}
$$

The gauge fields satisfy the boundary conditions:

$$
\left(E_\theta^+A_t - E_t^+A_\theta\right)|_{r\rightarrow\infty} = 0; \qquad \left(E_\theta^-\bar{A}_t - E_t^-\bar{A}_\theta\right)|_{r\rightarrow\infty} = 0. \tag{68}
$$

The boundary metric can be rewritten in terms of E,

$$
g_{\tilde{\mu}\tilde{\nu}} \equiv \frac{1}{2}\left(E_{\tilde{\mu}}^+E_{\tilde{\nu}}^- + E_{\tilde{\mu}}^-E_{\tilde{\nu}}^+\right). \tag{69}
$$

The indices of boundary spacetimes are labeled as $\tilde{\mu} = t,\theta$. When utilizing the Euclidean signature, the boundary condition corresponds to the torus manifold. The boundary condition of other manifolds can be derived from the Weyl transformation [50].

3.4. 2D Schwarzian theory

We derive the 2D Schwarzian theory from the AdS_3 Einstein gravity in this section. The asymptotic AdS_3 boundary condition provides the boundary constraint [41]. We then use the boundary description of Einstein gravity to show the 2D Schwarzian theory and discuss the modular symmetry. We demonstrate the dual of the 2D Schwarzian theory using chiral scalar fields.

3.4.1. Boundary constraint. The asymptotic behavior of the gauge fields is that:

$$
A|_{r\rightarrow\infty} = \begin{pmatrix}\frac{\mathrm{d}r}{2r} & 0\\ rE^+ & -\frac{\mathrm{d}r}{2r}\end{pmatrix};\ \bar{A}|_{r\rightarrow\infty} = \begin{pmatrix}-\frac{\mathrm{d}r}{2r} & -rE^-\\ 0 & \frac{\mathrm{d}r}{2r}\end{pmatrix}. \tag{70}
$$

We first parametrize the SL(2) transformations:

$$
\begin{aligned}
g_{\mathrm{SL}(2)} &= \begin{pmatrix}1 & 0\\ F & 1\end{pmatrix}\begin{pmatrix}\lambda & 0\\ 0 & \frac{1}{\lambda}\end{pmatrix}\begin{pmatrix}1 & \Psi\\ 0 & 1\end{pmatrix};\\
\bar{g}_{\mathrm{SL}(2)} &= \begin{pmatrix}1 & -\bar{F}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{\bar{\lambda}} & 0\\ 0 & \bar{\lambda}\end{pmatrix}\begin{pmatrix}1 & 0\\ -\bar{\Psi} & 1\end{pmatrix}.
\end{aligned}
\tag{71}
$$

By identifying the SL(2) transformations with the gauge fields:

$$
g_{\mathrm{SL}(2)}^{-1}\partial_\theta g_{\mathrm{SL}(2)}|_{r\rightarrow\infty} = A_\theta|_{r\rightarrow\infty}, \qquad \bar{g}_{\mathrm{SL}(2)}^{-1}\partial_\theta\bar{g}_{\mathrm{SL}(2)}|_{r\rightarrow\infty} = \bar{A}_\theta|_{r\rightarrow\infty}, \tag{72}
$$

we obtain the boundary constraint [41]:

$$
\lambda = \sqrt{\frac{rE_\theta^+}{\partial_\theta F}};\ \Psi = -\frac{1}{2rE_\theta^+}\frac{\partial_\theta^2F}{\partial_\theta F}, \qquad \bar{\lambda} = \sqrt{\frac{rE_\theta^-}{\partial_\theta\bar{F}}};\ \bar{\Psi} = -\frac{1}{2rE_\theta^-}\frac{\partial_\theta^2\bar{F}}{\partial_\theta\bar{F}}. \tag{73}
$$

3.4.2. SL(2) measure. The boundary constraint (73) has the symmetry generated by the composition of two SL(2) transformations:

$$g_{\mathrm{SL}(2)} \to h_{\mathrm{SL}(2)}(t)\, g_{\mathrm{SL}(2)},$$
$$h_{\mathrm{SL}(2)} \equiv \begin{pmatrix} a_1(t) & a_2(t) \\ a_3(t) & a_4(t) \end{pmatrix},\ a_1(t)a_4(t) - a_2(t)a_3(t) = 1. \tag{74}$$

The new transformation can be written explicitly

$$h_{\mathrm{SL}(2)} g_{\mathrm{SL}(2)} = \begin{pmatrix} a_1\lambda + a_2 F\lambda & a_1\lambda\Psi + a_2\left(F\lambda\Psi + \frac{1}{\lambda}\right) \\ a_3\lambda + a_4 F\lambda & a_3\lambda\Psi + a_4\left(F\lambda\Psi + \frac{1}{\lambda}\right) \end{pmatrix}. \tag{75}$$

Hence we obtain the transformation:

$$\lambda \to a_1\lambda + a_2 F\lambda, \qquad \lambda\Psi \to a_1\lambda\Psi + a_2\left(F\lambda\Psi + \frac{1}{\lambda}\right),$$
$$F\lambda \to a_3\lambda + a_4 F\lambda, \qquad F\lambda\Psi + \frac{1}{\lambda} \to a_3\lambda\Psi + a_4\left(F\lambda\Psi + \frac{1}{\lambda}\right). \tag{76}$$

This transformation implies that the transformed fields are given by:

$$\tilde{\lambda} = a_1\lambda + a_2 F\lambda,$$
$$\tilde{\Psi} = \Psi + \frac{a_2}{(a_1 + a_2 F)\lambda^2},$$
$$\tilde{F} = \frac{a_4 F + a_3}{a_2 F + a_1}, \tag{77}$$

We compute the SL(2) measure from the followings:

$$\frac{\partial g_{\mathrm{SL}(2)}}{\partial\lambda} = \begin{pmatrix} 1 & \Psi \\ F & F\Psi - \frac{1}{\lambda^2} \end{pmatrix},$$
$$\frac{\partial g_{\mathrm{SL}(2)}}{\partial F} = \begin{pmatrix} 0 & 0 \\ \lambda & \lambda\Psi \end{pmatrix},$$
$$\frac{\partial g_{\mathrm{SL}(2)}}{\partial\Psi} = \begin{pmatrix} 0 & \lambda \\ 0 & F\lambda \end{pmatrix}, \tag{78}$$

$$g^{-1}_{\mathrm{SL}(2)}\frac{\partial g_{\mathrm{SL}(2)}}{\partial\lambda} = \begin{pmatrix} \frac{1}{\lambda} + \Psi\lambda F & -\Psi\lambda \\ -\lambda F & \lambda \end{pmatrix}\begin{pmatrix} 1 & \Psi \\ F & F\Psi - \frac{1}{\lambda^2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda} & 2\frac{\Psi}{\lambda} \\ 0 & -\frac{1}{\lambda} \end{pmatrix},$$
$$g^{-1}_{\mathrm{SL}(2)}\frac{\partial g_{\mathrm{SL}(2)}}{\partial F} = \begin{pmatrix} \frac{1}{\lambda} + \Psi\lambda F & -\Psi\lambda \\ -\lambda F & \lambda \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \lambda & \lambda\Psi \end{pmatrix} = \begin{pmatrix} -\lambda^2\Psi & -\lambda^2\Psi^2 \\ \lambda^2 & \lambda^2\Psi \end{pmatrix},$$
$$g^{-1}_{\mathrm{SL}(2)}\frac{\partial g_{\mathrm{SL}(2)}}{\partial\Psi} = \begin{pmatrix} \frac{1}{\lambda} + \Psi\lambda F & -\Psi\lambda \\ -\lambda F & \lambda \end{pmatrix}\begin{pmatrix} 0 & \lambda \\ 0 & F\lambda \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{79}$$

$$
\begin{aligned}
G_{\lambda\lambda} &\sim \mathrm{Tr}\left(g_{\mathrm{SL}(2)}^{-1}\frac{\partial g_{\mathrm{SL}(2)}}{\partial\lambda}g_{\mathrm{SL}(2)}^{-1}\frac{\partial g_{\mathrm{SL}(2)}}{\partial\lambda}\right)=\frac{2}{\lambda^2},\\
G_{\lambda F}=G_{F\lambda} &\sim \mathrm{Tr}\left(g_{\mathrm{SL}(2)}^{-1}\frac{\partial g_{\mathrm{SL}(2)}}{\partial\lambda}g_{\mathrm{SL}(2)}^{-1}\frac{\partial g_{\mathrm{SL}(2)}}{\partial F}\right)=0,\\
G_{\lambda\Psi}=G_{\Psi\lambda} &\sim \mathrm{Tr}\left(g_{\mathrm{SL}(2)}^{-1}\frac{\partial g_{\mathrm{SL}(2)}}{\partial\lambda}g_{\mathrm{SL}(2)}^{-1}\frac{\partial g_{\mathrm{SL}(2)}}{\partial\Psi}\right)=0,\\
G_{FF} &\sim \mathrm{Tr}\left(g_{\mathrm{SL}(2)}^{-1}\frac{\partial g_{\mathrm{SL}(2)}}{\partial F}g_{\mathrm{SL}(2)}^{-1}\frac{\partial g_{\mathrm{SL}(2)}}{\partial F}\right)=0,\\
G_{F\Psi}=G_{\Psi F} &\sim \mathrm{Tr}\left(g_{\mathrm{SL}(2)}^{-1}\frac{\partial g_{\mathrm{SL}(2)}}{\partial F}g_{\mathrm{SL}(2)}^{-1}\frac{\partial g_{\mathrm{SL}(2)}}{\partial\Psi}\right)=\lambda^2,\\
G_{\Psi\Psi} &\sim \mathrm{Tr}\left(g_{\mathrm{SL}(2)}^{-1}\frac{\partial g_{\mathrm{SL}(2)}}{\partial\Psi}g_{\mathrm{SL}(2)}^{-1}\frac{\partial g_{\mathrm{SL}(2)}}{\partial\Psi}\right)=0.
\end{aligned}
\tag{80}
$$

Hence the determinant of the matrix $G_{\mu\nu}$ gives $-2\lambda^2$ term. The SL(2) measure is given by:

$$
\int Dg_{\mu\nu}\sim\int \mathrm{d}\lambda\wedge\mathrm{d}F\wedge\mathrm{d}\Psi\,\sqrt{-\det G_{\mu\nu}}\sim\int\mathrm{d}\lambda\wedge\mathrm{d}F\wedge\mathrm{d}\Psi\ \lambda. \tag{81}
$$

Including the boundary constraint into the measure, we obtain the following result

$$
\int\mathrm{d}\lambda\wedge\mathrm{d}F\wedge\mathrm{d}\Psi\ \lambda\delta\left(\lambda^2\left(\partial_\theta F\right)-rE_\theta^+\right)\delta\left(\frac{1}{2rE_\theta^+}\frac{\partial_\theta^2F}{\partial_\theta F}+\Psi\right)\sim\int\mathrm{d}F\ \frac{1}{\partial_\theta F}. \tag{82}
$$

We can also show that the measure is invariant under the SL(2) transformation

$$
\frac{\mathrm{d}F}{\partial_\theta F}=\frac{\mathrm{d}\tilde{F}}{\partial_\theta\tilde{F}}, \tag{83}
$$

where $\tilde{F}$ is the field F after the transformation. This result also implies that the boundary constraint (73) is invariant under the SL(2) transformation. Another SL(2) transformation $\bar{g}$ has a similar result.

3.4.3. 2D Schwarzian theory. Because the A_t and $\bar{A}_t$ only has a linear coupling term in S_G, we first integrate out the A_t, which is equivalent to using [41]:

$$
\begin{aligned}
&F_{r\theta}=\bar{F}_{r\theta}=0;\\
&g_{\mathrm{SL}(2)}^{-1}\partial_\theta g_{\mathrm{SL}(2)}=A_\theta,\ g_{\mathrm{SL}(2)}^{-1}\partial_r g_{\mathrm{SL}(2)}=A_r,\\
&\bar{g}_{\mathrm{SL}(2)}^{-1}\partial_\theta\bar{g}_{\mathrm{SL}(2)}=\bar{A}_\theta,\ \bar{g}_{\mathrm{SL}(2)}^{-1}\partial_r\bar{g}_{\mathrm{SL}(2)}=\bar{A}_r;
\end{aligned}
\tag{84}
$$

$$\begin{aligned}
S_{\mathrm{G1}} = &-\frac{k}{4\pi}\int \mathrm{d}^3x\, \epsilon^{tr\theta}\mathrm{Tr}\Big(-g_{\mathrm{SL}(2)}^{-1}\left(\partial_r g_{\mathrm{SL}(2)}\right)g_{\mathrm{SL}(2)}^{-1}\left(\partial_t g_{\mathrm{SL}(2)}\right)g_{\mathrm{SL}(2)}^{-1}\partial_\theta g_{\mathrm{SL}(2)}\\
&+g_{\mathrm{SL}(2)}^{-1}\left(\partial_r g_{\mathrm{SL}(2)}\right)g_{\mathrm{SL}(2)}^{-1}\left(\partial_\theta g_{\mathrm{SL}(2)}\right)g_{\mathrm{SL}(2)}^{-1}\left(\partial_t g_{\mathrm{SL}(2)}\right)\Big)\\
&+\frac{k}{4\pi}\int \mathrm{d}^3x\, \epsilon^{tr\theta}\mathrm{Tr}\Big(-\bar{g}_{\mathrm{SL}(2)}^{-1}\left(\partial_r \bar{g}_{\mathrm{SL}(2)}\right)\bar{g}_{\mathrm{SL}(2)}^{-1}\left(\partial_t \bar{g}_{\mathrm{SL}(2)}\right)\bar{g}_{\mathrm{SL}(2)}^{-1}\partial_\theta \bar{g}_{\mathrm{SL}(2)}\\
&+\bar{g}_{\mathrm{SL}(2)}^{-1}\left(\partial_r \bar{g}_{\mathrm{SL}(2)}\right)\bar{g}_{\mathrm{SL}(2)}^{-1}\left(\partial_\theta \bar{g}_{\mathrm{SL}(2)}\right)\bar{g}_{\mathrm{SL}(2)}^{-1}\left(\partial_t \bar{g}_{\mathrm{SL}(2)}\right)\Big)\\
&+\frac{k}{2\pi}\int \mathrm{d}t\mathrm{d}\theta\ \mathrm{Tr}\Big(g_{\mathrm{SL}(2)}^{-1}\left(\partial_\theta g_{\mathrm{SL}(2)}\right)g_{\mathrm{SL}(2)}^{-1}\left(D_- g_{\mathrm{SL}(2)}\right)\Big)\\
&-\frac{k}{2\pi}\int \mathrm{d}t\mathrm{d}\theta\ \mathrm{Tr}\Big(\bar{g}_{\mathrm{SL}(2)}^{-1}\left(\partial_\theta \bar{g}_{\mathrm{SL}(2)}\right)\bar{g}_{\mathrm{SL}(2)}^{-1}\left(D_+ \bar{g}_{\mathrm{SL}(2)}\right)\Big),
\end{aligned} \tag{85}$$

in which we define that:

$$D_+ = \frac{1}{2}\partial_t + \frac{1}{2}\frac{E_t^-}{E_\theta^-}\partial_\theta, \qquad D_- = \frac{1}{2}\partial_t + \frac{1}{2}\frac{E_t^+}{E_\theta^+}\partial_\theta. \tag{86}$$

We can do the further computation to rewrite the action in terms of the fields:

$$\begin{aligned}
&-\frac{k}{4\pi}\int \mathrm{d}^3x\, \epsilon^{tr\theta}\mathrm{Tr}\Big(-g_{\mathrm{SL}(2)}^{-1}\left(\partial_r g_{\mathrm{SL}(2)}\right)g_{\mathrm{SL}(2)}^{-1}\left(\partial_t g_{\mathrm{SL}(2)}\right)g_{\mathrm{SL}(2)}^{-1}\partial_\theta g_{\mathrm{SL}(2)}\\
&+g_{\mathrm{SL}(2)}^{-1}\left(\partial_r g_{\mathrm{SL}(2)}\right)g_{\mathrm{SL}(2)}^{-1}\left(\partial_\theta g_{\mathrm{SL}(2)}\right)g_{\mathrm{SL}(2)}^{-1}\left(\partial_t g_{\mathrm{SL}(2)}\right)\Big)\\
&\quad=\frac{k}{4\pi}\int \mathrm{d}t\mathrm{d}\theta\ \lambda^2\left(\partial_\theta F\partial_t\Psi - \partial_t F\partial_\theta\Psi\right);
\end{aligned} \tag{87}$$

$$\begin{aligned}
&\frac{k}{2\pi}\int \mathrm{d}t\mathrm{d}\theta\ \mathrm{Tr}\left(g^{-1}\left(\partial_\theta g\right)g^{-1}\left(D_- g\right)\right)\\
&\quad=\frac{k}{2\pi}\int \mathrm{d}t\mathrm{d}\theta\ \left(\frac{2}{\lambda^2}\left(\partial_\theta\lambda\right)\left(D_-\lambda\right)+\lambda^2\left(\left(D_-F\right)\left(\partial_\theta\Psi\right)+\left(\partial_\theta F\right)\left(D_-\Psi\right)\right)\right);
\end{aligned} \tag{88}$$

$$\begin{aligned}
&-\frac{k}{4\pi}\int \mathrm{d}^3x\, \epsilon^{tr\theta}\mathrm{Tr}\Big(-g_{\mathrm{SL}(2)}^{-1}\left(\partial_r g_{\mathrm{SL}(2)}\right)g_{\mathrm{SL}(2)}^{-1}\left(\partial_t g_{\mathrm{SL}(2)}\right)g_{\mathrm{SL}(2)}^{-1}\partial_\theta g_{\mathrm{SL}(2)}\\
&+g_{\mathrm{SL}(2)}^{-1}\left(\partial_r g_{\mathrm{SL}(2)}\right)g_{\mathrm{SL}(2)}^{-1}\left(\partial_\theta g_{\mathrm{SL}(2)}\right)g_{\mathrm{SL}(2)}^{-1}\left(\partial_t g_{\mathrm{SL}(2)}\right)\Big)\\
&+\frac{k}{2\pi}\int \mathrm{d}t\mathrm{d}\theta\ \mathrm{Tr}\Big(g_{\mathrm{SL}(2)}^{-1}\left(\partial_\theta g_{\mathrm{SL}(2)}\right)g_{\mathrm{SL}(2)}^{-1}\left(D_- g_{\mathrm{SL}(2)}\right)\Big)\\
&\quad=\frac{k}{2\pi}\int \mathrm{d}t\mathrm{d}\theta\ \left(\frac{2}{\lambda^2}\left(\partial_\theta\lambda\right)\left(D_-\lambda\right)+2\lambda^2\left(\partial_\theta F\right)\left(D_-\Psi\right)\right)\\
&\quad=\frac{k}{\pi}\int \mathrm{d}t\mathrm{d}\theta\ \left(\frac{\left(\partial_\theta\lambda\right)\left(D_-\lambda\right)}{\lambda^2}+\lambda^2\left(\partial_\theta F\right)\left(D_-\Psi\right)\right).
\end{aligned} \tag{89}$$

We can also get a similar result from the $\bar{g}$ [41]. Therefore, we acquire the boundary description [41]

$$S_{\mathrm{G1}} = \frac{k}{\pi}\int \mathrm{d}t\mathrm{d}\theta\left(\frac{(\partial_\theta\lambda)(D_-\lambda)}{\lambda^2} + \lambda^2(\partial_\theta F)(D_-\Psi)\right)$$
$$-\frac{k}{\pi}\int \mathrm{d}t\mathrm{d}\theta\left(\frac{(\partial_\theta\bar{\lambda})(D_+\bar{\lambda})}{\bar{\lambda}^2} + \bar{\lambda}^2(\partial_\theta\bar{F})(D_+\bar{\Psi})\right). \tag{90}$$

After inserting the boundary constraint (73), we can rewrite the boundary Lagrangian in terms of F and $\bar{F}$ [41],

$$S_{\mathrm{G1}} = \frac{k}{2\pi}\int \mathrm{d}t\mathrm{d}\theta\left(\frac{3}{2}\frac{(D_-\partial_\theta\mathcal{F})(\partial_\theta^2\mathcal{F})}{(\partial_\theta\mathcal{F})^2} - \frac{D_-\partial_\theta^2\mathcal{F}}{\partial_\theta\mathcal{F}}\right)$$
$$-\frac{k}{2\pi}\int \mathrm{d}t\mathrm{d}\theta\left(\frac{3}{2}\frac{(D_+\partial_\theta\bar{\mathcal{F}})(\partial_\theta^2\bar{\mathcal{F}})}{(\partial_\theta\bar{\mathcal{F}})^2} - \frac{D_+\partial_\theta^2\bar{\mathcal{F}}}{\partial_\theta\bar{\mathcal{F}}}\right), \tag{91}$$

where the new variables $\mathcal{F}$ and $\bar{\mathcal{F}}$ are given by [41]:

$$\mathcal{F} \equiv \frac{F}{E_\theta^+}; \qquad \bar{\mathcal{F}} \equiv \frac{\bar{F}}{E_\theta^-}. \tag{92}$$

The theory is called 2D Schwarzian theory after the Wick rotation (using the Euclidean time) and the identification ($z_t \sim z_t + 2\pi\tau$). The constant solutions of $E^\pm$ do not break the form of the 2D Schwarzian theory. However, the general Weyl transformation cannot guarantee the same form of the boundary Lagrangian.

3.4.4. Modular symmetry.

Because we have the boundary condition:

$$\lambda^2\partial_\theta F = E_\theta^+ r; \qquad \bar{\lambda}^2\partial_\theta\bar{F}2E_\theta^- r, \tag{93}$$

the terms that follow are considered as total derivatives in equation (90) [41]:

$$\frac{k}{\pi}\int \mathrm{d}t\mathrm{d}\theta\,\left(\lambda^2(\partial_\theta F)(D_-\Psi)\right), \qquad -\frac{k}{\pi}\int \mathrm{d}t\mathrm{d}\theta\,\left(\bar{\lambda}^2(\partial_\theta\bar{F})(D_+\bar{\Psi})\right). \tag{94}$$

Therefore, the boundary theory becomes [41]

$$S_{\mathrm{G1}} = \frac{k}{\pi}\int \mathrm{d}t\mathrm{d}\theta\left(\frac{(\partial_\theta\lambda)(D_-\lambda)}{\lambda^2} - \frac{(\partial_\theta\bar{\lambda})(D_+\bar{\lambda})}{\bar{\lambda}^2}\right). \tag{95}$$

Because $E_\theta^\pm$ is a constant, we can simplify the expression of λ and $\bar{\lambda}$ [41]:

$$\lambda = \sqrt{\frac{r}{\partial_\theta\mathcal{F}}}; \qquad \bar{\lambda} = \sqrt{\frac{r}{\partial_\theta\bar{\mathcal{F}}}}. \tag{96}$$

The derivative acting on the λ provides:

$$\partial_\theta\lambda = -\frac{\sqrt{r}}{2}\frac{\partial_\theta^2\mathcal{F}}{(\partial_\theta\mathcal{F})^{\frac{3}{2}}}, \qquad D_-\lambda = -\frac{\sqrt{r}}{2}\frac{D_-\partial_\theta\mathcal{F}}{(\partial_\theta\mathcal{F})^{\frac{3}{2}}}. \tag{97}$$

Therefore, we obtain:

$$(\partial_\theta\lambda)(D_-\lambda)=\frac{r}{4}\frac{\partial_\theta^2\mathcal{F}D_-\partial_\theta\mathcal{F}}{(\partial_\theta\mathcal{F})^3},\qquad \frac{(\partial_\theta\lambda)(D_-\lambda)}{\lambda^2}=\frac{1}{4}\frac{\left(\partial_\theta^2\mathcal{F}\right)(D_-\partial_\theta\mathcal{F})}{(\partial_\theta\mathcal{F})^2}. \tag{98}$$

Therefore, we acquire an alternative way to describe the boundary [41]

$$S_{\mathrm{G1}}=\frac{k}{4\pi}\int \mathrm{d}t\mathrm{d}\theta\left(\frac{\left(\partial_\theta^2\mathcal{F}\right)(D_-\partial_\theta\mathcal{F})}{(\partial_\theta\mathcal{F})^2}-\frac{\left(\partial_\theta^2\bar{\mathcal{F}}\right)\left(D_+\partial_\theta\bar{\mathcal{F}}\right)}{\left(\partial_\theta\bar{\mathcal{F}}\right)^2}\right). \tag{99}$$

Finally, we choose the field redefinition [41]:

$$\mathcal{F}\equiv\tan\left(\frac{\phi}{2}\right);\qquad \bar{\mathcal{F}}\equiv\tan\left(\frac{\bar{\phi}}{2}\right) \tag{100}$$

to rewrite the following term [41]

$$\frac{k}{4\pi}\int \mathrm{d}t\mathrm{d}\theta\,\frac{\left(\partial_\theta^2\mathcal{F}\right)(D_-\partial_\theta\mathcal{F})}{(\partial_\theta\mathcal{F})^2}=\frac{k}{4\pi}\int \mathrm{d}t\mathrm{d}\theta\left[\frac{\left(\partial_\theta^2\phi\right)(D_-\partial_\theta\phi)}{(\partial_\theta\phi)^2}-(\partial_\theta\phi)(D_-\phi)\right] \tag{101}$$

by using:

$$\begin{aligned}
\partial_\theta\mathcal{F}&=\frac{1}{2}\sec^2\left(\frac{\phi}{2}\right)(\partial_\theta\phi),\\
\partial_\theta^2\mathcal{F}&=\frac{1}{2}\sec^3\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)(\partial_\theta\phi)^2+\frac{1}{2}\sec^2\left(\frac{\phi}{2}\right)\left(\partial_\theta^2\phi\right),\\
\partial_t\partial_\theta\mathcal{F}&=\frac{1}{2}\sec^3\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)(\partial_t\phi)(\partial_\theta\phi)+\frac{1}{2}\sec^2\left(\frac{\phi}{2}\right)(\partial_t\partial_\theta\phi),\\
\partial_\theta D_-\mathcal{F}&=\frac{1}{2}\sec^3\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)(\partial_\theta\phi)(D_-\phi)+\frac{1}{2}\sec^2\left(\frac{\phi}{2}\right)(D_-\partial_\theta\phi),\\
\left(\partial_\theta^2\mathcal{F}\right)(D_-\partial_\theta\mathcal{F})&=\frac{1}{4}\sec^6\left(\frac{\phi}{2}\right)\sin^2\left(\frac{\phi}{2}\right)(\partial_\theta\phi)^3D_-\phi+\frac{1}{4}\sec^5\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)(\partial_\theta\phi)^2(D_-\partial_\theta\phi)\\
&\quad+\frac{1}{4}\sec^5\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)(\partial_\theta\phi)(\partial_\theta^2\phi)(D_-\phi)+\frac{1}{4}\sec^4\left(\frac{\phi}{2}\right)(D_-\partial_\theta\phi)(\partial_\theta^2\phi),\\
\frac{(\partial_\theta^2\mathcal{F})(D_-\partial_\theta\mathcal{F})}{(\partial_\theta\mathcal{F})^2}&=\sec^2\left(\frac{\phi}{2}\right)\sin^2\left(\frac{\phi}{2}\right)(\partial_\theta\phi)(D_-\phi)+\sec\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)(D_-\partial_\theta\phi)\\
&\quad+\sec\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)\frac{\partial_\theta^2\phi D_-\phi}{\partial_\theta\phi}\\
&\quad+\frac{(\partial_\theta^2\phi)(D_-\partial_\theta\phi)}{(\partial_\theta\phi)^2},
\end{aligned}$$

$$
\begin{aligned}
&\int \mathrm{d}t\mathrm{d}\theta\ \frac{\left(\partial_\theta^2\mathcal{F}\right)\left(D_-\partial_\theta\mathcal{F}\right)}{\left(\partial_\theta\mathcal{F}\right)^2}\\
&\quad=\int \mathrm{d}t\mathrm{d}\theta\ \left[\tan^2\left(\frac{\phi}{2}\right)\left(\partial_\theta\phi\right)\left(D_-\phi\right)-\frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)\left(\partial_\theta\phi\right)\left(D_-\phi\right)\right.\\
&\qquad-\frac{1}{2}\sec^2\left(\frac{\phi}{2}\right)\left(D_-\phi\right)\left(\partial_\theta\phi\right)+\tan\left(\frac{\phi}{2}\right)\frac{\left(\partial_\theta^2\phi\right)\left(D_-\phi\right)}{\partial_\theta\phi}-\tan\left(\frac{\phi}{2}\right)\left(D_-\partial_\theta\phi\right)\\
&\qquad\left.+\frac{\left(\partial_\theta^2\phi\right)\left(D_-\partial_\theta\phi\right)}{\left(\partial_\theta\phi\right)^2}\right]\\
&\quad=\int \mathrm{d}t\mathrm{d}\theta\ \left[\frac{\left(\partial_\theta^2\phi\right)\left(D_-\partial_\theta\phi\right)}{\left(\partial_\theta\phi\right)^2}-\left(\partial_\theta\phi\right)\left(D_-\phi\right)\right.\\
&\qquad\left.+\tan\left(\frac{\phi}{2}\right)\frac{(\partial_\theta^2\phi)(D_-\phi)}{\partial_\theta\phi}-\tan\left(\frac{\phi}{2}\right)\left(D_-\partial_\theta\phi\right)\right]\\
&\quad=\int \mathrm{d}t\mathrm{d}\theta\ \left[\frac{(\partial_\theta^2\phi)(D_-\partial_\theta\phi)}{(\partial_\theta\phi)^2}-(\partial_\theta\phi)(D_-\phi)\right.\\
&\qquad\left.+\frac{1}{2}\tan\left(\frac{\phi}{2}\right)\frac{(\partial_\theta^2\phi)(\partial_t\phi)}{\partial_\theta\phi}-\frac{1}{2}\tan\left(\frac{\phi}{2}\right)\left(\partial_t\partial_\theta\phi\right)\right]\\
&\quad=\int \mathrm{d}t\mathrm{d}\theta\ \left[\frac{(\partial_\theta^2\phi)(D_-\partial_\theta\phi)}{(\partial_\theta\phi)^2}-(\partial_\theta\phi)(D_-\phi)\right],
\end{aligned}
\tag{102}
$$

in which we use the following result in the last equality:

$$
\begin{aligned}
&\int \mathrm{d}t\mathrm{d}\theta\ \left[\tan\left(\frac{\phi}{2}\right)\frac{\left(\partial_\theta^2\phi\right)\left(\partial_t\phi\right)}{\partial_\theta\phi}-\tan\left(\frac{\phi}{2}\right)\left(\partial_t\partial_\theta\phi\right)\right]\\
&\quad=\int \mathrm{d}t\mathrm{d}\theta\ \left\{-\ln\left(\partial_\theta\phi\right)\partial_\theta\left[\partial_t\phi\tan\left(\frac{\phi}{2}\right)\right]+\ln\left(\partial_\theta\phi\right)\partial_t\left[\left(\partial_\theta\phi\right)\tan\left(\frac{\phi}{2}\right)\right]\right.\\
&\qquad\left.+\partial_\theta\left[\ln\left(\partial_\theta\phi\right)\left(\partial_t\phi\right)\tan\left(\frac{\phi}{2}\right)\right]-\partial_t\left[\ln\left(\partial_\theta\phi\right)\left(\partial_\theta\phi\right)\tan\left(\frac{\phi}{2}\right)\right]\right\}\\
&\quad=\int \mathrm{d}t\mathrm{d}\theta\ \left\{\partial_\theta\left[\ln\left(\partial_\theta\phi\right)\left(\partial_t\phi\right)\tan\left(\frac{\phi}{2}\right)\right]-\partial_t\left[\ln\left(\partial_\theta\phi\right)\left(\partial_\theta\phi\right)\tan\left(\frac{\phi}{2}\right)\right]\right\}\\
&\quad=0.
\end{aligned}
\tag{103}
$$

We can obtain a similar result from $\bar{F}$ [41]. Therefore, the Lagrangian description becomes [41]:

$$
\begin{aligned}
S_{\mathrm{G1}}&=\frac{k}{4\pi}\int \mathrm{d}t\mathrm{d}\theta\ \left(\frac{\left(\partial_\theta^2\mathcal{F}\right)\left(D_-\partial_\theta\mathcal{F}\right)}{\left(\partial_\theta\mathcal{F}\right)^2}-\frac{\left(\partial_\theta^2\bar{\mathcal{F}}\right)\left(D_+\partial_\theta\bar{\mathcal{F}}\right)}{\left(\partial_\theta\bar{\mathcal{F}}\right)^2}\right)\\
&=\frac{k}{4\pi}\int \mathrm{d}t\mathrm{d}\theta\ \left[\frac{\left(\partial_\theta^2\phi\right)\left(D_-\partial_\theta\phi\right)}{\left(\partial_\theta\phi\right)^2}-\left(\partial_\theta\phi\right)\left(D_-\phi\right)\right]\\
&\quad-\frac{k}{4\pi}\int \mathrm{d}t\mathrm{d}\theta\ \left[\frac{\left(\partial_\theta^2\bar{\phi}\right)\left(D_+\partial_\theta\bar{\phi}\right)}{\left(\partial_\theta\bar{\phi}\right)^2}-\left(\partial_\theta\bar{\phi}\right)\left(D_+\bar{\phi}\right)\right].
\end{aligned}
\tag{104}
$$

The measure becomes [41]

$$\int \frac{\mathrm{d}\phi}{\partial_\theta\phi}\frac{\mathrm{d}\bar{\phi}}{\partial_\theta\bar{\phi}} \tag{105}$$

The path integral over ϕ and $\bar{\phi}$ that we identify [41]:

$$\begin{aligned}
&\tan\left(\frac{\phi}{2}\right)\sim\frac{a_4(\psi)\tan\left(\frac{\phi}{2}\right)+a_3(\psi)}{a_2(\psi)\tan\left(\frac{\phi}{2}\right)+a_1(\psi)};\\
&\tan\left(\frac{\bar{\phi}}{2}\right)\sim\frac{\bar{a}_4(\psi)\tan\left(\frac{\bar{\phi}}{2}\right)+\bar{a}_3(\psi)}{\bar{a}_2(\psi)\tan\left(\frac{\bar{\phi}}{2}\right)+\bar{a}_1(\psi)}.
\end{aligned} \tag{106}$$

The on-shell-solution is unique up to the gauge redundancy [41]

$$\phi_0=\theta-\frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)}\psi. \tag{107}$$

We then obtain the boundary condition for the A-cycle of the torus [41]:

$$\begin{aligned}
&\phi(\psi,\theta+2\pi)=\phi(\psi,\theta)+2\pi;\\
&\phi(\psi+2\pi\mathrm{Im}(\tau),\theta+2\pi\mathrm{Re}(\tau))=\phi(\psi,\theta).
\end{aligned} \tag{108}$$

For the B-cycle, the bulk geometry is the Euclidean BTZ black hole. The solution is also unique [41]

$$\phi_0=\frac{\psi}{\mathrm{Im}(\tau)}. \tag{109}$$

The boundary condition is [41]:

$$\begin{aligned}
&\phi(\psi,\theta+2\pi)=\phi(\psi,\theta);\\
&\phi(\psi+2\pi\mathrm{Im}(\tau),\theta+2\pi\mathrm{Re}(\tau))=\phi(\psi,\theta)+2\pi.
\end{aligned} \tag{110}$$

The path integral over ϕ and $\bar{\phi}$ that we identify [41]:

$$\tan\left(\frac{\phi}{2}\right)\sim\frac{\tilde{a}_4\tan\left(\frac{\phi}{2}\right)+\tilde{a}_3}{\tilde{a}_2\tan\left(\frac{\phi}{2}\right)+\tilde{a}_1};\ \tan\left(\frac{\bar{\phi}}{2}\right)\sim\frac{\bar{\tilde{a}}_4\tan\left(\frac{\bar{\phi}}{2}\right)+\bar{\tilde{a}}_3}{\bar{\tilde{a}}_2\tan\left(\frac{\bar{\phi}}{2}\right)+\bar{\tilde{a}}_1(\psi)}, \tag{111}$$

where $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{a}_3$, $\tilde{a}_4$, $\bar{\tilde{a}}_1$, $\bar{\tilde{a}}_2$, $\bar{\tilde{a}}_3$, $\bar{\tilde{a}}_4$ are functions of

$$\theta-\frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)}\psi \tag{112}$$

satisfying [41]

$$\tilde{a}_1\tilde{a}_4 - \tilde{a}_2\tilde{a}_3 = \bar{\tilde{a}}_1\bar{\tilde{a}}_4 - \bar{\tilde{a}}_2\bar{\tilde{a}}_3 = 1. \tag{113}$$

$\bar{\psi}$ has a similar result for the A-cycle and B-cycle [41]. The modular transformation

$$\tau \to -\frac{1}{\tau}, \tag{114}$$

swaps the A-cycle and the B-cycle. The A-cycle and B-cycle partition functions are not invariant under modular transformation, indicating a lack of modular symmetry in this theory [41]. The loss of modular symmetry is due to the non-periodic boundary condition. The conventional CFT on the torus requires periodic boundary conditions for each direction. Hence the CFT_2 without the modular symmetry on the torus is the boundary theory of AdS_3 Einstein gravity theory.

3.4.5. Chiral scalar fields. Now we show that 2D Schwarzian theory is dual to the following action [42]

$$S_{2D1} = \frac{4k}{\pi}\int \mathrm{d}t\mathrm{d}\theta\ \left((D_-\phi)(\partial_\theta\phi) + \Pi\left(\partial_\theta\mathcal{F} - e^{4\phi}\right)\right). \tag{115}$$

The measure of path integration is

$$\int \mathrm{d}\phi\,\mathrm{d}\mathcal{F}\mathrm{d}\Pi. \tag{116}$$

If we first integrate out the Π and then integrate out the ϕ, equivalent to replacing ϕ by $\mathcal{F}$ with the following equality

$$\ln\partial_\theta\mathcal{F} = 4\phi, \tag{117}$$

and then we obtain

$$S_{2D2} = \frac{k}{4\pi}\int \mathrm{d}t\mathrm{d}\theta\ \frac{\partial_\theta^2\mathcal{F}}{(\partial_\theta\mathcal{F})^2}(D_-\partial_\theta\mathcal{F}). \tag{118}$$

The measure becomes

$$\int\frac{\mathrm{d}\mathcal{F}}{\partial_\theta\mathcal{F}}. \tag{119}$$

We can show that the dual theory is equivalent to the 2D Schwarzian theory up to a total derivative term (integration by part in θ, and the total derivative term vanishes for the torus manifold) [42]

$$\frac{k}{2\pi}\int \mathrm{d}t\mathrm{d}\theta\left(\frac{3}{2}\frac{(D_-\partial_\theta\mathcal{F})\left(\partial_\theta^2\mathcal{F}\right)}{(\partial_\theta\mathcal{F})^2} - \frac{D_-\partial_\theta^2\mathcal{F}}{\partial_\theta\mathcal{F}}\right) = \frac{k}{4\pi}\int \mathrm{d}t\mathrm{d}\theta\ \frac{\partial_\theta^2\mathcal{F}}{(\partial_\theta\mathcal{F})^2}(D_-\partial_\theta\mathcal{F}). \tag{120}$$

Now we integrate out the $\mathcal{F}$ in S_{2D1}, which is equivalent to introducing a constraint to the measure [42]

$$\delta(\Pi - f(t)). \tag{121}$$

We then can integrate out Π and perform a field redefinition

$$\phi \to \phi - \frac{1}{4}\ln f \tag{122}$$

to obtain the following action [42]

$$S_{2D3} = \frac{2k}{\pi}\int \mathrm{d}t\mathrm{d}\theta \left((\partial_t\phi)(\partial_\theta\phi) - \frac{E_t^+}{E_\theta^+}(\partial_\theta\phi)(\partial_\theta\phi) - e^{4\phi} \right), \tag{123}$$

up to a total derivative term. The measure becomes

$$\int \mathrm{d}\phi. \tag{124}$$

We can obtain a similar dual for the $\bar{A}$ [42]. Therefore, the dual of the 2D Schwarzian theory is represented by the chiral scalar fields below [42]

$$S_{CB} = \frac{4k}{\pi}\int \mathrm{d}t\mathrm{d}\theta \left((D_-\phi)(\partial_\theta\phi) - e^{4\phi} \right) - \frac{4k}{\pi}\int \mathrm{d}t\mathrm{d}\theta \left(\left(D_+\bar{\phi}\right)\left(\partial_\theta\bar{\phi}\right) + e^{4\bar{\phi}} \right). \tag{125}$$

The measure is

$$\int \mathrm{d}\phi\,\mathrm{d}\bar{\phi}. \tag{126}$$

We can observe the difference between the Liouville theory and 2D Schwarzian theory on the torus manifold [42].

3.5. Weyl transformation and Liouville theory

Because the Lagrangian description of 2D Schwarzian theory loses the covariant form, we cannot use the conventional result of Weyl anomaly from CFT_2 [41]. However, the direct calculation for the torus manifold shows the Liouville theory resulting from the Weyl transformation [50]. The spin connection ω^a satisfies the torsionless condition (or equation of motion of ω^a)

$$de_a + \epsilon_{abc}\omega^b \wedge e^c = 0. \tag{127}$$

We can solve the spin connection, and the asymptotic solution is:

$$\begin{aligned} &e^0|_{r\to\infty} = rdt,\ e^1|_{r\to\infty} = rd\theta,\ e^2|_{r\to\infty} = 0; \\ &\omega^0|_{r\to\infty} = rd\theta,\ \omega^1|_{r\to\infty} = rdt,\ \omega^2|_{r\to\infty} = 0. \end{aligned} \tag{128}$$

A general Weyl transformation

$$e^a \to \exp(\sigma(t,\theta))\, e^a \tag{129}$$

leads the asymptotic boundary condition to the gauge fields [50]:

$$A_\theta|_{r\to\infty} = \begin{pmatrix} -\frac{1}{2}\partial_t\sigma & 0 \\ re^\sigma E_\theta^+ & \frac{1}{2}\partial_t\sigma \end{pmatrix};\ \bar{A}_\theta|_{r\to\infty} = \begin{pmatrix} -\frac{1}{2}\partial_t\sigma & -re^\sigma E_\theta^- \\ 0 & \frac{1}{2}\partial_t\sigma \end{pmatrix}. \tag{130}$$

The boundary constraints after the Weyl transformation become [50]:

$$\begin{aligned}
&\lambda = e^{\frac{\sigma}{2}}\lambda^{+},\\
&\Psi = -\frac{e^{-\sigma}}{2E_\theta^{+}r}\left(\partial_\theta\left(\ln\mathcal{F}^{+}\right) - \partial_\theta\sigma - \partial_t\sigma\right),\\
&\mathcal{F} \equiv e^{-\sigma}\mathcal{F}^{+};\\
&\bar{\lambda} = e^{\frac{\sigma}{2}}\lambda^{-},\\
&\bar{\Psi} = -\frac{e^{-\sigma}}{2E_\theta^{-}r}\left(\partial_\theta\left(\ln\mathcal{F}^{-}\right) - \partial_\theta\sigma + \partial_t\sigma\right),\\
&\bar{\mathcal{F}} \equiv e^{-\sigma}\mathcal{F}^{-}.
\end{aligned} \tag{131}$$

where

$$\begin{aligned}
&\lambda^{+} \equiv \sqrt{\frac{rE_\theta^{+}}{\partial_\theta F}},\ \lambda^{-} \equiv \sqrt{\frac{rE_\theta^{-}}{\partial_\theta \bar{F}}};\\
&\mathcal{F}^{+} \equiv \frac{\partial_\theta F}{E_\theta^{+}},\ \mathcal{F}^{-} \equiv \frac{\partial_\theta \bar{F}}{E_\theta^{-}}.
\end{aligned} \tag{132}$$

The boundary constraints lead to the different boundary conditions [50]:

$$\begin{aligned}
&\left(E_\theta^{+}A_t - E_t^{+}A_\theta - E_\theta^{+}A_t^2 J_2 + E_t^{+}A_\theta^2 J_2\right)|_{r\to\infty} = 0;\\
&\left(E_\theta^{-}\bar{A}_t - E_t^{-}\bar{A}_\theta - E_\theta^{-}\bar{A}_t^2 + E_t^{-}\bar{A}_\theta^2 J_2\right)|_{r\to\infty} = 0,
\end{aligned} \tag{133}$$

where

$$A_\theta^2 = \bar{A}_\theta^2 = -\partial_t\sigma;\ A_t^2 = \bar{A}_t^2 = -\partial_\theta\sigma. \tag{134}$$

The upper index of A_θ^2 is the Lie algebra index.

The variation of A_θ and $\bar{A}_\theta$ [50]

$$\begin{aligned}
&-\frac{k}{8\pi}\int \mathrm{d}t\mathrm{d}\theta\ \left(\frac{E_t^{+}}{E_\theta^{+}}\delta\left(A_\theta^2 A_\theta^2\right) - \delta\left(A_t^2 A_t^2\right)\right)\\
&\quad + \frac{k}{8\pi}\int \mathrm{d}t\mathrm{d}\theta\ \left(\frac{E_t^{-}}{E_\theta^{-}}\delta\left(\bar{A}_\theta^2 \bar{A}_\theta^2\right) - \delta\left(\bar{A}_t^2 \bar{A}_t^2\right)\right)
\end{aligned} \tag{135}$$

shows the necessity of introducing the additional boundary term [50]:

$$\begin{aligned}
S_{\mathrm{B}1} &= \frac{k}{8\pi}\int \mathrm{d}t\mathrm{d}\theta\ \frac{E_t^{+}}{E_\theta^{+}}A_\theta^2 A_\theta^2 - \frac{k}{8\pi}\int \mathrm{d}t\mathrm{d}\theta\ A_t^2 A_t^2\\
&\quad - \frac{k}{8\pi}\int \mathrm{d}t\mathrm{d}\theta\ \frac{E_t^{-}}{E_\theta^{-}}\bar{A}_\theta^2 \bar{A}_\theta^2 + \frac{k}{8\pi}\int \mathrm{d}t\mathrm{d}\theta\ \bar{A}_t^2 \bar{A}_t^2\\
&= \frac{k}{8\pi}\int \mathrm{d}t\mathrm{d}\theta\ \frac{E_t^{+}}{E_\theta^{+}}A_\theta^2 A_\theta^2 - \frac{k}{8\pi}\int \mathrm{d}t\mathrm{d}\theta\ \frac{E_t^{-}}{E_\theta^{-}}\bar{A}_\theta^2 \bar{A}_\theta^2\\
&= \frac{k}{4\pi}\int \mathrm{d}t\mathrm{d}\theta\ \left(\partial_t\sigma\right)\left(\partial_t\sigma\right).
\end{aligned} \tag{136}$$

Applying the Weyl transformation to the boundary action (90) generates the additional term [50]:

$$\frac{(\partial_\theta\lambda)(D_-\lambda)}{\lambda^2}-\frac{\left(\partial_\theta\bar{\lambda}\right)\left(D_+\bar{\lambda}\right)}{\bar{\lambda}^2}$$
$$\to\frac{1}{4}(\partial_\theta\sigma)\left(\partial_t\ln\lambda^+\right)+\frac{1}{4}(\partial_t\sigma)\left(\partial_\theta\ln\lambda^+\right)-\frac{1}{2}(\partial_\theta\sigma)\left(\partial_\theta\ln\lambda^+\right)$$
$$-\frac{1}{4}(\partial_\theta\sigma)\left(\partial_t\ln\lambda^-\right)-\frac{1}{4}(\partial_t\sigma)\left(\partial_\theta\ln\lambda^-\right)-\frac{1}{2}(\partial_\theta\sigma)\left(\partial_\theta\ln\lambda^-\right)$$
$$-\frac{1}{4}(\partial_\theta\sigma)(\partial_\theta\sigma); \qquad (137)$$

$$\lambda^2(\partial_\theta F)(D_-\Psi)-\bar{\lambda}^2\left(\partial_\theta\bar{F}\right)\left(D_+\bar{\Psi}\right)$$
$$=\frac{1}{4}(\partial_t\sigma)\left(\partial_\theta\ln\mathcal{F}^+\right)-\frac{1}{4}(\partial_\theta\sigma)\left(\partial_\theta\ln\mathcal{F}^+\right)$$
$$-\frac{1}{4}(\partial_t\sigma)\left(\partial_\theta\ln\mathcal{F}^-\right)-\frac{1}{4}(\partial_\theta\sigma)\left(\partial_\theta\ln\mathcal{F}^-\right)$$
$$-\frac{1}{2}(\partial_t\sigma)(\partial_t\sigma)+\frac{1}{2}(\partial_\theta\sigma)(\partial_\theta\sigma)+\frac{1}{2}\left(\partial_t^2\sigma\right)-\frac{1}{2}\left(\partial_\theta^2\sigma\right), \qquad (138)$$

where

$$\mathcal{F}^+\equiv\frac{\partial_\theta F}{E_\theta^+},\ \mathcal{F}^-\equiv\frac{\partial_\theta\bar{F}}{E_\theta^-}. \qquad (139)$$

Because we consider the torus manifold, we can do the integration by part for each direction, and the coupling term between the background and dynamical fields vanishes through the integration by part [50]:

$$\frac{k}{\pi}\int \mathrm{d}t\mathrm{d}\theta\ \left(\frac{1}{4}(\partial_\theta\sigma)\left(\partial_t\ln\lambda^+\right)+\frac{1}{4}(\partial_t\sigma)\left(\partial_\theta\ln\lambda^+\right)\right.$$
$$-\frac{1}{2}(\partial_\theta\sigma)\left(\partial_\theta\ln\lambda^+\right)$$
$$+\frac{1}{4}(\partial_t\sigma)\left(\partial_\theta\ln\mathcal{F}^+\right)-\frac{1}{4}(\partial_\theta\sigma)\left(\partial_\theta\ln\mathcal{F}^+\right)$$
$$-\frac{1}{4}(\partial_\theta\sigma)\left(\partial_t\ln\lambda^-\right)-\frac{1}{4}(\partial_t\sigma)\left(\partial_\theta\ln\lambda^-\right)$$
$$-\frac{1}{2}(\partial_\theta\sigma)\left(\partial_\theta\ln\lambda^-\right)$$
$$\left.-\frac{1}{4}(\partial_t\sigma)\left(\partial_\theta\ln\mathcal{F}^-\right)-\frac{1}{4}(\partial_\theta\sigma)\left(\partial_\theta\ln\mathcal{F}^-\right)\right)$$
$$=\frac{k}{\pi}\int \mathrm{d}t\mathrm{d}\theta\ \left(\frac{1}{2}(\partial_\theta\sigma)\left(\partial_t\ln\lambda^+\right)-\frac{1}{2}(\partial_\theta\sigma)(\partial_\theta\ln\lambda^+)\right.$$
$$+\frac{1}{4}(\partial_t\sigma)(\partial_\theta\ln\mathcal{F}^+)-\frac{1}{4}(\partial_\theta\sigma)(\partial_\theta\ln\mathcal{F}^+)$$
$$-\frac{1}{2}(\partial_\theta\sigma)(\partial_t\ln\lambda^-)-\frac{1}{2}(\partial_\theta\sigma)(\partial_\theta\ln\lambda^-)$$
$$\left.-\frac{1}{4}(\partial_t\sigma)(\partial_\theta\ln\mathcal{F}^-)-\frac{1}{4}(\partial_\theta\sigma)(\partial_\theta\ln\mathcal{F}^-)\right)$$

$$
\begin{aligned}
&= \int dt d\theta \left[\frac{1}{4}(\partial_t\sigma)\left(\partial_\theta \ln\left(\mathcal{F}^+(\lambda^+)^2\right)\right)\right. \\
&- \frac{1}{4}(\partial_\theta\sigma)\left(\partial_\theta \ln\left(\mathcal{F}^+(\lambda^+)^2\right)\right) \\
&- \frac{1}{4}(\partial_t\sigma)\left(\partial_\theta \ln\left(\mathcal{F}^-(\lambda^-)^2\right)\right) \\
&\left. - \frac{1}{4}(\partial_\theta\sigma)\left(\partial_\theta \ln\left(\mathcal{F}^-(\lambda^-)^2\right)\right)\right] \\
&= 0.
\end{aligned} \tag{140}
$$

We use:

$$\mathcal{F}^+\left(\lambda^+\right)^2 = \mathcal{F}^-\left(\lambda^-\right)^2 = r \tag{141}$$

in the last equality. Hence the Weyl transformation generates the additional term from equation (90) as that [50]

$$\delta S_{\rm B} = \frac{k}{\pi}\int dt d\theta\left(-\frac{1}{2}(\partial_t\sigma)(\partial_t\sigma) + \frac{1}{4}(\partial_\theta\sigma)(\partial_\theta\sigma)\right). \tag{142}$$

The Weyl transformation induces the Liouville theory [50]

$$\delta S_{\rm B} + S_{\rm B1} = \frac{k}{\pi}\int dt d\theta\left(-\frac{1}{4}(\partial_t\sigma)(\partial_t\sigma) + \frac{1}{4}(\partial_\theta\sigma)(\partial_\theta\sigma)\right). \tag{143}$$

3.6. Partition function

Although the 2D Schwarzian theory is not supersymmetric, we can introduce the non-dynamical fermion field to obtain the supersymmetry [41]. Because the partition function is one-loop exact, the higher-loop terms do not have the contribution. The partition function of 2D Schwarzian theory is

$$Z(\tau) = |q|^{-\frac{c_{\rm cft_2}}{12}} \frac{1}{\prod_{n=2}^{\infty}|1-q^n|^2}, \tag{144}$$

where

$$q = e^{2\pi i\tau}. \tag{145}$$

Because the details of the computing partition function overlap with the calculation of EE, we only show the result here. With the help of the Dedekind η function,

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n), \tag{146}$$

we simplify the expression of the partition function

$$Z(\tau) = \frac{1}{|\eta(\tau)|^2}|q|^{-\frac{1}{12}\left(c_{\rm cft_2}-1\right)}|1-q|^2. \tag{147}$$

The path integration in the bulk for all asymptotic AdS_3 boundary conditions is equivalent to summing over the SL(2, $\mathbb{Z}$) group in the boundary theory [32]. Due to the summation of all modular transformations, the partition function becomes the manifest modular invariant form [32]

$$Z_M(\tau) = \sum_{c_1,d_1;(c_1,d_1)=1} Z\left(\frac{a_1\tau+b_1}{c_1\tau+d_1}\right), \tag{148}$$

where

$$a_1d_1 - b_1c_1 = 1,\ a_1,b_1,c_1,d_1 \in \mathbb{Z}. \tag{149}$$

The relative prime c_1 and d_1 implies that

$$(c_1,d_1) = 1. \tag{150}$$

If we have two solutions for a_1,b_1 and a_2,b_2, we then obtain that

$$(a_1 - a_2)d_1 = (b_1 - b_2)c_1. \tag{151}$$

We can further solve the above equation:

$$a_1 - a_2 = mc_1,\ b_1 - b_2 = md_1, \tag{152}$$

where m is an arbitrary integer. The transformation:

$$a_1 \to a_1 + mc_1;\ b_1 \to b_1 + md_1, \tag{153}$$

implies

$$\frac{a_1\tau+b_1}{c_1\tau+d_1} \to \frac{a_1\tau+b_1}{c_1\tau+d_1} + m. \tag{154}$$

and

$$q \to qe^{2\pi im} = q. \tag{155}$$

Because the partition function only depends on q, we have the redundancy from the different choices of m. Hence we only sum over $m = 0$ in the partition function, which is equivalent to not including a and b. We will further simplify the expression of Z_M by showing that $\sqrt{\mathrm{Im}(\tau)}|\eta(\tau)|^2$ is the modular invariant [32].

3.6.1. Modular invariant variable. There are two generators that produce the modular transformation: the T-transformation and the S-transformation. The T-transformation is:

$$\tau \to \tau + 1; \qquad a_1 = b_1 = d_1 = 1,\ c_1 = 0. \tag{156}$$

The S-transformation is:

$$\tau \to -\frac{1}{\tau}; \qquad a_1 = d_1 = 0,\ b_1 = -c_1 = 1. \tag{157}$$

q is invariant under the T-transformation. Therefore, $\sqrt{\mathrm{Im}(\tau)}|\eta(\tau)|^2$ is a modular invariant variable.

For discussing the S-transformation, we first define that

$$r_1 \equiv \exp\left(-\frac{2\pi i}{\tau}\right). \tag{158}$$

We then obtain

$$\eta\left(-\frac{1}{\tau}\right) = r_1^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - r_1^n). \tag{159}$$

By using the pentagonal number theorem (only when $|r|_1 < 1$, $\mathrm{Im}(r_1) > 0$, or $\mathrm{Im}(\tau) > 0$)

$$\prod_{n=1}^{\infty} (1 - x^n) = \sum_{k=-\infty}^{\infty} (-1)^k x^{\frac{k(3k-1)}{2}}, \tag{160}$$

we can rewrite the infinite product as an infinite summation in the Dedekind η function:

$$\eta\left(-\frac{1}{\tau}\right) = r_1^{\frac{1}{24}} \sum_{n=-\infty}^{\infty} (-1)^n r_1^{\frac{\left(3n^2-n\right)}{2}} = \sum_{n=-\infty}^{\infty} f(n), \tag{161}$$

where

$$f(n) \equiv \exp\left[\pi i \left(-\frac{1}{12\tau} + n - \frac{3n^2 - n}{\tau}\right)\right]. \tag{162}$$

We then do an analytical continuation from n to the real number x and find the Fourier transform as that:

$$\begin{aligned}\tilde{f}(k) &= \int_{-\infty}^{\infty} \mathrm{d}x\, \exp\left[\pi i \left(-\frac{1}{12\tau} + x - 2kx - \frac{3x^2 - x}{\tau}\right)\right] \\ &= \sqrt{\frac{-i\tau}{3}} \exp\left[\pi i \left(\frac{\tau\,(2k-1)^2}{12} - \frac{2k-1}{6}\right)\right].\end{aligned} \tag{163}$$

We use the result of Gaussian integration

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \exp\left(-a_2 x^2 + b_2 x + c_2\right) = \sqrt{\frac{\pi}{a_2}} \exp\left(\frac{b_2^2}{4a_2} + c_2\right) \tag{164}$$

with the choices of the parameters:

$$a_2 = \frac{3\pi i}{\tau},\ b_2 = \left(1 - 2k + \frac{1}{\tau}\right)\pi i,\ c_2 = -\frac{\pi i}{12\tau}. \tag{165}$$

The process of analytically continuing from an integer to a real number is not rigorous. This can be proven more easily and understandably by following a simpler and more intuitive method. Nonetheless, it is possible to obtain rigorous proof without using analytical continuation. We then apply equation (163) to the integer k case to obtain that

$$\eta\left(-\frac{1}{\tau}\right)=\sqrt{\frac{-i\tau}{3}}\sum_{k=-\infty}^{\infty}\exp\left[\pi i\left(\frac{\tau(2k-1)^2}{12}-\frac{2k-1}{6}\right)\right]. \tag{166}$$

Using the pentagonal number theorem also provides the following similar formula:

$$\eta(\tau)=q^{\frac{1}{24}}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{3n^2-n}{2}}=\sum_{n=-\infty}^{\infty}\left[\pi i\left(\frac{\tau(6n-1)^2}{12}+n\right)\right]. \tag{167}$$

The integer k can be decomposed as $3l, 3l+1, 3l+2$, where $l\in\mathbb{Z}$. Therefore, we obtain the following results:

$$\begin{aligned}
&\eta\left(-\frac{1}{\tau}\right)\\
&=\sqrt{\frac{-i\tau}{3}}\\
&\quad\times\sum_{l=-\infty}^{\infty}\left[\exp\left[\pi i\left(\frac{\tau(6l-1)^2}{12}-\frac{6l-1}{6}\right)\right]\right.\\
&\quad+\exp\left[\pi i\left(\frac{\tau(6l+1)^2}{12}-\frac{6l+1}{6}\right)\right]\\
&\quad\left.+\exp\left[\pi i\left(\frac{\tau(6l+3)^2}{12}-\frac{6l+3}{6}\right)\right]\right]\\
&=\sqrt{\frac{-i\tau}{3}}\\
&\quad\times\sum_{l=-\infty}^{\infty}\left[\exp\left[\pi i\left(\frac{\tau(6l-1)^2}{12}-l\right)\right]\exp\left(\frac{\pi i}{6}\right)\right.\\
&\quad+\exp\left[\pi i\left(\frac{\tau(6l-1)^2}{12}+l\right)\right]\exp\left(-\frac{\pi i}{6}\right)\\
&\quad\left.+\exp\left[\pi i\left(\frac{\tau(6l+3)^2}{12}-\frac{6l+3}{6}\right)\right]\right]\\
&=\sqrt{-i\tau}\\
&\quad\times\sum_{l=-\infty}^{\infty}\left[\exp\left[\pi i\left(\frac{\tau(6l-1)^2}{12}-l\right)\right]\right.\\
&\quad\left.+\exp\left[\pi i\left(\frac{\tau(6l+3)^2}{12}-\frac{6l+3}{6}\right)\right]\right].
\end{aligned} \tag{168}$$

In the last equality of equation (168), we use the further simplification:

$$\begin{aligned}&\sum_{l=-\infty}^{\infty}\exp\left[\pi i\left(\frac{\tau(6l+3)^2}{12}-\frac{6l+3}{6}\right)\right]\\&=\sum_{m\in2\mathbb{Z}+1}\exp\left[\pi i\left(\frac{3\tau m^2}{4}-\frac{m}{2}\right)\right]\\&=\sum_{m\in2\mathbb{Z}+1,m>0}\exp\left(\pi i\frac{3\tau m^2}{4}\right)\left(e^{\frac{\pi im}{2}}+e^{\frac{-\pi im}{2}}\right)\\&=0,\end{aligned} \tag{169}$$

in which we use:

$$e^{\pi im}=-1,\ m\in2\mathbb{Z}+1 \tag{170}$$

in the last equality. Combining equations (167) and (168) shows that:

$$\eta\left(-\frac{1}{\tau}\right)=\sqrt{-i\tau}\sum_{l=-\infty}^{\infty}\left[\exp\left[\pi i\left(\frac{\tau(6l-1)^2}{12}-l\right)\right]\right]=\sqrt{-i\tau}\eta(\tau). \tag{171}$$

The modular invariance property can be demonstrated in the following manner:

$$\sqrt{\mathrm{Im}\left(-\frac{1}{\tau}\right)}\left|\eta\left(-\frac{1}{\tau}\right)\right|^2=\sqrt{\mathrm{Im}\left(-\frac{1}{\tau}\right)|\tau|^2}|\eta(\tau)|^2=\sqrt{\mathrm{Im}(\tau)}|\eta(\tau)|^2. \tag{172}$$

Therefore, we can simplify the partition function even further

$$Z_M(\tau)=\frac{1}{\sqrt{\mathrm{Im}(\tau)}|\eta(\tau)|^2}\sum_{c,d}\left(\sqrt{\mathrm{Im}(\tau)}|q|^{-\frac{1}{12}\left(C_{\mathrm{cft}_2}-1\right)}|1-q|^2\right)|_M, \tag{173}$$

where $(\cdots)|_M$ means that

$$\tau\rightarrow\frac{a_1\tau+b_1}{c_1\tau+d_1}, \tag{174}$$

when $\mathrm{Im}(\tau)>0$.

3.6.2. Analytical expression. We will expand $|1-q|^2$ and express the $Z(\tau)$ as a sum of the Poincaré series [32]

$$P(\tau;n,m)\equiv\sum_{c_1,d_1;(c_1,d_1)=1}\left(\sqrt{\mathrm{Im}\tau}q^{-n}\bar{q}^{-m}\right)|_M. \tag{175}$$

In general, this series is divergent. We need to regularize the series through the analytical continuation [32]

$$P_1(\tau;s,n,m)\equiv\sum_{c_1,d_1;(c_1,d_1)=1}\left((\mathrm{Im}\tau)^s q^{-n}\bar{q}^{-m}\right)\big|_M. \tag{176}$$

In the domain $\mathrm{Re}(s) > 1$, we can get the following result

$$\begin{aligned}
&P_1(\tau;s,n,m)\\
&\quad= y_1^s e^{2\pi(\kappa y_1 + i\mu x_1)}\\
&\qquad+\sum_{c_1>0,d_1}\frac{y_1^s}{|c_1\tau+d_1|^2}\exp\left[\frac{2\pi\kappa y_1}{|c_1\tau+d_1|^2}+2\pi i\mu\left(\frac{a_1}{c_1}-\frac{c_1x_1+d_1}{c_1|c_1\tau+d_1|^2}\right)\right],
\end{aligned}\tag{177}$$

where

$$\kappa\equiv m+n,\ \mu\equiv m-n,\ \tau\equiv x_1+iy_1,\tag{178}$$

and then do the analytical continuation to $s=1/2$. The first term of equation (177) is given by $c_1=0$. Because $(0,d_1)=d_1$, we choose $d_1=1$ in the first term. The following fact is helpful for the calculation:

$$\begin{aligned}
\mathrm{Re}\left(\frac{a_1\tau+b_1}{c_1\tau+d_1}\right)&=\frac{a_1c_1|\tau|^2+b_1d_1+(a_1d_1+b_1c_1)x_1}{|c_1\tau+d_1|^2},\\
\mathrm{Im}\left(\frac{a_1\tau+b_1}{c_1\tau+d_1}\right)&=\frac{y_1}{|c_1\tau+d_1|^2}.
\end{aligned}\tag{179}$$

In the end, we obtain the analytical expression of the partition function after the analytical continuation:

$$\begin{aligned}
Z_M(\tau)=&\frac{1}{\sqrt{\mathrm{Im}(\tau)}|\eta(\tau)|^2}\\
&\times\left[P\left(\tau;\frac{C_{\mathrm{cft}_2}-1}{24},\frac{C_{\mathrm{cft}_2}-1}{24}\right)\right.\\
&-P\left(\tau;\frac{C_{\mathrm{cft}_2}-1}{24}-1,\frac{C_{\mathrm{cft}_2}-1}{24}\right)\\
&-P\left(\tau;\frac{C_{\mathrm{cft}_2}-1}{24},\frac{C_{\mathrm{cft}_2}-1}{24}-1\right)\\
&\left.+P\left(\tau;\frac{C_{\mathrm{cft}_2}-1}{24}-1,\frac{C_{\mathrm{cft}_2}-1}{24}-1\right)\right]\\
\sim&\frac{1}{\sqrt{\mathrm{Im}(\tau)}|\eta(\tau)|^2}\\
&\times\left[P_1\left(\tau;\frac{C_{\mathrm{cft}_2}-1}{24},\frac{C_{\mathrm{cft}_2}-1}{24}\right)\right.\\
&-P_1\left(\tau;\frac{C_{\mathrm{cft}_2}-1}{24}-1,\frac{C_{\mathrm{cft}_2}-1}{24}\right)\\
&-P_1\left(\tau;\frac{C_{\mathrm{cft}_2}-1}{24},\frac{C_{\mathrm{cft}_2}-1}{24}-1\right)\\
&\left.+P_1\left(\tau;\frac{C_{\mathrm{cft}_2}-1}{24}-1,\frac{C_{\mathrm{cft}_2}-1}{24}-1\right)\right].
\end{aligned}\tag{180}$$

4. EE and AdS_3/CFT_2 correspondence

To introduce EE, we begin with a discussion of classical Shannon entropy. We then introduce a reduced density matrix to define EE. Since the loss of a reduced density matrix in QFT, it is necessary to use the replica trick [9] to compute. We review the procedure of the replica trick. In the end, we review the holographic EE [6, 7] and the dual of EE from bulk Wilson line [47–49].

4.1. EE

We discuss the quantification of received information in the context of information theory, introducing a function $S(p)$ dependent on probability p. When one knows that the probability is one, it implies no surprise. Therefore, the S is

$$S(1) = 0. \tag{181}$$

The surprising level decreases as the probability increases:

$$S(p) > S(q)\,,\ p < q. \tag{182}$$

When two events are independent, the S has the additive property

$$S(pq) = S(p) + S(q)\,. \tag{183}$$

Finally, we assume that the S is a continuous function of the probability.

Using the additive property and the continuity of the S shows that

$$S(p^x) = xS(p) \tag{184}$$

with x as a positive rational number. Choosing

$$x = -\ln p, \tag{185}$$

where

$$0 < p \leqslant 1, \tag{186}$$

leads to

$$S(p) = S\left(e^{-x}\right) = x \cdot S\left(\frac{1}{e}\right) = -C \cdot \ln p, \tag{187}$$

where C is a constant,

$$C \equiv S\left(\frac{1}{e}\right) > S(1) = 0. \tag{188}$$

Therefore, we can derive a unique function that accurately defines the reception of information. The classical Shannon entropy is the expectation value of the S

$$S_c \equiv -\sum_j p_j \ln p_j. \tag{189}$$

When defining the von Neumann entropy, we replace the probability with a density matrix. For a pure quantum state in regions A and B, the density matrix is

$$\rho_{AB} \equiv |\psi\rangle\langle\psi|, \tag{190}$$

where $|\psi\rangle$ is a quantum state. The density matrix has already been normalized as

$$\mathrm{Tr}\left(\rho_{AB}\right) = 1, \tag{191}$$

which means that the trace of the density matrix is equal to one. The von Neumann entropy for the entire system is

$$S_{vn} \equiv -\mathrm{Tr}\left(\rho_{AB}\ln\rho_{AB}\right). \tag{192}$$

When one partial traces over a region B, one obtains a reduced density matrix for the region A

$$\rho_A \equiv \mathrm{Tr}_B\left(\rho_{AB}\right). \tag{193}$$

The EE for region A is

$$S_{EE,A} \equiv -\mathrm{Tr}\left(\rho_A\ln\rho_A\right). \tag{194}$$

4.2. Replica trick

We first introduce the field theory technique, replica trick [9], by the single interval case as in figure 1. The ground state wavefunctional is given by

$$\psi\left(\phi_0\left(\vec{x}\right)\right) = \int_{t_E=-\infty}^{\phi(t_E=0,\vec{x})=\phi_0(\vec{x})} \mathcal{D}\phi\; e^{-S(\phi)}, \tag{195}$$

where the flat Euclidean coordinates are $(t_E,\vec{x})$, and $S(\phi)$ is an action of a physical system.

A density matrix ρ_{AB} is given by

$$(\rho)_{\phi_0\tilde{\phi}_0} = \psi\left(\phi_0\right)\phi^\dagger\left(\tilde{\phi}_0\right). \tag{196}$$

The complex conjugate one $\psi^\dagger$ is given by a path integration from $t_E=\infty$ to $t_E=0$

$$\psi^\dagger\left(\tilde{\phi}_0\right) = \int_{t_E=\infty}^{\tilde{\phi}_0} \mathcal{D}\phi\; e^{-S(\phi)}. \tag{197}$$

A qubit state is a linear superposition of the different orthogonal states. Here ϕ_0 plays a similar role to the orthogonal state.

We integrate our ϕ_0 on the region B with that

$$\phi_0 = \tilde{\phi}_0 \tag{198}$$

for obtaining a reduced density matrix of the region A

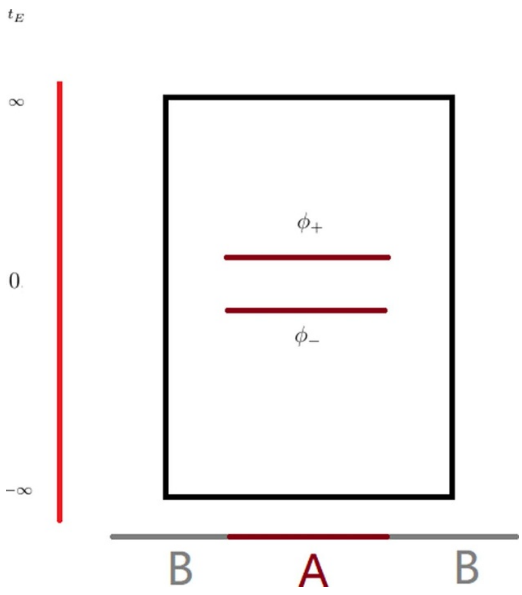

Figure 1. The path integration of a reduced density matrix ρ_{AB}.

$$(\rho_A)_{\phi_+\phi_-} = \frac{1}{Z_1}\int_{t_E=-\infty}^{t_E=\infty} \mathcal{D}\phi\, e^{-S(\phi)} \prod_{\vec{x}\in A} \delta\left(\phi\left(0^+,\vec{x}\right) - \phi_+\left(0,\vec{x}\right)\right) \delta\left(\phi\left(0^-,\vec{x}\right) - \phi_-\left(0,\vec{x}\right)\right), \tag{199}$$

where Z_1 is a partition function for a normalization

$$\mathrm{Tr}_A\rho_A = 1. \tag{200}$$

Performing a partial trace operation on one qubit results in the same level number for the orthogonal states of a qubit state. The condition (198) is similar to imposing the same level number.

Because it is hard to compute with the singularity at $t_E = 0$, we prepare n copies of the reduced density matrix of the region A as that

$$(\rho_A)_{\phi_{1+}\phi_{1-}} (\rho_A)_{\phi_{2+}\phi_{2-}} \cdots (\rho_A)_{\phi_{n+}\phi_{n-}} \tag{201}$$

with a boundary condition

$$\phi_{j-}(\vec{x}) = \phi_{(j+1)+}(\vec{x}),\ j = 1,2,\cdots,n, \tag{202}$$

where

$$\phi_{(n+1)+}(\vec{x}) \equiv \phi_{1+}(\vec{x}). \tag{203}$$

We then integrate each ϕ_{j+} to realize a partial trace operation. Therefore, a path-integral representation of $\mathrm{Tr}_A\rho_A^n$ can be provided:

$$\mathrm{Tr}_A\rho_A^n = \frac{1}{Z_1^n}\int_{(t_E,\vec{x})\in\mathcal{R}_n} \mathcal{D}\phi\, e^{-S(\phi)} \equiv \frac{Z_n}{Z_1^n}, \tag{204}$$

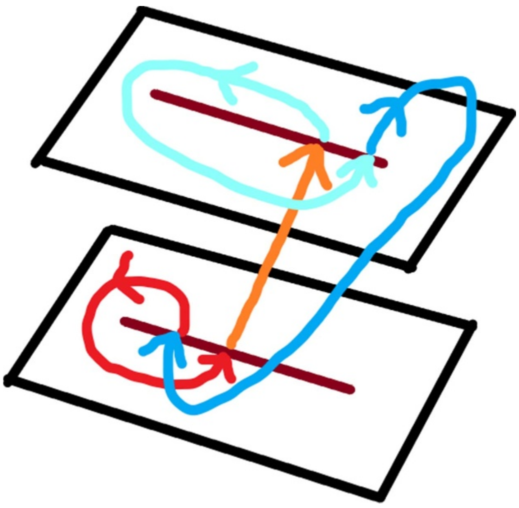

Figure 2. The demonstration of n-sheet manifold $\mathcal{R}_n$ for $n=2$.

where $\mathcal{R}_n$ is an n-sheet manifold as in figure 2, and Z_n is an n-sheet partition function (defined on an $\mathcal{R}_n$). In the end, we can take the one-sheet limit $n \to 1$ on Rényi entropy

$$S_n \equiv \frac{\ln \mathrm{Tr}_A \rho_A^n}{1-n} \tag{205}$$

to obtain EE as that

$$S_1 = \lim_{n\to 1} S_n. \tag{206}$$

4.3. RT conjecture

We introduce the RT conjecture [6, 7] in this section. The RT conjecture states that an AdS_{d+1} bulk minimum surface is dual to EE of CFT_d. We demonstrate the AdS_3 case.

The AdS_3 spacetime in the Poincaré coordinate is given by

$$\mathrm{d}s_{3P}^2 = -\frac{1}{\Lambda}\frac{\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}z^2}{z^2}. \tag{207}$$

The AdS_3 induced metric is given by:

$$\mathrm{d}s_{3b}^2 = h_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\frac{1}{\Lambda}\frac{1}{z^2}\left[1 + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2\right]\mathrm{d}x^2 \tag{208}$$

by choosing a time slice (t as a constant). Hence the area of this surface is given by

$$A_{AdS_3} = \sqrt{-\frac{1}{\Lambda}}\int \mathrm{d}x\, \frac{1}{z}\sqrt{1 + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2}. \tag{209}$$

The minimum area satisfies the relation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\delta A_{\mathrm{AdS}_3}}{\delta z'} = \frac{\delta A_{\mathrm{AdS}_3}}{\delta z},\ \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\frac{\mathrm{d}z}{\mathrm{d}x}}{z}\frac{1}{\sqrt{1+\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2}}\right] = -\frac{1}{z^2}\sqrt{1+\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2}, \tag{210}$$

where

$$z' \equiv \frac{\mathrm{d}z}{\mathrm{d}x}. \tag{211}$$

One solution is:

$$z(x) = \sqrt{L^2 - x^2},\ \frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{x}{z}. \tag{212}$$

We check the solution as in the following:

$$\begin{aligned}
-\frac{1}{z^2}\sqrt{1+\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2} &= -\frac{1}{z^2}\sqrt{1+\frac{x^2}{z^2}} = -\frac{L}{(L^2-x^2)^{\frac{3}{2}}},\\
\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\frac{\mathrm{d}z}{\mathrm{d}x}}{z}\frac{1}{\sqrt{1+\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2}}\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{x}{z^2}\frac{1}{\sqrt{1+\frac{2}{z^2}}}\right]\\
&= -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{z\sqrt{z^2+x^2}}\right) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{Lz}\right)\\
&= -\frac{1}{Lz} - \frac{x^2}{Lz^3} = -\frac{1}{L}\frac{z^2+x^2}{z^3} = -\frac{L}{z^3}\\
&= -\frac{L}{(L^2-x^2)^{\frac{3}{2}}} = -\frac{1}{z^2}\sqrt{1+\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2}.
\end{aligned} \tag{213}$$

The minimum area is given by:

$$\begin{aligned}
A_{\mathrm{AdS}_3} &= \sqrt{-\frac{1}{\Lambda}}\int_{-L+\delta}^{L-\delta}\mathrm{d}x\ \frac{1}{z}\sqrt{1+\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2} = \sqrt{-\frac{1}{\Lambda}}\int_{-L+\delta}^{L-\delta}\mathrm{d}x\ \frac{1}{z}\sqrt{1+\left(\frac{x}{z}\right)^2}\\
&= \sqrt{-\frac{1}{\Lambda}}\int_{-L+\delta}^{L-\delta}\mathrm{d}x\ \frac{L}{z^2} = \sqrt{-\frac{1}{\Lambda}}\int_{-L+\delta}^{L-\delta}\mathrm{d}x\ \frac{L}{L^2-x^2}\\
&= \frac{1}{2}\sqrt{-\frac{1}{\Lambda}}\int_{-L+\delta}^{L-\delta}\mathrm{d}x\ \left(\frac{1}{L-x}+\frac{1}{L+x}\right)\\
&= \sqrt{-\frac{1}{\Lambda}}\int_{0}^{L-\delta}\mathrm{d}x\ \left(\frac{1}{L-x}+\frac{1}{L+x}\right)\\
&= \sqrt{-\frac{1}{\Lambda}}\ln\left|\frac{L+x}{x-L}\right|\Bigg|_0^{L-\delta} = \sqrt{-\frac{1}{\Lambda}}\ln\frac{2L-\delta}{\delta}.
\end{aligned} \tag{214}$$

The integration range excludes the boundary sites L and $-L$ corresponding to CFT_2's entangling surface. We then define that:

$$\epsilon \equiv \sqrt{L^2-(L-\delta)^2} = \sqrt{2\delta L-\delta^2}. \tag{215}$$

Hence we can obtain two solutions

$$\delta = L \pm \sqrt{L^2 - \epsilon^2}. \tag{216}$$

If we assume $L \gg \epsilon > 0$, we should choose the solution

$$\delta = L - \sqrt{L^2 - \epsilon^2}. \tag{217}$$

Now minimum area becomes:

$$A_{AdS_3} = \sqrt{-\frac{1}{\Lambda}} \ln \frac{2L - \delta}{\delta} = \sqrt{-\frac{1}{\Lambda}} \ln \frac{L + \sqrt{L^2 - \epsilon^2}}{L - \sqrt{L^2 - \epsilon^2}}. \tag{218}$$

The holographic EE for the AdS$_3$ metric is shown as that [6, 7]:

$$\begin{aligned}
\frac{A_{\mathrm{AdS}_3}}{4G_3} &= \frac{1}{4\sqrt{-\Lambda}G_3} \ln \frac{L + \sqrt{L^2 - \epsilon^2}}{L - \sqrt{L^2 - \epsilon^2}} \\
&= \frac{c_{\mathrm{cft}_2}}{6} \ln \frac{L + \sqrt{L^2 - \epsilon^2}}{L - \sqrt{L^2 - \epsilon^2}} = \frac{c_{\mathrm{cft}_2}}{6} \ln \frac{4L^2}{\epsilon^2} + \cdots \\
&= \frac{c_{\mathrm{cft}_2}}{3} \ln \frac{2L}{\epsilon} + \cdots = \frac{c_{\mathrm{cft}_2}}{3} \ln \frac{L}{\epsilon} + \ldots,
\end{aligned} \tag{219}$$

the center charge of CFT$_2$ is given by [31]

$$c_{\mathrm{cft}_2} \equiv \frac{3}{2\sqrt{-\Lambda}G_3} \tag{220}$$

We use the following expansion:

$$\begin{aligned}
&L + \sqrt{L^2 - \epsilon^2} = 2L + \ldots, \\
&L - \sqrt{L^2 - \epsilon^2} = L\left(1 - \frac{\epsilon^2}{2L^2}\right) + \cdots = L - \frac{\epsilon^2}{2L}
\end{aligned} \tag{221}$$

in the third equality of equation (219). Therefore, we have demonstrated that the RT conjecture is capable of reproducing the EE of CFT$_2$.

4.4. EE in 2D Schwarzian theory

We apply the conformal mapping to transform a planar to a two-sphere (S^2) [8]. The removal of the top and bottom points from S^2 results in a cylinder manifold. This manifold serves as the asymptotic boundary for the Lorentzian AdS$_3$ in global coordinates. Once we identify the boundary and apply the Wick rotation to the AdS$_3$ spacetime, the cylinder transforms into a torus. This allows us to calculate the EE using the partition function of the n-sheet torus [49, 50]. However, this method may not account for the global effect resulting from the boundary identification.

4.4.1. Conformal mapping. The planar manifold is

$$\mathrm{d}s_P^2 = \mathrm{d}t^2 + \mathrm{d}x^2, \tag{222}$$

where

$$-\infty < t, x < \infty. \tag{223}$$

The entangling surface, a boundary of two subregions, is at $(t, |x|) = (0, L)$. The length of the interval is $2L$ at $t = 0$. We define new coordinate variables τ and u:

$$t = L\frac{\sin\left(\frac{\tau}{L}\right)}{\cosh(u) + \cos\left(\frac{\tau}{L}\right)}; \qquad x = L\frac{\sinh(u)}{\cosh(u) + \cos\left(\frac{\tau}{L}\right)}, \tag{224}$$

where

$$-\infty < u < \infty;\ 0 \leqslant \frac{\tau}{L} < 2\pi n. \tag{225}$$

on the n-sheet manifold, which gives the following terms:

$$\begin{aligned}
\mathrm{d}t &= \frac{\left(1 + \cos\left(\frac{\tau}{L}\right)\cosh(u)\right)\mathrm{d}\tau}{\left(\cosh(u) + \cos\left(\frac{\tau}{L}\right)\right)^2} - \frac{L\sin\left(\frac{\tau}{L}\right)\sinh(u)\,\mathrm{d}u}{\left(\cosh(u) + \cos\left(\frac{\tau}{L}\right)\right)^2},\\
\mathrm{d}x &= \frac{L\left(1 + \cos\left(\frac{\tau}{L}\right)\cosh(u)\right)\mathrm{d}u}{\left(\cosh(u) + \cos\left(\frac{\tau}{L}\right)\right)^2} + \frac{\sin\left(\frac{\tau}{L}\right)\sinh(u)\,\mathrm{d}\tau}{\left(\cosh(u) + \cos\left(\frac{\tau}{L}\right)\right)^2},\\
\mathrm{d}t^2 + \mathrm{d}x^2 &= \frac{\mathrm{d}\tau^2}{\left(\cosh^2(u) + \cos^2\left(\frac{\tau}{L}\right)\right)^2} + L^2\frac{\mathrm{d}u^2}{\left(\cosh^2(u) + \cos^2\left(\frac{\tau}{L}\right)\right)^2},
\end{aligned} \tag{226}$$

and then we obtain the metric in the new coordinate as that

$$ds_P^2 = \frac{\mathrm{d}\tau^2}{\left(\cosh^2(u) + \cos^2\left(\frac{\tau}{L}\right)\right)^2} + L^2\frac{\mathrm{d}u^2}{\left(\cosh^2(u) + \cos^2\left(\frac{\tau}{L}\right)\right)^2}. \tag{227}$$

In CFT, we can omit the common pre-factor, which generates the Liouville theory, and the new metric becomes

$$\mathrm{d}s_{P1}^2 = \frac{\mathrm{d}\tau^2}{L^2} + \mathrm{d}u^2. \tag{228}$$

The CFT partition function is invariant under a local rescaling of the metric (or the Weyl transformation). We then redefine

$$\sinh(u) = \cot(\psi), \tag{229}$$

where

$$0 \leqslant \psi < \pi, \tag{230}$$

and get

$$\mathrm{d}s_{P1}^2 = \frac{\mathrm{d}\tau^2}{L^2} + \frac{\mathrm{d}\psi^2}{\sin^2(\psi)}, \tag{231}$$

in which we use

$$\mathrm{d}u^2 = \frac{\mathrm{d}\psi^2}{\sin^2(\psi)}. \tag{232}$$

We omit the pre-factor to obtain the sphere manifold [8]

$$\mathrm{d}s_{P2}^2 = \mathrm{d}\psi^2 + \sin^2(\psi)\frac{\mathrm{d}\tau^2}{L^2}. \tag{233}$$

The sphere manifold is isomorphic to a cylinder manifold by removing the top and bottom points. We can use:

$$\mathrm{sech}(y) \equiv \sin(\psi);\ \mathrm{d}\theta \equiv \frac{\mathrm{d}\tau}{L} \tag{234}$$

to obtain

$$\mathrm{d}s_{P2}^2 = \mathrm{sech}^2(y)\left(\mathrm{d}y^2 + \mathrm{d}\theta^2\right). \tag{235}$$

After removing the pre-factor $\mathrm{sech}(y)$, we obtain the cylinder manifold

$$\mathrm{d}s_C^2 = \mathrm{d}y^2 + \mathrm{d}\theta^2, \tag{236}$$

where

$$0 \leqslant \theta < 2\pi n. \tag{237}$$

To compute the n-sheet partition function, we regularize the range of the y-direction

$$-\ln\left(\frac{L}{\epsilon}\right) < y < \ln\left(\frac{L}{\epsilon}\right), \tag{238}$$

where ϵ represents a number infinitely close to zero, and also identifies the boundary:

$$z_n \equiv \frac{\theta + iy}{n} \sim z_n + 2\pi\tau_n, \tag{239}$$

where τ_n is the complex structure of the n-sheet torus

$$\tau_n = \frac{i}{n\pi}\ln\left(\frac{L}{\epsilon}\right). \tag{240}$$

to obtain the torus manifold. The boundary fields in equation (104) satisfy the boundary conditions:

$$\begin{aligned}
&\phi\left(\frac{y}{n}, \frac{\theta}{n} + 2\pi\right) = \phi\left(\frac{y}{n}, \frac{\theta}{n}\right) + 2\pi, \\
&\phi\left(\frac{y}{n} + 2\pi\cdot\mathrm{Im}(\tau_n), \frac{\theta}{n} + 2\pi\cdot\mathrm{Re}(\tau_n)\right) = \phi\left(\frac{y}{n}, \frac{\theta}{n}\right); \\
&\bar{\phi}\left(\frac{y}{n}, \frac{\theta}{n} + 2\pi\right) = \bar{\phi}\left(\frac{y}{n}, \frac{\theta}{n}\right) + 2\pi, \\
&\bar{\phi}\left(\frac{y}{n} + 2\pi\cdot\mathrm{Im}(\tau_n), \frac{\theta}{n} + 2\pi\cdot\mathrm{Re}(\tau_n)\right) = \bar{\phi}\left(\frac{y}{n}, \frac{\theta}{n}\right).
\end{aligned} \tag{241}$$

4.4.2. EE for a single interval. We calculate the Rényi entropy

$$S_n = \frac{\ln Z_n - n\ln Z_1}{1-n}. \tag{242}$$

When $n \to 1$, we can extract EE from Rényi entropy. The n-sheet torus partition function is one-loop exact. Therefore, the contribution of the n-sheet partition function is up to the one-loop order [49, 50]. When applying the Weyl transformation from the cylinder manifold to a sphere manifold without the top and bottom points, the Liouville theory also appears and decouples from the 2D Schwarzian theory as in the torus case [50]. The Liouville theory does not provide the backreaction to affect the classical solution [50]. Therefore, the n-sheet partition function is a product of the classical n-sheet partition-function ($Z_{n,c}$) and the one-loop n-sheet partition-function ($Z_{n,q}$)

$$Z_n = Z_{n,c}\cdot Z_{n,q}. \tag{243}$$

The logarithm on the n-sheet partition function in calculating S_n,

$$\ln Z_n = \ln Z_{n,c} + \ln Z_{n,q}. \tag{244}$$

does not mix the classical and the one-loop terms. The boundary fields can be expanded as [41]:

$$\phi = \frac{\theta}{n} + \epsilon(y,\theta); \qquad \bar{\phi} = -\frac{\theta}{n} + \bar{\epsilon}(y,\theta), \tag{245}$$

where

$$\begin{aligned}
\epsilon(y,\theta) &\equiv \sum_{j_1,k_1}\epsilon_{j_1,k_1}e^{i\frac{j_1}{n}\theta - \frac{k_1}{\tau}y}; \qquad \epsilon^*_{j_1,k_1} \equiv \epsilon_{-j_1,-k_1},\\
\bar{\epsilon}(y,\theta) &\equiv \sum_{j_1,k_1}\bar{\epsilon}_{j_1,k_1}e^{i\frac{j_1}{n}\theta - \frac{k_1}{\tau}y}; \qquad \bar{\epsilon}^*_{j_1,k_1} \equiv \bar{\epsilon}_{-j_1,-k_1},
\end{aligned} \tag{246}$$

where

$$\tau_1 = \tau. \tag{247}$$

The saddle-points are the θ/n and the $-\theta/n$ for the ϕ and $\bar{\phi}$, respectively [41]. Each Fourier mode of the fluctuation, ϵ and $\bar{\epsilon}$, has three zero-modes:

$$\epsilon_{j_1,k_1} = 0; \qquad \bar{\epsilon}_{j_1,k_1} = 0,\ j_1 = -1,0,1 \tag{248}$$

due to the SL(2) gauge symmetry [41].

We first substitute the saddle points into the action. We then obtain

$$\ln Z_{n,c} = \frac{c_{\mathrm{cft}_2}}{6n}\ln\left(\frac{L}{\epsilon}\right) + n\frac{c_{\mathrm{cft}_2}}{6}\ln\left(\frac{L}{\epsilon}\right), \tag{249}$$

where the first term is from the 2D Schwarzian theory, and the second term is from the Liouville theory. Therefore, we derive the contribution of Rényi entropy from the saddle points:

$$S_{n,c} = \frac{1}{1-n}\left[\frac{c_{\mathrm{cft}_2}}{6n}\ln\left(\frac{L}{\epsilon}\right) - n\frac{c_{\mathrm{cft}_2}}{6}\ln\left(\frac{L}{\epsilon}\right)\right] = \frac{c_{\mathrm{cft}_2}(1+n)}{6n}\ln\frac{L}{\epsilon}. \tag{250}$$

When $n \to 1$, we reproduce the known result of CFT_2

$$\lim_{n\to 1} S_n = \frac{c_{\mathrm{cft}_2}}{3} \ln \frac{L}{\epsilon}. \tag{251}$$

Now we discuss the one-loop contribution of S_n from the $\epsilon(y,\theta)$ and $\bar{\epsilon}(y,\theta)$ [49, 50]. The expansion from the ϵ in the boundary action is [49, 50]

$$\begin{aligned}
&\frac{k}{4\pi}\int_{-\pi\mathrm{Im}(\tau)}^{\pi\mathrm{Im}(\tau)} \mathrm{d}y \int_0^{2\pi n} \mathrm{d}\theta \Bigg(n^2 \left(\partial_\theta^2 \epsilon(y,\theta)\right)\left(\bar{\partial}\partial_\theta \epsilon(y,\theta)\right) \\
&\quad - \left(\partial_\theta \epsilon(y,\theta)\right)\left(\bar{\partial}\epsilon(y,\theta)\right)\Bigg) \\
&= -i\frac{k}{4\pi} n\tau \cdot \Bigg\{ n^2 \sum_{j_1,k_1} \left[\left(-\frac{j_1^2}{n^2}\cdot\frac{1}{2}\cdot\left(i\frac{k_1}{\tau}+i\frac{j_1}{n}\right)\left(i\frac{j_1}{n}\right)\right]|\epsilon_{j_1,k_1}|^2 \\
&\quad - \sum_{j_1,k_1}\left[\left(i\frac{j_1}{n}\right)\cdot\frac{1}{2}\cdot\left(i\frac{k_1}{\tau}+i\frac{j_1}{n}\right)\right]|\epsilon_{j_1,k_1}|^2 \Bigg\} \\
&= -i\frac{k}{8\pi}\sum_{j_1,k_1} j_1\left(j_1^2-1\right)\left(k_1+\frac{j_1}{n}\tau\right)|\epsilon_{j_1,k_1}|^2,
\end{aligned} \tag{252}$$

where

$$\bar{\partial} \equiv \frac{1}{2}\left(-i\partial_y + \partial_\theta\right). \tag{253}$$

Taking the derivative of τ on the logarithm of the n-sheet one-loop partition gives

$$\partial_\tau \ln Z_{n,q} = -\sum_{j_1\neq 0,\pm 1}\sum_{k_1=-\infty}^{\infty} \frac{\frac{j_1}{n}}{k_1+\frac{j_1}{n}\tau}. \tag{254}$$

The following fact of the digamma function

$$\tilde{\psi}(1-x) - \tilde{\psi}(x) = \pi\cot(\pi x), \tag{255}$$

in which the digamma function is defined by

$$\tilde{\psi}(a) \equiv -\sum_{n_1=0}^{\infty}\frac{1}{n_1+a}, \tag{256}$$

is helpful to simplify the complicated summation in the n-sheet partition function:

$$\begin{aligned}
\sum_{m=-\infty}^{\infty}\frac{1}{m-x} &= -\sum_{m=0}^{\infty}\frac{1}{m+x} + \sum_{m=1}^{\infty}\frac{1}{m-x} \\
&= -\sum_{m=0}^{\infty}\frac{1}{m+x} + \sum_{m=0}^{\infty}\frac{1}{m+1-x} = \tilde{\psi}(x) - \tilde{\psi}(1-x) \\
&= -\pi\cdot\cot(\pi x).
\end{aligned} \tag{257}$$

Hence we obtain:

$$\partial_\tau \ln Z_{n,q} = -\sum_{j_1\neq 0,\pm 1}\sum_{k_1=-\infty}^{\infty}\frac{\frac{j_1}{n}}{k_1+\frac{j_1}{n}\tau} = -\sum_{j_1\neq 0,\pm 1}\left(\frac{j_1}{n}\pi\right)\cdot\cot\left(\frac{j_1}{n}\pi\tau\right)$$
$$= -2\pi\sum_{j=2}^{\infty}\left(\frac{j_1}{n}\right)\cdot\cot\left(\frac{j_1\pi\tau}{n}\right). \tag{258}$$

To obtain a universal term, we regularize the series:

$$\partial_\tau \ln Z_{n,q} = -2\pi\sum_{j_1=2}^{\infty}\left(\frac{j_1}{n}\right)\cdot\cot\left(\frac{j_1\pi\tau}{n}\right)$$
$$= -2\pi\sum_{j_1=2}^{\infty}\frac{j_1}{n}\cdot\left[\cot\left(\frac{j_1\pi\tau}{n}\right)+i\right]+2\pi i\sum_{j_1=2}^{\infty}\frac{j_1}{n}. \tag{259}$$

The series

$$\sum_{j_1=1}^{\infty} j_1 = 1+2\cdots \tag{260}$$

is divergent meaning that it does not converge to a finite value. We introduce the regularization to the summation:

$$\lim_{\epsilon\to 0}\sum_{n=1}^{\infty}e^{-n\epsilon}n = -\lim_{\epsilon\to 0}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\sum_{n=1}^{\infty}e^{-n\epsilon} = -\lim_{\epsilon\to 0}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\frac{e^{-\epsilon}}{1-e^{-\epsilon}}$$
$$= -\lim_{\epsilon\to 0}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\frac{1}{e^{\epsilon}-1} = -\lim_{\epsilon\to 0}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(\frac{1}{\epsilon}-\frac{1}{2}+\frac{\epsilon}{12}\right)$$
$$= -\frac{1}{12}+\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}. \tag{261}$$

The fourth equality uses:

$$e^{\epsilon}-1 = \epsilon+\frac{\epsilon^2}{2}+\frac{\epsilon^3}{6}+\ldots,$$
$$\frac{1}{e^{\epsilon}-1} = \frac{1}{\epsilon}\frac{1}{1+\frac{\epsilon}{2}+\frac{\epsilon^2}{6}+\cdots} = \frac{1}{\epsilon}\left(1-\frac{\epsilon}{2}-\frac{\epsilon^2}{6}+\frac{\epsilon^2}{4}+\cdots\right)$$
$$= \frac{1}{\epsilon}-\frac{1}{2}+\frac{\epsilon}{12}+\ldots. \tag{262}$$

We can apply the following result

$$\sum_{j_1=1}^{\infty} j_1 \to -\frac{1}{12}. \tag{263}$$

to obtain:

$$\partial_\tau \ln Z_{n,q} = -2\pi \sum_{j_1=2}^{\infty} \frac{j_1}{n} \cdot \left[\cot\left(\frac{j_1\pi\tau}{n}\right) + i\right] + 2\pi i \sum_{j_1=2}^{\infty} \frac{j_1}{n}$$
$$\to -2\pi \sum_{j_1=2}^{\infty} \frac{j_1}{n} \cdot \left[\cot\left(\frac{j_1\pi\tau}{n}\right) + i\right] - i\frac{13\pi}{6n}. \tag{264}$$

We integrate the τ, we obtain

$$\ln Z_{n,q} = -2\sum_{j_1=2}^{\infty}\left[\ln\sin\left(\frac{\pi j_1\tau}{n}\right) + i\frac{j_1\pi\tau}{n}\right] - i\frac{13\pi\tau}{6n} + \ldots, \tag{265}$$

where $\cdots$ is independent of the τ. Because the series is convergent in $\ln Z_{n,q}$ when $\mathrm{Im}(\tau) > 0$, we obtain

$$\ln Z_{n,q} = \frac{13}{3n}\ln\frac{L}{\epsilon} \tag{266}$$

when considering the limit

$$\frac{L}{\epsilon} \to \infty. \tag{267}$$

Hence we obtain

$$\ln Z_{n,q} = \frac{13}{6n}\ln\frac{L}{\epsilon}, \tag{268}$$

and the Rényi entropy for the one-loop correction is [49, 50]:

$$S_{n,q} = \frac{1}{1-n}\left(\ln Z_{n,q} - n\ln Z_{1,q}\right) = \frac{13(n+1)}{6n}\ln\frac{L}{\epsilon}. \tag{269}$$

This result implies that the quantum correction only shifts the value of the central charge

$$C_{\mathrm{cft}_2} = c_{\mathrm{cft}_2} + 13 \tag{270}$$

in the Rényi entropy [49, 50]

$$S_n = S_{n,c} + S_{n,q} \tag{271}$$

and EE [49, 50]

$$\lim_{n\to 1} S_n. \tag{272}$$

Hence the large central charge limit of CFT_2 includes the classical and one-loop contributions from AdS_3 bulk gravity. The result should be due to the different perturbation parameters between the bulk gravity and CFT.

4.5. Minimum surface = EE

We introduce Wilson line to the action for the n-sheet case [47, 48]

$$W_{\mathcal{R}}(C) = \int DUDP\ \exp\left[\int_C ds\ \left(\mathrm{Tr}\left(PU^{-1}D_sU\right) + \lambda(s)\left(\mathrm{Tr}\left(P^2\right) - c_2\right)\right)\right], \tag{273}$$

where

$$\sqrt{2c_2} = \frac{C_{cft_2}}{6}(1-n). \tag{274}$$

The U is an SL(2) element, P is its conjugate momentum, and the covariant derivative is defined as the following:

$$D_sU \equiv \frac{\mathrm{d}}{\mathrm{d}s}U + A_sU - U\bar{A}_s, \qquad A_s \equiv A_\mu\frac{\mathrm{d}x^\mu}{\mathrm{d}s}, \qquad \bar{A}_s \equiv \bar{A}_\mu\frac{\mathrm{d}x^\mu}{\mathrm{d}s}. \tag{275}$$

The endpoints of the Wilson line lie on the entangling surface of a single interval. We show the equation of motion of the P [47, 48]

$$U^{-1}D_sU = -2\lambda P; \tag{276}$$

the equation of motion of the U [47, 48]

$$\frac{\mathrm{d}P}{\mathrm{d}s} = 0; \tag{277}$$

the equation of motion of the $\lambda(s)$ [47, 48]

$$\mathrm{Tr}P^2 = c_2; \tag{278}$$

the equation of the gauge field A [47, 48]

$$i\frac{k}{2\pi}F_{\mu\nu} = -\int \mathrm{d}s\ \frac{\mathrm{d}x^\rho}{\mathrm{d}s}\epsilon_{\mu\nu\rho}\delta^3(x - x(s))\,UPU^{-1}; \tag{279}$$

the equation of the gauge field $\bar{A}$ [47, 48]

$$i\frac{k}{2\pi}\bar{F}_{\mu\nu} = -\int \mathrm{d}s\ \frac{\mathrm{d}x^\rho}{\mathrm{d}s}\epsilon_{\mu\nu\rho}\delta^3(x - x(s))\,P. \tag{280}$$

The solution can be written as follows [47, 48]:

$$\begin{aligned} A &= g^{-1}ag + g^{-1}dg, \qquad g = \exp(L_1z_y)\exp(\rho L_0);\\ \bar{A} &= \bar{g}^{-1}a\bar{g}^{-1} + \bar{g}^{-1}d\bar{g}, \quad \bar{g} = \exp(L_{-1}\bar{z}_y)\exp(-\rho L_0), \end{aligned} \tag{281}$$

where the gauge field is defined by that

$$a \equiv \sqrt{\frac{c_2}{2}}\frac{1}{k}\left(\frac{\mathrm{d}z_y}{z_y} - \frac{\mathrm{d}\bar{z}_y}{\bar{z}_y}\right)L_0, \tag{282}$$

corresponding to the choice:

$$\rho(s) = s, \qquad U(s) = 1, \qquad P(s) = \sqrt{2c_2}L_0. \tag{283}$$

ρ is a variable denoting the radial direction. The new coordinate variables are given by:

$$z_y = \theta + iy, \qquad \bar{z}_y = \theta - iy. \tag{284}$$

The gauge field a provides the holonomy [47, 48]

$$\oint a = 2\pi i \frac{\sqrt{2c_2}}{k} L_0. \tag{285}$$

We introduce a more convenient basis to the SL(2) algebras, L_{-1}; L_0; L_1, [47, 48]

$$[L_m, L_n] = (m-n) L_{m+n}, \qquad m, n = 0, \pm 1, \tag{286}$$

$$\mathrm{Tr}\left(L_0^2\right) = \frac{1}{2}, \qquad \mathrm{Tr}\left(L_{-1}L_1\right) = -1. \tag{287}$$

The trace of other bilinears is zero.
When we choose $c_2 = 0$ or $n = 1$, the solution of the gauge fields is [47, 48]:

$$\begin{aligned}
A &= g^{-1}\mathrm{d}g \\
&= \exp\left(-\rho L_0\right)\exp\left(-L_1 z_y\right) L_1 \exp\left(L_1 z_y\right)\exp\left(\rho L_0\right)\mathrm{d}z_y \\
&\quad + \exp\left(-\rho L_0\right)\exp\left(-L_1 z_y\right)\exp\left(L_1 z_y\right) L_0 \exp\left(\rho L_0\right)\mathrm{d}\rho \\
&= \exp\left(-\rho L_0\right) L_1 \exp\left(\rho L_0\right)\mathrm{d}z_y + L_0 \mathrm{d}\rho \\
&= \exp\left(\rho\right) L_1 \mathrm{d}z_y + L_0 \mathrm{d}\rho, \\
\bar{A} &= \bar{g}^{-1}\mathrm{d}\bar{g} \\
&= \exp\left(\rho L_0\right)\exp\left(-L_{-1}\bar{z}_y\right) L_{-1} \exp\left(L_{-1}\bar{z}_y\right)\exp\left(-\rho L_0\right)\mathrm{d}\bar{z}_y \\
&\quad - \exp\left(\rho L_0\right)\exp\left(-L_{-1}\bar{z}_y\right)\exp\left(L_{-1}\bar{z}_y\right) L_0 \exp\left(-\rho L_0\right)\mathrm{d}\rho \\
&= \exp\left(\rho L_0\right) L_{-1} \exp\left(-\rho L_0\right) d\bar{z}_y - L_0 \mathrm{d}\rho \\
&= \exp\left(\rho\right) L_{-1} \mathrm{d}\bar{z}_y - L_0 \mathrm{d}\rho.
\end{aligned} \tag{288}$$

Therefore, the vielbein becomes [47, 48]:

$$e = \frac{1}{2}\left(A - \bar{A}\right) = -\frac{1}{2}\exp\left(\rho\right) L_1 \mathrm{d}z_y + \frac{1}{2}\exp\left(\rho\right) L_{-1}\mathrm{d}\bar{z}_y + L_0 \mathrm{d}\rho. \tag{289}$$

We can use the vielbein to determine the metric [47, 48]

$$g_{\mu\nu} = 2\mathrm{Tr}\left(e_\mu e_\nu\right). \tag{290}$$

The spacetime interval is [47, 48]

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + e^{2\rho}\mathrm{d}z_y\mathrm{d}\bar{z}_y. \tag{291}$$

We then redefine z_y and $\bar{z}_y$:

$$z_y = re^{i\Phi}, \qquad \bar{z}_y = re^{-i\Phi}, \tag{292}$$

where

$$0 < r < \infty; 0 \geqslant \Phi < 2\pi, \tag{293}$$

to get the spacetime interval [47, 48]

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + e^{2\rho}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Phi^2\right). \tag{294}$$

We then redefine the coordinate variables as

$$\tilde{z} \equiv e^{-\rho}. \tag{295}$$

$\mathrm{d}s^2$ is precisely the AdS_3 solution in the Poincaré coordinate with the Euclidean signature [47, 48]

$$\mathrm{d}s^2 = \frac{1}{\tilde{z}^2}\left(\mathrm{d}\tilde{z}^2 + \mathrm{d}r^2 + \tilde{r}^2\mathrm{d}\Phi^2\right). \tag{296}$$

When n approaches 1, the geometry becomes the AdS_3 manifold.

Now we introduce the non-trivial gauge field a to consider the backreaction [47, 48]:

$$\begin{aligned} A &= \exp(\rho)L_1\mathrm{d}z_y + L_0\mathrm{d}\rho + \frac{1}{2k}\sqrt{\frac{c_2}{2}}\left(L_0 - z_y e^{\rho}L_1\right)\left(\frac{\mathrm{d}z_y}{z_y} - \frac{\mathrm{d}\bar{z}_y}{\bar{z}_y}\right), \\ \bar{A} &= \exp(\rho)L_{-1}\mathrm{d}\bar{z}_y - L_0\mathrm{d}\rho + \frac{1}{2k}\sqrt{\frac{c_2}{2}}\left(L_0 + \bar{z}_y e^{\rho}L_{-1}\right)\left(\frac{\mathrm{d}z_y}{z_y} - \frac{\mathrm{d}\bar{z}_y}{\bar{z}_y}\right). \end{aligned} \tag{297}$$

The vielbein is [47, 48]:

$$\begin{aligned} e &= \frac{1}{2}\left(A - \bar{A}\right) \\ &= \frac{1}{2}\exp(\rho)L_1\mathrm{d}z_y - \frac{1}{2}\exp(\rho)L_{-1}\mathrm{d}\bar{z}_y + L_0\mathrm{d}\rho \\ &\quad + \frac{1}{2k}\sqrt{\frac{c_2}{2}}\left(L_0 - z_y e^{\rho}L_1\right)\left(\frac{\mathrm{d}z_y}{z_y} - \frac{\mathrm{d}\bar{z}_y}{\bar{z}_y}\right) \\ &\quad - \frac{1}{2k}\sqrt{\frac{c_2}{2}}\left(L_0 + \bar{z}_y e^{\rho}L_{-1}\right)\left(\frac{\mathrm{d}z_y}{z_y} - \frac{\mathrm{d}\bar{z}_y}{\bar{z}_y}\right). \end{aligned} \tag{298}$$

The backreaction provides the n-sheet geometry [47, 48]:

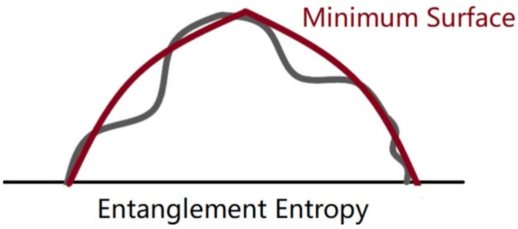

Figure 3. The minimum surface gives the on-shell contribution to the boundary theory's EE. The gauge fields cause surface fluctuations, which deform the central charge in EE.

$$\begin{aligned}
\mathrm{d}s^2 =& \mathrm{d}\rho^2 + e^{2\rho}\mathrm{d}z_y\mathrm{d}\bar{z}_y \\
&- \frac{1}{k}\sqrt{\frac{c_2}{2}}e^{2\rho}\bar{z}_y\mathrm{d}z_y\left(2i\mathrm{d}\Phi\right)\mathrm{Tr}\left(L_1L_{-1}\right) + \frac{1}{k}\sqrt{\frac{c_2}{2}}z_y\mathrm{d}\bar{z}_y\left(2i\mathrm{d}\Phi\right)\mathrm{Tr}\left(L_{-1}L_1\right) \\
&+ \frac{c_2}{2k^2}e^{2\rho}z_y\bar{z}_y\left(-4\mathrm{d}\Phi^2\right)\mathrm{Tr}\left(L_1L_{-1}\right) \\
=& \mathrm{d}\rho^2 + e^{2\rho}\mathrm{d}z_y\mathrm{d}\bar{z}_y + \frac{1}{k}\sqrt{\frac{c_2}{2}}e^{2\rho}\left(2ir^2\mathrm{d}\Phi\right)\left(2i\mathrm{d}\Phi\right) + \frac{2c_2}{k^2}e^{2\rho}r^2\mathrm{d}\Phi^2 \\
=& \mathrm{d}\rho^2 + e^{2\rho}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Phi^2\right) - e^{2\rho}\frac{2\sqrt{2c_2}}{k}r^2\mathrm{d}\Phi^2 + e^{2\rho}\frac{2c_2}{k^2}r^2\mathrm{d}\Phi^2 \\
=& \mathrm{d}\rho^2 + e^{2\rho}\left[\mathrm{d}r^2 + \left(\frac{\sqrt{2c_2}}{k} - 1\right)^2 r^2\mathrm{d}\Phi^2\right] \\
=& \mathrm{d}\rho^2 + e^{2\rho}\left(\mathrm{d}r^2 + n^2r^2\mathrm{d}\Phi^2\right). \qquad (299)
\end{aligned}$$

At the boundary $\rho \to \infty$, we obtain the n-sheet cylinder

$$\mathrm{d}s_b^2 = \mathrm{d}\tilde{y}^2 + n^2\mathrm{d}\Phi^2, \qquad (300)$$

by using

$$r \equiv e^{\tilde{y}} \qquad (301)$$

and omitting the pre-factor. The pre-factor generates the Liouville theory to compensate for the difference between the cylinder and sphere partition functions. Because the Wilson line $W_{\mathcal{R}}$ does not survive under the limit $n \to 1$, the boundary action only leaves the 2D Schwarzian theory and the Liouville theory [49, 50]. Hence computing the Wilson line in the AdS_3 Einstein gravity theory is equivalent to computing Z_n/Z_1^n in 2D Schwarzian theory and Liouville theory [49, 50]

$$\langle W_{\mathcal{R}}\rangle = \frac{Z_n}{Z_1^n} + \mathcal{O}\left(n-1\right), \qquad (302)$$

where $\langle W_{\mathcal{R}}\rangle$ is the expectation value of the Wilson line. The expression for EE can be restated using the Wilson line [49, 50]

$$S_{\mathrm{EE}} = \lim_{n\to 1}\frac{1}{1-n}\ln\langle W_{\mathcal{R}}\rangle, \qquad (303)$$

We conclude that the Wilson line should be a suitable operator playing the role of minimum surface (figure 3) [49, 50].

5. Outlook and future directions

We discussed various topics related to the context of AdS_3 Einstein gravity and its connection to CFT. AdS_3 pure Einstein gravity is a crucial tool for exploring the relationship between gravity and field theory. However, Einstein gravity with asymptotic AdS_3 boundary conditions is not considered a physical theory because it lacks a unitary dual CFT on the boundary. This was pointed out by [29]. A physical theory is unnecessarily needed to address the duality between the bulk and the boundary. 3D Einstein gravity theory can be made computable through renormalization techniques [28], which help deal with divergences. The Weyl transformation leads to the Liouville theory in 2D Schwarzian theory, which is not a conventional CFT_2. The Liouville theory simplifies computations in the context of AdS/CFT correspondence [50]. It is desirable to have the exact mapping on the bulk and boundary sides. String Theory suggests that introducing matter fields can lead to a dual unitary CFT. However, these matter fields can produce backreaction in the bulk, making the correspondence more complex. Given the close connection between 3D Einstein gravity and 2D dilaton gravity, we suggest starting with 2D gravity exploration to understand aspects of the correspondence better. We are interested in exploring the intricacies of the AdS/CFT correspondence, including the challenges related to unitarity, backreaction, and the role of matter fields. Theoretical physics is a continuously evolving field of research, and one area of particular interest is the holographic nature of the correspondence. Researchers are working hard to find explicit examples and mathematical frameworks to better understand these phenomena. By doing so, they hope to uncover new insights into the fundamental nature of gravity and QFT.

We discuss the relationship between central charges in CFT_2 and the bare gravitation constant in 3D Einstein gravity (G_3). In CFT_2, the central charge is an important parameter that characterizes the algebraic and geometric properties of the theory. It is not directly proportional to the inverse of G_3, which suggests that these two quantities are not simply related in this context [41]. The inverse central charge order corresponds to an infinite number of terms in the perturbation expansion of G_3. This means that when attempting to describe 3D Einstein gravity as a perturbative theory, there are an infinite number of terms that must be considered. These terms are captured by a proportional relationship to the inverse central charge order. It suggests that CFT_2 serves as a framework for a perturbative resummation of gravity, which can yield non-perturbative results. To avoid the non-renormalizability in the higher-dimensional gravity, the resummation of infinite expansion terms or the non-locality should be a simple solution. This is an interesting idea, as it hints that studying quantum gravity within the context of CFT_2 might provide insights that go beyond traditional perturbative approaches. CFT should be a better approach to studying quantum gravity. This could simplify the description of certain aspects of quantum gravity. We have established an interesting link between CFT_2 and AdS_3 Einstein gravity. Our findings indicate that the central charge in CFT_2 and the bare gravitational constant G_3 are not inversely proportional as previously thought [41]. However, we expect that there is a deeper connection that deserves further exploration as it could offer valuable insights into the field of quantum gravity.

We are currently discussing some key concepts at the intersection of quantum information and holography within the context of theoretical physics and quantum gravity [6]. The concepts from quantum information theory can be applied to classical gravity systems to analyze them. This analysis can potentially reveal new insights into emergent phenomena, as suggested by [6]. Quantum information measures can help in comprehending the emergence of spacetime in certain physical systems. This connection is intriguing because it indicates that the geometry of spacetime might be fundamentally linked to the quantum entanglement structure of the underlying degrees of freedom. The study of holography and related topics can benefit from quantum information measures as they can sometimes lead to simple and exact results, as shown by [49, 50]. Quantum information considerations possibly help resolve or shed light on the black hole information paradox. The paradox arises from apparent conflicts between the principles of quantum mechanics and general relativity in the context of black holes. Quantum information measures could offer novel approaches to comprehend black holes and spacetime beyond traditional methods. Researching quantum gravity from the perspective of quantum information is a promising and fascinating avenue. This approach possibly provides fresh insights into the nature of gravity and spacetime at the quantum level. We highlight the growing importance of quantum information measures in the study of holography, quantum gravity, and the black hole information paradox. It underscores the potential for quantum information theory to provide new tools and perspectives for addressing fundamental questions in theoretical physics.

Data availability statement

No new data were created or analyzed in this study.

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