Neumann Series based Voltage Sensitivity Analysis for Three Phase Distribution System

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***Abstract*—In this letter, a simplified voltage sensitivity analysis technique that can provide accurate estimates of voltage change across the network for a given change in bus power injections in a three-phase unbalanced distribution network is proposed. This technique is derived from the first-order approximation of the Neumann series, which allows maintaining the accuracy of the solution while the computational effort is reduced. The proposed technique is tested on a 559-bus unbalanced distribution system with multiple distributed generation resources. The results show that the average error in the voltage estimates with the proposed method is not more than 0.3% with the execution time of similar order relative to the state-of-the-art sensitivity analysis methods.**

***Index Terms*—Distributed Generation, Distribution System, Neumann series, Voltage Sensitivity Analysis.**

I. Introduction

INTERFACING distributed generation (DG) resources in modern distribution systems is a challenging task as it affects the voltage states across the network [1]. It is therefore important to evaluate the impact of DG on voltage profiles prior to accepting such interconnections. Voltage sensitivity analysis (VSA) can be used under such scenarios as it provides the approximate deviations in bus voltages for the speculated power variations in DG sources and updates the voltage estimates using the solution from the base case scenario. Such an approach takes reduced computation effort as it generally computes voltage deviations from the linear sensitivities.

VSA techniques in the existing literature can be broadly classified as: fitting based approaches and network model based approaches. Fitting based approaches [2]–[10] exploit network measurements or power flow analysis to determine a best fit for an assumed functional relation between the voltages and power levels. Obtaining the sensitivity factors in such fitting based VSA techniques are computationally intensive and so they are typically calculated off-line [4]. Hence, such techniques may not be directly incorporated in real time monitoring of distribution networks.

Approaches that use the network model can be implemented for on-line monitoring and operation of distribution systems since sensitivity factors are directly related to the solution of the base case scenario [11]–[14]. Since these methods linearize the sensitivities in the proximity of the base case solution, there is significant loss in accuracy if the changes in DG power levels are high. The governing power balance equations of distribution networks are non-linear in nature. To deal with such expressions, existing VSA techniques carry out linear approximations either in the network side (like [4]) or in the nodal power injections side (like [13]) in order to linearize the voltage sensitivities and reduce the computational effort.

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In this paper, a low complexity voltage sensitivity analysis technique is developed that exploits an approximate linear relation between voltage change and power injection changes without compromising on voltage magnitude estimation accuracy. In the proposed approach, for the first time, an attempt is made to model both power injections and network model as accurately as possible using the first order approximation of the Neumann series to execute the voltage sensitivity analysis. The proposed VSA is tested in 559-bus unbalanced distribution system and shown that the error in the computed voltage magnitudes is less than the technique given in [13] even for changes in DG power as high as 120% of the base value.

II. Proposed Voltage Sensitivity Technique

Consider a three phase unbalanced distribution system whose source node (point of interconnection to the bulk system) is numbered as Bus 0 and all the other nodes are numbered from 1. With this definition, the complex bus voltages (except the source node) can be written as:

$$\mathbf{v} = \begin{bmatrix} v_1^a & v_1^b & v_1^c & v_2^a & v_2^b & v_2^c & \dots \end{bmatrix}^T \quad (1)$$

Similarly, the complex power bus injections (except the source node) can be denoted in a vector form as:

$$\mathbf{s} = \begin{bmatrix} s_1^a & s_1^b & s_1^c & s_2^a & s_2^b & s_2^c & \dots \end{bmatrix}^T \quad (2)$$

With these definitions, the expressions for voltages of the three phase distribution system are given as follows:

$$\mathbf{v} = \mathbf{v}^S + \mathbf{Y}^{-1}\,\mathrm{diag}\left(\mathbf{v}^*\right)^{-1}\mathbf{s}^* \quad (3)$$

where, $\mathbf{Y}$ is the bus admittance matrix with the source node as reference and $\mathbf{v}^S$ is the voltage at all phases of source node.

VSA involves computing the change in voltage $\Delta\mathbf{v}$ for a given change in complex power injection from $\mathbf{s}$ to $\mathbf{s}+\Delta\mathbf{s}$. From (3), the amount of voltage change $\Delta\mathbf{v}$ is written as,

$$\Delta\mathbf{v} = \mathbf{Y}^{-1}\left(\mathrm{diag}\left(\mathbf{v}^*+\Delta\mathbf{v}^*\right)^{-1}\left(\mathbf{s}^*+\Delta\mathbf{s}^*\right) - \mathrm{diag}\left(\mathbf{v}^*\right)^{-1}\mathbf{s}^*\right) \quad (4)$$

A linear approximation for (4) is proposed in Theorem 1.

Theorem 1. *In an unbalanced power network with three phase bus impedance matrix, $\mathbf{Z}$, the voltage change, $\Delta\mathbf{v}$, for change in bus power injections, $\Delta\mathbf{s}$ can be approximated as:*

$$\Delta\mathbf{v} \approx \left(\mathbf{Z} + \mathbf{Z}\,\mathrm{diag}\left(|\mathbf{v}|^{\otimes -2}\otimes\left(\mathbf{s}^*+\Delta\mathbf{s}^*\right)\right)\mathbf{Z}\right)\left(\mathbf{v}^{\otimes -1}\otimes\Delta\mathbf{s}\right)^*, \quad (5)$$

where, $\otimes$ denotes element wise operation of the given vector.

Proof. To obtain a simplified formulation, (4) is subjected to algebraic manipulations and is rewritten as,

$$\mathbf{Y}\Delta\mathbf{v} = \tilde{\mathbf{Y}}\left(\mathbf{v}+\Delta\mathbf{v}\right) - \mathrm{diag}\left(\mathbf{v}^*\right)^{-1}\mathbf{s}^*, \tag{6}$$

where, $\tilde{\mathbf{Y}} = \mathrm{diag}\left(|\mathbf{v}+\Delta\mathbf{v}|\right)^{-2}\mathrm{diag}\left(\mathbf{s}^*+\Delta\mathbf{s}^*\right)$ which is a diagonal matrix with the value of load admittances. This definition can simplify the expression of voltage deviation as,

$$\Delta\mathbf{v} = \left(\mathbf{Y}-\tilde{\mathbf{Y}}\right)^{-1}\left(\tilde{\mathbf{Y}}\mathbf{v} - \mathrm{diag}\left(\mathbf{v}^*\right)^{-1}\mathbf{s}^*\right) \tag{7}$$

To adopt the Neumann Series, (7) is further rearranged as,

$$\Delta\mathbf{v} = \left(\mathbf{I}-\mathbf{Y}^{-1}\tilde{\mathbf{Y}}\right)^{-1}\mathbf{Y}^{-1}\left(\tilde{\mathbf{Y}}\mathbf{v} - \mathrm{diag}\left(\mathbf{v}^*\right)^{-1}\mathbf{s}^*\right) \tag{8}$$

where $\mathbf{I}$ is the identity matrix which has the same dimension as $\mathbf{Y}$. To apply the Neumann Series [15] to compute the inverse of $\mathbf{I}-\mathbf{Y}^{-1}\tilde{\mathbf{Y}}$, the following requirement must be satisfied:

$$\lim_{r\to\infty}\left(\mathbf{Y}^{-1}\tilde{\mathbf{Y}}\right)^r \to \mathbf{0} \tag{9}$$

It can be noted that the elements of $\mathbf{Y}$ which in turn are the line admittances of the distribution feeders usually have values much higher than the load admittance given in $\tilde{\mathbf{Y}}$ other than in fault conditions. Hence the condition given in (9) will be satisfied in normal operating conditions of the distribution network. Hence Neumann series [15] can be applied as

$$\left(\mathbf{I}-\mathbf{Y}^{-1}\tilde{\mathbf{Y}}\right)^{-1} = \sum_{r=0}^{\infty}\left(\mathbf{Y}^{-1}\tilde{\mathbf{Y}}\right)^r \tag{10}$$

With the condition given in (9), it can be noticed that the term $\left(\mathbf{Y}^{-1}\tilde{\mathbf{Y}}\right)^r$ tends to reduce rapidly as r increases. Hence, (10) can be approximated by discarding the higher order terms as

$$\left(\mathbf{I}-\mathbf{Y}^{-1}\tilde{\mathbf{Y}}\right)^{-1} \approxeq \mathbf{I}+\mathbf{Y}^{-1}\tilde{\mathbf{Y}} \tag{11}$$

This approximation is substituted in (8) to simplify as,

$$\Delta\mathbf{v} \approxeq \left(\mathbf{Y}^{-1}+\mathbf{Y}^{-1}\tilde{\mathbf{Y}}\mathbf{Y}^{-1}\right)\left(\tilde{\mathbf{Y}}\mathbf{v} - \mathrm{diag}\left(\mathbf{v}^*\right)^{-1}\mathbf{s}^*\right) \tag{12}$$

The bus impedance matrix (with source node as reference) is defined as $\mathbf{Z}=\mathbf{Y}^{-1}$. The elements of bus impedance matrix can be easily obtained for a radial distribution network using bus building algorithm and hence it can be incorporated in the approximation of voltage deviation which is given as,

$$\begin{aligned}\Delta\mathbf{v} \approxeq &\left(\mathbf{Z}+\mathbf{Z}\,\mathrm{diag}\left(|\mathbf{v}+\Delta\mathbf{v}|^{\otimes-2}\otimes\left(\mathbf{s}^*+\Delta\mathbf{s}^*\right)\right)\mathbf{Z}\right)\\ &\left(|\mathbf{v}+\Delta\mathbf{v}|^{\otimes-2}\otimes\left(\mathbf{s}^*+\Delta\mathbf{s}^*\right)\otimes\mathbf{v} - \left(\mathbf{v}^*\right)^{\otimes-1}\otimes\mathbf{s}^*\right).\end{aligned} \tag{13}$$

It is assumed that $|\mathbf{v}+\Delta\mathbf{v}|^{\otimes-2} \approxeq |\mathbf{v}|^{\otimes-2}$ and the approximate linearized expression of $\Delta\tilde{\mathbf{v}}$ corresponds to

$$\Delta\tilde{\mathbf{v}} = \tilde{\mathbf{Z}}\left(\mathbf{v}^{\otimes-1}\otimes\Delta\mathbf{s}\right)^* \tag{14}$$

$$\text{where } \tilde{\mathbf{Z}} = \mathbf{Z}+\mathbf{Z}\,\mathrm{diag}\left(|\mathbf{v}|^{\otimes-2}\otimes\left(\mathbf{s}^*+\Delta\mathbf{s}^*\right)\right)\mathbf{Z} \tag{15}$$

□

This approximate voltage change $\Delta\tilde{\mathbf{v}}$ given in (14) can be used to update the voltage estimates as $\mathbf{v}+\Delta\tilde{\mathbf{v}}$ for a speculated value of power injection $\mathbf{s}+\Delta\mathbf{s}$ with $\mathbf{s}$ and $\mathbf{v}$ being the power injections and bus voltages corresponding to the base case scenario, respectively. It can be seen in (15) that the power injections are modelled as equivalent impedance values and incorporated in network model given by the bus impedance matrix $\mathbf{Z}$. This significantly increases the accuracy of the approximation made in this proposed VSA approach.

The expression given in (14) is similar to the formulation given in [13]. Hence the proposed formulation has the same computation complexity as [13]. To show that the formulation given in (14) provides increased accuracy as compared to [13], the upper bound of the approximation error for the proposed voltage sensitivity analysis is derived in Corollary 1.1.

Corollary 1.1. *The approximation error of voltage magnitude in (14) at phase $m\in\{a,b,c\}$ of bus j is upper bounded by:*

$$\left|\delta v_j^m\right| \le \sum_k\sum_{n\in\{a,b,c\}}\left|\alpha_{jk}^{mn}\right| + \left|\beta_{jk}^{mn}\right| \tag{16}$$

where

$$\alpha_{jk}^{mn} = \frac{\kappa_{1,jk}^{mn}}{(1+\gamma_{1,k}^n)e_k^n} - \frac{\kappa_{2,jk}^{mn}}{(1+\gamma_{2,k}^n)f_k^n} \tag{17}$$

$$\beta_{jk}^{mn} = \frac{\kappa_{2,jk}^{mn}}{(1+\gamma_{1,k}^n)e_k^n} + \frac{\kappa_{1,jk}^{mn}}{(1+\gamma_{2,k}^n)f_k^n} \tag{18}$$

and

$$\kappa_{1,jk}^{mn} = \tilde{R}_{jk}^{mn}\Delta p_k^n + \tilde{X}_{jk}^{mn}\Delta q_k^n \tag{19}$$

$$\kappa_{2,jk}^{mn} = \tilde{X}_{jk}^{mn}\Delta p_k^n - \tilde{R}_{jk}^{mn}\Delta q_k^n \tag{20}$$

$$\gamma_{1,k}^n = (f_k^n/e_k^n)^2 \tag{21}$$

$$\gamma_{2,k}^n = (e_k^n/f_k^n)^2 \tag{22}$$

Proof. Using (14), the approximate voltage change at phase $m\in\{a,b,c\}$ of bus j can be expressed as,

$$\Delta\tilde{v}_j^m = \sum_k\sum_{n\in\{a,b,c\}}\frac{\tilde{Z}_{jk}^{mn}\Delta s_k^{n^*}}{v_k^{n^*}} \tag{23}$$

The approximation error of complex voltage value at phase $m\in\{a,b,c\}$ of bus j, δv_j^m can be written as:

$$\delta v_j^m = \Delta v_j^m - \Delta\tilde{v}_j^m \tag{24}$$

The complex values used in (14) are separated into real and imaginary part as $v_j^m = e_j^m + \mathbf{i}f_j^m$, $\Delta s_j^m = \Delta p_j^m + \mathbf{i}\Delta q_j^m$ and $\tilde{Z}_{jk}^{mn} = \tilde{R}_{jk}^{mn} + \mathbf{i}\tilde{X}_{jk}^{mn}$ where $n\in\{a,b,c\}$. With these definitions and by exploiting the similarity in voltage deviation formulation of [13], the upper bound of the error in the real and imaginary part of the approximation is written as:

$$\Re\{\delta v_j^m\} \le \sum_k\sum_{n\in\{a,b,c\}}\frac{\kappa_{1,jk}^{mn}}{(1+\gamma_{1,k}^n)e_k^n} - \frac{\kappa_{2,jk}^{mn}}{(1+\gamma_{2,k}^n)f_k^n} \tag{25}$$

$$\Im\{\delta v_j^m\} \le \sum_k\sum_{n\in\{a,b,c\}}\frac{\kappa_{2,jk}^{mn}}{(1+\gamma_{1,k}^n)e_k^n} + \frac{\kappa_{1,jk}^{mn}}{(1+\gamma_{2,k}^n)f_k^n} \tag{26}$$

With triangle inequality, the upper bound for the approximation error in (23) for voltage magnitude values is given in (27) which proves this corollary.

$$\left|\delta v_j^m\right| \le \left|\Re\{\delta v_j^m\}\right| + \left|\Im\{\delta v_j^m\}\right| \tag{27}$$

TABLE I
TOTAL LOCAL POWER GENERATION CORRESPONDING TO % CHANGE IN THE 559-BUS SYSTEM [16]

Change in Generation	-120%	-100%	-80%	-60%	-40%	-20%	0%	20%	40%	60%	80%	100%	120%
Real Power (MW)	0	0.1219	0.2438	0.3657	0.4876	0.6095	0.7314	0.8533	0.9752	1.0971	1.2190	1.3409	1.4628
Reactive Power (MVAr)	0	0.0590	0.1181	0.1771	0.2362	0.2952	0.3542	0.4133	0.4723	0.5313	0.5904	0.6494	0.7084

TABLE II
COMPARISON OF AVERAGE EXECUTION TIME OF PROPOSED METHOD (PM) WITH OTHER APPROACHES

Approach	PFA	VSA [13]	VSA [11]	PM
Avg. Exe. Time (s)	1.8636	0.0017	0.6593	0.0018

□

It is noted that if the cumulative active and reactive power injection in bus j is negative, then $\Re\{\tilde{Y}_{jk}^{mn}\} \leq 0$ and $\Im\{\tilde{Y}_{jk}^{mn}\} \leq 0$ for all k. Such a condition is valid in normal operating condition of three phase distribution network as the number of loads will be higher than the number of local generations and the cumulative power supplied by these local generators is usually less than the load demand. Under this condition, $\tilde{R}_{jk}^{uv} \leq \Re\{Z_{jk}^{mn}\}$ and $\tilde{X}_{jk}^{mn} \leq \Im\{Z_{jk}^{mn}\}$. Hence, it is worth noting that the upper bounds for the approximation error in real and imaginary part of voltage changes given in (25) and (26) is tighter than the bounds given in [13] when the total load demand is higher than the local power generations.

III. RESULTS AND DISCUSSION

The linear approximation developed in (14) is validated using a three phase unbalanced 559-bus system [16]. The base voltage is considered to be 4.8kV, which is its source node voltage. The total active and reactive power demand in the 559-bus system is 2.4379MW and 1.1807Mvar respectively. In addition to the source node, 120 1Φ local generators supply power a total of 0.7314 MW and 0.3542 Mvar in base case scenario. To evaluate the proposed VSA technique, the change in local generation is considered from -120% to 120% relative to the base case scenario. The total amount of real and reactive power produced by local generators are tabulated in Table I. Under 120% increase in local generation, the voltage magnitudes are computed with the sensitivities given in (14) and plotted in Fig. 1 along with the solutions provided by power flow analysis (PFA) and VSA techniques given in [11], [13]. The proposed approximation provides better accurate results than the VSA technique given in [11], [13] without the computational effort required for conventional PFA.

The approximation error of the proposed method (PM) is analyzed using the voltage magnitude values at phase ‘a’ of bus 504 under change in generation levels from -120% to 120% and plotted in Fig. 2 along with its upper bounds (UB) given in (16). These error values are compared against the results obtained from VSA techniques given in [11], [13]. Bus 504 phase ‘a’ was picked because it has the maximum estimation error and hence can be considered as a critical bus (from an estimation error standpoint). Fig. 2 indicates that the proposed method has lower approximation error as compared to the VSA techniques given in [11], [13]. Similar results were obtained for other buses. While VSA method of [11] provides better results than the proposed method for lower values of change in local generation, it requires significantly higher computation effort as compared to the proposed method.

Mean Absolute Percentage Error (MAPE) [17], used to quantify the overall estimation accuracy, is defined as:

$$MAPE = \frac{1}{3} \sum_{\phi \in \{a,b,c\}} \left(\frac{1}{N} \sum_{j=1}^{N} \frac{\left|\Delta \mathbf{v}_j^{\phi}\right| - \left|\Delta \tilde{\mathbf{v}}_j^{\phi}\right|}{\left|\Delta \mathbf{v}_j^{\phi}\right|} \right). \quad (28)$$

MAPE results plotted in Fig. 3 indicate that the proposed method provides better estimates of voltage changes as its MAPE is lower than that of the techniques proposed in [11], [13]. It can be noticed in Fig. 3 that the increase in the total amount of local generation not only increases the nodal voltage levels, it also decreases the MAPE with the VSA given in [13]. It is because the accuracy of VSA given in [13] increases as the ratio between change in voltage and nodal voltages decreases and vice versa. The average execution time of the proposed method and the the techniques given in [11], [13] are computed for 100 Monte Carlo runs and tabulated in Table II. PM and the technique given in [13] have the same order of execution time, but both are much faster than the other two approaches.

IV. CONCLUSION

This paper developed a simplified technique to obtain voltage sensitivities for change in power injections at multiple buses in three phase unbalanced distribution networks. This technique is approximated from the first order terms of Neumann series. The approximation is shown to be highly accurate over a wide range of power injection variations. The proposed method is tested on a 559-bus unbalanced distribution system and shown that it provides higher accuracy than the other VSA methods without any loss in computational efficiency. To obtain the voltage estimates, the proposed VSA requires solution of the base case scenario which may not be directly available in low observable distribution networks. Hence, as part of our future work, the VSA approach will be extended to unobservable networks using sparsity based state estimators.

REFERENCES

[1] “IEEE standard for interconnection and interoperability of distributed energy resources with associated electric power systems interfaces,” *IEEE Std 1547-2018*, pp. 1–138, 2018.

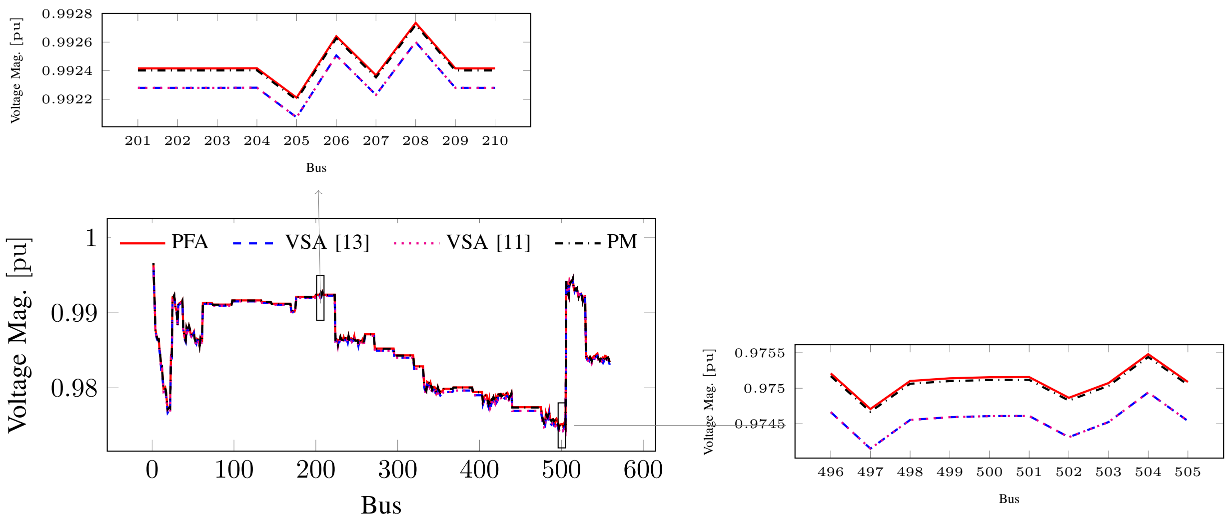

Fig. 1. Phase a Voltage Magnitude for 559-bus system [16] after 120% increase in local generation.

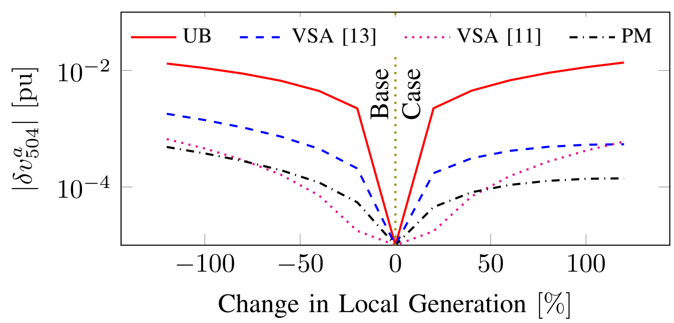

Fig. 2. Voltage Approximation Error at bus 504 phase ‘a’ for the proposed method (PM), and its Upper Bounds (UB), and VSA given in [13] and [11].

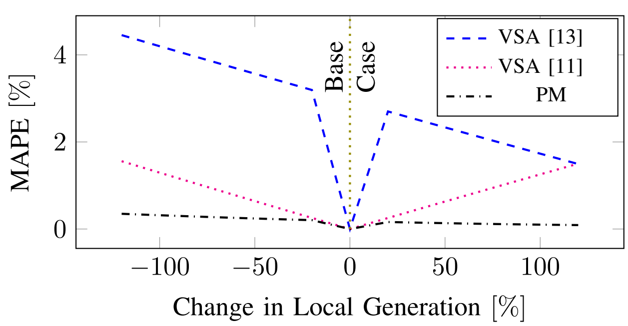

Fig. 3. Voltage Magnitude estimation performance of the proposed method (PM) compared to VSA given in [13] and [11].

[2] K. Christakou, J.-Y. LeBoudec, M. Paolone, and D.-C. Tomozei, “Efficient computation of sensitivity coefficients of node voltages and line currents in unbalanced radial electrical distribution networks,” *IEEE Transactions on Smart Grid*, vol. 4, no. 2, pp. 741–750, 2013.

[3] F. Tamp and P. Ciufo, “A sensitivity analysis toolkit for the simplification of mv distribution network voltage management,” *IEEE Trans. Smart Grid*, vol. 5, no. 2, pp. 559–568, 2014.

[4] S. Weckx, R. D’Hulst, and J. Driesen, “Voltage sensitivity analysis of a laboratory distribution grid with incomplete data,” *IEEE Trans. Smart Grid*, vol. 6, no. 3, pp. 1271–1280, 2015.

[5] Z. Zhang, L. F. Ochoa, and G. Valverde, “A novel voltage sensitivity approach for the decentralized control of dg plants,” *IEEE Trans. Power Syst.*, vol. 33, no. 2, pp. 1566–1576, 2018.

[6] B. B. Zad, J. Lobry, and F. Vallée, “A new voltage sensitivity analysis method for medium-voltage distribution systems incorporating power losses impact,” *Electric Power Components and Systems*, vol. 46, 2018.

[7] M. U. Qureshi, S. Grijalva, M. J. Reno, J. Deboever, X. Zhang, and R. J. Broderick, “A fast scalable quasi-static time series analysis method for pv impact studies using linear sensitivity model,” *IEEE Trans. Sustain. Energy*, vol. 10, no. 1, pp. 301–310, 2019.

[8] H. Su, P. Li, P. Li, X. Fu, L. Yu, and C. Wang, “Augmented sensitivity estimation based voltage control strategy of active distribution networks with pmu measurement,” *IEEE Access*, vol. 7, pp. 44 987–44 997, 2019.

[9] G. Gruosso, R. S. Netto, L. Daniel, and P. Maffezzoni, “Joined probabilistic load flow and sensitivity analysis of distribution networks based on polynomial chaos method,” *IEEE Trans. Power Syst.*, vol. 35, 2020.

[10] F. J. Zarco-Soto, J. L. Martínez-Ramos, and P. J. Zarco-Periñán, “A novel formulation to compute sensitivities to solve congestions and voltage problems in active distribution networks,” *IEEE Access*, vol. 9, 2021.

[11] A. Safayet, P. Fajri, and I. Husain, “Reactive power management for overvoltage prevention at high pv penetration in a low-voltage distribution system,” *IEEE Trans. Ind. Appl.*, vol. 53, 2017.

[12] K. Jhala, B. Natarajan, and A. Pahwa, “Probabilistic voltage sensitivity analysis (PVSA)—a novel approach to quantify impact of active consumers,” *IEEE Trans. Power Syst.*, vol. 33, no. 3, pp. 2518–2527, 2018.

[13] S. Munikoti, B. Natarajan, K. Jhala, and K. Lai, “Probabilistic voltage sensitivity analysis to quantify impact of high pv penetration on unbalanced distribution system,” *IEEE Trans. Power Syst.*, vol. 36, 2021.

[14] S. Maharjan, A. M. Khambadkone, and J. C.-H. Peng, “Enhanced z-bus method for analytical computation of voltage sensitivities in distribution networks,” *IET Gener. Transm. Distrib*, vol. 14, no. 16, 2020.

[15] M. Wu, B. Yin, A. Vosoughi, C. Studer, J. R. Cavallaro, and C. Dick, “Approximate matrix inversion for high-throughput data detection in the large-scale mimo uplink,” in *IEEE ISCAS*, 2013.

[16] A. R. Malekpour and A. Pahwa, “Radial test feeder including primary and secondary distribution network,” in *NAPS*, 2015.

[17] S. Dahale, H. S. Karimi, K. Lai, and B. Natarajan, “Sparsity based approaches for distribution grid state estimation - a comparative study,” *IEEE Access*, vol. 8, pp. 198 317–198 327, 2020.