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Factorial growth at low orders in perturbative QCD: control over truncation uncertainties

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Abstract: A method, known as "minimal renormalon subtraction" [*Phys. Rev. D* **97** (2018) 034503, *JHEP* **08** (2017) 62], relates the factorial growth of a perturbative series (in QCD) to the power p of a power correction Λ^p/Q^p. (Λ is the QCD scale, Q some hard scale.) Here, the derivation is simplified and generalized to any p, more than one such correction, and cases with anomalous dimensions. Strikingly, the well-known factorial growth is seen to emerge already at low or medium orders, as a consequence of constraints on the Q dependence from the renormalization group. The effectiveness of the method is studied with the gluonic energy between a static quark and static antiquark (the "static energy"). Truncation uncertainties are found to be under control after next-to-leading order, despite the small exponent of the power correction ($p = 1$) and associated rapid growth seen in the first four coefficients of the perturbative series.

Keywords: Large-Order Behaviour of Perturbation Theory, Renormalons, Renormalization Group

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1 Introduction

In 2018, the Fermilab Lattice, MILC, and TUMQCD collaborations [1] used lattice-QCD calculations of heavy-light meson masses to obtain results for renormalized quark masses in the modified minimal subtraction ($\overline{\text{MS}}$) scheme. The total uncertainty ranges from below 1% (for bottom, charm, and strange) to 1–2% (for down and up). The $\overline{\text{MS}}$ scheme inevitably entails perturbation theory. Usually a top source of uncertainty would come from truncating the perturbative series in the strong coupling α_s. In ref. [1], however, the error budgets exhibit negligible uncertainty from truncation (cf., figure 4 [1]). The associated uncertainty was estimated by omitting the highest-order coefficient (of α_s^4) in the relation between the pole mass and the $\overline{\text{MS}}$ mass. It was found to be comparable to the statistical uncertainty and much smaller than the parametric uncertainty in α_s.

Essential to ref. [1] is a reinterpretation of the perturbation series [2] that in turn relies crucially on a formula for the normalization of the leading renormalon ambiguity of the pole mass [3]. Readers who are not familiar with renormalons are encouraged to indulge the jargon for a moment: clearly it is worth pursuing how to generalize refs. [2, 3], in the hope of controlling the truncation uncertainty in further applications. This paper takes up that pursuit.

The coefficients of many perturbative series in quantum mechanics [4, 5] and quantum field theory [6–8] are known to grow factorially. In QCD and other asymptotically free theories, a class of leading and subleading growths arises from soft loop momenta in Feynman diagrams. Details of the growth can be obtained from studying implications of the renormalization group. At the same time, the growth is related to power-law corrections to the perturbation series. For now, let us characterize the growth of the l^th coefficient as $Ka^l l^b l!$ for some K, a, and b. A basic renormalization-group analysis (e.g., ref. [9]) determines a and b but not the normalization K. There are, however, at least three expressions in the literature for K [3, 10–13]. The expressions in refs. [3] and [13] bear some resemblance to each other, but the one in refs. [10–12] is different.

The generalizations initially sought in the present work started modest: I wanted to look at scale dependence of α_s to see (as a co-author of refs. [1, 2]) whether our quoted uncertainties held up, and I wanted to treat arbitrary power corrections. Dissatisfaction with my understanding of the normalization derived in ref. [3] led to a simple way of analyzing the problem with interesting findings:

- the normalization of ref. [3] is reproduced, at least in practical terms;
- the standard factorial growth starts at low orders, not just at asymptotically large l;
- the second coefficient of the β function and the exponent of the power correction determine the order at which the factorial growth becomes a practical matter;
- the way to deal with a sequence of power corrections becomes clear.

The third item is well known, but, even so, many analyses of large-order effects use a one-term β function. The last item was mentioned in v1 and v2 on arXiv.org of ref. [3], but the discussion was removed from the final publication. The derivation of the factorial growth presented below is so straightforward, it is almost surprising that it has not been known for decades. If it has appeared in the literature before, it is obscure.

The rest of this paper is organized as follows. Section 3 recalls ref. [2] and generalizes its ideas to an arbitrary (single) power correction. Section 4 considers cases with more than one power-suppressed contribution. Sections 3 and 4 rely on a special renormalization scheme that simplifies the algebra; other schemes are discussed in section 5. Section 6 considers the complication of anomalous dimensions. Proposals to improve perturbation theory should study at least one example, so section 7 applies section 3 to the static energy between a heavy quark-antiquark pair, for which four terms in the perturbation series are known (like the pole-mass–$\overline{\rm MS}$-mass relation). A summary and some outlook is offered in section 8. A modification of the Borel summation used in sections 3 and 4 is given in appendix A.

2 Notation and setup

The problem at hand is to compute in QCD, or other asymptotically free quantum field theory, a physical quantity that depends on a high-energy scale Q (or, as in section 7, short distance $r = 1/Q$). The hard scale Q can be used to obtain a dimensionless version of the physical quantity.

The dimensionless quantity can be approximated order-by-order in perturbation theory up to power corrections:

$$\mathscr{R}(Q) = r_{-1} + R(Q) + C_p \frac{\Lambda^p}{Q^p}, \qquad R(Q) = \sum_{l=0} r_l(\mu/Q)\alpha_s(\mu)^{l+1}, \tag{2.1}$$

where the term r_{-1} can be 0 or not, C_p is (for now) independent of Q, $\Lambda \sim \mu e^{-1/2\beta_0\alpha_s(\mu)}$ is the scale arising from dimensional transmutation, α_s is the gauge coupling in some scheme, and μ is the renormalization scale. The power p can be deduced from the operator-product expansion, an effective field theory, or other considerations. For now, let us consider the case with only one power correction, postponing until section 4 the more general case. Laboratory measurements or the continuum limit of lattice gauge theory can be used to provide a nonperturbative determination of $\mathscr{R}(Q)$. Fits of data for $\mathscr{R}(Q)$ could, ideally, be used to determine α_s with nuisance parameter C_p. As an asymptotic expansion, the sum representing $R(Q)$ in eq. (2.1) diverges, however, so an upper summation limit does not make sense without further discussion. Indeed, the definition of the power correction rests on how the sum is treated.

$\mathscr{R}$ and R do not depend of μ, so the μ dependence of the coefficients is intertwined with the μ dependence of α_s and, thus, dictated by

$$\dot{\alpha}_s(\mu) \equiv 2\beta(\alpha_s) = -2\alpha_s(\mu) \sum_{k=0}^{\infty} \beta_k \alpha_s(\mu)^{k+1}, \tag{2.2}$$

where $\dot{g} = \mathrm{d}g/\mathrm{d}\ln\mu$. The derivatives of the coefficients must satisfy

$$\dot{r}_l(\mu/Q) = 2\sum_{j=0}^{l-1}(j+1)\beta_{l-1-j} r_j(\mu/Q). \tag{2.3}$$

Integrating these equations (in a mass-independent renormalization scheme) one after the other leads to

$$r_0(\mu/Q) = r_0, \tag{2.4a}$$
$$r_1(\mu/Q) = r_1 + 2\beta_0 \ln(\mu/Q) r_0, \tag{2.4b}$$
$$r_2(\mu/Q) = r_2 + 2\ln(\mu/Q)\,(2\beta_0 r_1 + \beta_1 r_0) + [2\beta_0\ln(\mu/Q)]^2 r_0, \tag{2.4c}$$
$$\begin{aligned} r_3(\mu/Q) = r_3 &+ 2\ln(\mu/Q)\,(3\beta_0 r_2 + 2\beta_1 r_1 + \beta_2 r_0) + 3[2\beta_0\ln(\mu/Q)]^2 r_1 \\ &+ 10\beta_0\beta_1 \ln^2(\mu/Q) r_0 + [2\beta_0\ln(\mu/Q)]^3 r_0, \end{aligned} \tag{2.4d}$$

and so on, with constants of integration $r_l \equiv r_l(1)$. The dependence of $R(Q)$ on Q is, thus, tied to the renormalization-dictated dependence on μ.

Eq. (2.3) is a matrix equation, $\dot{\boldsymbol{r}} = 2\mathbf{D}\cdot\boldsymbol{r}$, with $D_{lj} = (j+1)\beta_{l-1-j}$ if $l > j$ and $D_{lj} = 0$ otherwise. For sections 3 to 5, it is convenient to develop this matrix notation further, for instance writing

$$R = \mathfrak{A}_s \cdot \boldsymbol{r}_s = \left\lceil \alpha_s \quad \alpha_s^2 \quad \alpha_s^3 \quad \alpha_s^4 \quad \cdots \right\rceil \left\lceil \begin{matrix} r_0 \\ r_1 \\ r_2 \\ r_3 \\ \vdots \end{matrix} \right\rceil. \tag{2.5}$$

Floorless delimiters ⌈ ⌉ are used instead of brackets [ ] or parentheses as a reminder that the vectors are infinite sequences. Below it will be useful to think of the subscript "s" as standing for "starting scheme", in practice $\overline{\text{MS}}$.

The matrix notation makes scheme and scale dependence manifest and eases derivations. For example, if

$$\alpha_b = \alpha_\mathrm{s} + b_1\alpha_\mathrm{s}^2 + b_2\alpha_\mathrm{s}^3 + b_3\alpha_\mathrm{s}^4 + \cdots\,, \tag{2.6}$$

then $\mathfrak{A}_b = \mathfrak{A}_\mathrm{s}\cdot\mathbf{b}^{-1}$ with scheme-conversion matrix

$$\mathbf{b}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ b_1 & 1 & 0 & 0 & \cdots \\ b_2 & 2b_1 & 1 & 0 & \cdots \\ b_3 & b_1^2+2b_2 & 3b_1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -b_1 & 1 & 0 & 0 & \cdots \\ 2b_1^2-b_2 & -2b_1 & 1 & 0 & \cdots \\ 5b_1b_2-5b_1^3-b_3 & 5b_1^2-2b_2 & -3b_1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}. \tag{2.7}$$

The coefficients in the "b" scheme are $\boldsymbol{r}_b = \mathbf{b}\cdot\boldsymbol{r}_\mathrm{s}$. The lower-triangular structure of these and other matrices is the key to the forthcoming analysis.

The $\overline{\mathrm{MS}}$ scheme can be thought of as the "laboratory frame", where $\boldsymbol{r}_\mathrm{s}$ is most easily obtained. The "center-of-mass frame", which reduces subsequent labor, is the "geometric scheme" defined by [14]

$$\beta(\alpha_\mathrm{g}) = -\frac{\beta_0\alpha_\mathrm{g}^2}{1-(\beta_1/\beta_0)\alpha_\mathrm{g}}. \tag{2.8}$$

Equivalently, $\beta_k = \beta_0(\beta_1/\beta_0)^k$, so the β-function series, eq. (2.2), is geometric. In eq. (2.6), $b_1 = 2\beta_0\ln\Lambda_\mathrm{g}/\Lambda_{\overline{\mathrm{MS}}}$; taking $b_1=0$ not only eliminates or simplifies many entries in the scheme-conversion matrix but also means $\Lambda_\mathrm{g} = \Lambda_{\overline{\mathrm{MS}}}$ requires no conversion. Expressions for the b_i connecting the geometric and $\overline{\mathrm{MS}}$ schemes are less interesting than the entries of the conversion matrix:

$$\mathbf{b}_\mathrm{g} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ \delta_2 & 0 & 1 & 0 & 0 & \cdots \\ \tfrac{1}{2}\delta_3 & 2\delta_2 & 0 & 1 & 0 & \cdots \\ \tfrac{1}{3}\delta_4 - \tfrac{1}{6}\delta_3\check\beta_1 + \tfrac{5}{3}\delta_2^2 + \tfrac{1}{3}\delta_2\check\beta_1^2 & \delta_3 & 3\delta_2 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}, \tag{2.9}$$

where $\delta_k = \check\beta_k - \check\beta_1^k$, $\check\beta_k = \beta_k/\beta_0$, with the nonuniversal β_k $(k>1)$ of the original scheme. The geometric scheme can be reached from any starting point: first introduce a scale change to align, say, Λ_lat with $\Lambda_{\overline{\mathrm{MS}}}$; then the coefficient vector $\boldsymbol{r}_\mathrm{g} = \mathbf{b}_\mathrm{g}\cdot\boldsymbol{r}$ is independent of the ultraviolet regulator and renormalization used to obtain $\boldsymbol{r}$.

3 One power correction

Let us recall how refs. [2, 3] handle the pole mass. The heavy-quark effective theory provides an expression for a heavy-light hadron mass [15–17] along the lines of eq. (2.1):

$$\mathscr{M} = \bar m\left(1 + \sum_{l=0} r_l\alpha_\mathrm{s}^{l+1}(\bar m)\right) + \bar\Lambda + \mathrm{O}(1/\bar m), \tag{3.1}$$

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where $\bar{m} = m_{\overline{\rm MS}}(\mu)$ evaluated at $\mu = \bar{m}$, and $\bar{\Lambda}$, which is of order Λ, is the energy of gluons and light quarks. The series times $\bar{m}$ is known as the pole (or on-shell) mass. The coefficients r_l are obtained from the quark self-energy by putting the quark on shell iteratively at each order in perturbation theory. The coefficients are infrared finite and gauge independent at every order of the iteration [18], but they grow factorially with the order l [19–22]. The series thus diverges, rendering its interpretation ambiguous. A hadron mass cannot be ambiguous, so the ambiguity in the series must be canceled by $\bar{\Lambda}$ (and higher-power terms) [23].

Komijani [3] exploited the fact that the leading factorial growth in the series, being related to $\bar{\Lambda}$, is independent of $\bar{m}$. Therefore, taking a derivative with respect to $\bar{m}$ generates a quantity without $\bar{\Lambda}$. The derivative yields

$$1 + \sum_{l=0} r_l \alpha_{\rm s}^{l+1}(\bar{m}) + 2\beta\left(\alpha_{\rm s}(\bar{m})\right) \sum_{l=0} (l+1) r_l \alpha_{\rm s}^{l}(\bar{m}) \equiv 1 + \sum_{k=0} f_k \alpha_{\rm s}^{k+1}(\bar{m}), \tag{3.2}$$

where the f_k are obtained by expanding out $\beta(\alpha_{\rm s})$ on the left-hand side:

$$f_k = r_k - 2\sum_{l=0}^{k-1} (l+1)\beta_{k-1-l} r_l. \tag{3.3}$$

Eq. (3.3) is eq. (2.3) of ref. [3].

Komijani recast eqs. (3.2) and (3.3) as a differential equation (eq. (1.6) of ref. [3]),

$$r(\alpha) + 2\beta(\alpha) r'(\alpha) = f(\alpha), \tag{3.4}$$

where the prime denotes a derivative with respect to α. The appendix of ref. [3] derives an asymptotic solution to eq. (3.4) that pins down the normalization of the large-order coefficients r_l, $l \gg 1$, i.e., the quantity denoted K in section 1. Note that ref. [3] obtains a particular solution to eq. (3.4). A general solution consists of any particular solution plus a solution to the corresponding homogeneous equation with 0 instead of $f(\alpha)$ on the right-hand side. The solution of the homogeneous equation is a constant of order Λ. In this paper, eq. (3.3) is used instead of eq. (3.4) as the starting point in search of a particular solution.

Before presenting the solution, let us generalize Komijani's idea to eq. (2.1): multiply $\mathscr{R}$ by Q^p so the Λ^p term no longer depends on Q, differentiate once with respect to Q, and then divide by pQ^{p-1}:

$$\mathscr{F}^{(p)}(Q) \equiv \hat{Q}^{(p)}\mathscr{R}(Q) \equiv \frac{1}{pQ^{p-1}} \frac{{\rm d}\, Q^p \mathscr{R}}{{\rm d}Q} = r_{-1} + F^{(p)}(Q). \tag{3.5}$$

In this case $F^{(p)} = \hat{Q}^{(p)} R$ also, and a nonzero r_{-1} cancels out just like the 1 in eq. (3.2). Introducing a series for $F^{(p)}$ and collecting like powers of $\alpha_{\rm s}$,

$$F^{(p)}(Q) = \sum_{k=0} f_k^{(p)}(\mu/Q)\alpha_{\rm s}(\mu)^{k+1}, \qquad f_k^{(p)} = r_k - \frac{2}{p}\sum_{l=0}^{k-1}(l+1)\beta_{k-1-l} r_l. \tag{3.6a}$$

In matrix notation,

$$\boldsymbol{f}^{(p)} = \mathbf{Q}^{(p)} \cdot \boldsymbol{r}, \qquad \mathbf{Q}^{(p)} = \mathbf{1} - \frac{2}{p}\mathbf{D}, \tag{3.6b}$$

with $\mathbf{D}$ defined above.

Eq. (3.6) can be derived either by keeping $\alpha_s(\mu)$ independent of Q and taking the derivative of the coefficients or by setting $\mu = Q$, as in eq. (3.2), so the coefficients are constant with $\alpha_s(Q)$ encoding the Q dependence. Eq. (3.6) generalize eqs. (3.3) and (3.4) to arbitrary p; the differential equation à la eq. (3.4) corresponding to eq. (3.6) has $2/p$ multiplying $\beta(\alpha)$. The particular solution to the differential equation is simply obtained by solving eq. (3.6b): $\boldsymbol{r} = \mathbf{Q}^{(p)^{-1}} \cdot \boldsymbol{f}^{(p)}$.

At this point, one might wonder what could be gained this way. For some L, the r_l, $l < L$, are available in the literature. Via eq. (3.6a), just as many $f_k^{(p)}$ are obtained from these L terms and the first L coefficients β_j (eq. (2.2)). Solving eq. (3.6b) should just return the original information. That is, of course, correct, but the solution, spelled out below, *also* yields information about the r_l for $l \geq L$. Exploiting this additional information is the gist of this analysis.

The solution of eq. (3.6b) is easiest in the geometric scheme. Let $b \equiv \beta_1/2\beta_0^2$, so that $2\beta_k = (2\beta_0)^{k+1}b^k$ (in the geometric scheme), and let $\tau \equiv 2\beta_0/p$. Then $\mathbf{Q}_\mathrm{g}^{(p)} = \mathbf{b}_\mathrm{g} \cdot \mathbf{Q}^{(p)} \cdot \mathbf{b}_\mathrm{g}^{-1}$ has elements

$$\left[Q_\mathrm{g}^{(p)}\right]_{kl} = \begin{cases} 0, & k < l, \\ 1, & k = l, \\ -(l+1)\tau^{k-l}(pb)^{k-l-1}, & k > l, \end{cases} \tag{3.7a}$$

which looks like

$$\mathbf{Q}_\mathrm{g}^{(p)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -\tau & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -\tau^2 pb & -2\tau & 1 & 0 & 0 & 0 & 0 & \cdots \\ -\tau(\tau pb)^2 & -2\tau^2 pb & -3\tau & 1 & 0 & 0 & 0 & \cdots \\ -\tau(\tau pb)^3 & -2\tau(\tau pb)^2 & -3\tau^2 pb & -4\tau & 1 & 0 & 0 & \cdots \\ -\tau(\tau pb)^4 & -2\tau(\tau pb)^3 & -3\tau(\tau pb)^2 & -4\tau^2 pb & -5\tau & 1 & 0 & \cdots \\ -\tau(\tau pb)^5 & -2\tau(\tau pb)^4 & -3\tau(\tau pb)^3 & -4\tau(\tau pb)^2 & -5\tau^2 pb & -6\tau & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{3.7b}$$

$\mathbf{Q}_\mathrm{g}^{(p)}$ exhibits geometric but not factorial growth.

The inverse is easily obtained row-by-row:

$$\mathbf{Q}_\mathrm{g}^{(p)^{-1}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \tau & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \tau^2\frac{\Gamma(3+pb)}{\Gamma(2+pb)} & 2\tau & 1 & 0 & 0 & 0 & 0 & \cdots \\ \tau^3\frac{\Gamma(4+pb)}{\Gamma(2+pb)} & 2\tau^2\frac{\Gamma(4+pb)}{\Gamma(3+pb)} & 3\tau & 1 & 0 & 0 & 0 & \cdots \\ \tau^4\frac{\Gamma(5+pb)}{\Gamma(2+pb)} & 2\tau^3\frac{\Gamma(5+pb)}{\Gamma(3+pb)} & 3\tau^2\frac{\Gamma(5+pb)}{\Gamma(4+pb)} & 4\tau & 1 & 0 & 0 & \cdots \\ \tau^5\frac{\Gamma(6+pb)}{\Gamma(2+pb)} & 2\tau^4\frac{\Gamma(6+pb)}{\Gamma(3+pb)} & 3\tau^3\frac{\Gamma(6+pb)}{\Gamma(4+pb)} & 4\tau^2\frac{\Gamma(6+pb)}{\Gamma(5+pb)} & 5\tau & 1 & 0 & \cdots \\ \tau^6\frac{\Gamma(7+pb)}{\Gamma(2+pb)} & 2\tau^5\frac{\Gamma(7+pb)}{\Gamma(3+pb)} & 3\tau^4\frac{\Gamma(7+pb)}{\Gamma(4+pb)} & 4\tau^3\frac{\Gamma(7+pb)}{\Gamma(5+pb)} & 5\tau^2\frac{\Gamma(7+pb)}{\Gamma(6+pb)} & 6\tau & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \tag{3.8a}$$

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or, expressed as in eq. (3.7a),

$$\left[Q_{\mathrm{g}}^{(p)\,-1}\right]_{lk} = \begin{cases} 0, & l < k, \\ 1, & l = k, \\ (k+1)\dfrac{\tau^l \Gamma(l+1+pb)}{\tau^k \Gamma(k+2+pb)}, & l > k. \end{cases} \tag{3.8b}$$

From one row to the next, the entries increase both in a factorial way and by powers of τ. As stated in section 1, the growth starts at low orders. From one column to the next, the entries *decrease* factorially (and by powers of τ). Both factorials grow rapidly only once $l \gg pb$, $k \gg pb$, so — again as stated in section 1 — the higher the power p, the longer the growth need not be apparent from explicit expressions for the coefficients. Growth is also postponed for large b, which happens if β_0 is small but β_1 is not.

Reexpressing eq. (3.8) as series coefficients,

$$r_l = f_l^{(p)} + \left(\frac{2\beta_0}{p}\right)^l \Gamma(l+1+pb) \sum_{k=0}^{l-1} \frac{k+1}{\Gamma(k+2+pb)} \left(\frac{p}{2\beta_0}\right)^k f_k^{(p)}, \tag{3.9}$$

which holds (in the geometric scheme) for all l. Eq. (3.9) is similar to eq. (2.22) of ref. [3], except for three details: eq. (2.22) of ref. [3] omits the first term $f_l^{(p)}$, has ∞ as the upper limit of the sum, and holds only asymptotically (i.e., the relation is $\sim$ instead of $=$). $\boldsymbol{f}^{(p)}$ grows more slowly than $\Gamma(l+1+pb)$ or $\Gamma(k+2+pb)$, so for $l \gg 1$ it is accurate to neglect the first term and to extend the sum to ∞. The crucial difference is that eq. (3.9) holds for all l, starting with the next few orders beyond the known r_l.

Recall that L terms are available. Nowadays, $L = 4$ for some problems (e.g., eq. (3.1) and section 7) and $L = 3$ for others. For $l < L$, eq. (3.9) returns the r_l available at the outset. For $l \geq L$, eq. (3.9) suggests estimating r_l (in the geometric scheme) by

$$r_l \approx R_l^{(p)} \equiv R_0^{(p)} \left(\frac{2\beta_0}{p}\right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)}, \quad l \geq L, \tag{3.10a}$$

$$R_0^{(p)} \equiv \sum_{k=0}^{L-1} (k+1) \frac{\Gamma(1+pb)}{\Gamma(k+2+pb)} \left(\frac{p}{2\beta_0}\right)^k f_k^{(p)}. \tag{3.10b}$$

The expression for $R_0^{(1)}$ is the same as that for $N_{k_{\max}}$ (with $k_{\max} = L-1$) in eq. (2.23) of ref. [3]. It is also resembles the formula (taken in the geometric scheme) for $P_{1/2}$ in eqs. (17) of ref. [13]. Applying eq. (3.10) to the series $R(Q)$ yields

$$R(Q) \approx \sum_{l=0}^{L-1} r_l \alpha_{\mathrm{g}}^{l+1}(Q) + \sum_{l=L}^{\infty} R_l^{(p)} \alpha_{\mathrm{g}}^{l+1}(Q). \tag{3.11a}$$

The first L terms are as usual and the others are estimated via their fastest growing part. For subsequent analysis, it is better to start the second sum at $l = 0$,

$$R(Q) \approx \sum_{l=0}^{L-1} \left(r_l - R_l^{(p)}\right) \alpha_{\mathrm{g}}^{l+1}(Q) + \sum_{l=0}^{\infty} R_l^{(p)} \alpha_{\mathrm{g}}^{l+1}(Q), \tag{3.11b}$$

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which follows from subtracting and adding $\sum_{l=0}^{L-1} R_l^{(p)} \alpha_{\rm g}^{l+1}(Q)$. For convenience below, let

$$R_{\rm RS}^{(p)}(Q) \equiv \sum_{l=0}^{L-1} \left(r_l - R_l^{(p)}\right) \alpha_{\rm g}(Q)^{l+1}, \qquad R_{\rm B}^{(p)}(Q) \equiv \sum_{l=0}^{\infty} R_l^{(p)} \alpha_{\rm g}(Q)^{l+1}. \tag{3.12}$$

$R_{\rm RS}^{(p)}$ is similar to the truncation to L terms of the "renormalon subtracted" (RS) scheme for R [12]. Here, $R_{\rm RS}^{(p)}$ arises not by intentional subtraction but from rearranging terms. In the examples of the pole mass [2] and the static energy (section 7), $r_l - R_l^{(p)}$ is smaller than r_l, especially for $l = 3$, 4.

Because of the factorial growth of the $R_l^{(p)}$, the series $R_{\rm B}^{(p)}$ does not converge. It can be assigned meaning through Borel summation, however. Using the integral representation of $\Gamma(l+1)$,

$$\begin{aligned} R_{\rm B}^{(p)}(Q) &= R_0^{(p)} \sum_{l=0}^{\infty} \left[\frac{\Gamma(l+1+pb)}{\Gamma(1+pb)\Gamma(l+1)} \int_0^\infty \left(\frac{2\beta_0 t}{p}\right)^l \mathrm{e}^{-t/\alpha_{\rm g}(Q)} \mathrm{d}t\right], \\ &\to R_0^{(p)} \int_0^\infty \frac{\mathrm{e}^{-t/\alpha_{\rm g}(Q)}}{(1-2\beta_0 t/p)^{1+pb}} \mathrm{d}t, \end{aligned} \tag{3.13}$$

where the second line is obtained by swapping the order of summation and integration. Strictly speaking, the swap is not allowed because the integrand has a branch point at $t = p/2\beta_0$. This singularity is known as a renormalon [8]. It is customary to place the cut on the real axis from the branch point to $+\infty$. In ref. [2], we split the integral into two parts, over the intervals $[0, p/2\beta_0)$ before the cut and $[p/2\beta_0, \infty)$ along the cut. The first integral is unambiguous and given below.

For the interval $[p/2\beta_0, \infty)$, the contour must be specified. Taking it slightly above or below the cut, for example, yields

$$\delta R^{(p)} \equiv R_0^{(p)} \int_{p/2\beta_0 \pm \mathrm{i}\varepsilon}^{\infty \pm \mathrm{i}\varepsilon} \frac{\mathrm{e}^{-t/\alpha_{\rm g}(Q)}}{(1-2\beta_0 t/p)^{1+pb}} \mathrm{d}t = -R_0^{(p)} \mathrm{e}^{\pm \mathrm{i}pb\pi} \frac{p^{1+pb}}{2^{1+pb}\beta_0} \Gamma(-pb) \left[\frac{\mathrm{e}^{-1/[2\beta_0\alpha_{\rm g}(Q)]}}{[\beta_0\alpha_{\rm g}(Q)]^b}\right]^p, \tag{3.14}$$

and the factor $\mathrm{e}^{\pm \mathrm{i}pb\pi}$ illustrates the ambiguity. The quantity inside the bracket is identically $\Lambda_{\rm g}/Q = \Lambda_{\overline{\rm MS}}/Q$, so without loss $\delta R^{(p)} \propto (\Lambda/Q)^p$ can be lumped into the solution of the homogeneous differential equation à la eq. (3.4) or, equivalently, the power correction $C_p \Lambda^p/Q^p$ in eq. (2.1) [2].

Because the interchange of summation and integration in eq. (3.13) is not allowed, $R_{\rm B}^{(p)}$ can be *assigned* to be (taking $b < 0$ at first and then applying analytic continuation)

$$R_{\rm B}^{(p)}(Q) = R_0^{(p)} \int_0^{p/2\beta_0} \frac{\mathrm{e}^{-t/\alpha_{\rm g}(Q)}}{(1-2\beta_0 t/p)^{1+pb}} \mathrm{d}t = R_0^{(p)} \frac{p}{2\beta_0} \mathscr{J}(pb, 1/2\beta_0\alpha_{\rm g}(Q)), \tag{3.15a}$$

$$\mathscr{J}(c, y) = e^{-y}\Gamma(-c)\gamma^\star(-c, -y), \tag{3.15b}$$

which is acceptable because the asymptotic (small $\alpha_{\rm g}$) expansion of $\mathscr{J}$ returns the original series in eq. (3.12). Here $\gamma^\star(a, x) \equiv [1/\Gamma(a)] \int_0^1 \mathrm{d}t\, t^{a-1} \mathrm{e}^{-xt}$ is known as the limiting function of the incomplete gamma function [24]. It is analytic in a and x and has a convergent expansion

$$\gamma^\star(a,-y) = \frac{1}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{y^n}{n!(n+a)}, \quad \forall y, \tag{3.16}$$

which saturates quickly, also when $a = -pb < 0$.

Combining the various ingredients leads to the prescription

$$\mathscr{R}(Q) \equiv r_{-1} + R^{(p)}_{\mathrm{RS}}(Q) + R^{(p)}_{\mathrm{B}}(Q) + C_p\frac{\Lambda^p}{Q^p} \tag{3.17}$$

for estimating $\mathscr{R}(Q)$. Here, $R^{(p)}_{\mathrm{RS}}(Q)$ is introduced in eq. (3.12) and $R^{(p)}_{\mathrm{B}}(Q)$ is defined by the right-hand side of eq. (3.15a). Eq. (3.17) is just eq. (2.25) of ref. [2], generalized to p different from 1.

For the relation between the pole mass and $\overline{\mathrm{MS}}$ mass, ref. [2] referred to eq. (3.17) as "minimal renormalon subtraction" (MRS) in analogy with the RS mass of ref. [12]. The derivation given here arguably does not subtract anything but instead adds new information to the usual truncated perturbation series, rearranges a few terms, and then assigns meaning to an otherwise ill-defined series expression. Even so, this paper continues to refer to the procedure as MRS. For example, it is often convenient to consider $R^{(p)}_{\mathrm{RS}}(Q) + R^{(p)}_{\mathrm{B}}(Q) \equiv R_{\mathrm{MRS}}(Q)$ as a single object. The asymptotic (small α_{g}) expansion of $R_{\mathrm{MRS}}(Q)$ is identical to the original series $R(Q)$.

Starting with eq. (3.11), the renormalization scale has been chosen to be $\mu = Q$. If $\mu = sQ$ is chosen instead, the derivations do not change. The coupling $\alpha_{\mathrm{g}}(Q)$ simply becomes $\alpha_{\mathrm{g}}(sQ)$ and the coefficients $r_l = r_l(1)$ and $f^{(p)}_k = f^{(p)}_k(1)$ become $r_l(s)$ and $f^{(p)}_k(s)$. How these effects play out in practice is discussed in section 7. In $\delta R^{(p)}$, the bracket in eq. (3.14) becomes $[\Lambda_{\mathrm{g}}/sQ]^p$, so the overall change is to replace $R^{(p)}_0(1)$ with $R^{(p)}_0(s)/s^p$.

4 Cascade of power corrections

In general, problems like eq. (2.1) have more than one power correction. If there are two, with $p_2 > p_1$, $\mathscr{F}^{p_1}$ still contains $(p_1 - p_2)C_{p_2}\Lambda^{p_2}/p_1Q^{p_2}$, which can be removed with $\hat{Q}^{(p_2)}$:

$$\boldsymbol{f}^{\{p_1,p_2\}} \equiv \mathbf{Q}^{(p_2)}\cdot\boldsymbol{f}^{(p_1)} \quad\Rightarrow\quad \boldsymbol{f}^{(p_1)} = \mathbf{Q}^{(p_2)^{-1}}\cdot\boldsymbol{f}^{\{p_1,p_2\}}. \tag{4.1}$$

These coefficients could then be used in eq. (3.9). A similar idea was mentioned in v1 and v2 on arXiv.org of ref. [3]. With the early onset of the "large-l" behavior not yet clear when ref. [3] was written, the utility of eq. (4.1) was also not clear. For whatever reason, the discussion was removed from the final publication.

More concretely and in general, if the set of powers is $\{p_1, p_2, \ldots, p_n\}$, the operator (with $\hat{Q}^{(p_1)}$ rightmost)

$$\hat{Q}^{\{p_i\}} = \prod_{j=0}^{n-1}\hat{Q}^{(p_{n-j})} \tag{4.2}$$

$\{p_i\} \setminus j$	1	2	3	4	6	8
{1,2}	2	-1	–	–	–	–
{1,3}	3/2	–	$-1/2$	–	–	–
{1,2,3}	3	-3	1	–	–	–
{1,2,4}	8/3	-2	–	1/3	–	–
{1,2,3,4}	4	-6	4	-1	–	–
{2,4}	–	2	–	-1	–	–
{2,4,6,8}	–	4	–	-6	4	-1
{4,6}	–	–	–	3	-2	–
{4,6,8}	–	–	–	6	-8	3
{1,2,4,6}	16/5	–	-3	1	$-1/5$	–
{1,3,4,6,8}	96/35	–	$-32/5$	6	$-8/5$	9/35

Table 1. Partition coefficients $h_j^{\{p_i\}}$ for various sets of powers p_i.

fully removes the power corrections associated with these powers. In matrix notation, the F-series coefficients

$$\boldsymbol{f}^{\{p_i\}} = \mathbf{Q}^{\{p_i\}} \cdot \boldsymbol{r} = \prod_{j=0}^{n-1} \mathbf{Q}^{(p_{n-j})} \cdot \boldsymbol{r} \tag{4.3}$$

are obtained with $\mathbf{Q}^{\{p_i\}}$, which is the obvious matrix representation of $\hat{Q}^{\{p_i\}}$. This equation can be solved for

$$\boldsymbol{r} = \mathbf{Q}^{\{p_i\}^{-1}} \cdot \boldsymbol{f}^{\{p_i\}} = \prod_{j=1}^{n} \mathbf{Q}^{(p_j)^{-1}} \cdot \boldsymbol{f}^{\{p_i\}}, \tag{4.4}$$

and, as above, the series $R(Q)$ is approximated by using the L known terms of $\boldsymbol{r}$ while using the rest of them from this solution.

Because the $\mathbf{Q}^{(p_i)}$ commute, their inverses do, so a partial-fraction decomposition turns the product into a sum,

$$\prod_{j=1}^{n} \mathbf{Q}^{(p_j)^{-1}} = \sum_{j=1}^{n} h_j^{\{p_i\}} \mathbf{Q}^{(p_j)^{-1}}, \qquad h_j^{\{p_i\}} = \prod_{k=1, k\neq j}^{n} \frac{p_k}{p_k - p_j}. \tag{4.5}$$

Note that $\sum_{j=1}^{n} h_j^{\{p_i\}} = 1$, $\sum_{j=1}^{n} p_j h_j^{\{p_i\}} = 0$; table 1 shows the $h_j^{\{p_i\}}$ for various sets $\{p_i\}$. The solution is thus,

$$\boldsymbol{r} = \sum_{j=1}^{n} h_j^{\{p_i\}} \mathbf{Q}^{(p_j)^{-1}} \cdot \boldsymbol{f}^{\{p_i\}}, \tag{4.6}$$

which generalizes eq. (3.9). The prescription is again to take the first L r_l as computed in the literature and approximate the rest with the leading factorials in eq. (4.6). That means

$$\mathscr{R}(Q) \equiv r_{-1} + R_{\mathrm{RS}}^{(p)}(Q) + R_{\mathrm{B}}^{(p)}(Q) + \sum_{i=1}^{n} C_{p_i} \frac{\Lambda^{p_i}}{Q^{p_i}}, \tag{4.7a}$$

$$R_{\mathrm{RS}}(Q) \equiv \sum_{l=0}^{L-1} \left(r_l - R_l^{\{p_i\}} \right) \alpha_{\mathrm{g}}^{l+1}(Q), \tag{4.7b}$$

$$R_{\rm B}(Q) \equiv \sum_{j=1}^{n} h_j^{\{p_i\}} R_0^{\{p_i\}(p_j)} \frac{p_j}{2\beta_0} \mathscr{J}\left(p_j b, p_j/2\beta_0 \alpha_{\rm g}(Q)\right), \tag{4.7c}$$

where

$$R_l^{\{p_i\}} = \sum_{j=1}^{n} h_j^{\{p_i\}} R_0^{\{p_i\}(p_j)} \left(\frac{2\beta_0}{p_j}\right)^l \frac{\Gamma(l+1+p_j b)}{\Gamma(1+p_j b)}, \tag{4.7d}$$

$$R_0^{\{p_i\}(p_j)} = \sum_{k=0}^{L-1} (k+1) \frac{\Gamma(1+p_j b)}{\Gamma(k+2+p_j b)} \left(\frac{p_j}{2\beta_0}\right)^k f_k^{\{p_i\}}. \tag{4.7e}$$

The same $f_k^{\{p_i\}}$ appear in all $R_0^{\{p_i\}(p_j)}$, hence the somewhat fussy notation. For lack of a better name, MRS now stands for "multiple renormalon subtraction" even though, again, the procedure as developed here adds information. A possible notation to distinguish how many power corrections have been removed from a given series is "MRS$\{p_1, p_2, \ldots, p_n\}$".

5 Other renormalization schemes

While the geometric scheme simplifies the solution of the matrix equations, it is useful to generalize MRS to arbitrary schemes. Given the algebra of section 3, the simplest way to solve to eq. (3.6b) is to combine eqs. (2.9) and (3.8a), yielding

$$\mathbf{Q}^{(p)\,-1} = \mathbf{b}_{\rm g}^{-1} \cdot \mathbf{Q}_{\rm g}^{(p)\,-1} \cdot \mathbf{b}_{\rm g} = \mathbf{Q}_{\rm g}^{(p)\,-1} + \boldsymbol{\Delta}^{(p)}. \tag{5.1}$$

The lower-triangular matrix $\boldsymbol{\Delta}^{(p)}$ contains the δ_k introduced immediately after eq. (2.9), which parametrize the deviation from the geometric scheme of the arbitrary-scheme β-function coefficients. The same result is obtained, of course, by solving eq. (3.6b) directly and eliminating the β_k in favor of the δ_k.

Another way to express eq. (5.1) is

$$\mathbf{Q}^{(p)\,-1} = \mathbf{Q}_{\rm g}^{(p)\,-1} \left(\mathbf{1} + \mathbf{K}^{(p)}\right), \tag{5.2}$$

where $\mathbf{K}^{(p)}$ looks like

$$\mathbf{K}^{(p)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \tau\delta_2 & 0 & 0 & 0 & 0 & 0 \\ 2\tau^2\delta_2 + \tau\delta_3 & 2\tau\delta_2 & 0 & 0 & 0 & 0 \\ 3\tau^3(2+pb)\delta_2 + 2\tau^2\delta_3 + \tau\delta_4 & 6\tau^2\delta_2 + 2\tau\delta_3 & 3\tau\delta_2 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{bmatrix}. \tag{5.3}$$

The matrix $\mathbf{K}^{(p)}$ can be decomposed into matrix coefficients of δ_i, $\delta_i\delta_j$, etc. The matrix multiplying single powers of δ_i possess an easily seen pattern:

$$\left.\frac{\partial K_{lk}}{\partial \delta_i}\right|_{\forall j, \delta_j = 0} = \begin{cases} 0, & l < k+i+1 \\ (k+1)\tau, & l = k+i+1 \\ (k+1)(l-i)\dfrac{\tau^{l-i}\Gamma(l-i+pb])}{\tau^k \Gamma(k+2+pb)}, & l > k+i+1. \end{cases} \tag{5.4}$$

For example, the term $3\tau^3(2+pb)\delta_2$ in $K_{50}^{(p)}$ is the first nontrivial term. Starting on the $l = 6$ row (not shown in eq. (5.3)), $\mathbf{K}^{(p)}$ contains pieces proportional to δ_i^2; similarly, starting on the $l = 7$ row, $\mathbf{K}^{(p)}$ contains pieces proportional to $\delta_i\delta_j$. In neither case is any pattern to the matrix coefficient apparent.

The original correction $\mathbf{\Delta}^{(p)}$ looks similar to the right-hand side of eq. (5.3), but its structure, which is most easily constructed from $\mathbf{Q}_\mathrm{g}^{(p)\,-1} \cdot \mathbf{K}^{(p)}$, is less illuminating than $\mathbf{K}^{(p)}$'s. The terms in r_l, $l \geq L$, stemming from $\mathbf{\Delta}^{(p)}$ are smaller than those from $\mathbf{Q}_\mathrm{g}^{(p)\,-1}$. In previous work on the large-l behavior of the r_l [3, 9, 22, 25], the δ_j appear in a way that does not look like the medium-l pattern accessible by the matrix derivation.

In practice, however, the details of $\mathbf{K}^{(p)}$ may not matter. Only the first few δ_j are known. In the geometric scheme they enter the coefficients $\boldsymbol{r}_\mathrm{g}$ and $\boldsymbol{f}_\mathrm{g}$. Thus, they may as well be absorbed into $\boldsymbol{r}_\mathrm{s}$ and $\boldsymbol{f}_\mathrm{s}$ by introducing

$$\boldsymbol{f}_\mathrm{Ks}^{(p)} \equiv \left(\mathbf{1} + \mathbf{K}^{(p)}\right) \cdot \boldsymbol{f}_\mathrm{s}^{(p)}, \qquad \mathfrak{A}_\mathrm{s} \cdot \boldsymbol{r}_\mathrm{s} = \mathfrak{A}_\mathrm{s} \cdot \mathbf{Q}_\mathrm{g}^{(p)\,-1} \cdot \boldsymbol{f}_\mathrm{Ks}^{(p)}. \tag{5.5}$$

Then Borel summation can be applied by combining the growing part of $\mathbf{Q}_\mathrm{g}^{(p)\,-1}$ with $\mathfrak{A}_\mathrm{s}$ and combining the diminishing part of $\mathbf{Q}_\mathrm{g}^{(p)\,-1}$ with $\boldsymbol{f}_\mathrm{Ks}^{(p)}$ to form the normalization factor. Indeed, if L orders are available, and the scheme is chosen so that $\delta_j = 0$ for all $j \leq L-2$, then the upper-left $L \times L$ block of $\mathbf{K}^{(p)}$ vanishes, and the knowable part of $\boldsymbol{f}_\mathrm{Ks}^{(p)}$ coincides with $\boldsymbol{f}_\mathrm{s}^{(p)}$.

A reason to consider schemes other than the geometric coupling is that $\alpha_\mathrm{g}(\mu)$ runs into a branch point of the Lambert-W function [26] at $\mu = (\mathrm{e}/2b)^b\Lambda$. (For $N_c = n_f = 3$, $(\mathrm{e}/2b)^b \approx 1.629$.) Figure 1 shows the running of α_g and α_2 (α_n for $n = 2$), in SU(3) gauge theory with three massless flavors. The pole in the geometric β function, which is the source of the problem, can be removed while retaining a closed-form relation between $\ln(\mu/\Lambda)$ and a family of schemes α_n:

$$\beta(\alpha_n) = -\frac{\beta_0\alpha_n^2}{1 - (\beta_1/\beta_0)\alpha_n + n(\beta_1\alpha_n/\beta_0)^{n+1}}, \tag{5.6a}$$

$$\ln(\mu/\Lambda) = \frac{1}{2\beta_0\alpha_n} + b\ln(\beta_0\alpha_n) - b\left(\frac{\beta_1\alpha_n}{\beta_0}\right)^n. \tag{5.6b}$$

In section 7, α_2 is used to study how MRS works in practice. Like α_g, α_2 has $\delta_2 = 0$, so that $\mathbf{K}^{(p)}$ can be neglected (for $L \leq 4$). $\alpha_{\overline{\mathrm{MS}}}$ can be formulated by integrating the $\overline{\mathrm{MS}}$ β function with either $1/\beta(\alpha_\mathrm{s})$ or $\beta(\alpha_\mathrm{s})$ expanded to fixed order. Both have an undesirable fixed point à la α_g. Truncating with β_3, the former choice — also used in section 7 — is valid only for $\mu \geq 2.1797\Lambda$, at which point $\alpha_\mathrm{s} = 0.97601$ (cf., figure 1). The latter (again truncating with β_3) is valid only for $\mu \geq 0.87645\Lambda$, asymptotically as $\alpha_\mathrm{s} \to \infty$ and in practice for $\alpha_\mathrm{s} \gtrsim 50$.

6 Anomalous dimensions

The Q dependence is not always as simple as the power law in eq. (2.1), because C_p can depend on Q via $\alpha_\mathrm{s}(Q)$. In the operator-product expansion, for example, power corrections

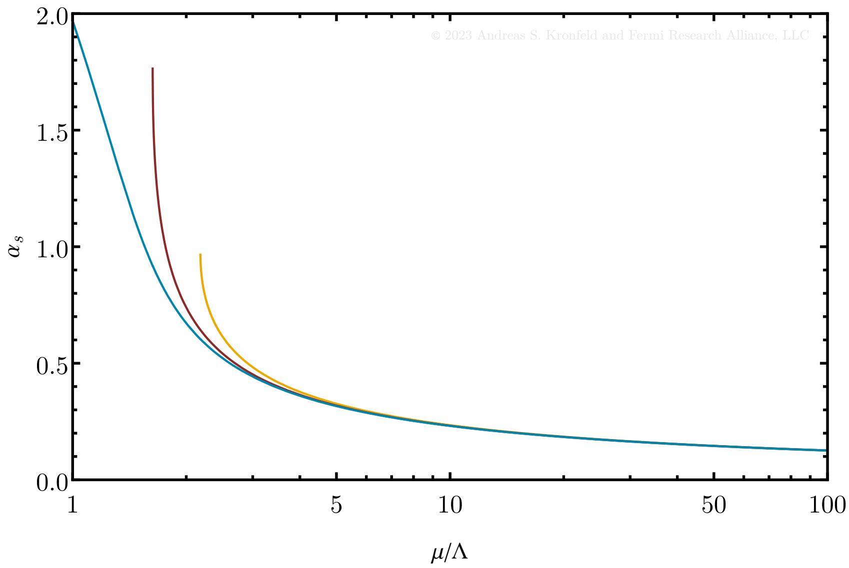

Figure 1. Gauge coupling α_s vs. μ/Λ in various schemes for $N_c = n_f = 3$: $\alpha_{\overline{\mathrm{MS}}}(\mu)$ (gold), $\alpha_{\mathrm{g}}(\mu)$ (red), and $\alpha_2(\mu)$ (blue). $\alpha_{\overline{\mathrm{MS}}}$ (with $1/\beta(\alpha)$ expanded to fixed order) and α_{g} run into branch points at $\mu \approx 2.1797\Lambda$ and $(\mathrm{e}/2b)^b\Lambda \approx 1.629\Lambda$, respectively. For smaller μ, they are undefined. At these points, $\alpha_{\overline{\mathrm{MS}}}(2.1797\Lambda) = 0.97601$, $\alpha_{\mathrm{g}}(1.629\Lambda) = 1.76715$. $\alpha_2(\mu)$ does not behave this way and like α_{g} has $\delta_2 = 0$.

take the form

$$C_p(\mu/Q, \alpha_{\mathrm{s}}(\mu))\frac{\langle \mathscr{O}(\mu)\rangle}{Q^p} = \widehat{C}_p(\alpha_{\mathrm{s}}(Q))\frac{\langle \mathscr{O}_{\mathrm{RGI}}\rangle}{Q^p}. \tag{6.1}$$

On the right-hand side, the renormalization group has been used to factor the μ dependence, such that $\langle \mathscr{O}_{\mathrm{RGI}}\rangle \propto \Lambda^p$. The renormalization-group-invariant (RGI) Wilson coefficient can be written

$$\widehat{C}_p(\alpha_{\mathrm{s}}) = (2\beta_0\alpha_{\mathrm{s}})^{\psi} \sum_{l=-1} c_l \alpha_{\mathrm{s}}^{l+1}, \tag{6.2}$$

where $\psi = \gamma_0/2\beta_0$ and γ_0 is the one-loop anomalous dimension of $\mathscr{O}$. Some of the leading coefficients may (for some reason) vanish, and the series is known in practice only to some order. Strategies for truncating the series in eq. (6.2) lie beyond the scope of this paper.

Let us assume $c_{-1} \neq 0$. It is convenient to extend the matrix notation to $\mathfrak{A}_{\mathrm{g}} = \begin{bmatrix} 1 & \alpha_{\mathrm{g}} & \alpha_{\mathrm{g}}^2 & \alpha_{\mathrm{g}}^3 & \alpha_{\mathrm{g}}^4 & \cdots \end{bmatrix}$, $\boldsymbol{r}_{\mathrm{g}} = \begin{bmatrix} r_{-1} & r_0 & r_1 & r_2 & r_3 & \cdots \end{bmatrix}^{\mathrm{T}}$, and so on. The r_{-1} entry is useful for bookkeeping; it cannot influence the final result, so below it can be set to 0, which is equivalent to changing to physical quantity to $\mathscr{R}(Q) - r_{-1}$.

To isolate Λ^p so that it can be differentiated away, it is necessary to multiply by $Q^p/\widehat{C}_p$. Division by the Wilson coefficient changes the series $\mathfrak{A}_{\mathrm{g}} \cdot \boldsymbol{r}_{\mathrm{g}}$ to $(2\beta_0\alpha_{\mathrm{s}})^{-\psi}\mathfrak{A}_{\mathrm{g}} \cdot \mathbf{C}^{-1} \cdot \boldsymbol{r}_{\mathrm{g}}$, where

$$\mathbf{C} = \begin{bmatrix} c_{-1} & 0 & 0 & 0 & \cdots \\ c_0 & c_{-1} & 0 & 0 & \cdots \\ c_1 & c_0 & c_{-1} & 0 & \cdots \\ c_2 & c_1 & c_0 & c_{-1} & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{6.3}$$

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The operation $\hat{Q}^{(p)}$ is applied to $(2\beta_0\alpha_s)^{-\psi}\mathfrak{A}_g \cdot \mathbf{C}^{-1}\boldsymbol{r}_g$, followed by multiplication by $\widehat{C}_p$. These steps yield $\mathfrak{A}_s \cdot \boldsymbol{f}_g^{(p,\psi,\widehat{C})}$ with

$$\boldsymbol{f}_g^{(p,\psi,\widehat{C})} = \mathbf{C} \cdot \mathbf{Q}_g^{(p,\psi)} \cdot \mathbf{C}^{-1} \cdot \boldsymbol{r}_g, \tag{6.4}$$

and $\mathbf{Q}_g^{(p,\psi)}$ has the same entries as in eq. (3.7) but with $l \to l - \psi$ and $k \to k - \psi$. The inverse $\mathbf{Q}_g^{(p,\psi)\,-1}$ is given by eq. (3.8) with the same substitutions.

Eq. (6.4) can be solved for

$$\boldsymbol{r}_g = \mathbf{C} \cdot \mathbf{Q}_g^{(p,\psi)\,-1} \cdot \mathbf{C}^{-1} \cdot \boldsymbol{f}_g^{(p,\psi,\widehat{C})}, \tag{6.5}$$

which has the same structure as the scheme change eq. (5.1). Thus,

$$\mathbf{C} \cdot \mathbf{Q}_g^{(p,\psi)\,-1} \cdot \mathbf{C}^{-1} = \mathbf{Q}_g^{(p,\psi)\,-1} \cdot \left(\mathbf{1} + \mathbf{K}^{(p,\psi,\widehat{C})}\right), \tag{6.6}$$

and $\mathbf{K}^{(p,\psi,\widehat{C})}$ can be absorbed into the coefficients $\boldsymbol{f}_g^{(p,\psi,\widehat{C})}$, as in eq. (5.5) when estimating r_l, $l \geq L$. In the basic formulas for the improved series, eq. (3.10), it makes sense (along with $l \to l - \psi$ and $k \to k - \psi$) to change the conventional factor $\Gamma(1 + pb)$ to $\Gamma(1 + pb - \psi)$ and to omit ψ in the powers of $2\beta_0/p$. The change to the normalization factor, $R_0^{(p,\psi)}$, is straightforward. In the Borel summation leading up to eq. (3.15), ψ always appears as $pb - \psi$; the sum over l and splitting of the integration follow exactly as in section 3.

If more than one power correction has an anomalous dimension, they still can be removed successively. Now every step affects all subsequent steps. The case of removing two power corrections reveals how complications ensue. Let the two power terms be $\widehat{C}_i\Lambda^{p_i}/Q^{p_i}$, $i = 1, 2$. The first step converts the second Wilson coefficient

$$\widehat{C}_2 = (2\beta_0\alpha_s)^{\psi_2}\mathfrak{A}_s \cdot \boldsymbol{c}_2 \mapsto \frac{p_1 - p_2}{p_1}(2\beta_0\alpha_s)^{\psi_2}\mathfrak{A}_s \cdot \boldsymbol{c}_{2/1}, \tag{6.7a}$$

$$\boldsymbol{c}_{2/1} = \mathbf{C}_1 \cdot \mathbf{Q}^{(p_1-p_2,\psi_1-\psi_2)} \cdot \mathbf{C}_1^{-1} \cdot \boldsymbol{c}_2. \tag{6.7b}$$

The second step then leads to

$$\boldsymbol{f}_g^{\{(p_1,\psi_1,\widehat{C})_1,(p_2,\psi_2,\widehat{C})_2\}} = \mathbf{C}_{2/1} \cdot \mathbf{Q}_g^{(p_2,\psi_2)} \cdot \mathbf{C}_{2/1}^{-1} \cdot \mathbf{C}_1 \cdot \mathbf{Q}_g^{(p_1,\psi_1)} \cdot \mathbf{C}_1^{-1} \cdot \boldsymbol{r}_g. \tag{6.8}$$

Note that the same outcome is obtained if the Λ^{p_2} term is removed first, i.e.,

$$\mathbf{C}_{2/1} \cdot \mathbf{Q}_g^{(p_2,\psi_2)} \cdot \mathbf{C}_{2/1}^{-1} \cdot \mathbf{C}_1 \cdot \mathbf{Q}_g^{(p_1,\psi_1)} \cdot \mathbf{C}_1^{-1} = \mathbf{C}_{1/2} \cdot \mathbf{Q}_g^{(p_1,\psi_1)} \cdot \mathbf{C}_{1/2}^{-1} \cdot \mathbf{C}_2 \cdot \mathbf{Q}_g^{(p_2,\psi_2)} \cdot \mathbf{C}_2^{-1}, \tag{6.9}$$

and similarly for their inverses. A decomposition of the right-hand side of eq. (6.8) along the lines of eq. (4.5) seems possible by isolating $\mathbf{Q}_g^{(p_1,\psi_1)}$ and $\mathbf{Q}_g^{(p_2,\psi_2)}$ and pragmatically absorbing the rest into the coefficients (as in eq. (6.6)), but an elegant arrangement has (so far) eluded me.

Suppose the c_l vanish for $l < n$. The first nonzero term, c_n, should not be connected to the r_l, $l \leq n$. A possible route forward is to subtract $\sum_{l=-1}^{n} r_l\alpha_s^{l+1}$ from $\mathscr{R}$, and the difference is still a valid observable. The factorially growing contributions can then be treated as before. If $p_2 = p_1$ but $\psi_2 \neq \psi_1$, the vector $\boldsymbol{c}_{2/1}$ in eq. (6.7) must be redefined as $-2\mathbf{C}_1 \cdot \mathbf{D}^{(\psi_1-\psi_2)} \cdot \mathbf{C}_1^{-1} \cdot \boldsymbol{c}_2$ with $c_{2/1,-1} = 0$, so the second step will have to be tweaked in a similar way.

7 The static energy

To see MRS in action, the procedure is applied in this section to the gluonic energy stored between a static quark and a static antiquark, $E_0(r)$, called the "static energy" for short. It is computed in lattice gauge theory from the exponential fall-off at large t of a $t \times r$ Wilson loop [27, 28]. The lattice quantity is the sum of a physical quantity plus twice the linearly divergent self-energy of a static quark. Dimensional regularization has no linear divergence, but on general grounds a constant of order Λ is possible. Setting $\mathscr{R}(1/r) = -rE_0(r)/C_F$ yields a quantity of the form given in eq. (2.1) with $r_{-1} = 0$ and $p = 1$.

The static energy is a good candidate to test MRS because four orders in perturbation theory are known, thus enabling a thorough test. Beyond the tree-level result of order α_s, $\overline{\text{MS}}$-scheme results are available at order α_s^2 [29, 30], α_s^3 [31–33], and α_s^4 [34–37]. The one-loop [38, 39], two-loop [40, 41], three-loop [42, 43], and four-loop [44–46] coefficients of the $\overline{\text{MS}}$ β function are also needed. The five-loop coefficient β_4 [47–49] is not needed here.

References [29–37] compute the static potential, $V(q)$, in momentum space, finding it to be infrared divergent starting at order α_s^4 [50]. This behavior reflects the emergence of an "ultrasoft" scale $\alpha_s r^{-1}$ in addition to the hard scale r^{-1}. Ultrasoft contributions can be described in a multipole expansion and thereby demonstrated to render the static energy infrared finite [51–53]. If $\alpha_s r^{-1} \gg \Lambda$, the ultrasoft part can be calculated perturbatively [51, 53], and the total static energy is explicitly seen to be infrared finite [35, 51, 53]. A remnant of the cancellation remains in logarithms of the ratio of the two scales, $\ln[(\alpha_s r^{-1})/r^{-1}] = \ln \alpha_s$.

Following the exposition of Garcia i Tormo [54], a momentum-space quantity, here denoted $\tilde{\mathscr{R}}(q)$, poses a second problem à la eq. (2.1), again with $r_{-1} = 0$ but now with $p > 1$. To distinguish the series coefficients associated with $\tilde{\mathscr{R}}(q)$ and $\mathscr{R}(1/r)$ from each other and the distance r, the notation used here is

$$\tilde{R}(q) = \sum_{l=0} a_l(\mu/q)\alpha_s(\mu)^{l+1}, \qquad R(1/r) = \sum_{l=0} v_l(\mu r)\alpha_s(\mu)^{l+1}. \tag{7.1}$$

The coefficients $a_l(1)$ are available in the literature [29–37] and can be found in a consistent notation in the accompanying Mathematica [55] notebook. Each $v_l(\mu r)/r$ is 4π times the Fourier transform of $a_l(\mu/q)/q^2$. Indeed, the $p = 1$ factorial growth of the v_l arises from the Fourier transform of the logarithms (cf., eq. (2.4)) in $a_l(\mu/q)$. The series $F^{(1)}(1/r)$, derived as in section 3 from $R(1/r)$, is related to the "static force", $\mathfrak{F}(r) = -\mathrm{d}E_0/\mathrm{d}r$, by $\mathscr{F}(r) = F^{(1)}(1/r) = -r^2\mathfrak{F}(r)/C_F$. Note that $\mathfrak{F}(r)$ — and, hence $\mathscr{F}(r)$ and $F^{(1)}(1/r)$ — is expected to be free of renormalon ambiguities [52, 56], because the change in static energy from one distance to another is physical. The series f_l should eventually exhibit factorial growth owing to instantons, i.e., with $p \geq 4\pi\beta_0 = \frac{11}{3}C_A - \frac{4}{3}\sum_f T_f$.

The remainder of this section gives numerical and graphical results for SU(3) gauge theory with three massless flavors. For brevity, the superscript "(1)" on $F^{(1)}$, $R_0^{(1)}$, etc., is omitted. To obtain numerical results and prepare plots, α_s in the ultrasoft logarithm, $\ln \alpha_s$, must be specified. This α_s can be taken to run, namely taken to be the same as the expansion parameter $\alpha_s(\mu)$. Alternatively, α_s can be held fixed. Below, $\alpha_s(s/r)$ (or $\alpha_s(sq)$), for various fixed s is used as an expansion parameter, and the ultrasoft α_s is chosen either to be the same or, for comparison, a fixed value $\alpha_s = \frac{1}{3}$. This value arises at scales where

l	$\overline{\text{MS}}$ $a_l(1)$	$\overline{\text{MS}}$ $f_l(1)$	geometric $a_l(1)$	geometric $f_l(1)$	eq. (5.6), $n=2$ $a_l(1)$	eq. (5.6), $n=2$ $f_l(1)$
0	1	1	1	1	1	1
1	0.557 042	−0.048 552	0.557 042	−0.048 552	0.557 042	−0.048 552
2	1.702 18	0.687 291	1.834 97	0.820 079	1.834 97	0.820 079
3	2.436 87	0.323 257	2.832 68	0.558 242	3.013 89	0.739 452

Table 2. Perturbation series coefficients with $s=1$ for $\tilde{R}(q)$ (a_l) and $F(r)$ (f_l). Here $\alpha_s = \frac{1}{3}$ for a_3 and f_3.

l	$\overline{\text{MS}}$ $v_l(1)$	$\overline{\text{MS}}$ $v_l(1)-V_l(1)$	geometric $v_l(1)$	geometric $v_l(1)-V_l(1)$	eq. (5.6), $n=2$ $v_l(1)$	eq. (5.6), $n=2$ $v_l(1)-V_l(1)$
0	1	0.206 061	1	0.182 531	1	0.177 584
1	1.383 84	−0.202 668	1.383 84	−0.249 689	1.383 84	−0.259 574
2	5.462 28	0.019 479	5.595 07	−0.009 046	5.595 07	−0.042 959
3	26.6880	0.219 262	27.3034	0.050 179	27.4846	0.066 468

Table 3. Perturbation series coefficients with $s=1$ for $R(r)$ and R_{RS} (with V_l derived from v_l as R_l from r_l in section 3). Here $\alpha_s = \frac{1}{3}$ for v_3 and $v_l - V_l$.

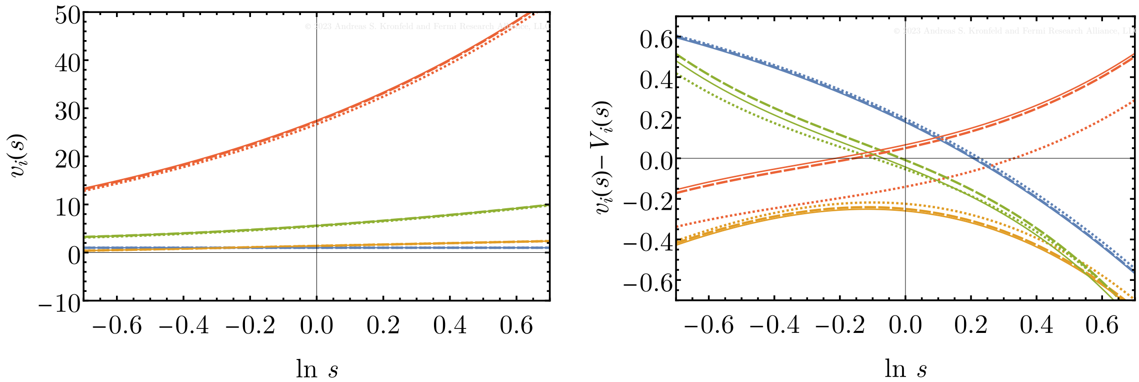

Figure 2. Scale dependence of $v_l(s)$ (left) and $v_l(s) - V_l(s)$ (right) vs. $\ln s$. Note the difference in vertical scale. Blue, gold, green, and red correspond to $l = 0$, 1, 2, and 3, respectively. Dotted, dashed, and solid curves correspond to the $\overline{\text{MS}}$, geometric, and α_2 schemes, respectively.

perturbation theory starts to break down, making it a reasonable alternative. Resummation of logarithms $\alpha_s^{3+n} \ln^n \alpha_s$ [57] and $\alpha_s^{4+n} \ln^n \alpha_s$ [58, 59] is not considered here.

Table 2 shows the first four a_l and f_l in three different renormalization schemes, $\overline{\text{MS}}$, geometric, and eq. (5.6) with $n = 2$. The scheme dependence in the two- and three-loop coefficients is about 10%. The (non)growth in l conforms with expectations: a_l is perhaps growing slowly and f_l is not growing yet. (Recall, $p > 1$ for a_l and $p \geq 9$ for f_l.) Table 3 shows the first four v_l in the same three schemes. The growth is obvious. Table 3 also shows the subtracted coefficients $v_l(1) - V_l(1)$. The cancellation is striking.

The cancellation at $s = 1$ is robust, as shown in figure 2 over an illustrative interval of $\ln s$. The range of $v_3(s)$ and even $v_2(s)$ dwarfs that of all $v_l(s) - V_l(s)$: $v_3(s) - V_3(s)$

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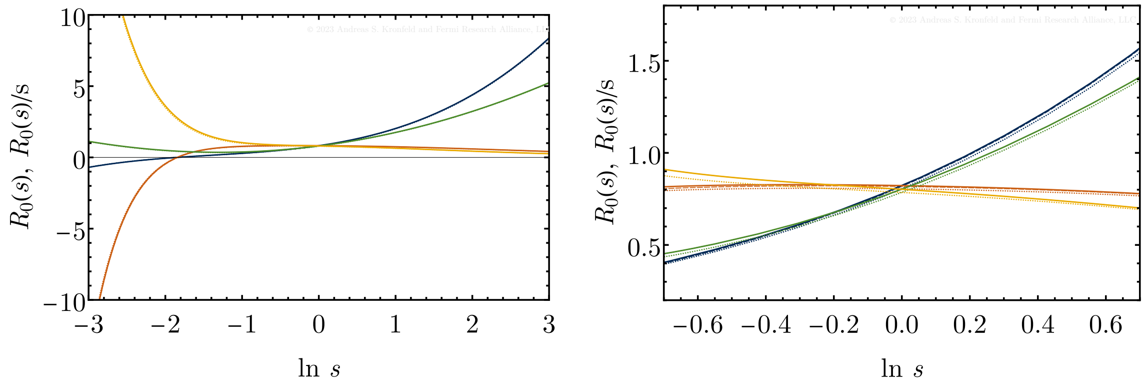

Figure 3. Scale dependence of $R_0(s)$ and $R_0(s)/s$ over a very wide range (left) and a relevant range (right). Blue and orange (green and yellow) curves corresponds to $L = 4$ ($L = 3$) in eq. (3.10b). Blue and green (orange and yellow) curves corresponds to $R_0(s)$ ($R_0(s)/s$). Dotted, dashed, and solid curves correspond to the $\overline{\text{MS}}$, geometric, and α_2 schemes, respectively.

	$\overline{\text{MS}}$		geometric		eq. (5.6), $n = 2$	
s	$L = 4$	$L = 3$	$L = 4$	$L = 3$	$L = 4$	$L = 3$
$\frac{1}{2}$	0.386 864	0.437 281	0.403 196	0.454 397	0.397 605	0.437 281
1	0.793 939	0.785 114	0.817 469	0.802 230	0.801 081	0.785 114
2	1.523 44	1.387 07	1.554 17	1.404 19	1.526 98	1.387 07

Table 4. Normalization factor $R_0(s)$ of the $p = 1$ factorial growth in three schemes for $s \in \{\frac{1}{2}, 1, 2\}$ at three ($L = 4$) and two ($L = 3$) loops. Here $\alpha_s = \frac{1}{3}$ for $L = 4$.

$(v_2(s) - V_2(s))$ is 50–100 (5–10) times smaller than $v_3(s)$ ($v_2(s)$). Near $\ln s = 0$, these two subtracted coefficients are unusually small. Overall, the cancellation is best for $\ln s \approx \frac{1}{4}$, where $|v_0 - V_0|$ is especially small, while the others are of typical size.

Interestingly, as $\ln s$ is taken negative both factors in the first term $[v_l(s) - V_l(s)]\alpha_s(s/r)$ increase. This behavior can be traced to the normalization factor $R_0(s)$, which is plotted in figure 3 for the three schemes. There is not much scheme dependence. Curves for $L = 4$ and $L = 3$ in eq. (3.10b) are shown. They are close, or even very close, to each other for $|\ln s| \leq \ln 2$. Sample numerical values are given in table 4, again using both four and three terms in eq. (3.10b). The shape of $R_0(s)$ follows from the positivity of the highest-power logarithmic term in eq. (2.4) and the positivity of the coefficients in eq. (3.10b). Near $\ln s = -2$, the four-term $R_0(s)$ goes negative, which is a reflection of $v_3(s)$ being run to an absurd extreme while omitting (unknown) higher orders. Indeed, the three-term approximation to $R_0(s)$ turns up near $\ln s = -2$, which is a reflection of $v_2(s)$ being run to an absurd extreme. Figure 3 also shows $R_0(s)/s$, which multiplies the term absorbed into the power correction (cf., last sentence in section 3). It is nearly constant over a wide range, especially once $L = 4$.

The coefficients' variation with s is set up to compensate that of $\alpha_s(sq)$ or $\alpha_s(s/r)$. Figure 4 shows how $\tilde{R}(q)$, $F(1/r)$, $R(1/r)$, and $R_{\text{MRS}}(1/r)$ depend on Λ/q or $r\Lambda$ for $s \in \{\frac{1}{2}, 1, 2\}$. (Plotted this way, the high-q, short-r domain, where perturbation theory

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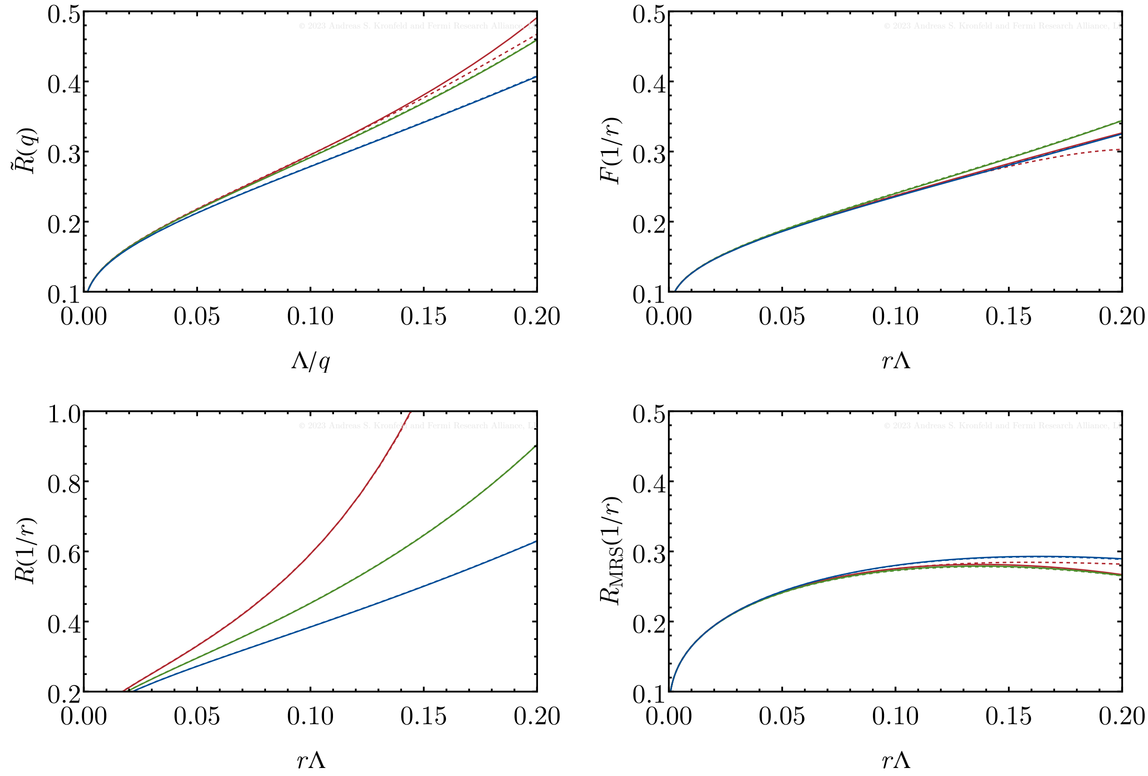

Figure 4. Scale variation in the α_2 scheme up to and including the α_s^4 term. Top: $\tilde{R}(q)$ and $F(1/r)$; neither suffers the $p = 1$ renormalon. Bottom: $R(1/r)$ (with $p = 1$ renormalon) and $R_{\mathrm{MRS}}(1/r)$ (after MRS). Red, green, and blue curves correspond to $s = \frac{1}{2}$, $s = 1$, and $s = 2$, respectively. Solid (dashed) curves correspond to a running (fixed) α_s in the ultrasoft $\ln \alpha_s$. Note that the vertical scale for $R(1/r)$ is twice that of the other three plots.

works best without any effort, is shrunk into a small region.) The variation with s is mild for $\tilde{R}(q)$, even milder for $F(1/r)$, and catastrophic for $R(1/r)$. After MRS, however, the scale variation is as mild for $R_{\mathrm{MRS}}(1/r)$ as for the renormalon-free $F(1/r)$. As shown in figure 5, the fractional difference of both remains a few percent for $r\Lambda \lesssim 0.1$ (with $s = 1$ and running ultrasoft $\ln \alpha_s$ as the baseline).

The mild variation with s is a pleasant outcome given the s dependence of the subtracted coefficients (cf., figure 2). Figure 6 shows the variation with s as a function of r of the Borel sum $R_{\mathrm{B}}(1/r)$ (left, eq. (3.17)) and the subtracted series $R_{\mathrm{RS}}(1/r)$ for $L = 4$ (right). Both are quite sensitive to s, but their sum (bottom right of figure 4) is not.

The first two orders suffice to lift the s dependence, as shown in figure 7. Here, $R_{\mathrm{B}}(1/r)$ is shown (dotted curve) and each term $(v_l - V_l)\alpha_s^{l+1}$, $l = 0, 1, 2, 3$, in $R_{\mathrm{RS}}(1/r)$ is accumulated (dashed curves with longer dashes as the order increases) until the total $L = 4$ (solid) result $R_{\mathrm{MRS}}(1/r)$ is reached.

Adding the tree-level term $(v_0 - V_0)\alpha_s$ to the Borel sum overshoots the full (solid) result, but adding the one-loop term $(v_1 - V_1)\alpha_s^2$ yields a curve almost indistinguishable from $R_{\mathrm{MRS}}(1/r)$. Indeed, it is hard to distinguish the longer-dashed curves from the solid ones, underscoring that the two-loop term $(v_2 - V_2)\alpha_s^3$ makes a small change while the

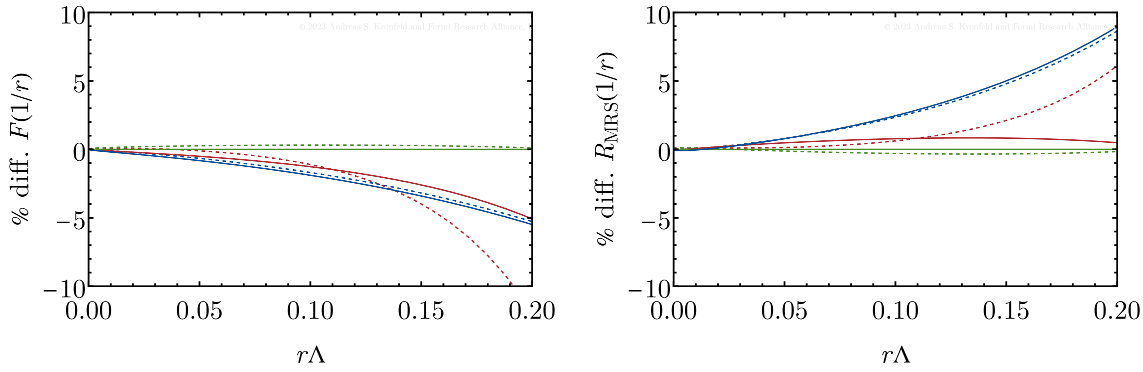

Figure 5. Scale variation in the α_2 scheme of the fractional difference of $F(1/r)$ (left) and $R_{\mathrm{MRS}}(1/r)$ (right), with respect to $s = 1$ with running ultrasoft α_{s}. Curve and color code as in figure 4.

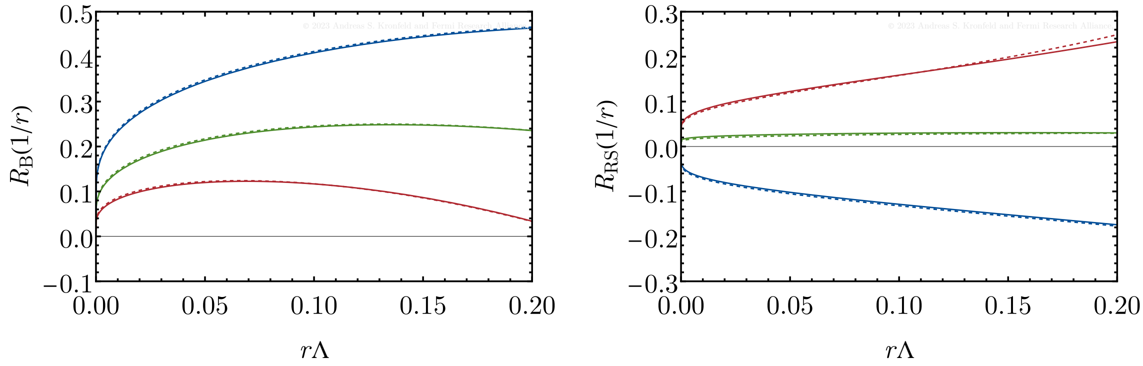

Figure 6. Scale variation in the α_2 scheme of the Borel sum $R_{\mathrm{B}}(1/r)$ (left) and the $L = 4$-subtracted series $R_{\mathrm{RS}}(1/r)$ (right). Curve and color code as in figure 4.

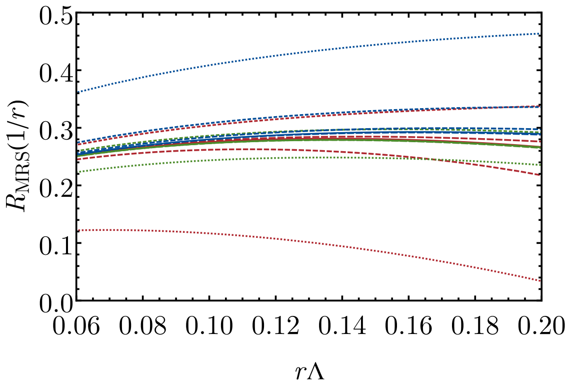

Figure 7. Borel sum $R_{\mathrm{B}}(1/r)$ (dotted curves) accumulating successively each term $(v_l - V_l)\alpha_{\mathrm{s}}^{l+1}$ (dashed curves with longer dashes for larger l) in the three schemes. Solid curves for the full R_{MRS}. Color code as in figure 4.

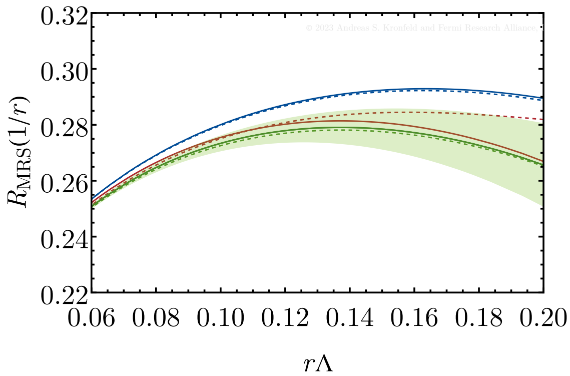

Figure 8. Same as figure 4 (bottom right) but with a band stemming from the uncertainty in R_0 (taken equal to the last term in eq. (3.10b)) and an expanded vertical scale. Curve and color code as in figure 4.

three-loop term $(v_3 - V_3)\alpha_s^4$ makes hardly any change. As with the pole mass [2], MRS perturbation theory converges (in the practical sense) quickly.

Let us return to the term-by-term change in R_0 (cf., figure 3 and table 4). The highest-order term in eq. (3.10b) can be used to estimate the uncertainty in R_0 from omitting even higher orders [2]. In the case at hand, the term with f_3 yields the estimate, which is around 10% or less (cf., table 4). Figure 8 (right) overlays the resulting $s = 1$ uncertainty band on $R_{\text{MRS}}(1/r)$ on the curves (at various s) of figure 4 (bottom right). The uncertainty propagated to R_{MRS} is smaller than 10% because changes in R_0 push $R_{\text{B}}(1/r)$ and $R_{\text{RS}}(1/r)$ in opposite directions. Indeed, the uncertainty in R_{MRS} stemming from R_0 is smaller than the difference between the $s = 1$ and the $s = \frac{1}{2}$ and 2 curves. Note, however, that the R_0-uncertainty, as defined here, is smaller at $s = 1$ than at $s = \frac{1}{2}$ and 2 (cf., figure 3 and table 4). The uncertainty bands of these other choices (not shown) cover all three.

8 Summary and outlook

The initial aim of this work was to study and extend the discussion of factorial growth and renormalons started in refs. [2, 3]. I found, however, that the perspective, derivation, and interpretation could be simplified: a straightforward analysis extracts information from the renormalization-group constraints on the series coefficients. The only other ingredient is the knowledge (or assumption) of the powers $\{p_i\}$ of the power-suppressed corrections to the perturbative series of a physical observable. A by-product of adding this information to the series is to subtract the leading factorial growth (aka "renormalon effects") from the first few series coefficients. Remarkably, the factorial growth is not just a large-order phenomenon: it starts at low orders. How it comes to dominate the coefficients depends of the power of the power correction. (The lower the power, the more powerful the factorial!)

The worked example of the static energy (section 7) seems successful in removing a power correction of order $r\Lambda$ from (a dimensionless version of the) static energy, $-rE_0/C_F$. The conventional choice of $\mu = 1/r$ (in the $\overline{\text{MS}}$ scheme) seems near an optimum: perturbation theory converges with the MRS treatment as well as it does for the static force, which is thought to suffer corrections only of high power. Varying μ by a factor of two rearranges contributions between the tree and one-loop fixed order contributions, on the one hand, and (a specific definition of) the Borel sum of the factorial growth, on the other. At *very* short distances, $r\Lambda \lesssim \frac{1}{16}$, the total result (using all information through order α_s^4) does not vary over a wide range of μ. It will therefore be interesting to fit to lattice-QCD data (e.g., that of ref. [28]) and compare with other approaches to taming the series. (Some other approaches are described in refs. [56, 60–64] and earlier work cited there.)

Some practical issues remain before applying the MRS procedure to, say, an α_s determinations. The MRS method, like standard perturbation theory, does not say what scale s to choose: starting in the $\overline{\text{MS}}$ scheme with $s = 1$ and varying by a factor of 2 is conventional. When the factorial growth of coefficients matters, i.e., when MRS has something to offer, the $\ln^n s$ contributions associated with scale setting cannot tame the coefficients. Scale setting in light of MRS may warrant a closer look. Another issue is that in many applications, some of the quarks cannot be taken massless. A nonzero quark mass in a loop alters the

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loop's growth, removing the factorials from the infrared. It is probably best to add massive quark-loop effects at fixed order and not to use the massless result as a stand-in for the massive one [65]. Last, when anomalous dimensions are an important feature for more than one round of MRS, the method (as presented in section 6) remains to cumbersome to be appealing. It may suffice to neglect the anomalous dimensions, but only practical experience will tell.

A Modified Borel summation

Alternatives to the standard Borel resummation are possible [14], and a natural variant is pursued here, leading to the same endpoint. Start with eq. (3.12):

$$R_{\mathrm{B}}^{(p)} \equiv \sum_{l=0}^{\infty} R_l^{(p)} \alpha^{l+1} = R_0^{(p)} \alpha \sum_{l=0}^{\infty} \left(\frac{2\beta_0\alpha}{p}\right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)}. \tag{A.1}$$

The l-dependent Γ function can be expressed as $\Gamma(l+1+pb) = \int_0^\infty t^{l+pb} \mathrm{e}^{-t} \mathrm{d}t$. Swapping the order of summation and integration

$$R_{\mathrm{B}}^{(p)} = \frac{R_0^{(p)}\alpha}{\Gamma(1+pb)} \int_0^\infty \sum_{l=0}^{\infty} \left(\frac{2\beta_0\alpha t}{p}\right)^l t^{pb}\mathrm{e}^{-t}\mathrm{d}t = \frac{R_0^{(p)}\alpha}{\Gamma(1+pb)} \int_0^\infty \frac{t^{pb}\mathrm{e}^{-t}}{1-2\beta_0\alpha t/p}\mathrm{d}t, \tag{A.2}$$

which only has a simple pole instead of a branch point. After integrating

$$R_{\mathrm{B}}^{(p)} = R_0^{(p)} \frac{p}{2\beta_0} \mathscr{J}(pb, p/2\beta_0\alpha) - R_0^{(p)} \mathrm{e}^{\pm \mathrm{i}pb\pi} \frac{p^{1+pb}}{2^{1+pb}\beta_0} \Gamma(-pb) \left[\frac{\mathrm{e}^{-1/2\beta_0\alpha}}{(\beta_0\alpha)^b}\right]^p, \tag{A.3}$$

where the factor $\mathrm{e}^{\pm \mathrm{i}pb\pi}$ in the second term corresponds to passing the contour below or above the pole. As before, the second term can be absorbed into the power correction, and the first — the principal part — is taken to define $R_{\mathrm{B}}^{(p)}$, the same as eq. (3.15).

Acknowledgments

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