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ABSTRACT

Dynamics of electrons subjected to a constant amplitude classical electromagnetic (EM) wave is investigated as a fundamental, representative problem in the physics of interacting quantum and classical waves. In the nonrelativistic regime (electrons as Schrödinger waves), the electron energy acquires a constant and a time dependent part. Driven by EM waves, both parts scale strongly with the amplitude, but we expect no resonant enhancement since the parallel electron "speed" of nonrelativistic electrons could never match the wave phase velocity. In the relativistic regime (electron as a Klein–Gordon wave), however, a class of electron waves (with parallel speed matching the EM phase speed) are resonantly excited to extremely high energies. Such a direct resonant energy transfer from intense electromagnetic waves constitutes a mechanism that could, in principle, power the most energetic of cosmic rays (this mechanism will work on protons just as well). Some predictions of the theory will, hopefully, be tested in laboratory laser experiments. The nonrelativistic calculations will also be examined in the context of recent experiments using photon-induced near-field electron microscopy in detail.

I. INTRODUCTION

Wave–wave and wave–particle interactions/resonances are the two fundamental processes that dominate energy exchange between various collective motions of many-body systems, in particular, of classical plasma physics. The celebrated phenomenon of Landau damping (sometimes Landau growth), one of the pillars of theoretical physics, describes how a particle traversing near the phase speed of a wave can, very effectively, exchange energy with the wave. The particles, traveling a bit faster than the wave, impart energy to the wave (instability) while the particles traveling a bit slower, may gain energy from the wave (particle acceleration).

Cherenkov radiation, discovered even earlier, is an "instability" where a class of particles, moving faster than light (in a dielectric medium with a refractive index greater than unity), emit light, converting particle energy into a light wave. In nonrelativistic (NR) systems, such a direct interaction of particles with light waves is not possible although extreme relativistic particles could do Cherenkov-like emission in a tenuous plasma where the speed of light is little less than c.

In quantum mechanics, the erstwhile classical particle (say an electron) is described by a wave equation. The interaction between an electron and a classical electromagnetic (EM) wave, then, will translate to an interaction between a quantum wave (QW) and a classical wave (CW). Let us give the generic name QW–CW to this class of wave–wave processes. An investigation of QW–CW in both the nonrelativistic (electrons obeying Schrödinger and Pauli–Schrödinger equations) and relativistic regimes [Klein–Gordon (KG) and Dirac equations], possibly, with and without an ambient magnetic field, is the theme of this paper.

The interest in QW–CW is further stimulated by the recent experimental work[1] in photon-induced near-field electron microscopy (PINEM),[1–8] demonstrating the resonant phase-matching exchange between photons and electrons. In the most theoretical work on PINEM, the photon field is approximated by an effective longitudinal electric field that interacts strongly with the electron whose velocity is essentially along the direction of propagation, that is, in the paraxial approximation. Though the parallel electron motion may be near relativistic, it is shown that the nonrelativistic treatment with appropriate correction to the electron mass is adequate.[9] The resonant phase-matching effect is achieved by using a medium in order to reduce the velocity of light so that electron–photon resonance becomes possible.

Though inspired, partially, by the PINEM results, the current QW–CW formulation is quite different; in fact, our photon field is purely transverse same as the vacuum-like EM field in a tenuous plasma. Despite the fact that the most profound manifestation of QW–CW is found to occur for highly relativistic electrons in a transverse EM field, we will begin our investigation with a nonrelativistic formulation. It will be assumed that the EM wave is circularly polarized (for algebraic simplicity) and its amplitude is maintained constant; thus, the efficacy of the resonant interaction will be measured by the energy associated with the electron states.

We will develop a systematic theory of energy exchange between a quantum and a classical wave, demonstrating when the energy transfer may be efficient. We will concentrate on investigating the dynamics of the quantum electron wave (EW) in the field of a circularly polarized electromagnetic (CPEM) wave (with and without an ambient magnetic field). The nature of the EW–EM interaction, of course, can drastically change as we go from the nonrelativistic to relativistic electrons. The nonrelativistic EW is parabolic, while the relativistic EW is hyperbolic like the EM waves; one would expect much stronger resonance effects in the latter combination. In either case, high energy states emerge in the relativistic EW that can be catapulted to extreme energies.

In our calculations, the EM field is not quantized; consequently, it does not directly deal with the experimental setup in which two (or any finite number) photons are simultaneously interacting with a quantum electron. A classical EM field, however, is a many-photon entity. Thus, this work studying the electron quantum dynamics in this classical EM field is an example of the electron–multiple photon interaction.

We first calculate the spectrum of the nonrelativistic EW embodied in the Schrödinger and Pauli–Schrödinger equations embedded in a circularly polarized EM wave (Sec. II A). In addition to the strict nonrelativistic problem, we also simulate aspects of a PINEM-like configuration by assuming that the parallel electron motion may be near relativistic and show that a resonance interaction does become possible; such a resonant interaction is clearly not available in a fully nonrelativistic electron. It also turns out that the spin contribution to the electron energy is negligible. This system is a representative of coupled parabolic and hyperbolic waves.

In Sec. II B, the system is generalized to include a constant external magnetic field. It is found that for a given CPEM amplitude and frequency, one can always find B_0 so that the effective energy of even the lowest landau level can be substantially modified. Precise values of B_0 for the particular case of visible light will be displayed.

When the EW–EM system is in the relativistic regime, we must resort to the Klein–Gordon/Dirac equation; both interacting waves, then, are hyperbolic. The hyperbolic waves display very strong coupling that could "resonantly" enhance the electron energy to extreme values (we will always assume a fixed amplitude for the EM wave). The Klein–Gordon (KG)/Dirac (mostly the simpler KG) system is solved and interpreted in Sec. III.

In Sec. IV, we would collect and summarize the most important findings of this effort and dwell on the possible implications and applications of our results.

II. QW–CW: NONRELATIVISTIC ELECTRON WAVES–ELECTROMAGNETIC WAVES

For analytic simplicity, the EM wave is assumed to be circularly polarized (CPEM). The contravariant components of electromagnetic four potential A^μ of a CPEM wave propagating in the z-direction are (the Minkowski signature tensor $\eta^{\mu\nu} = \mathrm{diag}[1, -1, -1, -1]$)

$$A^0 = 0 = A^z, \quad A^x = A\cos(\omega t - kz), \quad A^y = -A\sin(\omega t - kz), \tag{1}$$

where A is its constant amplitude and the four-wave vector in the lab frame is $k^\mu = [\omega, 0, 0, k]$. Notice that $k^\mu A_\mu = 0 = \mathbf{k}\cdot\mathbf{A}$, and the Lorentz invariant $A^\mu A_\mu = A^2$ has no space time dependence; it is the latter property that makes CPEM waves most amenable to analytical calculations. The CPEM waves in the electric $\mathbf{E} = \omega A(\hat{e}_x \sin(\omega t - kz) + \hat{e}_y \cos(\omega t - kz))$ and magnetic $\mathbf{B} = kA(-\hat{e}_x\cos(\omega t - kz) + \hat{e}_y \sin(\omega t - kz))$ fields imply a *constant* Poynting flux $(\mathbf{E}\times\mathbf{B} = k\omega A^2\hat{e}_z)$ along the z-direction. For most details, readers are referred to Ref. 12.

The preceding description of CPEM waves pertains to arbitrary values of the amplitude A, and it can be nonrelativistic $(qA/mc^2 \ll 1)$ or highly relativistic $(qA/mc^2 \gg 1)$. Here, $q(m)$ is the electron charge (mass) and c is the speed of light in vacuum. The CPEM wave will be treated as provided, and its dynamics is not calculated—only electron quantum mechanics subjected to this wave will be explored.

A. Schrödinger–Pauli quantum waves

We start with the simplest problem based on the Schrödinger equation embedded the in CPEM wave ($\hbar = 1 = c$)

$$i\frac{\partial\Psi}{\partial t} = \frac{1}{2m}(\mathbf{p} + q\mathbf{A})^2\Psi, \quad \mathbf{p} = -i\nabla, \tag{2}$$

where Ψ is the wave function, $\mathbf{p} = -i\nabla$ is the momentum operator, and $\mathbf{A}$ is a function of $\zeta = \omega t - kz$ and defined in (1), which is purely transverse and aligned perpendicular to the direction of propagation (z). After trivial manipulation, (2) becomes

$$\begin{aligned} -2im\frac{\partial\Psi}{\partial t} - (\nabla_\perp^2 + \partial_z^2)\psi - 2iqA[\cos(\omega t - kz)\partial_x\Psi \\ -\sin(\omega t - kz)\partial_y\Psi] + q^2A^2\Psi = 0, \end{aligned} \tag{3}$$

with the ζ dependent terms implying that while the perpendicular momentum components are conserved, the energy and the z-momentum are not. The wave function, then, allows independent Fourier modes

$$\Psi(x, y, z, t) = e^{iK_\perp(x\cos\varphi + y\sin\varphi)}\psi(t, z) \tag{4}$$

that obey

$$-2im\frac{\partial\psi}{\partial t} - \partial_z^2\psi + 2qAK_\perp\cos(\omega t - kz)\psi + (K_\perp^2 + q^2A^2)\psi = 0, \tag{5}$$

where $K_x = K_\perp\cos\varphi$, $K_y = K_\perp\sin\varphi$, and the phase φ has been eliminated in a trivial redefinition of the variable z. In Sec. III, we will seek more general solutions of the KG equation, but here we pursue the minimum solution that will reveal the interesting features of EW–EM coupling. If the t and z dependence come only via $\zeta = \omega t - kz$, i.e., $\psi(t, z) \equiv \psi(\zeta)$, then the Schrödinger equation becomes

$$-2im\omega\frac{d\psi}{d\zeta} - k^2\frac{d^2\psi}{d^2\zeta} + 2qAK_\perp \cos\zeta\psi + (K_\perp^2 + q^2A^2)\psi = 0. \quad (6)$$

Notice that every term in the preceding equation is "composed" from attributes of the two interacting waves, the EW (ψ, $K_\perp$, m, and q) and the EM wave (A, ω, and k).

Equation (6) could be trivially solved but for the inhomogeneous term proportional to $\cos\zeta$. This term, however, is the most profound expression of the EW–EM coupling; it goes to zero when the electron perpendicular momentum goes to zero. Although (6) can be converted to a Mathieu equation, it is much better to aim for an approximate solution. Notice that the ratio of the first two derivative terms is an effective dimensionless momentum in the nonrelativistic (NR) limit (where $K_\zeta = 1/L_\zeta$ is the characteristic length of the ζ-derivative with $K_\zeta \ll 1$),

$$R = \frac{m\omega}{k^2K_\zeta} \approx \frac{m}{kK_\zeta} \gg 1, \quad \omega \approx k \ll m. \quad (7)$$

Thus, the second derivative is negligibly small for the NR electron. The remaining equation is readily integrated as

$$\begin{aligned}\psi &= \psi_0 \exp\left(-\frac{i}{2m\omega}(\beta\zeta + \alpha\sin\zeta)\right)\\ &= \psi_0\sum_n J_n\left(\frac{\alpha}{2m\omega}\right)\exp\left(-i\left(\frac{\beta}{2m\omega}+n\right)\zeta\right),\end{aligned} \quad (8)$$

where $\alpha = 2qAK_\perp$ and $\beta = K_\perp{}^2 + q^2A^2$. We must emphasize that the preceding solution and the results to follow are correct only to the leading order ($\sim\alpha \approx \beta$); all terms of the kind $[(qA/m)\alpha$, $(\omega/m)\alpha$, $\ldots]$ are neglected.

We will discuss this solution in more detail:

- The normalized expectation values of the time dependent energy and the z-momentum associated with (8),

$$\begin{aligned}&E = i\left\langle\psi^*\frac{\partial\psi}{\partial t}\right\rangle, \quad K_z = -i\left\langle\psi^*\frac{\partial\psi}{\partial z}\right\rangle,\\ &\langle\ \ \rangle = \frac{1}{2L}\int_{-L}^{L}dz, \quad \frac{\partial}{\partial t} = -\frac{\omega}{k}\frac{\partial}{\partial z},\end{aligned} \quad (9)$$

  do, indeed, satisfy the phase matching condition

$$E = \frac{\omega}{k}K_z, \quad \frac{E}{K_z} = \frac{\omega}{k}, \quad (10)$$

  that is, the group velocity of the electron wave (in the direction of EM wave propagation) is the same as that of the EM wave. This is by design and not by accident. The wave function is, of course, appropriately normalized; the domain $[-L, L]$ defines the z-extent of the system.
- Equation (10) implies that, given the conserved perpendicular conserved $K_\perp$, it is sufficient to examine only the energy spectrum E,

$$E = \frac{\beta}{2m} + \frac{\alpha}{2m}\langle\cos\zeta\rangle \equiv \frac{K_\perp^2 + q^2A^2}{2m} + \frac{2qAK_\perp}{2m}\langle\cos\zeta\rangle, \quad (11)$$

  for a complete description of the state of the electron. Notice that a simple classical calculation to the leading order (see Appendix A), and for the same constraint (energy/z-momentum $= \omega$/k), gives

$$E_c = \frac{mv^2}{2} = \frac{q^2A^2}{2m}, \quad (12)$$

  which is simply the $K_\perp = 0$ limit of the quantum result. Thus, the most profound quantum modification is through the interaction of the electron perpendicular momentum with the CPEM propagating in the z-direction. The z-averaging is readily done to obtain the time dependent energy (expectation value of),

$$E = \frac{K_\perp^2 + q^2A^2}{2m} + \frac{2qAK_\perp}{2m}\frac{\sin(kL)}{kL}\cos(\omega t), \quad (13)$$

- the electron energy depends on a combination of kinematic ($K_\perp$) and field parts (A). Most interestingly, a part of the leading order electron energy oscillates at the EM wave frequency (ω); the higher harmonics do not appear in (13) because of our having neglected higher order terms [see the sentence after Eq. (8)]. In order to demonstrate that higher harmonics are, indeed, present, we calculate the amplitude of the second harmonic in Appendix B.
- Although we have calculated interesting synergy between the nonrelativistic EW and the EM wave, this interaction is not what would qualify as a "resonant" effect. It is of course expected since, for a nonrelativistic electron, $\langle v_z\rangle = \langle p_z\rangle/m \ll c$, and there is no direct parallel synchronization with the EM wave. There is, however, a strong wave–wave interaction that manifest through electron oscillations at the EM frequency (and its harmonics). It is essential to point out that the condition (10) is not a resonance condition; it simply reflects our decision to study only a limited class of solutions whose space time dependence is exactly the same as that of the wave. Resonant or not, the electrons do gain energy from the wave–wave interaction.
- We will find in Sec. III that, for relativistic electrons, extremely strong resonant transfer of energy will take place
- The wave function in Eq. (8) has also been expressed as a sum of n terms; each of the n terms is weighted by a Bessel function whose argument reflects the quantum interaction between the two waves. Although it appears that we have a spectrum with a quantum number n, it is not the entire sum or an individual n component that constitutes the electron wave function. This feature is a pure quantum effect.
- An essential message of this calculation that must be emphasized is that the "explicit" wave–wave interaction is fully controlled by the electron momentum in the direction perpendicular to the EM propagation. If the electron motion was essentially parallel to EM propagation (paraxial approximation), this effect would be missing.
- Spin effects—In order to explore the spin induced modification to the NR electron spectrum, we solve the Pauli–Schrödinger equation in Appendix C. It is shown that the spin-dependent contribution for energy is negligibly small,

$$\frac{E_{spin}}{q^2A^2/2m} \sim \frac{k}{2m} \ll 1,$$

making it unnecessary to pursue spin corrections.

- When the EM driver is a CPEM wave, we could still work out a problem similar in spirit to PINEM theories. To incorporate electrons moving with near relativistic parallel speeds, we write

$$\psi = \exp(ik_c z)\varphi, \tag{14}$$

where the initial z-momentum is allowed to be large as compared to other momenta, i.e., $|k_c| \gg |\partial_z \varphi|/\varphi$. The Schrödinger equation (5), then, reduces in the leading order to

$$\frac{\partial \varphi}{\partial t} + v_0 \frac{\partial \varphi}{\partial z} = -i(\mu + \lambda \cos\zeta)\varphi, \tag{15}$$

with

$$v_0 = \frac{k_c}{m}, \quad \mu = \frac{q^2A^2 + K_\perp^2 + k_c^2}{2m}, \quad \lambda = \frac{qAK_\perp}{m}. \tag{16}$$

Assuming $\varphi = \varphi(\zeta)$, we derive

$$\varphi = \exp\left(-i\frac{\mu\zeta}{\omega - kv_0} - i\frac{\lambda \sin\zeta}{\omega - kv_0}\right). \tag{17}$$

Notice the appearance of the resonant denominator; strong resonance enhancement in energy is achieved when $v_0 \to \omega/k = c$. With k_c being dominant, the energy is approximated by

$$E \sim \frac{q^2A^2 + K_\perp^2 + k_c^2}{1 - kv_0/\omega} \sim \frac{k_c^2}{1 - kv_0/\omega}, \tag{18}$$

it could be much larger than the initial electron energy as one approaches the resonance. We will find later that QW–CW resonance is way stronger for relativistic electrons.

B. Schrödinger electron in the CPEM wave and an external axial magnetic field

Let us now calculate the energy spectrum for an electron interacting with a CPEM wave when a constant external axial magnetic field $B_0\hat{e}_z$ is present. Such a field is derivable from a vector potential

$$\mathbf{A}_0 = B_0 x \hat{e}_y. \tag{19}$$

We have chosen the Landau gauge for simplicity. This addition will make the vector potential dependent on x in addition to the $\zeta = \omega t - kz$ dependence of the $\mathbf{A}$ associated with the CPEM wave. The modified Schrödinger equation (again neglecting the $\partial^2/\partial\zeta^2$ term) becomes

$$-2im\omega\frac{\partial\psi}{\partial\zeta} - \frac{\partial^2\psi}{\partial x^2} + \left(q^2A^2 + \alpha^2x^2 - 2\alpha qAx\sin\zeta - 2iqA\cos\zeta\frac{\partial}{\partial x}\right)\psi = 0, \tag{20}$$

with $\alpha = qB_0 = m\Omega$, where Ω is the gyrofrequency of the particle. It is easy to see that in the limit of the vanishing external field ($\alpha = 0$), Eq. (20) reduces to (3) provided ($\partial/\partial y = 0$). By the same token, when $A = 0$, it is reduced to the equation that is used to calculate the Landau levels.

Through a tedious pursuit, Eq. (20) can be exactly solved so that the Landau levels of the electron (dressed in CPEM waves) can be obtained. We will summarize the crucial steps of the calculation in Appendix D. However, we will display, here, the leading order modifications to the results of Sec. II A. The approximate wave function and expectation value of the energy are

$$\psi = \psi_0 \exp\left(-i\left((l+1/2)\frac{\Omega}{\omega} + \frac{q^2A^2 + K_x^2}{2m\omega}\right)\zeta - i\frac{2qAK_x}{2m\omega}\sin\zeta\right), \tag{21}$$

$$E = \left(l + \frac{1}{2}\right)\Omega + \frac{K_x^2 + q^2A^2}{2m} + \frac{2qAK_x}{2m}\frac{\sin(kL)}{kL}\cos(\omega t); \tag{22}$$

the latter, now, contains the energy corresponding to the Landau level l. To this order, the interference of the CPEM wave and the ambient field does not appear.

Under what conditions does the effect of B_0 become noticeable? Let us compare the dimensionless quantities,

$$\frac{\Omega}{\omega} \quad : \quad \frac{q^2A^2}{2m\omega},$$

remembering that all energies are nonrelativistic. Let us take the CPEM wave $\omega \sim 10^{15}$ (for visible light) and $qA \sim \omega$. Then, the two terms become comparable when

$$\Omega \sim 10^{-6}\omega = 10^9 \quad \Rightarrow B_0 \sim 60G.$$

Notice that as the CPEM intensity (A^2) is increased, larger B_0 will be needed to make a substantial difference to the energy "eigenvalue." When qA is 100 times larger (still very nonrelativistic), one would need a 10^4-fold increase. Evidently, the higher Landau states will be affected even at lower ambient fields.

In our opinion, the preceding analysis suggests an entire experimental program to map out the ($\Omega - qA$) domain for the interacting quantum and classical waves. With a little extra effort, interesting interference effects can be derived; we plan to do that in a follow-up paper.

III. QW–CW, RELATIVISTIC ELECTRONS—RELATIVISTIC AMPLITUDE CPEM WAVE

In Sec. II, we explored some very interesting aspects of QW–CW interactions, but a conventional resonant interaction was not possible because the electrons could not traverse close to the speed of light. It is precisely why the light-like EM waves show no Landau damping in standard nonrelativistic plasma physics.

The situation, however, changes drastically when we graduate to a fully relativistic system. In what follows, both the electrons and the EM waves have relativistic energies. Although there will be some overlap with Sec. II, particularly, in general, methodology, we will present the relativistic calculations in full detail. We will, especially, dwell on the origin and nature of the resonant solutions.

A. Preliminaries

Let us begin with a deep but relatively trivial realization that the group velocity of the "quantum wave" associated with a relativistic particle (with the energy–momentum relation, $E^2 = c^2P^2 + m^2c^4$),

$$v_p = \frac{\partial E}{\partial P} = c^2 \frac{P}{E} = c \frac{P}{(P^2 + m^2c^2)^{1/2}}, \tag{23}$$

tends to approach c as the particle momentum becomes much greater than its rest mass. Under appropriate conditions, then, such a particle could resonantly exchange energy/momentum with a propagating electromagnetic (EM) wave. To resonantly interact with the CPEM wave propagating in the z-direction, it is the class of "particles" with

$$v_{pz} = \frac{\partial E}{\partial P_z} = c^2 \frac{P_z}{E} = c \frac{P_z}{(P_z^2 + P_\perp^2 + m^2c^2)^{1/2}} \to c \tag{24}$$

that will meet the criterion. Obviously for such particles, $P_z^2 \gg P_\perp^2, m^2$. We will find that extremely high energy states become accessible to a quantum particle of this genre when it is immersed in a CPEM wave of constant magnitude. It is suggested that this process, *inter alia*, could manifest in a cosmic setting where relativistic particles find themselves in the vicinity of an intense source of EM energy (emanating from some cataclysmic event); most energetic of cosmic rays could be generated by this mechanism.

What is remarkable is that such as the nonrelativistic system, the principal qualitative, and some quantitative aspects of this possible spectacular phenomenon can be derived, illustrated, and demonstrated in a simple straightforward semi-classical calculation (describing the particle quantum mechanically while the intense EM wave is treated classically). In fact, a parallel calculation dealing with the resonant interaction of the KG and gravitational waves is already published.[10]

The quantum treatment of the particle dynamics is crucial since the entire phenomenon results from the interaction of two waves—the EM and the quantum wave associated with the particle.

Much of the following presentation is based on work began several years ago.[11] In a recent paper,[12] we developed the basic framework for the physics of relativistic quantum fluids interacting with intense EM waves; the first step is to solve the eigenvalue problem embodied in the relativistic quantum equations—the Klein–Gordon (KG) and the Dirac equation—in the presence of a given arbitrary amplitude EM wave. It is important to point out that the literature is quite full of exact and approximate solutions similar to those described in Ref. 12. In fact, the attempts to solve the KG-EM and Dirac-EM systems began almost 90 years ago.[13,14] The Volkov vacuum solution of the Dirac-EM system becomes the reference for a variety of later papers in which the solution was extended to a plasma;[15–26] aside from the Volkov-type results, other solutions have also been derived for these systems.[27,28] There is a considerable overlap between the KG/Dirac solutions presented in Refs. 12 and 21 that both could be considered as basic references. The scope of the work developed in Ref. 12, however, is much larger than just the solutions of the wave equations; it is the construction of a relativistic quantum fluid based on these solutions.[29]

Although the solutions of the Dirac equation are considerably more complicated (and richer) than the KG solutions, the essence of the phenomenon of resonant energization, barring some inessential details, can be fully captured in a KG analysis. We will, therefore, present only the much simpler KG calculation in some detail and simply state the corresponding results for the Dirac system. The wave function of the KG quantum particle of rest mass m and charge q interacting with the electromagnetic field ($\hbar = 1 = c$) is described by

$$(p_\mu + qA_\mu)(p^\mu + qA^\mu)\Psi = (i\partial_\mu + qA_\mu)(i\partial^\mu + qA^\mu)\Psi = m^2\Psi, \tag{25}$$

where $p^\mu = [p^0 = E, \mathbf{p}]$ is the energy momentum four vector of the particle. For the CPEM of (1), Eq. (25) expands as

$$\partial_t^2\Psi - \nabla^2\Psi - (2iqA\left[\cos(\omega t - kz)\partial_x\Psi - \sin(\omega t - kz)\partial_y\Psi\right] + \mathcal{M}^2\Psi = 0, \tag{26}$$

where $\mathcal{M} = \sqrt{m^2 + q^2A^2}$ is the field-renormalized effective mass, a concept of great significance emphasizing that the EM wave contributes profoundly to the effective inertia of the quasi-quantum particle (QPF) born out of the KG/EM union. Equation (26), then, can be viewed as the wave equation of the QPF of mass $\mathcal{M}$, still in interaction with the CPEM wave through the term proportional to qA.

Out of the four space-time dimensions, the two space dimensions orthogonal to the propagation direction (z) are ignorable. We can, then, define the independent Fourier mode as

$$\Psi(x, y, z, t) = e^{iK_\perp(\cos\varphi x + \sin\varphi y)}\psi(t, z) \tag{27}$$

that obeys

$$\partial_t^2\psi - \partial_z^2\psi + 2qAK_\perp\cos(\omega t - kz)\psi + ({K_\perp}^2 + \mathcal{M}^2)\psi = 0, \tag{28}$$

where $K_x = K_\perp\cos\varphi$, $K_y = K_\perp\sin\varphi$, and the phase φ has been eliminated in a trivial redefinition of the variable z. One immediately notes that the perpendicular momentum $K_\perp$ is a good quantum number. Thus, A and $K_\perp$ are simply specified numbers.

Notice the difference between the KG (28) and the Schrödinger (6) equations. The composite QPF obeys a hyperbolic wave equation for the former while the Schrödinger equation is parabolic. This has profound consequences as we will explore now.

Since both the KG QPF (28) and the EM operators are hyperbolic ($\partial_t^2 - \nabla^2$ for EM), the analogous mathematical structure is precisely what can make the two waves strongly resonate; it provides the necessary condition for the existence of the resonant wave–wave interaction, the principal mechanism for energy transfer. Importantly, the dependence of the interaction term in (28) on z and t implies that neither the energy E nor the z-momentum K_z of the QPF is a constant of the motion. Consequently, we will deal with the respective expectation values, $\langle E\rangle$ and $\langle K_z\rangle$.

The explicit interaction term in (28) vanishes for $K_\perp = 0$. In this limit, both E and K_z become good quantum numbers, and (28) allows the exact solution

$$\Psi(z, t) = \Psi(0)e^{-iEt + iK_z} \tag{29}$$

provided the EM field modified relativistic dispersion relation

$$E^2 = K_z^2 + \mathcal{M}^2 = K_z^2 + m^2 + q^2A^2 \tag{30}$$

is satisfied. This exact result, though rather simple, is quite noteworthy. When the motion of the quantum particle is limited to the direction of the EM propagation, the particle "energy" increases by q^2A^2 over its field-free value; no other discernible change can be seen. This special solution will be used as a fiducial/reference solution in the rest of this paper.

Of all the classes of solutions accessible to (28), we concentrate here only on those that have a direct bearing on the resonant energy exchange between the two waves—the quantum particle wave and the classical EM wave. Though the phenomenon of phase-matching resonance has some similarities with the well-known resonant process—Landau damping[30]—it is quite different in its nature, context, as well as in its consequences. Landau damping is generally a linear process caused when a self-consistently generated wave (often electrostatic) damps on a conventional classical particle (imparting energy to the particle in the process); both the wave and the particle are strictly classical. In the resonant energization process due to phase-matching, reported in this paper, the energy exchange takes place between two waves: the quantum KG wave and the classical EM wave. As pointed out earlier, the mathematical roots for the resonant energy exchange lie in the hyperbolic wave operator $(\partial_t^2 - \nabla^2)$ that controls the dynamics of both the KG and the Maxwell equations.

The best route to finding the resonant solution is to stipulate that ψ depends on t and z through the combination $\zeta = \omega t - kz$, the phase factor of the EM wave. Equation (28), then, converts to $[\psi = \psi(\zeta)]$

$$(\omega^2 - k^2)\frac{d^2\psi}{d\zeta^2} + [\mu + \lambda\cos\zeta]\psi = 0, \tag{31}$$

where $\mu = K_\perp^2 + m^2 + q^2A^2$ and $\lambda = 2qAK_\perp$ $(\lambda/\mu < 1)$. Equation (28) is a simple Mathieu equation and can be rigorously analyzed. However, a physically motivated "approximate" analysis is much more accessible and transparent.

The most striking characteristic of (31) is that the highest derivative term is multiplied by $\omega^2 - k^2$; this coefficient is strictly zero for an EM wave propagating in vacuum. In that case, the exact system allows only the trivial solution $\psi = 0$. However, if the EM dispersion relation was of the form $\omega^2 - k^2 = \epsilon^2$ with $\epsilon \ll \omega$ (as for a tenuous plasma), Eq. (31) is a perfectly well-defined but a singular differential equation. It is singular in the sense that it will support a solution for which $d^2\psi/d\zeta^2$ is sufficiently large to balance the other terms. Since large $d\psi/d\zeta$ implies larger energy and z-momentum, this will constitute the Resonant solution that we set out to seek.

Let us first study the $K_\perp = 0$ (the fiducial case) limit of Eq. (31). In this case, $\lambda = 0$ and $\mu = m^2 + q^2A^2$. From the (exact) positive energy solution,

$$\psi = \psi(0)e^{-i\sqrt{\frac{m^2+q^2A^2}{\omega^2-k^2}}(\omega t - kz)}, \tag{32}$$

one readily calculates the particle energy and z-momentum

$$\langle E\rangle = \frac{i}{\psi}\frac{\partial\psi}{\partial t} = (m^2 + q^2A^2)^{1/2}\frac{\omega}{\sqrt{\omega^2 - k^2}}, \tag{33}$$

$$\langle K_z\rangle = -\frac{i}{\psi}\frac{\partial\psi}{\partial z} = (m^2 + q^2A^2)^{1/2}\frac{k}{\sqrt{\omega^2 - k^2}}. \tag{34}$$

As before, $\langle\ \rangle$ implies that we are dealing with expectation values despite the fact in the limit of $K_\perp = 0$, both energy and momentum are exactly conserved. Evidently, both $\langle E\rangle$ and $\langle K_z\rangle$ become very large as $\omega \to k$ (or $\Omega \to 0$); the explicit expression of the energy/momentum enhancement (boost) factor,

$$\frac{\omega}{\sqrt{\omega^2 - k^2}} \gg 1. \tag{35}$$

The expressions (33) and (34) are reassuring since both $\langle E\rangle$ and $\langle K_z\rangle$ are much greater than $\sqrt{m^2 + q^2A^2}$ so that $(E/K_z) \to 1$ is consistent with (30).

It may appear somewhat odd that the two exact solutions (29)–(30) and (32)–(34) of the same $(K_\perp = 0)$ equation seem to look so different. Solutions (32)–(34) are entirely consistent with (29)–(30) but has deeper content. From (33) and (34), one may readily derive

$$\langle E\rangle^2 - \langle K_z\rangle^2 = (m^2 + q^2A^2)\left(\frac{\omega^2}{\omega^2 - k^2} - \frac{k^2}{\omega^2 - k^2}\right) = m^2 + q^2A^2, \tag{36}$$

exactly the same energy–momentum relationship as in Eq. (30). This, of course, is simply the generic relativistic energy momentum relation in the presence of a CPEM wave. However, the remarkable attribute of the solution (32)–(34) is that we have been able to calculate explicit, independent expressions for both $\langle E\rangle$ and $\langle K_z\rangle$ where (29)–(30) is limited only to the energy–momentum relationship (30). It is these explicit formulas that demonstrate the fundamental phenomenon of resonantly boosted energy/momentum of the QPF.

Further insight into the deeper content of the solution (32)–(34) is obtained when one finds that the ratio of the QPF energy to its z-momentum,

$$\frac{\langle E\rangle}{\langle K_z\rangle} = \frac{\omega}{k}, \tag{37}$$

is exactly equal to the ratio of the energy of the EM wave to its z-momentum. The equality (7) leads to the same exponents in (29) and (32) establishing formal equivalence. This is the resonant condition for effective phase-matching; the resonant energization occurs when the phase velocity of the EM wave equals to the axial (in the propagation direction) phase velocity of the relativistic quantum particle.[1]

We can now briefly deal with Eq. (31) fully to derive somewhat a detailed structure of the resonant solution that reduces to (32)–(34) in the limit $K_\perp = 0 = \lambda$. Let us recall that the resonant energy (measured in units of the rest mass m) $E(K_\perp = 0) = \sqrt{(m^2 + (qA)^2}\omega/\sqrt{\omega^2 - k^2}$ is much greater than unity which means that the ζ variation contained in (32) is much stronger than the ζ variation of the interaction term $(\cos\zeta)$ in (31). This suggests a WKB analysis (see Ref. 12), but we will follow a simple perturbation approach to show that essentials of the resonant solution are already contained in (32)–(34). Let us consider the wavefunction (see the definition of μ),

$$\psi = \psi_s(\zeta)e^{-i\frac{\sqrt{\mu}}{\sqrt{\omega^2-k^2}}\zeta} \equiv \psi_s(\zeta)e^{-iS\zeta}, \tag{38}$$

where $\psi_s(\zeta)$ represents the slow variation [compared to the eikonal, i.e., $(d\psi_s/d\zeta)/(\psi_s) \ll S = \sqrt{\mu/(\omega^2 - k^2)}$] of the wave function. Substituting (38) into (31) and neglecting $d^2\psi_s/d\zeta^2$, we derive

$$\frac{1}{\psi_s}\frac{d\psi_s}{d\zeta} = \frac{i\lambda}{2S(\omega^2 - k^2)}\cos\zeta \tag{39}$$

that integrates to

$$\psi_s = e^{i\frac{qAK_\perp}{[\mu(\omega^2-k^2)]^{1/2}}\sin\zeta} = e^{i\alpha\sin\zeta} \tag{40}$$

from which it is trivial to verify that $d\psi_s/d\zeta = |\alpha| \ll S$ even when $K_\perp$ is not small. Thus, the approximate but complete solution

$$\psi = e^{-iS\zeta + i\alpha \sin\zeta} = \sum_n J_n(\alpha)\psi_n\,, \qquad \psi_n = e^{-iE_n t + iK_{zn} z}\,, \tag{41}$$

could be seen as a sum of n terms. The nth component occurs with a weight $J_n(\alpha)$ and has characteristic energy and z-momentum,

$$E_n = \sqrt{\mu}\frac{\omega}{\sqrt{\omega^2-k^2}} - n\omega\,, \quad K_{zn} = \sqrt{\mu}\frac{k}{\sqrt{\omega^2-k^2}} - nk. \tag{42}$$

Since $\sqrt{\mu/(\omega^2-k^2)} \gg 1$, E_n (K_{zn}) can substantially differ from $\langle E\rangle$ [$\langle K_z\rangle$] only for very large n where the Bessel functions tend to be rather small for moderate values of α. Extensive numerical solutions of the Mathieu equation (31) reveal that apart from some minor modulations (evidently reflecting the higher n components discussed above), the basic structure of the resonant solution is well represented by the set (32)–(34) with eikonal S going from $\sqrt{\mu/(\omega^2-k^2)}$ toward $\sqrt{(\mu+\lambda)/(\omega^2-k^2)}$ as $K_\perp$ becomes larger, and λ approaches μ.

The reader must have noticed that the preceding discussion is the relativistic translation of the nonrelativistic calculations of Sec. II.

In order to advance further, we must find a reasonable evaluation/estimate for $\Omega^2 = (\omega^2 - k^2)$, the quantity that controls the energy amplification factor of the resonant solution, remembering that in this analysis, the arbitrary amplitude EM wave is not only externally specified but also been treated as a monochromatic wave with ω and k as parameters determining the wave phase. The preceding analysis is not valid in pure vacuum propagation, $\omega = k$. Thus, we assume that the CPEM wave is passing through a charged gas (plasma) of the weakly interacting QPF and compute the induced current by adding together the individual contributions of these KG QPF particles. With this current substituted in Maxwell equations, then, we can find the self-consistent solution to determine the dispersion relation obeyed by the EM wave; this dispersion relation would relate ω and k. For the relativistic KG plasma (with rest frame density n_R and temperature T), such a dispersion relation was calculated, for instance, in Ref. 12 [Eq. (46)],

$$\omega^2 - k^2 = \frac{{\omega_p}^2}{\sqrt{(1+q^2A^2/m^2)}\Gamma_{th}}, \tag{43}$$

where $\omega_p = \sqrt{4\pi q^2 n_R/m}$ is the invariant (rest frame) plasma frequency and Γ_{th} signifies the thermal enhancement of the particle inertia [$\Gamma_{th} = K_2(g)/K_1(g) \geq 1$ for a relativistic Maxwellian plasma]. The argument $g = \sqrt{(m^2+q^2A^2)}/T$ is the inverse of the normalized temperature, but normalized to the field enhanced inertia of the QPF and not to the bare rest mass. In the rest of this paper, $\Gamma_{th} = 1$.

The dispersion relation (43) gives us the necessary input to derive explicit expressions for normalized energy Γ and normalized z-momentum (with $K_\perp \neq 0$) fully in terms of the specified quantities (plasma density, intensity, and the frequency of the CPEM field),

$$\Gamma = \frac{\langle E\rangle}{m} = \left(1+\frac{q^2A^2+K_\perp^2}{m^2}\right)^{1/2}\left(1+\frac{q^2A^2}{m^2}\right)^{1/4}\frac{\omega}{\omega_p}, \tag{44}$$

$$\frac{\langle K_z\rangle}{m} = \left(1+\frac{q^2A^2+K_\perp^2}{m^2}\right)^{1/2}\left(1+\frac{q^2A^2}{m^2}\right)^{1/4}\frac{k}{\omega_p}. \tag{45}$$

The EM field boosts up the normalized energy and z-momentum through two distinct processes: (1) by non-resonantly contributing, the field energy perpendicular to the kinematic energy (the first bracket) and (2) through the resonant enhancement epitomized in the factor $[1+q^2/A^2)/m^2]^{1/4}\omega/\omega_p$; the latter can become arbitrarily large (even for moderate values of qA/m) since in many cases of interest, $\omega \gg \omega_p$. There is an equally spectacular gain (through the combined effects of the resonant and non-resonant energization) for strong EM fields ($qA \gg m$), the normalized energy eventually going up as $(qA/m)^{3/2}$.

It is worthwhile to reiterate that only those QPF waves that satisfy the resonant condition $\langle E\rangle/\langle K_z\rangle = \omega/k$ (propagating in unison with the EM wave along z-direction) will, preferentially, gain large amount of z-momentum from the EM wave. It is this gain that translates into a commensurate gain in QPF energy while maintaining the relativistic energy momentum relationship $\langle E\rangle^2 = \langle K_z\rangle^2 + K_\perp^2 + \mathcal{M}^2$, where $\mathcal{M}^2 = (m^2 + q^2A^2)$ reflects the field-modified effective mass.

It must be pointed out that, though we have found extremely high energy resonant solutions, we have not explicitly shown the accessibility of these states, i.e., whether such states are the results of a demonstrated energy gain from the wave.

We will, now, describe some plausible steps in the acceleration process that might catapult the resonant particles to high energies. In Appendix E, we derive, quite approximately, an explicit formula for the RMS value ($|d\langle E\rangle/dt|$) of the rate of change of particle energy,

$$\left|\frac{d\langle E\rangle}{dt}\right| = \frac{\omega}{4(\omega^2-k^2)^{1/2}}\frac{2qAK_\perp}{(m^2+K_\perp^2+q^2A^2)^{1/2}}\frac{\sin(kL)}{kL}. \tag{46}$$

The preceding expression is both interesting and revealing. The first and foremost message is that this rate goes to zero for $K_\perp = 0$; thus, the energy gain is possible only for (quasi) particles with finite perpendicular energy. This comes as no surprise since the wave–particle coupling is fully contained in the term proportional to $K_\perp$ (or λ). It is to be noted that this proportionality is strictly true only for small $K_\perp$; the gain becomes independent of $K_\perp$ for large values. Exactly the same pattern holds for the A dependence. It is the comparison of $K_\perp^2 + q^2A^2$ with m^2 that determines the appropriate limit. Furthermore, as expected, the rate of energy increase is resonantly enhanced by the factor $\omega/(\sqrt{\omega^2-k^2})$. On the other hand, if the interaction length L is much longer than the EM wavelength ($kL \gg 1$), the energy transfer rate is deeply suppressed due to phase mixing. The strongest energy transfer takes place when $kL \approx 1$. The size L is, indeed, the size of the interaction region and could represent the astrophysical object of interest. The instantaneous rate of energy increase (46) scales as $1/kL$, but this lowering of the instantaneous rate does not determine the final energy which is fully given by the energy eigenvalue (44). All it implies is that it will take longer to get there.

What we have demonstrated in this paper is the resonant energization of a single relativistic quantum wave–particle in a high amplitude classical electromagnetic wave. It should be viewed as a first calculation (like working out the single particle orbits in electromagnetic fields) in the development of a possible sub-field, investigating the resonant dynamics of a quantum KG (Dirac) gas embedded in a high amplitude electromagnetic wave. The very primitive system shows, what we believe, is a profound resonance effect accessible in the minimal model; it follows simply from the fact that the quantum waves associated with highly relativistic particles (and under special conditions, for near relativistic particles) are physically (dispersion) and mathematically (hyperbolic equation) so similar to the electromagnetic wave that strong resonance take place without invoking any collective phenomena—fluid or kinetic.

We emphasize that the phenomena of the characteristic resonances that can be studied within the more complex system involving a gas of relativistic particles must contend with this elementary intrinsic mode of energy exchange. How will it affect the larger dynamics and how, in fact, the larger dynamics will affect the nature of this resonance is a very interesting question that is on our agenda. There are, perhaps, two obvious approaches: use something similar to the statistical model for a plasma constructed in Ref. 12 or invoke a quantum kinetic model formulated through the time evolution of a Wigner function for the Klein–Gordon field.[33] In the latter model, the particle–antiparticle nature of the wave and zitterbewegung effects are displayed in an explicit form. For a large amplitude CPEM wave, it is expected that the primitive resonance of this work, will play an important role in the overall dynamics embodied, for example, in the quantum kinetic model.

IV. DISCUSSION

Let us now put in perspective the demonstration of the phenomenon of direct resonant transfer of energy–momentum from an intense EM wave to a relativistic quantum particle by spelling out the basic assumptions and stating the most important potential consequences of this work. First, the reverse process of the EM wave gaining energy from the relativistic particles (analog of Landau growth) is, theoretically, just as possible under appropriate conditions. An interesting example of the inverse mechanism at work can be seen in Ref. 31 where an EM wave is shown to feed off the free energy in a sheared velocity field.

In addition, although the calculations presented in Sec. III were derived for arbitrary parameters (m, qA, ω, k, and $\langle K_\perp \rangle$), they are, subliminally, geared toward highly relativistic particles and intense EM fields, i.e., $\langle E\rangle/\sqrt{m^2+q^2A^2} \simeq \langle K_z\rangle/\sqrt{m^2+q^2A^2} \gg 1$ along with $qA/m \gg 1$. For typical EM waves of interest (intense optical lasers, for example), on the other hand, the energy "per quantum" (normalized to, say, the electron rest mass) is rather small: $\omega/m \simeq k/m \ll 1$; clearly, the typical particle or field energy is much more larger than ω. These ratios and the enhancement factor $(1+(qA/m)^2)^{1/4}\omega/\omega_p \gg 1$ are needed to estimate the "extent" of resonant energization.

Since no energy inventory is presented, it is assumed that the EM wave acts as an intense constant energy pump. Such a situation could, readily, be created in very high energy laser experiments. One can also imagine that this calculation will pertain to superstrong energization of cosmic ray particles that happen to be in the vicinity of a cataclysmic event (colliding blackholes, for example) that is accompanied by prodigious emissions of electromagnetic energy. Since the basic mechanism is the resonance between a class of relativistic particle-waves and a wave that propagates essentially with the speed of light, it is hardly a stretch to conclude that intense gravitational waves (produced again in some cataclysmic event) could just as well substitute EM waves. This was already proved in Ref. 10, showing that it is possible for the resonant energization of relativistic quantum particles due to the phase-matching resonance with gravitational waves.

The quantum wave investigated in this paper is the KG wave that pertains to a spinless particle. The analysis is easily extended to a half spin Dirac particle (calculations are much more involved). However, the essentials of the resonant energization process are equivalent (there is very little difference in the KG and Dirac cases in the aforementioned regime when $\langle E\rangle/\omega \gg 1$). More detailed solution of the KG, as well as a complete solution of the Dirac equation, will be the subject matter of forthcoming papers. Mostly, analytical calculations will be supplemented by numerical solutions.

One must, at least, comment on the probability of finding a particle that will be subjected to such resonant energization. The answer to this question requires considerable thought but the resonant condition $\langle E\rangle/\langle K_z\rangle = \omega/k$ does provide a clue, the resonant particles will be found only a small but non-zero sliver of the $K_\perp$–K_z phase space. In fact, this sliver is even narrower if we remember that K_z has to be much greater than $\sqrt{m^2+q^2A^2+K_\perp^2}$; the electron motion must be essentially paraxial as in PINEM experiments

This paper constitutes a conceptual formulation and delineation of the dynamics of a quantum wave and a classical wave in interaction. Despite interesting wave–wave processes, the nonrelativistic electrons do not show a conventional resonance; the latter, however, strongly manifests the relativistic KG electrons. We will make some simple estimates for the efficacy of the resonant process when the normalized energies are high for $qA/m \gg 1$, $qA \gg K_\perp$, $\Gamma \approx (qA/m)^{3/2}(\omega/\omega_p)$. Let us examine the consequences of this formula in the following two different contexts (where conditions for resonant energization might prevail):

- First, in high-intensity laser plasma experiments. Let us consider a wave of amplitude $qA/m \approx 10^2 - 10^4$ propagating in a barely under-dense plasma with $\omega/\omega_p \geq 1$. The resonant solutions then correspond to a total normalized energy $\Gamma = E/m \sim (qA/m)^{3/2} \sim 10^3 - 10^6$ of which the resonant boost factor $\Gamma_{res} \approx (qA/m)^{1/2} \approx 10 - 10^2$. Two straightforward experiments can be proposed to look for the signatures of the proposed mechanism: (a) qualitative, by searching for particles with unexpectedly high energies, and (b) quantitative, by testing the predicted scaling of the resonant energy with the wave amplitude $\Gamma \approx (qA/m)^{3/2}$.
- Second, in a cosmic setup, the plasma densities will be much smaller and $\omega/\epsilon \gg 1$. One could, then, expect to generate a proton in the PeV energy range $\Gamma_{proton} \approx 10^6$ by the combination $qA/M_p = 10$ and $\omega/\epsilon = 10^5$.[32] With this mechanism, it is eminently possible to catapult protons to much higher energies since EM fields associated with cataclysmic events could be of a prodigious magnitude.

One must confront the fact that this calculation is only semi-classical (semi-quantum) and does not, for instance, include primary loss mechanisms such as synchrotron losses, inverse Compton scattering, and curvature radiation. Secondary loss mechanisms, such as pair production, will also happen at sufficiently high photon energies—if pair production was intense, then the resonant energy could be shared among many particles and the energy carried by a given particle need not be so high. However, remembering that our quantum particle is very "heavy" when it is in the intense EM field (its effective mass), then the process of pair production will be deeply suppressed. There are two reasons: (1) the elementary process of a single photon producing this heavy particle is totally negligible ($\hbar\omega \ll 2\mathcal{M}c^2$) and (2) even a multiple photon process (due to very high amplitudes) will also be ineffective in crossing a threshold as high as $2\mathcal{M}c^2 \gg 2mc^2$. Therefore, it is more likely that the resonant effect due to the dispersive properties of the wave occur, expecting only a rather small number of pairs sharing the total resonance energy. However, a proper estimate of the energy loss must be done and will be the subject matter of a forthcoming paper. In addition, the use of Klein–Gordon equation may be questioned because of its so-called non-definite probability density. However, that is not quite relevant here. In fact, the Dirac equation leads essentially to the same results. The calculation is rather robust, and its principal results are sound.

Finally, as pointed out earlier, this work should be viewed as a first calculation in the development of a possible sub-field investigating the resonant dynamics of a quantum KG (Dirac) gas embedded in a high amplitude electromagnetic wave. It can provide a motivation as well as the basis for studying the resonance dynamics in more comprehensive systems such as the statistical model for quantum plasma fluid theory (constructed in Ref. 12 and a quantum kinetic model formulated through the time evolution of a Wigner function for the Klein–Gordon field[33]).

We think this mechanism, along with its sister channels (such as the energization via gravitational waves), should be seriously investigated as one of the processes that may be powering the most energetic cosmic rays.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors state that they have no conflicts of interest to disclose.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX A: CLASSICAL DYNAMICS OF THE ELECTRON

The classical, nonrelativistic dynamics of the electron subjected to the CPEM fields [defined in Eq. (1)] is contained in the Lorentz equation,

$$\frac{\partial \mathbf{v}}{\partial \zeta} = g\left[-\frac{\partial \mathbf{a}}{\partial \zeta} - \frac{k}{\omega}\mathbf{v}\times\left(\hat{e}_z \times \frac{\partial \mathbf{a}}{\partial \zeta}\right)\right]; \quad g = qA/m; \qquad \mathbf{a} = \hat{e}_x \cos\zeta - \hat{e}_y \sin\zeta, \tag{A1}$$

where following the quantum case, we have converted the t–z derivatives in terms of ζ. Two immediate consequences are

$$\frac{1}{2}\frac{\partial v^2}{\partial \zeta} = -g\mathbf{v}\cdot\frac{\partial \mathbf{a}}{\partial \zeta}, \quad \mathbf{a}\cdot\frac{\partial \mathbf{v}}{\partial \zeta} = 0, \tag{A2}$$

because $\mathbf{a}\cdot\mathbf{a} = 1$ and is a constant. The second condition allows the rewriting of the first as

$$\frac{1}{2}\frac{\partial v^2}{\partial \zeta} = -g\frac{\partial(\mathbf{v}\cdot\mathbf{a})}{\partial \zeta} \Rightarrow \quad \frac{v^2}{2} = -g\mathbf{v}\cdot\mathbf{a}, \tag{A3}$$

where the constant of integration is chosen to be zero. Equation (A1) is split into z and perpendicular components,

$$\frac{\partial \mathbf{v}_\perp}{\partial \zeta} = g\left[-\frac{\partial \mathbf{a}}{\partial \zeta} + \frac{k}{\omega}v_z\frac{\partial \mathbf{a}}{\partial \zeta})\right], \quad \frac{\partial v_z}{\partial \zeta} = -g\frac{k}{\omega}\frac{\partial(\mathbf{v}\cdot\mathbf{a})}{\partial \zeta}. \tag{A4}$$

From (A3) and (A4), we find the relation

$$\frac{v^2}{2} = \frac{k}{\omega}v_z, \tag{A5}$$

precisely the relation that exists between energy and z-momentum in the quantum case that we considered. Equation (A4) allow the exact solution

$$\mathbf{v}_\perp = \lambda\mathbf{a}, v_z = v_0 \quad \Rightarrow v_\perp^2 = \lambda^2, \lambda = -g\left(1 - \frac{k}{\omega}v_0\right), \tag{A6}$$

substituting the expression for λ in (A5), we find that v_0 obeys the quadratic

$$\frac{k}{\omega}v_o = g^2\left(1 - \frac{k}{\omega}v_0\right)^2. \tag{A7}$$

This relation is exact. However, since for nonrelativistic motion, $v_0 \ll 1$, $g = qA/m \ll 1$, we have, to leading order,

$$\frac{k}{\omega}v_o \approx g^2, \quad E = \frac{mv^2}{2} \approx \frac{q^2A^2}{2m}, \tag{A8}$$

as referenced in the main text. Next order corrections are easily obtained.

APPENDIX B: HIGHER ORDER SOLUTIONS

Our leading order solutions to the Schrödinger equation (6) were calculated (in Sec. II A) by neglecting the second derivative term. We now calculate the first higher order correction to the solution (8) by including the second order term perturbatively. We seek a solution of the form

$$\Psi = \psi\phi, \tag{B1}$$

where ψ is the solution (8) and ϕ is a slow (in ζ) modulation field, such that $d_\zeta\phi/\phi \ll d_\zeta\psi/\psi$. Let us rewrite (6) as

$$-i\frac{d\Psi}{d\zeta} + (s_1\cos\zeta + s_2)\Psi = h\frac{d^2\Psi}{d^2\zeta}, \qquad h = \frac{k^2}{2m\omega}, \quad s_1 = \frac{qAK_\perp}{m\omega}, \quad s_2 = \frac{(K_\perp^2 + q^2A^2)}{2m\omega}. \tag{B2}$$

Substituting (B1) into (B2), under the above assumptions, we find

$$-i\frac{1}{\phi}\frac{d\phi}{d\zeta} = \frac{h}{\psi}\frac{d^2\psi}{d^2\zeta}. \tag{B3}$$

After straightforward but longish algebra, one next order corrections are contained in the solution

$$\Psi = \exp(hs_2 + hs_1\cos\zeta) \times \exp\left(-is_2\zeta - is_1\sin\zeta - ihs_2^2\zeta - i\frac{h}{2}s_1^2\zeta - 2ihs_1s_2\sin\zeta - ih\frac{s_1^2}{4}\sin 2\zeta\right). \tag{B4}$$

The total energy associated with the system

$$E_T = \frac{\beta}{2m} + \frac{\alpha}{2m}\langle\cos\zeta\rangle + h\omega\left[\left(s_2^2 + \frac{s_1^2}{2}\right) - 2s_1s_2\langle\cos\zeta\rangle - \frac{s_1^2}{4}\langle\cos 2\zeta\rangle\right]. \tag{B5}$$

The energy modifications are all rather small,

$$\Delta E \sim \frac{k^2}{2m}\left(\frac{q^2A^2}{m\omega}\right)^2 \ll E \sim \frac{q^2A^2}{m}. \tag{B6}$$

One does, however, see the contribution from the second harmonic through $\langle\cos 2\zeta\rangle$.

APPENDIX C: THE EFFECT OF SPIN

When electron spin is included, the relevant Pauli equation takes the form ($\zeta = \omega t - kz$)

$$-i\frac{\partial\psi}{\partial t} - \frac{1}{2m}\nabla^2\psi - \frac{2qAK_\perp}{2m}\cos\varrho\,\psi + \frac{q^2A^2}{2m}\psi + \frac{qAk}{2m}\chi\psi = 0, \tag{C1}$$

where

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{C2}$$

is the two-component wave function and

$$\chi = \begin{pmatrix} 0 & \exp(i\zeta)\\ \exp(-i\zeta) & 0\end{pmatrix} \tag{C3}$$

is the matrix coupling the spin up and spin down states. Seeking (as before and in the same approximation) the solutions of the form $\psi = \psi(\varrho)$, we manipulate (C1) to

$$\begin{aligned}\frac{d\psi_1}{d\zeta} + ig(\zeta)\psi_1 &= i\mu e^{i\zeta}\psi_2,\\ \frac{d\psi_2}{d\zeta} + ig(\zeta)\psi_2 &= i\mu e^{-i\zeta}\psi_1,\end{aligned} \tag{C4}$$

with

$$g = \frac{qAK_\perp}{2m\omega}\cos\zeta + \frac{q^2A^2 + K_\perp^2}{2m\omega},$$
$$\mu = \frac{qAk}{2m\omega}.$$

To the required accuracy, we find the NR solution to be

$$\psi_1 = \exp -i\left[h(\zeta) - \mu^2\zeta\right], \tag{C5}$$

$$\psi_2 = \exp -i\left[h(\zeta) + \mu^2\zeta\right], \tag{C6}$$

where

$$h(\zeta) = \frac{qAK_\perp}{2m\omega}\sin\zeta + \left[\frac{K_\perp^2}{2m\omega} + \frac{q^2A^2}{2m\omega}\right]\zeta. \tag{C7}$$

Notice that the spin correction to the Schrödinger solution

$$\frac{\mu^2}{q^2A^2/2m\omega} = \frac{k^2}{2mw} \approx \frac{k}{2m} \ll 1$$

for most of the relevant EM spectrum.

APPENDIX D: LANDAU LEVELS

The system is described by Eq. (20). Let us look for a solution with the form

$$\psi = \varphi(\zeta, x)\exp\left(i\beta(\zeta)x - \alpha x^2/2\right), \tag{D1}$$

such that the function β satisfy the equation

$$2m\omega\frac{d\beta}{d\zeta} = -2\alpha i\left(\beta + qAe^{i\zeta}\right), \tag{D2}$$

and then

$$\beta = -\frac{\alpha qA}{m\omega + 2\alpha}e^{i\zeta}. \tag{D3}$$

In this way, the equation becomes

$$2im\omega\frac{\partial\varphi}{\partial\zeta} = \frac{\partial^2\varphi}{\partial x^2} - \left(\alpha + \beta^2 + q^2A^2 + 2\beta qA\cos\zeta\right)\varphi - 2(\alpha x - i\beta - iqA\cos\zeta)\frac{\partial\varphi}{\partial x}. \tag{D4}$$

The solution of this equation has the form

$$\varphi(\zeta, x) = \chi(\zeta)H_n\left(\alpha^{1/2}(x + f(\zeta))\right), \tag{D5}$$

where H_n is the nth-order Hermite polynomial, f is

$$f(\zeta) = \frac{1}{m\omega}e^{i\alpha\zeta/m\omega}\int d\zeta\, e^{-i\alpha\zeta/m\omega}(\beta + qA\cos\zeta), \tag{D6}$$

and

$$\chi(\zeta) = \exp\left[-\frac{i}{2m\omega}\left(\alpha(2n+1) - q^2A^2\right)\zeta - \frac{i}{m\omega}\int d\zeta\left(\beta^2 + 2\beta qA\cos\zeta\right)\right]. \tag{D7}$$

APPENDIX E: REATE OF CHANGE OF PARTICLE ENERGY

Let us begin with the approximate wave function (41)

$$\Psi = e^{-iS\zeta + i\alpha\sin\zeta}, \tag{E1}$$

where $S = \sqrt{\mu/(\omega^2 - k^2)}$, $\alpha = qAK_\perp/[\mu(\omega^2 - k^2)]^{1/2}$, and $\zeta = \omega t - kz$, The wave function is normalized in the region $-L < z < L$, that is,

$$\langle\Psi^*\Psi\rangle = \frac{1}{2L}\int_{-L}^{L}\Psi^*\Psi dz = 1. \tag{E2}$$

The expectation value of the energy can be readily evaluated,

$$\begin{aligned}\langle E\rangle = i\left\langle \Psi^* \frac{\partial \Psi}{\partial t}\right\rangle &= \frac{\omega}{2L}\int_{-L}^{L}(S+\alpha\cos\zeta)dz \\ &= -\frac{\omega}{2kL}\int_{\omega t+kL}^{\omega t-kL}(S+\alpha\cos\zeta)d\zeta \\ &= \omega S + \alpha\omega\cos\omega t\,\frac{\sin kL}{kL}. \end{aligned} \tag{E3}$$

We can now compute

$$\frac{d\langle E\rangle}{dt} = \alpha\omega^2\sin(\omega t)\frac{\sin(kL)}{kL}, \tag{E4}$$

whose root mean square value (averaged over a period 2π of ωt) may be explicitly displayed as

$$\begin{aligned}\left|\frac{d\langle E\rangle}{dt}\right| &= \left\langle \frac{1}{2\pi}\int_{-\pi}^{\pi} d(\omega t)\left(\frac{d\langle E\rangle}{dt}\right)^2\right\rangle^{1/2} \\ &= \frac{\omega}{4(\omega^2-k^2)^{1/2}}\frac{2qAk_\perp}{(m^2+k_\perp^2+q^2A^2)^{1/2}}\frac{\sin(kL)}{kL}\end{aligned} \tag{E5}$$

in terms of the basic parameters of the system.

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