RESEARCH ARTICLE | DECEMBER 20 2021

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Rev Sci Instrum 92, 124707 (2021)
https://doi.org/10.1063/5.0063057

Review of Scientific Instruments | ARTICLE | scitation.org/journal/rsi

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Cite as: Rev. Sci. Instrum. 92, 124707 (2021); doi: 10.1063/5.0063057
Submitted: 11 July 2021 • Accepted: 3 December 2021 •
Published Online: 20 December 2021

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ABSTRACT

We describe an improvement on the magnetic scalar potential approach to design an electromagnet, which incorporates the need to wind the coil as a helix. Any magnetic field that can be described by a magnetic scalar potential is produced with high fidelity within a Target region; all fields are confined within a larger Return. The helical winding only affects the field in the Return.

I. USING THE MAGNETIC SCALAR POTENTIAL TO DESIGN A COIL

In practical uses of magnetism, it is sometimes desirable to be able to create a very well characterized magnetic field, for example, a field that is uniform within a specified region (hereafter, the Target) and with a surrounding specified region (the Return) that confines the field so that there is no magnetic disturbance outside the device. This goal is made more complicated by the need to introduce an external current source, connected to a single length of wire that winds around the surfaces. We will describe a set of rules that solves this problem in general for any time-independent magnetic field allowed by Maxwell's equations, within a Target and Return of almost any shape.

Reference 1 had described an algorithm to do this that makes use of the magnetic scalar potential to determine the surface current densities on the interfaces. Here is how the algorithm works: the specified magnetic field inside the Target is represented by a scalar potential such that $\vec{H}_{target}(\vec{r}) = -\vec{\nabla}U_{target}$. The Return envelops the Target. The normal component of the field is continuous across the boundary between the Target and the Return, and it is required to be zero at the exterior surface. Then the magnetic scalar potential U_{return} in the Return is determined by Neumann boundary conditions.

The component of the field parallel to the surface is in general not continuous across the boundary. There are current fields $\vec{K}(\vec{r})$ on these surfaces that redirect the field there. It can be shown[1] that this surface current density flows along lines such that $U_{target} - U_{return}$ is constant and that the current between any two such lines is equal to the difference in these values. This establishes the physical interpretation of magnetic scalar potential as a "source potential" for electrical current in analogy, with charge being the source of electric flux lines.

The result is a complete description of how to construct surface current distributions that are exactly produced within the Target in any field configuration that is consistent with Maxwell's equations. Choosing a family of lines with different constant values for $U_{target} - U_{return}$ differing by constant increments divides the surfaces into ribbons, which can be turned into physical wires, all carrying the same current.

In the limit of very fine wires, there is a current sheet on the surface that is determined by the gradient of the difference between the two scalar potentials,

$$\vec{K}(\vec{r}) = -\hat{n} \times \vec{\nabla}\delta U, \tag{1}$$

where the stream function[2–4] $\delta U = U_{return} - U_{target}$ is the difference between the scalar potentials on the two sides of the surface and $\hat{n}$ is the outward-directed normal.

A small source of dissatisfaction with this algorithm for constructing designed-field coils is that what it prescribes is the current that should flow *in closed loops* around the surface of the Target and on the surfaces of the Return. However, a magnet coil is usually wound as a helix so that there is one wire that advances down the coil as it wraps around it. The effect of breaking each loop and connecting it to its neighbors to make a series circuit introduces the equivalent of a single wire lying transverse to the loops; the modification to the Target field will be of order 1/N, which is usually less than 1% (and there are ways of partially canceling the effect), but magnets being designed for metrology and precision physics experiments must meet far higher standards. Since the high quality field envisioned by the algorithm is otherwise only spoiled by small corrections due to wire discreteness (these are localized near the edge of the Target on a length scale set by the wire spacing), this would appear to be the largest design error in the constructed field.[5]

Here, we will describe a modification of the algorithm that removes this defect, allowing the construction of a more nearly perfect coil. In Sec. II, we will work though a problem that is exactly solvable but not quite trivial; later, we will generalize this to arbitrary geometry.

II. SPHERICAL ELECTROMAGNET

We need to figure out how to accommodate a "longitudinal" current (roughly parallel to the field in the Target).

Winding an infinite solenoid as a helix also introduces a longitudinal current, but this does not affect the field inside it at all. This can be readily generalized to any azimuthally symmetric object. Let the rotational symmetry axis be z and the shape of the object be $\rho(z)$. First, consider the case where there is only a longitudinal (i.e., along $\hat{z}$) current symmetrically distributed on the surface of the Target. There cannot be a field inside with a $\hat{\phi}$ component (due to Ampere's law), nor with a $\hat{\rho}$ component (due to Gauss's law), nor with a $\hat{z}$ component (caused by the current, which is in the $\hat{z}$ and $\hat{\rho}$ directions): there is no field inside at all.

Current conservation requires that the current along $\hat{z}$ through any cross section be the same. Then Ampere's law (in the integral form) shows that the field outside is indistinguishable from that of a long straight wire along the axis (this wire will have to actually exist beyond the object), which has only a $\hat{\phi}$ component. This adds no field component normal to the surface since the field has only a $\hat{\phi}$ component, which is consistent with the claim that the field inside is not affected since the surface currents will prevent its entry.

The currents and fields can be explicitly calculated for the case of a sphere of radius P with only an axial current. The surface current density [according to Eq. (1)] is $\vec{K}_{axial} = -\hat{\theta}I/(2\pi P \sin\theta)$ (in spherical coordinates), and the field outside is $\vec{H} = \hat{\phi}I/2\pi r \sin\theta$. We can represent this as a magnetic scalar potential $\vec{H} = -\vec{\nabla}U_{axial}$, where

$$U_{axial} = -\phi I/2\pi. \tag{2}$$

This is a multivalued function since $\phi = 0$ and $\phi = 2n\pi$ are physically indistinguishable; for the purpose of clearly stating a boundary value problem for U_{axial}, we can make it single valued by introducing a branch surface, interrupting any path that wraps around the sphere or the wire extension (for example, the plane $\phi = 0$ outside the sphere). Ampere's law requires the multivalued character and shows that the discontinuity I is everywhere across the branch surface. Despite the discontinuity, the gradient of U_{axial} (the magnetic field) is continuous everywhere except at the poles $\theta = 0$ and $\theta = \pi$ and at the wire extension, and it is useful to think of U_{axial} as a multivalued function in what follows.

Now we will combine this with the surface currents that *do* cause fields inside. This will give a "helical" flow pattern to the surface current.

When we wish to have a field inside the Target, proceed as described in Sec. I to find the corresponding surface current density and field in the Return, and add these to the currents and fields just constructed to find a consistent set of fields produced by wires that wind around the surfaces.

For example, consider how the magnetic potential construction describes the sphere discussed above when it has a uniform magnetic field inside (but no axial current) and that all magnetic flux is enclosed within a Return of radius Q. The magnetic potential within each region is a solution to the Laplace equation; these are connected by the continuity of the normal component, which reduces to $-\frac{\partial}{\partial r}U_{target} = 0$ in this special case; similarly, the normal component of the magnetic field vanishes at the outer surface of the Return. These conditions imply that inside the target, the magnetic potential is

$$U_{target} = -zH_0 = -H_0 r \cos\theta. \tag{3}$$

The potential in the Return is

$$U_{return} = H_0 \frac{P^3(r + Q^3/2r^2)\cos\theta}{Q^3 - P^3}. \tag{4}$$

The surface current density that matches up the tangential magnetic fields on interface between the Target and the Return is

$$\begin{aligned}\vec{K}_{inner} &= -\hat{n} \times \nabla(U_{return} - U_{target})|_{r=P} \\ &= -\hat{\phi}\frac{\partial}{r\partial\theta}(U_{target} - U_{return}) \\ &= \hat{\phi}\frac{3}{2}H_0\frac{Q^3}{Q^3 - P^3}\sin\theta \equiv \hat{\phi}S_{inner}\sin\theta,\end{aligned} \tag{5}$$

and on the outer surface of the Return, it is

$$\begin{aligned}\vec{K}_{outer} &= -\hat{n} \times \nabla(0 - U_{return})|_{r=Q} \\ &= -\hat{\phi}\frac{3}{2}H_0\frac{P^3}{Q^3 - P^3}\sin\theta \equiv -\hat{\phi}S_{outer}\sin\theta,\end{aligned} \tag{6}$$

where $\hat{n}$ is the outward normal $\hat{r}$.

If we slice up the interface surface at equal intervals of the potential difference $U_{outer} - U_{inner}$ (which for the sphere will be at equal intervals of the coordinate z—call this interval D), each of the resulting rings on the surface of the Target is carrying the same current $\vec{I} = \hat{\phi}DS_{inner}$. The rings have the same extent in the coordinate z but varying width on the surface of the sphere.

Now we make this set of rings helical so that after one turn, the bottom of this ring is higher (measured along the z axis) by the amount D. This adds an axial current $\vec{I}_{helix} = \hat{z}DS_{inner}$ in the form of the surface current density $\vec{K}_{axial}$. This modifies the potential in the Return U_{return} (4) by adding an axial potential of the form U_{axial} (2) given above; this has no effect on the field inside the sphere.

The interesting point is that in moving up one turn, the scalar potential difference $U_{return} - U_{target}$ decreases by DS and in going around one turn, the potential outside due to the axial current U_{axial} increases by DS. This shows that the condition $U_{return} + U_{axial} - U_{target}$ = constant generates a helical winding for a spherical electromagnet that completely and efficiently covers the surface.

The same construction applies to the outer surface of the Return (except that there is no field outside the Return so that there is no potential $U_{outside}$ to contend with); observe that all of the current that was moving along the z axis on the Target will return along the outer surface, in agreement with both Ampere's law and current conservation.

The complete magnetic potential in the Return for the coil-made-helical is given by

$$U_{chiral} = H_0 \frac{P^3(r + Q^3/2r^2)\cos(\theta)}{Q^3 - P^3} - I\frac{1}{2\pi}\phi. \tag{7}$$

This is the potential that has the correct normal components at the surfaces of the Return and the discontinuity at the branch surface.

The last term has no effect inside the sphere (by construction). The effect on the field outside can be seen in Fig. 1, which compares one set of field lines when there is no longitudinal current to the case when there is a small one.

The potential difference across the surface of the target and across the outer surface of the return determines the surface current density. Slicing each boundary surface along lines of constant potential difference in increments of I turns each one into a ribbon that winds around the surface, connecting up with the next ribbon. Explicitly, the potential differences are

$$\delta U_{inner} = \frac{3}{2}H_0 P\frac{Q^3}{Q^3 - P^3}\cos\theta - I\frac{1}{2\pi}\phi \tag{8}$$

at the interface between the Target and the Return and

$$\delta U_{outer} = -\frac{3}{2}H_0 Q\frac{P^3}{Q^3 - P^3}\cos\theta + I\frac{1}{2\pi}\phi \tag{9}$$

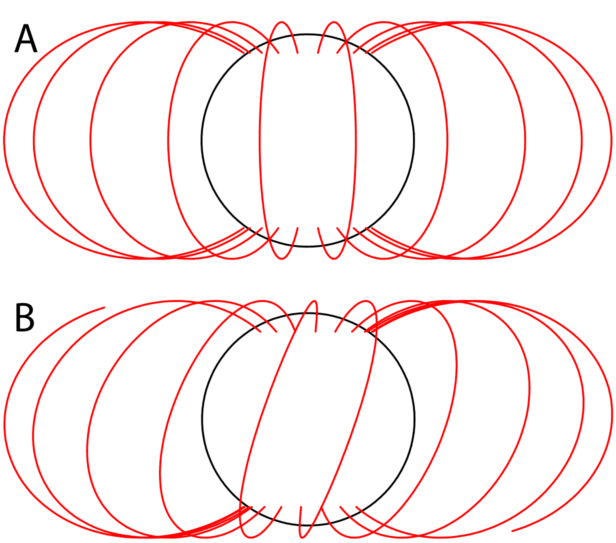

FIG. 1. Field lines outside the sphere: (a) $I = 0$ (no axial current); (b) $I/2\pi H_0 P = 0.02$. These are the lines that exit the sphere 36° from the axis for the case $Q = \infty$ (an unbounded return).

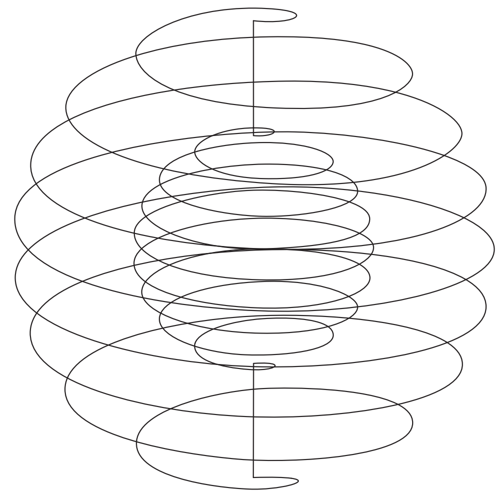

FIG. 2. Illustration of the helical winding for the spherical electromagnet. An actual coil would use a much smaller current and far more turns.

on the exterior surface of the Return. The resulting electromagnetic coil wound along a line of constant value of Eq. (8) or Eq. (9) is shown in Fig. 2. The tangential (along the surface) gradient of the potential differences corresponds to a surface current density [Eq. (1)] that twists about the sphere in such a way that the tangential component of the field inside is related to the tangential field outside according to Ampere's law. Beyond the Target and within the Return, there is a pair of wires along the axis carrying the current I from one surface to the other (and back again). In constructing the spherical magnet, the field H_0 inside and the current I are independent parameters. Once the potentials have been chosen, the path of the windings is determined by the condition of constant potential difference across the surface; the width D of the ribbons (or spacing of the wires) is given by the ratio that approaches I/H_0 in the limit that D is small compared to the cross section of the coil.

The combined potential U_{chiral} is multivalued. In numerical solutions, this will appear as a nonphysical discontinuity of magnitude I at each crossing of the branch cut, which may confuse the graphical display algorithm. However, the current density computed from U_{axial} has no discontinuities; on the Target surface, the winding is traced out by a single contour along which $U_{chiral} - U_{target}$ is constant (and similarly for the outer surface of the Return).

III. THE GENERAL CASE

The construction described above will work with little modification for any system with an axis of rotation; the only difference is that the construction of U_{return} will entail solving the Laplace problem with Neumann boundary conditions in a more complicated geometry. However, it can also be generalized to any Target and Return regions of arbitrary shape and physically allowable magnetic field in the Target. The first step is to follow the basic algorithm[1] to learn how to produce a designed field by finding the scalar potentials U_{target} and U_{return}. Equal intervals of δU describe ribbon loops on the surfaces, which carry equal current. Although this does not yet specify how to make a coil, it approximates its form. The points where δU is maximal and minimal on each surface (the four black blobs on the two surfaces in Fig. 3) are the places where the coil on a surface must start and end. For fields of dipolar character in a suitable Target and Return, there will be just one maximal and minimal point on each surface.

Now we need to choose a "wiring diagram" as illustrated in Fig. 3: jumper wires connecting the point on the Target with minimal $U_{return} - U_{target}$ to the point on the Return with maximal U_{target} and similarly for the other pair of extrema, with the current supply injected at one of these paths via a twisted pair. The exact path of the jumper wires does not matter except that they must start and end at the places indicated. At the same time, we decide how large a current I to use relative to the size of the field H_0 to be constructed, which will determine the width of the ribbons, just as in the case of a solenoid.

Assuming that the Return wraps around the Target, choose a branch surface whose boundary edge includes the jumper wires and traverses across the surfaces of the Target and the Return so that the surface interrupts any path through the Return that encircles the Target. This is the branch surface for U_{axial}. Treating this as a special kind of boundary, the Return is now singly connected.

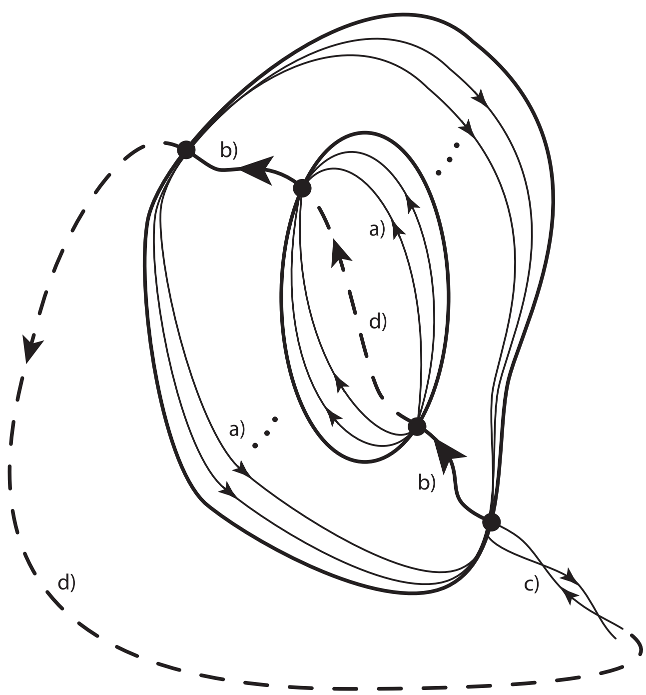

FIG. 3. Illustration of generalized axial windings used to connect all equipotential contours of the Target and Return fields into a continuous helical winding. (a) Axial currents flow from the maximum to minimum of $U_{return} - U_{target}$ on the inner surface of the Return and back from maximum to minimum of U_{return} on the outer surface. (b) Two jumpers connect the axial currents to make a complete circuit. The corresponding toroidal winding field is confined to the Return, so it does not disturb the inner or outer specified fields. (c) The entire coil is energized from a twisted pair inserted into one jumper on the outer surface. (d) The two jumpers can be extended outside the Return to form a filament circuit with calculable potential U_{wire} with the same singularity as U_{axial} so that the latter can be solved from the difference using standard Neumann boundary conditions.

As in the case of the spherical electromagnet, we now construct a second scalar potential U_{axial} in the Return that has a vanishing normal derivative at the surface of the Target and the Return and a discontinuity of magnitude I across the branch surface. This is not a standard Neumann boundary condition, although perfectly well defined.

Here is a way to turn it into a more nearly standard Neumann boundary value problem, as illustrated in Fig. 3: choose a wire path that carries the chosen current such that it incorporates the two jumpers, and add wires that lie outside the Return and inside the Target (arbitrarily) to make a complete circuit. The Biot–Savart integral for the magnetic field of this circuit corresponds to a scalar potential $U_{wire}(\vec{r}) = I\Omega/4\pi$, where Ω is the solid angle subtended by the circuit viewed from the point $\vec{r}$ and has a branch surface, which should be chosen to coincide with that of U_{axial}; by construction, the difference between U_{axial} and U_{wire} is a solution to Laplace's problem inside the Return and with a normal derivative, which is the difference between those of the two potentials on the surfaces of the Return (which are calculable from what has been done already) and continuous across the branch plane. Thus, this is a kind of Neumann boundary value problem in a singly connected region.

As mentioned above, we construct $U_{chiral} = U_{return} + U_{axial}$ in the Return, now using the version of U_{chiral} that is continuous across the branch surface and thus multiple valued in steps of size I. The difference $U_{chiral} - U_{target}$ is constant along a path that slices the surface of the Target into one long ribbon that defines the appropriate winding; similarly, U_{chiral} is constant along a path on the outer surface of the return that defines the winding there.

The wires connecting the Target to the outer surface of the Return should join the extrema of the relevant potential differences, but the positioning of these has already been determined earlier in the algorithm. We claim that the wire positions continue to be appropriate: the wires themselves are singular points of the fields where U_{chiral} takes on different values when the wire is approached along different paths. Then the "perturbation" due to the tangential fields will automatically be absorbed in the determination of the surface windings.

The axial potential U_{axial} produces an additional magnetic "winding field." In the specific and general cases described above, this was a toroidal field circling about the jumpers. The boundary conditions were constructed to confine the winding field completely inside the Return so that it did not perturb either the Target or external fields.

The effect of replacing this with discrete windings is localized near the surfaces on a length scale that is set by the winding spacing.[1] This is largest near the wire connection points where the current density vanishes when the surfaces are smooth (this can be readily seen in the case of the spherical magnet) and has the consequence that the largest discreteness error occurs near the poles.[6] We propose that the Target and Return be deformed near the wire attachments to make them somewhat conical about the wire (the Target becomes a lemon, and the Return has the shape of an apple). The linear (rather than quadratic) variation of the potential near the connection point implies that the width of the spiraling ribbon will not grow so much in that limit, thus decreasing the healing length and making the field in the Target closer to the design field.

IV. CONCLUSION

The rules described above provide a general method to convert the series of disjoint equipotential contour loops described in Ref. 1, each carrying current I, into a continuous helical winding through use of an auxiliary axial potential U_{axial} representing the current I flowing from the lowest to the highest equipotential along each surface of the Return, with the corresponding toroidal field confined to the Return. This method is very general in the sense that the choice of the wiring circuit, current, and equipotential constants classifies all possible helical windings, which approximate the specified Target and external field with a discrete wire or trace winding.

ACKNOWLEDGMENTS

This work was supported, in part, by the U.S. Department of Energy, Office of Nuclear Physics, under Contract No. DE-SC0014622.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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