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# Bootstrap current and parallel ion velocity in imperfectly optimized stellarators

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## Abstract

A novel derivation of the parallel ion velocity, and the bootstrap and Pfirsch-Schlüter currents in an imperfectly optimized (that is, almost omnigenous) stellarator magnetic field,  $\vec{B}$ , is presented. It is shown that, when the conventional radially local form of the drift kinetic equation is employed, the flow velocity and the bootstrap current acquire a spurious contribution proportional to  $\omega/\nu$ , where  $\omega$  denotes the  $\vec{E} \times \vec{B}$  rotation frequency (due to the radial electric field  $\vec{E}$ ) and  $\nu$  the collision frequency. This contribution is particularly large in the  $\sqrt{\nu}$  regime and at smaller collisionalities, where  $\omega/\nu \gtrsim 1$ , and is presumably present in most numerical calculations, but it disappears if a more accurate drift kinetic equation is used.

## 1. Introduction

In stellarators and tokamaks, the flow velocity of each plasma species along the magnetic field is affected by the cross field (radial) derivatives of the density, electrostatic potential and temperature, and so is, therefore, the parallel current. The existence of this “bootstrap” current is of great practical importance, but in stellarators it is still incompletely understood. It can be calculated numerically from the drift kinetic equation, but analytical expressions have only been derived in the so called  $1/\nu$  regime of moderately small collisionality (Shaing *et al.* 1989; Helander *et al.* 2011), the omnigenous limit (Landreman & Catto 2012), and, more recently, in the  $\sqrt{\nu}$  regime of lower collisionality (Helander *et al.* 2017). In all these regimes, the theory predicts a current (or parallel flow velocity) that is independent of collisionality, but numerical simulations (Beidler *et al.* 2011; Kernbichler *et al.* 2016) suggest otherwise down to very small collisionalities. They also find that the bootstrap current depends sensitively on the radial electric field – an effect not included in the analytical expressions.

In the present paper, a possible reason for this result is proposed, showing that the standard drift kinetic equation solved by most numerical codes leads to a term in the parallel flow velocity proportional to  $\omega/\nu$ , where  $\omega$  denotes the poloidal  $\vec{E} \times \vec{B}$  rotation frequency and  $\nu$  the collision frequency. (Here  $\vec{E}$  denotes the radial electric field and  $\vec{B}$  the magnetic field strength.) This contribution to the bootstrap current is particularly important in the  $\sqrt{\nu}$  regime and at smaller collisionalities, where  $\omega/\nu \gg 1$ , and does not appear in earlier analytical formulas, but it should be present in numerical simulations. However, it is also shown that it is, in fact, spurious. It results from the use of a radially local approximation of the drift kinetic equation, which is nearly always employed in the neoclassical theory of stellarators, both in

analytical theory and in numerical simulations. It usually seems to be admissible, but apparently not in the present case.

In the absence of turbulence, optimized stellarators attempt to confine all collisionless charged particle orbits by clever design of the magnetic equilibrium. If the magnetic field strength on a flux surface depends on the toroidal ( $\zeta$ ) and poloidal ( $\vartheta$ ) angles in a single linear combination ( $M\vartheta - N\zeta$ ) the field is said to be quasisymmetric (Nührenberg & Zille 1988, Boozer 1995), as in the Helically Symmetric Experiment (HSX) stellarator (Anderson *et al.* 1995), and the collisionless orbits are all confined. Here  $M$  and  $N$  are integers corresponding to the number of times a line of constant  $B$  on a flux surface must traverse the torus toroidally and poloidally, respectively, before closing on itself. Quasisymmetric behavior is thought to be possible on only an isolated flux or constant pressure surface (Garren & Boozer 1991; Plunk & Helander 2018; Landreman & Sengupta 2018; Landreman *et al.* 2019). More generally, confined collisionless charged particle orbits are possible if the magnetic field has the property normally referred to as omnigenity (Cary & Shasharina 1997a&b), with the quasi-isodynamic behavior of the Wendelstein 7-X (W7-X) stellarator (Nührenberg 2010; Beidler *et al.* 1990; Grieger *et al.* 1992) being the most notable example (although it is only approximately realized). (Presumably this more general property can also only be satisfied on or in the vicinity of a single flux surface.) Stellarator optimization schemes try to approach these idealized limits over a large fraction of the cross section (Henneberg *et al.* 2019), while maintaining good stability properties and avoiding substantial turbulent transport, because significant departures from omnigenity result in unacceptably large collisional transport.

The neoclassical properties of perfectly omnigenous fields are well understood (Helander & Nührenberg 2009; Landreman & Catto 2012), and the effect of departures from omnigenity on radial particle and heat transport are becoming better documented as simulations and analysis improve. However, understanding of the effect of departures from omnigenity on the bootstrap current at low collisionalities is less clear. While it is likely that transitional particles that spend time trapped in multiple wells are responsible for much of the puzzling behavior (Beidler *et al.* 2011; Kernbichler *et al.* 2016), the procedure used here finds that in the presence of an  $\vec{E} \times \vec{B}$  drift tangential to the magnetic surfaces there is a spurious modification to the ion flow that becomes large at low collisionality on non-omnigenous flux surfaces. It affects both the bootstrap current and the parallel flow velocity.

For larger collisionalities, the treatment here results in expressions for the bootstrap current and parallel ion flow that are independent of collision frequency. Like certain earlier results, the expressions obtained here are valid in the  $\sqrt{v}$  and standard  $1/v$  regimes of collisionality (Galeev *et al.* 1969; Ho & Kulsrud 1987; Calvo *et al.* 2017). These expressions fail in the superbanana plateau regime of collisionality (Shaing 2015; Calvo *et al.* 2017), but this regime is not considered in many neoclassical stellarator simulations (Beidler *et al.* 2011; Kernbichler *et al.* 2016) and is usually not of interest for the background species unless the  $\vec{E} \times \vec{B}$  drift due to the radial electric field is smaller than the magnetic drift. In the superbanana plateau regime the magnetic tangential frequency depends on pitch angle. If the magnetic field is

omnigenous or  $\omega/\nu \ll 1$ , the results found here for the bootstrap current and parallel ion velocity are consistent with the expressions of Helander *et al.* (2017), but contain a slightly more explicit and compact geometrical coefficient that is obtained by a very different procedure. The result is also consistent with Shaing *et al.* (1989), Helander *et al.* (2011), and Landreman & Catto (2012).

The next two sections give background material for non-optimized stellarators (sec. 2) and drift kinetics (sec. 3), while section 4 discusses equilibrium properties. Section 5 reformulates stellarator drift kinetics to extract the terms odd in the parallel velocity as the lowest order corrections to the Maxwellian. In section 6 the drift kinetic equation is solved by ignoring a certain term related to the tangential drift, and section 7 evaluates the resulting parallel ion flow velocity and parallel current density. At low collisionality, the specious tangential  $\vec{E} \times \vec{B}$  term is considered and found to give rise to a modification of the ion flow velocity and bootstrap current, which is evaluated in section 8. A brief discussion follows in section 9.

## 2. Non-optimized stellarator background

To treat collisional transport in a non-omnigenous stellarator it is normally assumed that the electrons and ions are in the  $1/\nu$  and  $\sqrt{\nu}$  regimes, respectively. In both regimes collisions must be retained, but ion streaming is smaller than electron streaming so the ions are sensitive to the tangential  $\vec{E} \times \vec{B}$  drift and the electrons are not. Consequently, as  $\nu$  gets small, the electron diffusivities increase as  $1/\nu$ , while the ion diffusivities decrease as  $\sqrt{\nu}$  (Galeev *et al.* 1969; Ho & Kulsrud 1987). The radial ion and electron particle losses balance and set the electric field to maintain ambipolarity, so both collisionality regimes must be considered when evaluating the bootstrap current.

At higher collisionalities, the lowest order radial electric field,  $\vec{E} \simeq -\nabla\Phi(\psi)$ , is roughly at the ion diamagnetic level (to bring ion transport down to the electron level), then

$$Zen_i \partial\Phi/\partial\psi \sim -T_i \partial n_i/\partial\psi, \quad (2.1)$$

with the poloidal flux function  $\psi$  related to the toroidal flux function  $\Psi$  by  $d\Psi = qd\psi$ , and the safety factor  $q$  and rotational transform  $\iota$  related by  $\iota q = 1$ . The tangential  $\vec{E} \times \vec{B}$  drift frequency is then

$$\omega_E = -c \frac{\partial\Phi}{\partial\psi} \sim \frac{cT_i}{Zen_i} \frac{\partial n_i}{\partial\psi} \sim \frac{v_i^2}{\Omega_i a^2} = \frac{\rho_i}{a^2} v_i, \quad (2.2)$$

while the tangential  $\nabla B$  drift frequency is

$$\omega_{\nabla B} \sim \rho_i v_i / aR, \quad (2.3)$$

where  $\rho_i$  and  $v_i$  are the ion gyroradius and thermal speed,  $\Omega_i = ZeB/M_i c$ , and  $r$ ,  $a$ , and  $R$  are the distance from the magnetic axis, the nominal radial scale, and the major radius. The ion charge number and mass are  $Z$  and  $M_i$ , while  $c$  is the speed of light and  $e$  is the charge of a proton. Normally  $\omega_E \sim R\omega_{\nabla B}/a \gg \omega_{\nabla B}$ , so the magnetic drift is often neglected as small. Then, there is no superbanana plateau regime as the vanishing of the combined magnetic and electric drift is unlikely for the background ions (at any pitch angle), unless the  $\vec{E} \times \vec{B}$  becomes small.

As already noted, when all collisionless orbits are confined, a stellarator is referred to as omnigenous, and the special case of the magnitude of the magnetic field  $B$  depending on only a

single helical variable is referred to as quasisymmetry. For an omnigenous, but non-quasisymmetric, stellarator, ambipolarity requires that the radial electric field reduce the ion particle transport to the typically smaller electron level by setting it at the ion diamagnetic level as already estimated. More precisely, the ion root is given by (Landreman & Catto 2012)

$$Zen_i \frac{\partial \Phi}{\partial \psi} \simeq -\frac{\partial p_i}{\partial \psi} + \frac{1.17}{T_i} \frac{\partial T_i}{\partial \psi}. \quad (2.4)$$

As flux surfaces become non-omnigenous the ion particle diffusivity enters the  $\sqrt{\nu}$  regime and decreases as the collisionality is reduced. The electron particle diffusivity increases as  $1/\nu$  as the tangential drift is unimportant. At some point the omnigenous ion root may be altered and become sensitive to collisions as it nears the transition to the electron root,

$$en_e \partial \Phi / \partial \psi \sim T_e \partial n_e / \partial \psi, \quad (2.5)$$

to keep the electron particle transport at the now lower ion level. If a change in the sign of  $\partial \Phi / \partial \psi$  occurs, then there are flux surfaces with very small  $\vec{E} \times \vec{B}$  drift where the  $\nabla B$  drift matters (Matsuoka *et al.* 2015). As the  $\nabla B$  drift vanishes at some pitch angle, there can be flux surfaces for which superbanana plateau transport becomes important. The treatment here does not allow for this possibility, and neither do many simulations (Beidler *et al.* 2011; Kernbichler *et al.* 2016).

Typically the  $1/\nu$  and  $\sqrt{\nu}$  collisionality regimes are evaluated by a transit averaged kinetic equation for the trapped particles of the form,

$$\overline{\vec{v}_d \cdot \nabla \psi} \frac{\partial f_0}{\partial \psi} + \overline{\vec{v}_d \cdot \nabla \alpha} \frac{\partial \bar{h}_t}{\partial \alpha} = C\{\bar{h}_t\}, \quad (2.6)$$

with  $f_0$  a Maxwellian and  $\bar{h}_t$  a trapped particle modification to it. Here  $\alpha$  is the angle variable in the Clebsch representation of the magnetic field,  $\vec{B} = \nabla \alpha \times \nabla \psi$ . The overbar notation indicates transit or bounce average and will be defined carefully in the next section. The tangential drift is

$$\overline{\vec{v}_d \cdot \nabla \alpha} \rightarrow -\omega(\psi) = -c \partial \Phi / \partial \psi, \quad (2.7)$$

and, unless noted otherwise, is assumed to be a flux function for the treatment in the following sections. In the transit averaged equation at least the radial drift term  $\overline{\vec{v}_d \cdot \nabla \psi}$  must retain the departure from omnigenity. Two basic approximations have been made in Eqs. (2.6) – (2.7), and in Eq. (3.1) below. The radial component of the drift acting on  $\bar{h}_t$  has been neglected, and the tangential part of the drift has been approximated by its  $\vec{E} \times \vec{B}$  component. Although these approximations are common in neoclassical stellarator theory, it is shown that they can, in fact, be misleading.

For the electrons the tangential drift is negligible ( $\omega \ll \nu_e$ ), giving

$$\overline{\vec{v}_d \cdot \nabla \psi} \partial f_0 / \partial \psi \sim C\{\bar{h}_t\} \sim \nu_e \bar{h}_t, \quad (2.8)$$

$\bar{h}_t / f_M \propto 1/\nu_e$ , and electron diffusivities proportional to  $1/\nu_e$ , with  $\nu_e$  the electron collision frequency. The tangential drift frequency  $\omega$  matters for the ions ( $\omega \gg \nu_i$ ). In the  $\sqrt{\nu}$  regime, the collisionality is assumed weak enough to satisfy

$$\rho_i / a \gg R \nu_i / v_i = R / \lambda, \quad (2.9)$$

where  $\lambda$  is the mean free path, with  $\lambda \sim v_i/v_i \sim v_e/v_e$  and  $v_e$  the electron thermal speed.

For  $T_i = 10$  keV,  $n_i = 10^{14}$  cm $^{-3}$ , and  $B = 5$  T;  $v_i \approx 10^8$  cm/sec,  $v_i \approx 50$  sec $^{-1}$ , and  $\rho_i \approx 0.3$  cm. Then  $a = 100$  cm and  $R = 1000$  cm give  $Rv_i/v_i = 5 \times 10^{-4}$  and  $\rho_i/a = 3 \times 10^{-3}$ , so the resulting inequality is typically satisfied.

The basics of drift kinetics in stellarator geometry are sketched in the next section.

### 3. Drift kinetics in a stellarator

To retain both the  $1/v$  and  $\sqrt{v}$  regimes, the ion drift kinetic equation is employed (Hazeltine 1973; Simakov & Catto 2005),

$$v_{\parallel} \bar{b} \cdot \nabla (f_1 + \frac{Ze\phi f_0}{T}) + \bar{v}_d \cdot \nabla \alpha \frac{\partial}{\partial \alpha} (f_1 + \frac{Ze\phi f_0}{T}) + \bar{v}_d \cdot \nabla f_0 = C\{f_1\}, \quad (3.1)$$

where  $f_1$  is the perturbed distribution function,  $C\{f_1\}$  is the linearized collision operator,  $f_0$  is the Maxwellian

$$f_0 = f_0(\psi, E) = n(\psi) \left[ \frac{M_i}{2\pi T(\psi)} \right]^{3/2} e^{-M_i v^2 / 2T(\psi)} = n(\psi) \left[ \frac{M_i}{2\pi T(\psi)} \right]^{3/2} e^{-[M_i E - Ze\Phi(\psi)] / T(\psi)}, \quad (3.2)$$

and the drift velocity (ignoring the unimportant parallel velocity correction) is

$$\bar{v}_d = \frac{c}{B^2} \bar{B} \times \nabla \Phi + \frac{\mu}{\Omega} \bar{b} \times \nabla B + \frac{v_{\parallel}^2}{\Omega} \bar{b} \times (\bar{b} \cdot \nabla \bar{b}) \approx \frac{v_{\parallel}}{\Omega} \nabla \times (v_{\parallel} \bar{b}). \quad (3.3)$$

The kinetic equation is written in  $\psi$ ,  $\alpha = \zeta - q\vartheta$ ,  $B$ ,  $E = v^2/2 + Ze\Phi(\psi)/M_i$ , and  $\mu = v_{\perp}^2/2B$  variables, with the lowest order electrostatic potential  $\Phi$  taken as a flux function. The remaining spatial variables are  $\vartheta$  and  $\zeta$ , the poloidal and toroidal angle variables, respectively, and the full electrostatic electric field is

$$\bar{E} = -\nabla[\Phi(\psi) + \phi(\psi, \vartheta, \zeta)]. \quad (3.4)$$

The magnetic field is  $\bar{B} = B\bar{b}$  and the parallel velocity is  $v_{\parallel} = \sigma v \xi$ , with  $\xi = \sqrt{1 - \lambda B/B_0}$ , pitch angle defined as  $\lambda = 2\mu B_0/v^2$ ,  $\sigma = \pm 1$ , and  $B_0$  a flux function ( $B_0^2 = \langle B^2 \rangle$ ). Here and throughout  $\partial f_0/\partial \psi$  is evaluated holding  $E$  fixed.

Using Boozer (1981) and Clebsch representations for the magnetic field gives

$$\bar{B} = B\bar{b} = \nabla \alpha \times \nabla \psi = K(\psi, \vartheta, \zeta) \nabla \psi + G(\psi) \nabla \vartheta + I(\psi) \nabla \zeta, \quad (3.5)$$

with  $K$  periodic in  $\vartheta$  and  $\zeta$ , and  $RG/rI \sim B_p/B$ . The preceding give  $B = (G + qI)\bar{b} \cdot \nabla \vartheta$ , as well as  $\bar{B} \cdot \nabla \alpha = 0 = \bar{B} \cdot \nabla \psi$  and  $q\bar{B} \cdot \nabla \vartheta = q\nabla \alpha \times \nabla \psi \cdot \nabla \vartheta = q\nabla \psi \times \nabla \vartheta \cdot \nabla \zeta = \bar{B} \cdot \nabla \zeta$ .

The well depth  $\delta$  of the omnigenous magnetic field is

$$\delta = (B_{\max} - B_{\min}) / (B_{\max} + B_{\min}). \quad (3.6)$$

Here  $B_{\max}$  and  $B_{\min}$  denote the maximum and minimum field strength on the magnetic surface in question. As shown by Cary and Shasharina (1997a), in an omnigenous field these values are reached not in isolated points but along lines that close toroidally, poloidally or helically on the flux surface. Any departure from omnigenicity is assumed to result in smaller non-omnigenous well depths of

$$\delta_{\text{no}} \ll \delta. \quad (3.7)$$

The omnigenous portion of the magnetic field is assumed to be  $N$  toroidal cells ( $N = 0$  is quasiaxisymmetry) of well depth  $\delta$ , with mod  $B$  contours closing on themselves after  $M$  toroidal turns and  $N$  poloidal turns. In  $\psi$ ,  $\alpha$ , and  $B$  variables  $d\vartheta d\zeta = (\vec{b} \cdot \nabla \vartheta / \vec{b} \cdot \nabla B) dB d\alpha$ . When performing flux surface averages it is convenient to be aware of the periodic variable

$$\tilde{\alpha} = \alpha / (M - qN) \quad (3.8)$$

since  $\tilde{\alpha} \rightarrow 2\pi + \tilde{\alpha} = \tilde{\alpha}$  for  $M$  toroidal circuits and  $N$  poloidal circuits on a constant  $B$  curve. As there are two points on either side of the  $B$  minimum,  $\vec{B}$ , in each of the  $N$  cells, both sides must be summed or integrated over and are referred to as branches (Landreman & Catto 2012).

In  $\psi$ ,  $\alpha$ ,  $B$  variables the divergence of an arbitrary vector  $\vec{A}$  is

$$\nabla \cdot \vec{A} = \vec{B} \cdot \nabla \left( \frac{\vec{A} \cdot \nabla B}{\vec{B} \cdot \nabla B} \right) + \vec{B} \cdot \nabla B \left[ \frac{\partial}{\partial \alpha} \left( \frac{\vec{A} \cdot \nabla \alpha}{\vec{B} \cdot \nabla B} \right) + \frac{\partial}{\partial \psi} \left( \frac{\vec{A} \cdot \nabla \psi}{\vec{B} \cdot \nabla B} \right) \right]. \quad (3.9)$$

The tangential drift is then

$$\vec{v}_d \cdot \nabla \alpha = v_{\parallel} \vec{b} \cdot \nabla \left( \frac{v_{\parallel} \vec{B} \cdot \nabla \alpha \times \nabla B}{\Omega \vec{B} \cdot \nabla B} \right) + v_{\parallel} \vec{b} \cdot \nabla B \frac{\partial}{\partial \psi} \left( \frac{v_{\parallel} B^2}{\Omega \vec{B} \cdot \nabla B} \right) \simeq \frac{B v_{\parallel}}{\Omega} \frac{\partial v_{\parallel}}{\partial \psi} \simeq -c \frac{\partial \Phi}{\partial \psi} \equiv -\omega(\psi), \quad (3.10)$$

with  $\omega$  positive for the ion root, and the radial drift is

$$\vec{v}_d \cdot \nabla \psi = v_{\parallel} \vec{b} \cdot \nabla \left( \frac{v_{\parallel} \vec{B} \cdot \nabla \psi \times \nabla B}{\Omega \vec{B} \cdot \nabla B} \right) - \frac{B v_{\parallel}^2}{\Omega} \vec{b} \cdot \nabla B \frac{\partial}{\partial \alpha} \left( \frac{1}{\vec{b} \cdot \nabla B} \right). \quad (3.11)$$

Using pressure balance to find  $\vec{b} \times \nabla \psi \cdot (\nabla B - \vec{b} \cdot \nabla \vec{b}) = 0$  as in Helander & Nührenberg (2009) and Landreman & Catto (2012), gives the alternate radial drift form

$$\vec{v}_d \cdot \nabla \psi = \frac{\vec{B} \cdot \nabla \psi \times \nabla B}{\vec{B} \cdot \nabla B} v_{\parallel} \vec{b} \cdot \nabla \left( \frac{v_{\parallel}}{\Omega} \right), \quad (3.12)$$

where for a quasisymmetric flux surface with  $B_{qs} = B_0 [1 - \delta \cos(M\vartheta - N\zeta)]$ ,

$$\frac{\vec{B} \cdot \nabla \psi \times \nabla B}{\vec{B} \cdot \nabla B} \rightarrow \frac{MI + NG}{M - qN} \lesssim RB, \quad (3.13)$$

with the tokamak and quasiaxisymmetric cases given by  $N = 0$ .

The transit average of an arbitray function  $A$  is defined variously as

$$\bar{A} = \frac{\oint_{\alpha} d\ell A / v_{\parallel}}{\oint_{\alpha} d\ell / v_{\parallel}} = \frac{\oint_{\alpha} d\tau A}{\oint_{\alpha} d\tau} = \frac{\oint_{\alpha} d\vartheta A / v_{\parallel} \vec{b} \cdot \nabla \vartheta}{\oint_{\alpha} d\vartheta / v_{\parallel} \vec{b} \cdot \nabla \vartheta} = \frac{\Sigma \oint_{\alpha} dB A / v_{\parallel} \vec{b} \cdot \nabla B}{\Sigma \oint_{\alpha} dB / v_{\parallel} \vec{b} \cdot \nabla B}, \quad (3.16)$$

with  $d\tau = d\ell / v_{\parallel} = d\vartheta / v_{\parallel} \vec{b} \cdot \nabla \vartheta = dB / v_{\parallel} \vec{b} \cdot \nabla B$ . The integrals are over a full bounce for trapped particles and over a complete poloidal circuit for passing. A sum  $\Sigma$  is inserted in integrals over all allowed  $B$  as a reminder that both "branches" or sides of the magnetic well must be integrated over (the two "branches" are on either side of the minimum  $\vec{B}$  in each identical stellarator cell). The subscript  $\alpha$  on an integral means that it is to be held fixed while performing the integration. In a transit averaged description passing particles trace out flux surfaces, and it can take a passing particle many toroidal circuits in a non-omnigenous magnetic field to return to close to its starting field line point (Helander 2014). In particular, the passing flux surface average of  $v_{\parallel}$  can be viewed as

$$\langle v_{\parallel} \rangle_p = \lim_{L \rightarrow \infty} \left( \int_0^L d\ell v_{\parallel} B^{-1} / \int_0^L d\ell B^{-1} \right) = \left( \oint d\alpha \Sigma \oint dB v_{\parallel} / \bar{\mathbf{B}} \cdot \nabla B \right) / \left( \oint d\alpha \Sigma \oint dB / \bar{\mathbf{B}} \cdot \nabla B \right). \quad (3.17)$$

This expression is consistent with the usual definition of the flux surface average of any quantity  $A$  sampling the entire flux surface:

$$\langle A \rangle = \frac{\oint d\zeta \oint d\vartheta A / \bar{\mathbf{B}} \cdot \nabla \vartheta}{\oint d\zeta \oint d\vartheta / \bar{\mathbf{B}} \cdot \nabla \vartheta} = \frac{\oint d\zeta \oint d\vartheta A / B^2}{\oint d\zeta \oint d\vartheta / B^2} = \frac{\oint d\alpha \Sigma \oint dB A / \bar{\mathbf{B}} \cdot \nabla B}{\oint d\alpha \Sigma \oint dB / \bar{\mathbf{B}} \cdot \nabla B}. \quad (3.18)$$

A few useful properties associated with a MHD equilibrium are briefly discussed next.

#### 4. MHD equilibrium properties and radial drift

Some useful relations follow from conditions associated with force balance and ambipolarity. Force balance ( $\bar{\mathbf{J}} \times \bar{\mathbf{B}} = c \nabla p$ ) and charge conservation ( $\nabla \cdot \bar{\mathbf{J}} = 0$ ) require there be no radial current ( $\bar{\mathbf{J}} \cdot \nabla \psi = 0$ ), where  $p = p_i + p_e$  is the total pressure and the current density is  $\bar{\mathbf{J}} = c B^{-2} \bar{\mathbf{B}} \times \nabla p(\psi) + J_{\parallel} \bar{\mathbf{b}}$ . Ambipolarity,  $\bar{\mathbf{J}} \cdot \nabla \psi = 0$ , requires  $\nabla \psi \cdot \nabla \times \bar{\mathbf{B}} = 0 = \nabla \cdot (\bar{\mathbf{B}} \times \nabla \psi)$ , giving

$$\bar{\mathbf{B}} \cdot \nabla \left( \frac{\bar{\mathbf{B}} \cdot \nabla \psi \times \nabla B}{\bar{\mathbf{B}} \cdot \nabla B} \right) = \bar{\mathbf{B}} \cdot \nabla B \frac{\partial}{\partial \alpha} \left( \frac{B^2}{\bar{\mathbf{B}} \cdot \nabla B} \right), \quad (4.1)$$

Integrating along a field line (fixed  $\alpha$ ) on the flux surface traced out by the field line gives

$$\frac{\bar{\mathbf{B}} \cdot \nabla \psi \times \nabla B}{\bar{\mathbf{B}} \cdot \nabla B} - \frac{\partial}{\partial \alpha} \int_{B_0}^B \frac{dB' B'}{\bar{\mathbf{b}}' \cdot \nabla B'} = X(\psi) = \left\langle \frac{\bar{\mathbf{B}} \cdot \nabla \psi \times \nabla B}{\bar{\mathbf{B}} \cdot \nabla B} \right\rangle - \left\langle \frac{\partial}{\partial \alpha} \int_{B_0}^B \frac{dB' B'}{\bar{\mathbf{b}}' \cdot \nabla B'} \right\rangle, \quad (4.2)$$

where  $B_0^2 = \langle B^2 \rangle$  is chosen so that the lower limit will satisfy  $\bar{\mathbf{b}} \cdot \nabla B_0 = 0$  as desired. The  $\alpha$  derivatives are always at fixed  $B$  unless otherwise noted, and  $\partial B_0 / \partial \alpha = 0$ . In an omnigenous system the lower limit  $B_0$  of the integral over  $B$  can be replaced by another  $\alpha$  independent value such as  $\hat{B} = B_{\max}$  or  $\check{B} = B_{\min}$ . The last equality in (4.2), follows upon flux surface averaging.

As a result, ambipolarity always requires

$$\frac{\bar{\mathbf{B}} \cdot \nabla \psi \times \nabla B}{\bar{\mathbf{B}} \cdot \nabla B} = \left\langle \frac{\bar{\mathbf{B}} \cdot \nabla \psi \times \nabla B}{\bar{\mathbf{B}} \cdot \nabla B} \right\rangle + \frac{\partial F_0}{\partial \alpha} - \left\langle \frac{\partial F_0}{\partial \alpha} \right\rangle, \quad (4.3)$$

where the integrals over  $B$  are at fixed  $\alpha$ , and  $F_0$  is defined as

$$F_0 = \int_{B_0}^B \frac{dB' B'}{\bar{\mathbf{b}}' \cdot \nabla B'}. \quad (4.4)$$

The flux surface average  $\langle \partial F_0 / \partial \alpha \rangle$  vanishes for an omnigenous flux surface thanks to Eq. (5.13) below.

In Eqs. (4.2) and (4.4), the integrals are taken at fixed  $\psi$  and  $\alpha$  (therefore following a field line) starting from a point where  $B' = B_0$  and proceeding along the field to the point where  $B' = B$ . As already mentioned, in an omnigenous field, all level curves  $B = B_0$  wrap around the torus poloidally, toroidally, or helically (Cary & Shasharina 1997a), so that it is indeed possible to find a point where  $B = B_0$  on each field line within one period of the device. The field is allowed to be non-omnigenous, but only slightly so, in order that it still possesses this property. Otherwise it would be difficult to ascribe any precise meaning to these integrals.

Force balance and charge conservation, require a geometrical function  $U$  proportional to  $J_{\parallel}$  to exist satisfying

$$\vec{B} \cdot \nabla(U/B) = \vec{B} \cdot \nabla\psi \times \nabla B^{-2} = \nabla \cdot (B^{-2} \vec{B} \times \nabla\psi). \quad (4.5)$$

Rewriting using the divergence in  $\psi, \alpha$  and  $B$  variables,

$$\vec{B} \cdot \nabla \left( \frac{U}{B} - \frac{\vec{B} \cdot \nabla\psi \times \nabla B}{B^2 \vec{B} \cdot \nabla B} \right) = -\vec{b} \cdot \nabla B \frac{\partial}{\partial \alpha} \left( \frac{1}{\vec{b} \cdot \nabla B} \right). \quad (4.6)$$

Using the same procedure to integrate over the flux surface traced out by a field line leads to

$$BU - \frac{\vec{B} \cdot \nabla\psi \times \nabla B}{\vec{B} \cdot \nabla B} + B^2 \frac{\partial}{\partial \alpha} \int_{B_0}^B \frac{dB'}{\vec{B}' \cdot \nabla B'} = Y(\psi) B^2. \quad (4.7)$$

Flux surface averaging gives  $Y$

$$Y(\psi) \langle B^2 \rangle = \langle BU \rangle - \left\langle \frac{\vec{B} \cdot \nabla\psi \times \nabla B}{\vec{B} \cdot \nabla B} \right\rangle + \left\langle B^2 \frac{\partial}{\partial \alpha} \int_{B_0}^B \frac{dB'}{\vec{B}' \cdot \nabla B'} \right\rangle. \quad (4.8)$$

Eliminating  $Y$  yields

$$\frac{U}{B} - \frac{\langle BU \rangle}{\langle B^2 \rangle} = \left( \frac{\vec{B} \cdot \nabla\psi \times \nabla B}{B^2 \vec{B} \cdot \nabla B} - \frac{\partial}{\partial \alpha} \int_{B_0}^B \frac{dB'}{\vec{B}' \cdot \nabla B'} \right) - \frac{1}{\langle B^2 \rangle} \left( \left\langle \frac{\vec{B} \cdot \nabla\psi \times \nabla B}{\vec{B} \cdot \nabla B} \right\rangle - \left\langle B^2 \frac{\partial}{\partial \alpha} \int_{B_0}^B \frac{dB'}{\vec{B}' \cdot \nabla B'} \right\rangle \right). \quad (4.9)$$

Terms in  $U/B$  that are only flux functions do not contribute to  $\vec{B} \cdot \nabla(U/B)$ . Therefore, picking

$$\langle BU \rangle = \left\langle \frac{\vec{B} \cdot \nabla\psi \times \nabla B}{\vec{B} \cdot \nabla B} \right\rangle - \left\langle B^2 \frac{\partial}{\partial \alpha} \int_{B_0}^B \frac{dB'}{\vec{B}' \cdot \nabla B'} \right\rangle, \quad (4.10)$$

(where the last term vanishes for an omnigenous flux surface) leaves

$$BU = \frac{\vec{B} \cdot \nabla\psi \times \nabla B}{\vec{B} \cdot \nabla B} - B^2 \frac{\partial}{\partial \alpha} \int_{B_0}^B \frac{dB'}{\vec{B}' \cdot \nabla B'} = \left\langle \frac{\vec{B} \cdot \nabla\psi \times \nabla B}{\vec{B} \cdot \nabla B} \right\rangle - \left\langle \frac{\partial F_0}{\partial \alpha} \right\rangle + \frac{\partial}{\partial \alpha} \int_{B_0}^B \frac{dB' (B'^2 - B^2)}{\vec{B}' \cdot \nabla B'}, \quad (4.11)$$

and the last form follows by using ambipolarity. As a result,  $BU$  is a flux function for a quasisymmetric flux surface, or when summed over both branches on an omnigenous surface.

Otherwise,  $BU$  is nearly a flux function when  $B^2 \gg \widehat{B}^2 - \widetilde{B}^2$ , where  $\widehat{B} = B_{\max}$  and  $\widetilde{B} = B_{\min}$ .

The next section formulates the drift kinetic equation in a particularly convenient form.

## 5. Drift kinetic equations for imperfectly optimized stellarators

To find the modifications to the Maxwellian it is convenient to let

$$f_1 = h(\vec{r}, \vec{v}) - Ze\phi_0/T, \quad (5.1)$$

then using form (3.12) for the radial drift,  $h$  satisfies

$$v_{\parallel} \vec{b} \cdot \nabla h - \omega \frac{\partial h}{\partial \alpha} - C\{h\} = - \frac{\vec{B} \cdot \nabla\psi \times \nabla B}{\vec{B} \cdot \nabla B} \frac{\partial f_0}{\partial \psi} v_{\parallel} \vec{b} \cdot \nabla \left( \frac{v_{\parallel}}{\Omega} \right). \quad (5.2)$$

Rewriting the preceding using (4.1) gives the form

$$v_{\parallel} \vec{b} \cdot \nabla \left( h + \frac{v_{\parallel} \vec{B} \cdot \nabla\psi \times \nabla B}{\Omega \vec{B} \cdot \nabla B} \frac{\partial f_0}{\partial \psi} \right) - \omega \frac{\partial h}{\partial \alpha} - C\{h\} = \frac{B \partial f_0}{\Omega \partial \psi} v_{\parallel} \vec{b} \cdot \nabla B \frac{\partial}{\partial \alpha} \left( \frac{v_{\parallel}}{\vec{b} \cdot \nabla B} \right), \quad (5.3)$$

and, thereby, the particularly convenient form

$$\omega \frac{\partial h}{\partial \alpha} + C\{h\} = v_{\parallel} \vec{b} \cdot \nabla H = v_{\parallel} \vec{b} \cdot \nabla h + v_{\parallel} \vec{b} \cdot \nabla \Delta \frac{\partial f_0}{\partial \psi} = v_{\parallel} \vec{b} \cdot \nabla h + \vec{v}_d \cdot \nabla \psi \frac{\partial f_0}{\partial \psi}, \quad (5.4)$$

where

$$H \equiv h + \Delta \partial f_0 / \partial \psi, \quad (5.5)$$

$$\Delta \equiv \frac{v_{\parallel} \bar{\mathbf{B}} \cdot \nabla \psi \times \nabla \mathbf{B}}{\Omega \bar{\mathbf{B}} \cdot \nabla \mathbf{B}} - \frac{\partial \tilde{\mathcal{J}}}{\partial \alpha} = \frac{v_{\parallel}}{\Omega} \left( \left\langle \frac{\bar{\mathbf{B}} \cdot \nabla \psi \times \nabla \mathbf{B}}{\bar{\mathbf{B}} \cdot \nabla \mathbf{B}} \right\rangle - \left\langle \frac{\partial F_0}{\partial \alpha} \right\rangle + \frac{\partial F_0}{\partial \alpha} \right) - \frac{\partial \tilde{\mathcal{J}}}{\partial \alpha}, \quad (5.6)$$

and

$$\tilde{\mathcal{J}} \equiv \frac{Mc}{Ze} \int_{\bar{\mathbf{B}}}^{\mathbf{B}} \frac{dB' v'_{\parallel}}{\bar{\mathbf{b}}' \cdot \nabla \mathbf{B}'}, \quad (5.7)$$

with  $v'_{\parallel} = v_{\parallel}(\mathbf{B}', v, \lambda)$ . The function  $\tilde{\mathcal{J}}$  is related to, but not the same as, the second adiabatic invariant

$$J \equiv (Mc/Ze) \oint_{\alpha} d\ell v_{\parallel}, \quad (5.8)$$

for which  $\partial J / \partial \alpha = 0$  on an omnigenous flux surface. The lower limit  $\bar{\mathbf{B}}$  in  $\tilde{\mathcal{J}}$  remains unspecified except that it must satisfy  $\partial \bar{\mathbf{B}} / \partial \alpha = 0$  to allow

$$\left( \int_{\bar{\mathbf{B}}}^{\mathbf{B}} dB' v'_{\parallel} \frac{\partial}{\partial \alpha} \left( \frac{1}{\bar{\mathbf{b}}' \cdot \nabla \mathbf{B}'} \right) \right) = \frac{\partial}{\partial \alpha} \left( \int_{\bar{\mathbf{B}}}^{\mathbf{B}} \frac{dB' v'_{\parallel}}{\bar{\mathbf{b}}' \cdot \nabla \mathbf{B}'} \right). \quad (5.9)$$

The kinetic equation (5.4) agrees with the omnigenous form obtained from (3.12) by Helander & Nührenberg (2009) and Landreman & Catto (2012) for  $\omega = 0$ . They also write

$$\bar{\mathbf{v}}_{\perp} \cdot \nabla \psi = v_{\parallel} \bar{\mathbf{b}} \cdot \nabla \Delta, \quad (5.10)$$

but make the replacement

$$\Delta \rightarrow \Delta_{\text{om}} \equiv \frac{v_{\parallel}}{\Omega} \left\langle \frac{\bar{\mathbf{B}} \cdot \nabla \psi \times \nabla \mathbf{B}}{\bar{\mathbf{B}} \cdot \nabla \mathbf{B}} \right\rangle + \int_{\bar{\mathbf{B}}}^{\mathbf{B}} dB' \frac{\partial F_0'}{\partial \alpha} \frac{\partial}{\partial B'} \left( \frac{v'_{\parallel}}{\Omega'} \right), \quad (5.11)$$

where  $\bar{\mathbf{B}} = \hat{\mathbf{B}} = B_{\text{max}}$  when  $\lambda < B_0 / \hat{\mathbf{B}}$ , and  $\bar{\mathbf{B}} = B_0 / \lambda$  otherwise, as their choice of the lower limit. To see the results here are consistent with their omnigenous results, integrate by parts to obtain

$$\frac{v_{\parallel}}{\Omega} \frac{\partial F_0}{\partial \alpha} - \frac{\partial \tilde{\mathcal{J}}}{\partial \alpha} = \int_{\bar{\mathbf{B}}}^{\mathbf{B}} dB' \frac{\partial F_0'}{\partial \alpha} \frac{\partial}{\partial B'} \left( \frac{v'_{\parallel}}{\Omega'} \right) = - \frac{Bv}{2\Omega} \int_{\bar{\mathbf{B}}}^{\mathbf{B}} \frac{dB' \partial F_0'}{\partial \alpha} \frac{2 - \lambda B' / B_0}{B'^2 \sqrt{1 - \lambda B' / B_0}}, \quad (5.12)$$

where  $v'_{\parallel} = v_{\parallel}(\mathbf{B}' = \bar{\mathbf{B}}, v, \lambda) = 0$  is needed. For an omnigenous flux surface the local condition

$$\frac{\partial}{\partial \alpha} \Sigma \frac{1}{|\bar{\mathbf{b}} \cdot \nabla \mathbf{B}|} = 0, \quad (5.13)$$

must be satisfied, where  $\Sigma$  denotes a sum over both branches. This condition, first found by Cary & Shasharina (1997a,b), requires that the sum of the incremental lengths associated with the same  $\mathbf{B}$  on the two connected branches must be alpha independent [for a more detailed explanation see Helander (2014) following Eq. (79)]. Therefore, integrating over both branches on either side of the the single minimum  $\bar{\mathbf{B}} = B_{\text{min}}$  gives  $\langle \partial F_0 / \partial \alpha \rangle = 0$ . Moreover, omnigenicity requires  $\Sigma \partial \tilde{\mathcal{J}} / \partial \alpha = 0$ .

The omnigenous limit is recovered only if the lower limits used by Helander & Nührenberg (2009) and Landreman & Catto (2012) can be employed. The alternate form given here only reduces to their form (5.10) and (5.11) for an omnigenous flux surface. More generally,  $\Delta$  must satisfy (5.6) with limits nearly equal to these omnigenous ones.

In the next section the corrections to the Maxwellian are evaluated for a non-optimized stellarator by assuming that the population of transitional particles is small.

## 6. Solution for a weak departure from omnigenity

The drift kinetic equation will be solved separately for the trapped (t) and passing (p), using subscripts t and p as necessary for clarity. A solution to the passing kinetic equation (5.4) is obtained by recalling (5.5), writing

$$H_p = \bar{H}_p + \tilde{H}_p, \quad (6.1)$$

where  $\bar{H}_p \gg \tilde{H}_p$ . Assuming streaming dominates, then to lowest order

$$v_{\parallel} \bar{\mathbf{b}} \cdot \nabla \bar{H}_p = 0, \quad (6.2)$$

where

$$\bar{H}_p = \bar{H}_p(\psi, v, \lambda, \sigma). \quad (6.3)$$

To next order

$$v_{\parallel} \bar{\mathbf{b}} \cdot \nabla \tilde{H}_p = \omega \frac{\partial h_p}{\partial \alpha} + C\{h_p\}, \quad (6.4)$$

and, upon transit averaging,

$$\overline{\omega \frac{\partial h_p}{\partial \alpha} + C\{h_p\}} = 0, \quad (6.5)$$

where to lowest order

$$h_p = \bar{H}_p - \Delta_p \partial f_0 / \partial \psi. \quad (6.6)$$

Using  $\bar{\mathbf{B}} \cdot \nabla \psi \times \nabla \mathbf{B} / \bar{\mathbf{B}} \cdot \nabla \mathbf{B} \sim BU \lesssim RB$ , the transit averaged equation suggests that for the ions

$$\frac{\bar{H}_p}{f_0} \sim \frac{\Delta_p}{RB_p a} \lesssim \frac{\rho_{pi}}{a}, \quad (6.7)$$

while the equation non-averaged equation gives

$$\frac{\tilde{H}_p}{f_0} \sim \frac{(\omega + v_i)R}{v_i} \frac{\bar{H}_p}{f_0} \lesssim \frac{(\omega + v_i)R \rho_{pi}}{v_i a}, \quad (6.8)$$

where  $(\omega + v_i)R \ll v_i$ . Here and elsewhere,  $\rho_{pi} = \rho_i B / B_p$ , and  $\rho_i$  the ion gyroradius and  $B_p$  the poloidal magnetic field. As  $\bar{H}_p$  and  $\Delta_p$  are odd in  $v_{\parallel}$  in (6.6),  $\tilde{H}_p$  from (6.4) will be even in  $v_{\parallel}$ .

The relation between transit and flux surface averages for the passing,

$$\bar{Q}_p = \langle BQ / v_{\parallel} \rangle_p / \langle B / v_{\parallel} \rangle_p, \quad (6.9)$$

is useful, along with (6.5), to obtain

$$\left\langle \frac{B}{v_{\parallel}} C\{h_p\} \right\rangle = \omega \frac{\partial f_0}{\partial \psi} \left\langle \frac{B \partial \Delta_p}{v_{\parallel} \partial \alpha} \right\rangle = \omega \frac{\partial f_0}{\partial \psi} \left[ \frac{B}{\Omega} \left\langle \frac{\partial^2 F_0}{\partial \alpha^2} \right\rangle - \left\langle \frac{B \partial^2 \tilde{J}}{v_{\parallel} \partial \alpha^2} \right\rangle \right]_p. \quad (6.10)$$

As the departure from omnigenity is assumed weak ( $\delta_{no} \ll \delta$ ) and the tangential drift  $\omega$  and collision terms are small, it is tempting to perform the flux surface averages by assuming the flux surface is omnigenous to lowest order to make  $\langle B v_{\parallel}^{-1} \partial \Delta_p / \partial \alpha \rangle$  vanish. However, as  $\bar{\mathbf{v}}_d \cdot \nabla h_p$  in the drift kinetic equation has been approximated by  $-\omega(\psi) \partial h_p / \partial \alpha \approx \omega(\psi) (\partial \Delta_p / \partial \alpha) \partial f_0 / \partial \psi$ , as in most numerical simulations, integrating the right side of (6.10) by parts in  $\alpha$ , results in B integrals of  $\alpha$  derivatives of the  $\bar{\mathbf{b}} \cdot \nabla \mathbf{B}$  from the flux surface averages. Therefore, any departure

from omnigenity gives  $\langle \mathbf{B}v_{\parallel}^{-1}\partial\Delta_p/\partial\alpha \rangle \neq 0$ , which will contribute at very low collisionalities as discussed further in section 8.

A more careful drift kinetic treatment retaining the full nonlocal behavior of the drift departure from a flux surface results in the more precise replacement

$$\begin{aligned} \omega \frac{\partial f_0}{\partial \psi} \left\langle \frac{\mathbf{B} \partial \Delta_p}{v_{\parallel} \partial \alpha} \right\rangle &\Rightarrow \left\langle \frac{\mathbf{B}}{v_{\parallel}} \bar{\mathbf{v}}_d \cdot \nabla h_p \right\rangle = \left\langle \nabla \cdot \left( \frac{\mathbf{B}}{v_{\parallel}} \bar{\mathbf{v}}_d h_p \right) \right\rangle \approx \frac{1}{V'} \frac{\partial}{\partial \psi} V' \left[ \left\langle (\bar{H}_p - \Delta_p) \frac{\partial f_0}{\partial \psi} \frac{\mathbf{B}}{v_{\parallel}} \bar{\mathbf{v}}_d \cdot \nabla \psi \right\rangle \right], \\ &= \frac{1}{V'} \frac{\partial}{\partial \psi} V' \left[ \left\langle (\bar{H}_p - \Delta_p) \frac{\partial f_0}{\partial \psi} \right\rangle \bar{\mathbf{B}} \cdot \nabla \Delta_p \right] = \frac{1}{V'} \frac{\partial}{\partial \psi} V' \left[ \bar{H}_p \langle \bar{\mathbf{B}} \cdot \nabla \Delta_p \rangle - \frac{\partial f_0}{\partial \psi} \bar{\mathbf{B}} \cdot \nabla \left( \frac{\Delta_p^2}{2} \right) \right] = 0 \end{aligned} \quad (6.11)$$

$V' = \oint d\zeta \oint d\vartheta / B^2$ . Consequently, the right side of (6.10) does indeed vanish! However, most simulations do not use this more precise treatment and therefore retain effects associated with solving (6.10). These effects are considered in detail in section 8 to investigate their effects on flow and the bootstrap current.

Here and in the next section, the more accurate constraint equation is solved by taking the right side of (6.10) to vanish and considering

$$0 = \left\langle \frac{\mathbf{B}}{v_{\parallel}} C \left\{ \bar{H}_p - \frac{v_{\parallel}}{\Omega} \left\langle \frac{\bar{\mathbf{B}} \cdot \nabla \psi \times \nabla \mathbf{B}}{\Omega \bar{\mathbf{B}} \cdot \nabla \mathbf{B}} \right\rangle \frac{\partial f_0}{\partial \psi} \right\} \right\rangle = \left\langle \frac{\mathbf{B}}{v_{\parallel}} C \left\{ \bar{H}_p - \frac{v_{\parallel} f_0}{\Omega T} \left( \frac{M_i v^2}{2T} - \frac{5}{2} \right) \left\langle \frac{\bar{\mathbf{B}} \cdot \nabla \psi \times \nabla \mathbf{B}}{\Omega \bar{\mathbf{B}} \cdot \nabla \mathbf{B}} \right\rangle \frac{\partial T}{\partial \psi} \right\} \right\rangle, \quad (6.12)$$

where conservation of momentum for ion-ion collisions,  $C\{v_{\parallel} f_0\} = 0$ , is employed to obtain the final form. Within a geometric factor the preceding constraint equation is the same found as for a tokamak and for an omnigenous flux surface (Landreman & Catto, 2012).

To form the parallel flow and parallel current density the trapped equation must also be solved. In this case, the kinetic equation must be written in a form with the  $\Delta$  terms appearing in the large parallel streaming term, for which a non-omnigenous flux surface yields

$$\overline{\bar{\mathbf{v}}_d \cdot \nabla \psi} = \overline{v_{\parallel} \bar{\mathbf{b}} \cdot \nabla \Delta_t} \neq 0, \quad (6.13)$$

so  $\overline{v_{\parallel} \bar{\mathbf{b}} \cdot \nabla \Delta_t}$  must be retained. Therefore, using

$$H_t = h_t + \Delta_t \partial f_0 / \partial \psi, \quad (6.14)$$

gives

$$\bar{H}_t = \bar{h}_t + \bar{\Delta}_t \partial f_0 / \partial \psi, \quad (6.15)$$

(and not  $\bar{H}_t = \bar{h}_t$ ), where both  $\bar{H}_t = \bar{H}_t(\psi, \alpha, v, \lambda)$  and  $\bar{h}_t = \bar{h}_t(\psi, \alpha, v, \lambda)$  are even functions of  $v_{\parallel}$ , with  $\bar{\mathbf{b}} \cdot \nabla \bar{H}_t = 0 = \bar{\mathbf{b}} \cdot \nabla \bar{h}_t$ ,  $h_t - \bar{h}_t = \tilde{h}_t$ , and  $H_t - \bar{H}_t = \tilde{H}_t$ . Moreover,

$$\tilde{H}_t = \tilde{h}_t + (\Delta_t - \bar{\Delta}_t) \partial f_0 / \partial \psi, \quad (6.16)$$

where

$$\tilde{h}_t \sim \Delta_t \partial f_0 / \partial \psi \lesssim \sqrt{\delta} \rho_{pi} f_0 / a. \quad (6.17)$$

The trapped equation,

$$\omega \partial h_t / \partial \alpha + C\{h_t\} = v_{\parallel} \bar{\mathbf{b}} \cdot \nabla \tilde{H}_t = v_{\parallel} \bar{\mathbf{b}} \cdot \nabla (h_t + \Delta_t \partial f_0 / \partial \psi), \quad (6.18)$$

can be transit averaged to obtain the combined  $\sqrt{v}$  and  $1/v$  form

$$\omega \partial \bar{h}_t / \partial \alpha + \overline{C\{\bar{h}_t\}} = \overline{v_{\parallel} \bar{\mathbf{b}} \cdot \nabla \Delta_t \partial f_0 / \partial \psi}, \quad (6.19)$$

where in the small tangential drift and collision terms on the left the lowest order omnigenous trapped orbits are used in the averages so  $\tilde{h}_t$  does not contribute. The assumption that streaming dominates means  $(\omega + \nu) \ll v_{\parallel}/qR \sim v_{\parallel}/R$ . As a result, the function  $\bar{h}_t$  is even in  $v_{\parallel}$  and of order

$$\frac{\bar{h}_t}{f_0} \lesssim \frac{\delta_{\text{no}} \delta \rho_i v_i B}{(\omega + \nu_i) R B_p a} \sim \frac{\delta_{\text{no}} \delta v_i \rho_{pi}}{(\omega + \nu_i) R a} \ll 1, \quad (6.20)$$

where  $\overline{v_{\parallel} \bar{b} \cdot \nabla \Delta_t} \sim \delta_{\text{no}} v_{\parallel} \Delta / R$  and  $v_{\parallel}^2 \sim \delta v_i^2$ , with  $\Delta \lesssim v_{\parallel} R B / \Omega$  as (3.13) is used. The unaveraged form of the trapped equation (6.18) suggests the estimate

$$\frac{\tilde{H}_t}{\tilde{h}_t} \sim \frac{(\omega + \nu_i) R}{v_i \sqrt{\delta}} \ll 1, \quad (6.21)$$

so that  $\tilde{H}_t$  can be neglected as small in (6.16), giving

$$\tilde{h}_t = -(\Delta_t - \bar{\Delta}_t) \partial f_0 / \partial \psi \simeq -\Delta_t \partial f_0 / \partial \psi, \quad (6.22)$$

as  $\bar{\Delta}_t$  is small. Notice that  $\bar{h}_t \sim \tilde{h}_t$  is allowed when

$$\delta_{\text{no}} \sqrt{\delta} \sim \frac{(\omega + \nu_i) R}{v_i} \sim \frac{\rho_{pi}}{a} + \frac{R}{\lambda} \ll 1, \quad (6.23)$$

where the mean free path is  $\lambda \sim v_i / \nu_i \gg R$  and  $v_i \gg \omega R$ , but  $\delta_{\text{no}} \ll \delta$ .

Just as the departure from an omnigenous flux surface makes  $\overline{v_{\parallel} \bar{b} \cdot \nabla \Delta_t} \neq 0$ , the transit averaged radial trapped step  $\bar{\Delta}_t$  no longer vanishes as

$$\bar{\Delta}_t = \frac{\overline{v_{\parallel} \bar{B} \cdot \nabla \psi \times \nabla B}}{\Omega \bar{B} \cdot \nabla B} - \frac{\partial \tilde{J}}{\partial \alpha} = \frac{v_{\parallel} \partial F_0}{\Omega \partial \alpha} - \frac{\partial \tilde{J}}{\partial \alpha} \neq 0. \quad (6.24)$$

The trapped  $F_0$  and  $\tilde{J}$  now depend on the field line label  $\alpha$  so that  $\langle B v_{\parallel}^{-1} \partial F_0 / \partial \alpha \rangle \neq 0$  and  $\langle \partial \tilde{J} / \partial \alpha \rangle \neq 0$ . Moreover, the omnigenous choice for the lower limit  $\bar{B}$  for the trapped is no longer quite right when the maximum field on a field line varies from one end to the other and also depends on the field line.

As  $\bar{h}_t$  is even in  $v_{\parallel}$  it will not contribute to the parallel ion flow or the parallel current density, and there is no need to evaluate it to determine the bootstrap current.

To find the contributions to the parallel ion flow and parallel current density the passing constraint (6.12) must be solved. The solution procedure uses the linearized model like particle collision operator of Kovrizhnikh & Connor (Rosenbluth *et al.* 1972; Connor *et al.* 1973),

$$C\{g\} = \nu Q(\nu) [L\{g\} + \frac{M_i}{T} u v_{\parallel} f_0] = \nu Q(\nu) L\{g - \frac{M_i}{T} u v_{\parallel} f_0\}, \quad (6.25)$$

where  $L$  is the Lorentz operator

$$L\{g\} = \frac{1}{2} \nabla_{\nu} \cdot [(\nu^2 \bar{I} - \bar{v} \bar{v}) \cdot \nabla_{\nu} g] = \frac{2B_0}{Bv^2} v_{\parallel} \frac{\partial}{\partial \lambda} (\lambda v_{\parallel} \frac{\partial g}{\partial \lambda}), \quad (6.26)$$

with  $L\{v_{\parallel} f_0\} = -v_{\parallel} f_0$ , and  $u$  the term retained to preserve collisional momentum conservation,

$$u = 3T \int d^3 v Q v_{\parallel} g / M_i \int d^3 v Q v^2 f_0. \quad (6.27)$$

The function  $Q$  is defined as

$$Q(x) = \frac{1}{x^3} \left[ \left(1 - \frac{1}{2x^2}\right) \text{erf}(x) + \frac{1}{2x} \frac{d}{dx} \text{erf}(x) \right], \quad (6.28)$$

with  $x = v(M_i/2T)^{1/2}$ , and  $\text{erf}(x) = 2\pi^{-1/2} \int_0^x dt e^{-t^2}$  the error function. For ions

$$v_i \equiv \frac{\sqrt{2} \pi Z^4 e^4 n_i \ell n \Lambda}{M_i^{1/2} T_i^{3/2}} \rightarrow \frac{3\sqrt{2}\pi}{4} v_{ii}, \quad (6.29)$$

with  $v_{ii} = 4\sqrt{\pi} Z^4 e^4 n_i \ell n \Lambda / 3M_i^{1/2} T_i^{3/2}$ .

The flux surface averaged passing form is then

$$\left\langle \frac{\mathbf{B}}{v_{\parallel}} C\{g\} \right\rangle = \frac{2v_{\parallel} Q B_0}{v^2} \frac{\partial}{\partial \lambda} \left[ \lambda \left\langle v_{\parallel} \frac{\partial}{\partial \lambda} \left( g - \frac{M_i}{T} u v_{\parallel} f_0 \right) \right\rangle \right], \quad (6.30)$$

where based on constraint (6.12)

$$g = \bar{H}_p - \frac{v_{\parallel} f_0}{\Omega T} \left( \frac{M_i v^2}{2T} - \frac{5}{2} \right) \left\langle \frac{\vec{\mathbf{B}} \cdot \nabla \psi \times \nabla \mathbf{B}}{\Omega \vec{\mathbf{B}} \cdot \nabla \mathbf{B}} \right\rangle \frac{\partial T}{\partial \psi}. \quad (6.31)$$

Integrating from  $\lambda = 0$ , and using  $v_{\parallel} \partial v_{\parallel} / \partial \lambda = -Bv^2/2B_0$  leads to

$$\frac{\langle v_{\parallel} \rangle_p}{f_0} \frac{\partial \bar{H}_p}{\partial \lambda} = -\frac{M_i v^2}{2TB_0} \langle \mathbf{B}u \rangle - \frac{v^2}{2\Omega_0 T} \left( \frac{M_i v^2}{2T} - \frac{5}{2} \right) \left\langle \frac{\vec{\mathbf{B}} \cdot \nabla \psi \times \nabla \mathbf{B}}{\vec{\mathbf{B}} \cdot \nabla \mathbf{B}} \right\rangle \frac{\partial T}{\partial \psi}, \quad (6.32)$$

where  $\Omega_0 = ZeB_0/M_i c$  and

$$\langle \mathbf{B}u \rangle = 3TM_i^{-1} B^{-1} \int d^3v Q v_{\parallel} \langle B^2 g \rangle / \int d^3v Q v^2 f_0. \quad (6.33)$$

The second  $\lambda$  integration is performed to make  $g$  vanish at the trapped-passing boundary  $\lambda = B_0/\hat{B}$ , with  $\hat{B}$  the maximum value of  $B$  on a flux surface, to find

$$\bar{H}_{pi} = \left\langle \frac{\vec{\mathbf{B}} \cdot \nabla \psi \times \nabla \mathbf{B}}{\vec{\mathbf{B}} \cdot \nabla \mathbf{B}} \right\rangle \left( \frac{M_i v^2}{2T_i} - 1.33 \right) \frac{\sigma v f_{0i}}{2\Omega_0 T_i} \frac{\partial T_i^{B_0/\hat{B}}}{\partial \psi} \int_{\lambda}^{\lambda'} \frac{d\lambda'}{\langle \sqrt{1 - \lambda' B/B_0} \rangle}. \quad (6.34)$$

Using (6.6), (6.14), (6.22), and (6.34) gives

$$\frac{f_{li}}{f_{0i}} - \frac{Ze\phi}{T_i} = \frac{h_i}{f_{0i}} = \frac{\bar{H}_{pi} + \bar{h}_{ti} + \tilde{H}_{ti} + \tilde{H}_{pi}}{f_{0i}} - \left[ \frac{\Delta_p}{f_{0i}} + \frac{(\Delta_t - \bar{\Delta}_t)}{f_{0i}} \right] \frac{\partial f_{0i}}{\partial \psi} \simeq \frac{\bar{H}_{pi} + \bar{h}_{ti} + \tilde{H}_{pi}}{f_{0i}} - \frac{\Delta}{f_{0i}} \frac{\partial f_{0i}}{\partial \psi}, \quad (6.35)$$

where the last form neglects  $\tilde{H}_{ti}$  and  $\bar{\Delta}_t$ , and uses  $\Delta = \Delta_p + \Delta_t$ . Inserting  $\Delta$  and the solution for  $\bar{H}_{pi}$  obtained by Landreman and Catto (2012) with the geometric and notational replacements

$$\frac{MI + NG}{M - qN} \rightarrow \left\langle \frac{\vec{\mathbf{B}} \cdot \nabla \psi \times \nabla \mathbf{B}}{\vec{\mathbf{B}} \cdot \nabla \mathbf{B}} \right\rangle, \quad (6.36)$$

yields

$$\begin{aligned} \frac{f_{li}}{f_{0i}} = & \frac{\bar{H}_{ti} + \tilde{H}_{pi}}{f_{0i}} + \frac{Ze\phi}{T_i} - \left( \frac{v_{\parallel} \vec{\mathbf{B}} \cdot \nabla \psi \times \nabla \mathbf{B}}{\Omega \vec{\mathbf{B}} \cdot \nabla \mathbf{B}} - \frac{\partial \tilde{J}}{\partial \alpha} \right) \left[ \frac{1}{p_i} \frac{\partial p_i}{\partial \psi} + \frac{Ze}{T_i} \frac{\partial \Phi}{\partial \psi} + \left( \frac{M_i v^2}{2T_i} - \frac{5}{2} \right) \frac{1}{T_i} \frac{\partial T_i}{\partial \psi} \right] \\ & + \left\langle \frac{\vec{\mathbf{B}} \cdot \nabla \psi \times \nabla \mathbf{B}}{\vec{\mathbf{B}} \cdot \nabla \mathbf{B}} \right\rangle \left( \frac{M_i v^2}{2T_i} - 1.33 \right) \frac{\sigma v H(B_0/\hat{B} - \lambda)}{2\Omega_0 T_i} \frac{\partial T_i^{B_0/\hat{B}}}{\partial \psi} \int_{\lambda}^{\lambda'} \frac{d\lambda'}{\langle \sqrt{1 - \lambda' B/B_0} \rangle} \end{aligned}, \quad (6.37)$$

where  $\bar{H}_{ti}$  and  $\tilde{H}_{pi}$  are unimportant for parallel flow as they must be even functions of  $v_{\parallel}$  with  $\tilde{H}_{pi} \ll \bar{H}_{pi}$  with  $\bar{H}_{pi}$  odd in  $v_{\parallel}$ . A Heaviside step function  $H(B_0/\hat{B} - \lambda)$ , with value one for  $\lambda < B_0/\hat{B}$  and vanishing otherwise, is inserted to combine the trapped and passing forms.

The next section evaluates the parallel ion flow velocity and parallel current density associated with the perturbed distribution function  $f_{ti}$  given by (6.37).

## 7. Flow and current density for a weak departure from omnigenity

Determining the parallel ion velocity requires evaluating  $\int d^3v v_{\parallel} h_i = \int d^3v v_{\parallel} (h_{pi} + h_{ti})$ . The procedure is straightforward to summarize. First, as  $\bar{H}_{ti}$  and  $\tilde{H}_{pi}$  are even in  $v_{\parallel}$ ,

$$\int d^3v v_{\parallel} (\bar{H}_{ti} + \tilde{H}_{pi}) = 0. \quad (7.1)$$

Also, integrating

$$M_i \int d^3v v_{\parallel}^2 \frac{\partial f_0}{\partial \psi} = \left( \frac{\partial p}{\partial \psi} + Z e n \frac{\partial \Phi}{\partial \psi} \right). \quad (7.2)$$

In addition, recalling the omnigenous result (5.12), using  $d^3v \rightarrow 2\pi(Bv^2/B_0\xi)dvd\lambda$  as all integrals are even in  $v_{\parallel}$ , and following the procedure below (49) in Landreman & Catto (2012),

$$\begin{aligned} M_i \int d^3v v_{\parallel} \frac{\partial \tilde{J}}{\partial \alpha} \frac{\partial f_0}{\partial \psi} &= M_i \int d^3v v_{\parallel} \frac{\partial f_0}{\partial \psi} \left[ \frac{v_{\parallel}}{\Omega} \frac{\partial F_0}{\partial \alpha} + v^2 \int_B^{\hat{B}} dB' \frac{\partial F_0'}{\partial \alpha} \frac{(2 - \lambda B'/B_0)}{2\Omega' B' v'_{\parallel}} \right] = \frac{1}{\Omega} \frac{\partial F_0}{\partial \alpha} \left( \frac{\partial p}{\partial \psi} + Z e n \frac{\partial \Phi}{\partial \psi} \right) \\ &- \frac{\pi M_i B}{\Omega_0} \int_0^{\infty} dv v^4 \frac{\partial f_0}{\partial \psi} \int_B^{\hat{B}} \frac{dB' \partial F_0'}{B'^2 \partial \alpha} \int_0^{B_0/B'} d\lambda \frac{(2 - \lambda B'/B_0)}{\sqrt{1 - \lambda B'/B_0}} = \left( \frac{1}{\Omega} \frac{\partial F_0}{\partial \alpha} - \frac{W}{\Omega} \right) \left( \frac{\partial p}{\partial \psi} + Z e n \frac{\partial \Phi}{\partial \psi} \right) \end{aligned} \quad (7.3)$$

where near omnigenity is used to write  $W$  as

$$W \equiv 2B^2 \int_B^{\hat{B}} \frac{dB' \partial F_0'}{B'^3 \partial \alpha}. \quad (7.4)$$

with the upper limit of  $\hat{B} = B_{\max}$  to integrate over all  $B'$ . Moreover, (6.34) yields

$$M_i \int d^3v v_{\parallel} \bar{H}_{pi} = \left\langle \frac{\bar{\mathbf{B}} \cdot \nabla \psi \times \nabla B}{\bar{\mathbf{B}} \cdot \nabla B} \right\rangle \frac{B f_c n_i \partial T_i}{B_0 \Omega_0 \partial \psi}, \quad (7.5)$$

where  $f_c$  is the usual effective passing fraction defined by

$$f_c = \frac{3}{4} \int_0^{B_0/\hat{B}} \frac{d\lambda \lambda}{\langle \sqrt{1 - \lambda B/B_0} \rangle_{OS}} \rightarrow 1 - 1.46\sqrt{\delta}. \quad (7.6)$$

The preceding evaluation makes use of

$$\int_0^{B_0/\hat{B}} d\lambda \int_{\lambda}^{B_0/\hat{B}} \frac{d\lambda'}{\langle \sqrt{1 - \lambda' B/B_0} \rangle} = \int_0^{B_0/\hat{B}} \frac{d\lambda'}{\langle \sqrt{1 - \lambda' B/B_0} \rangle} \int_0^{\lambda'} d\lambda = \int_0^{B_0/\hat{B}} \frac{d\lambda' \lambda'}{\langle \sqrt{1 - \lambda' B/B_0} \rangle} = \frac{4}{3} f_c, \quad (7.7)$$

and

$$\int d^3v v_{\parallel} \sigma v f_{oi} \left( \frac{M_i v^2}{2T_i} - 1.33 \right) = \frac{2\pi B}{B_0} \int_0^{B_0/\hat{B}} d\lambda \int_0^{\infty} dv f_{oi} v^4 \left( \frac{M_i v^2}{2T_i} - 1.33 \right) = 1.17 \frac{3p_i B}{2M_i B_0}. \quad (7.8)$$

Combining these results, the parallel ion velocity,  $V_{\parallel i}$ , for a non-optimized stellarator is

$$n_i V_{\parallel i} = \int d^3 v v_{\parallel} f_{i1} = \frac{-c}{ZeB} \left[ \left\langle \frac{\vec{B} \cdot \nabla \psi \times \nabla B}{\vec{B} \cdot \nabla B} \right\rangle + W \right] \left( \frac{\partial p_i}{\partial \psi} + Z n_i \frac{\partial \Phi}{\partial \psi} \right) - 1.17 \left\langle \frac{\vec{B} \cdot \nabla \psi \times \nabla B}{\vec{B} \cdot \nabla B} \right\rangle \frac{B^2 f_c n_i}{\langle B^2 \rangle} \frac{\partial T_i}{\partial \psi}, \quad (7.9)$$

where  $\langle \partial F_0 / \partial \alpha \rangle$  is neglected as the departure from omnigenity is weak. This result agrees Helander *et al.* (2017), but expresses the geometrical factors in a more compact way thanks to the assumption of near-omnigenity. It thus also reduces to the omnigenous limit found by Landreman and Catto (2012) (within minor notational differences).

Checking  $\nabla \cdot \vec{V} = 0$  by recalling (4.6), and using

$$\vec{B} \cdot \nabla \left( \int d^3 v \frac{v_{\parallel}}{B} \frac{\partial \tilde{J}}{\partial \alpha} \frac{\partial f_0}{\partial \psi} \right) = \vec{B} \cdot \nabla B \frac{\partial}{\partial \alpha} \int d^3 v \frac{v_{\parallel}}{B} \frac{\partial f_0}{\partial \psi} \frac{\partial \tilde{J}}{\partial B} = \frac{M_i c}{Ze} \vec{b} \cdot \nabla B \frac{\partial}{\partial \alpha} \left( \frac{1}{\vec{b} \cdot \nabla B} \right) \int d^3 v v_{\parallel}^2 \frac{\partial f_0}{\partial \psi}, \quad (7.10)$$

gives the required result

$$n_i \vec{B} \cdot \nabla (V_{\parallel i} / B) = -\frac{c}{Ze} \left( \frac{\partial p_i}{\partial \psi} + Z n_i \frac{\partial \Phi}{\partial \psi} \right) \vec{B} \cdot \nabla (U/B) = -\frac{c}{Ze} \left( \frac{\partial p_i}{\partial \psi} + Z n_i \frac{\partial \Phi}{\partial \psi} \right) \nabla \cdot (B^{-2} \vec{B} \times \nabla \psi). \quad (7.11)$$

Writing

$$V_{\parallel i} = V_{\parallel i}^{\text{PS}} + B \langle B V_{\parallel i} \rangle / \langle B^2 \rangle, \quad (7.12)$$

and using lowest order omnigenity gives

$$n_i \langle B V_{\parallel i} \rangle = -\frac{c}{Ze} \left\langle \frac{\vec{B} \cdot \nabla \psi \times \nabla B}{\vec{B} \cdot \nabla B} \right\rangle \left( \frac{\partial p_i}{\partial \psi} + Z n_i \frac{\partial \Phi}{\partial \psi} - 1.17 f_c n_i \frac{\partial T_i}{\partial \psi} \right), \quad (7.13)$$

so that the Pfirsch-Schlüter flow is

$$n_i V_{\parallel i}^{\text{PS}} = -\frac{c}{ZeB} \left( \frac{\partial p_i}{\partial \psi} + Z n_i \frac{\partial \Phi}{\partial \psi} \right) \left[ \left( 1 - \frac{B^2}{\langle B^2 \rangle} \right) \left\langle \frac{\vec{B} \cdot \nabla \psi \times \nabla B}{\vec{B} \cdot \nabla B} \right\rangle + W \right]. \quad (7.14)$$

Nearly all the complicated geometric behavior is in the Pfirsch-Schlüter flow. The relative simplicity of  $\langle B V_{\parallel i} \rangle$  in the preceding parallel ion flow expression along with a similar treatment for the electrons, means that the bootstrap current for weakly non-optimized stellarators is most easily found from the Landreman & Catto (2012) form to be

$$\langle B J_{\parallel} \rangle = \frac{c(Z^2 + 2.2Z + 0.75)(1 - f_c)}{Z(Z + \sqrt{2})} \left\langle \frac{\vec{B} \cdot \nabla \psi \times \nabla B}{\vec{B} \cdot \nabla B} \right\rangle \left[ \frac{\partial p_i}{\partial \psi} + \frac{\partial p_e}{\partial \psi} - \frac{(2.1Z + 0.88)n_e}{(Z^2 + 2.2Z + 0.75)} \frac{\partial T_e}{\partial \psi} - \frac{1.17 n_e}{Z} \frac{\partial T_i}{\partial \psi} \right]. \quad (7.15)$$

The Pfirsch-Schlüter current must be consistent with  $\nabla \cdot \vec{J} = 0$  and

$$J_{\parallel} = J_{\parallel}^{\text{PS}} + B \langle B J_{\parallel} \rangle / \langle B^2 \rangle, \quad (7.16)$$

giving

$$J_{\parallel}^{\text{PS}} = -\frac{c}{B} \left( \frac{\partial p_i}{\partial \psi} + \frac{\partial p_e}{\partial \psi} \right) \left[ \left( 1 - \frac{B^2}{\langle B^2 \rangle} \right) \left\langle \frac{\vec{B} \cdot \nabla \psi \times \nabla B}{\vec{B} \cdot \nabla B} \right\rangle + W \right], \quad (7.17)$$

where integrating by parts gives

$$\langle W \rangle \propto \Sigma \oint dB \oint \frac{d\alpha W}{\vec{B} \cdot \nabla B} \propto \Sigma \oint dB B \oint d\alpha \frac{\partial}{\partial \alpha} \left( \frac{1}{|\vec{b} \cdot \nabla B|} \right) \int_{\vec{B}}^B \frac{dB'}{B'^3} F'_0, \quad (7.18)$$

which indeed vanishes for an omnigenous device as required by Eqs. (7.15) to (7.17). In addition,  $J_{\parallel}^{\text{PS}}$  satisfies

$$\bar{\mathbf{B}} \cdot \nabla \left( \frac{J_{\parallel}^{\text{PS}}}{B} \right) = -c \left( \frac{\partial p_i}{\partial \psi} + \frac{\partial p_e}{\partial \psi} \right) \bar{\mathbf{B}} \cdot \nabla \left( \frac{1}{B^2} \left\langle \frac{\bar{\mathbf{B}} \cdot \nabla \psi \times \nabla B}{\bar{\mathbf{B}} \cdot \nabla B} \right\rangle + \frac{W}{B^2} \right) = -\nabla \cdot \left[ \frac{c}{B^2} \bar{\mathbf{B}} \times \nabla (p_i + p_e) \right], \quad (7.19)$$

since using (4.5) and (4.6) leads to

$$\bar{\mathbf{B}} \cdot \nabla \left( \frac{1}{B^2} \left\langle \frac{\bar{\mathbf{B}} \cdot \nabla \psi \times \nabla B}{\bar{\mathbf{B}} \cdot \nabla B} \right\rangle + \frac{W}{B^2} \right) = \bar{\mathbf{B}} \cdot \nabla \left( \frac{U}{B} \right) = \nabla \cdot \left( \frac{1}{B^2} \bar{\mathbf{B}} \times \nabla \psi \right). \quad (7.20)$$

Only the geometrical coefficients of the bootstrap current differ from the forms of Landreman & Catto (2012) and Helander *et al.* (2017). When normalized to the axisymmetric tokamak result ( $N = 0$  in the quasisymmetric limit) a geometric ratio is obtained,

$$\frac{\langle \text{BJ}_{\parallel} \rangle}{\langle \text{BJ}_{\parallel} \rangle_{\text{tok}}} = \frac{(1-f_c)}{(1-f_c)_{\text{tok}}} \left\langle \frac{\bar{\mathbf{B}} \cdot \nabla \psi \times \nabla B}{\bar{\mathbf{B}} \cdot \nabla B} \right\rangle \simeq \frac{\sqrt{\delta}}{\sqrt{\varepsilon} I} \left\langle \frac{\bar{\mathbf{B}} \cdot \nabla \psi \times \nabla B}{\bar{\mathbf{B}} \cdot \nabla B} \right\rangle \sim \frac{1}{I(\psi)} \left\langle \frac{\bar{\mathbf{B}} \cdot \nabla \psi \times \nabla B}{\bar{\mathbf{B}} \cdot \nabla B} \right\rangle, \quad (7.21)$$

where the last two forms assume a dominant  $N$  and  $M$  of amplitude  $\delta \sim r/R = \varepsilon$ . For a quasi-poloidally symmetric or, more generally, a quasi-isodynamic omnigenous, surface  $\langle \text{BJ}_{\parallel} \rangle = 0$ .

So far no collisionality dependence enters, unlike the simulations (Beidler *et al.* 2011; Kernbichler *et al.* 2016) which typically exhibit dependence on  $\nu$  and  $\omega$  as  $\nu \rightarrow 0$ . The geometric **ratio** obtained here, using momentum conserving like particle collision operators, is the same as that of the simulations which retain only pitch angle scattering (see Appendix). For more details on differences in coefficients of  $\langle \text{BJ}_{\parallel} \rangle$  and  $V_{\parallel}$  see the results of Helander *et al.* (2017) and Landreman *et al.* (2014). The second reference also shows that at about  $\omega R/v_i \sim 0.3$  strong electric field effects begin to modify the trapped region of phase space as found by Kagan & Catto (2010).

The preceding expressions extend the quasisymmetric results of Helander *et al.* (2017) by having more compact and explicit (though less general) geometric coefficients while retaining tangential drift. They reduce to the omnigenous limit (Landreman & Catto 2012) which is consistent with other earlier results (Shaing *et al.* 1989; Helander *et al.* 2011).

Collisional modifications in the presence of only a tangential  $\bar{\mathbf{E}} \times \bar{\mathbf{B}}$  drift as employed in most simulations are evaluated in the next section. This spurious drive in (6.10) modifies the ion flow and bootstrap current at weak collisionality and for finite  $\bar{\mathbf{E}} \times \bar{\mathbf{B}}$  tangential drift in a manner similar to most simulations and offers a possible explanation for some of the discrepancy between simulations and the bootstrap result of (7.15).

## 8. Non-omnigenous collisional tangential drift modification

At small collisionality and large tangential  $\bar{\mathbf{E}} \times \bar{\mathbf{B}}$  drift, when the approximate drive term on the right side of (6.10) is retained as in simulations, a spurious ion flow and bootstrap current is generated as will be shown in this section. In this limit an additional contribution to  $h_p$ , defined as  $h_p^{\omega/\nu}$ , is found by solving

$$\omega \frac{\partial f_0}{\partial \psi} \left\langle \frac{B \partial \Lambda_p}{v_{\parallel} \partial \alpha} \right\rangle = \omega \frac{\partial f_0}{\partial \psi} \left\langle \left( \frac{B}{v_{\parallel}} \frac{\partial}{\partial \alpha} \left( \frac{v_{\parallel}}{\Omega} \frac{\partial F_0}{\partial \alpha} - \frac{\partial \tilde{J}}{\partial \alpha} \right) \right) \right\rangle = \left\langle \frac{B}{v_{\parallel}} C \{ h_p^{\omega/\nu} \} \right\rangle, \quad (8.1)$$

for  $h_p^{\omega/\nu} = \bar{H}_p^{\omega/\nu}$ . Inserting  $\tilde{J}$

$$\frac{2\nu Q B_0}{v^2} \frac{\partial}{\partial \lambda} [\lambda \langle v_{\parallel} \frac{\partial}{\partial \lambda} (h_p^{\omega/\nu} - \frac{M_i}{T} u v_{\parallel} f_0) \rangle] = \frac{\omega B_0}{\Omega_0} \frac{\partial f_0}{\partial \psi} \left\{ \langle \frac{\partial^2 F_0}{\partial \alpha^2} \rangle - \langle \frac{\partial^2}{\partial \alpha^2} \left[ \frac{B}{\sqrt{1-\lambda B/B_0}} \int_{\bar{B}}^B dB' \frac{\sqrt{1-\lambda B'/B_0}}{\bar{b}' \cdot \nabla B'} \right] \rangle \right\}. \quad (8.2)$$

The right side can be written as a  $\lambda$  derivative by using Gradshteyn & Ryzhik (2007):

$$\frac{\sqrt{1-\lambda b'}}{\sqrt{1-\lambda b}} = \frac{1-\lambda b'}{\sqrt{1-\lambda(b+b')+\lambda^2 b b'}} = \quad (8.3)$$

$$\frac{\partial}{\partial \lambda} \left\{ \left( \frac{b-b'}{2b\sqrt{b b'}} \right) \ln [2\sqrt{b b' (1-\lambda b)(1-\lambda b')} + 2b b' \lambda - (b+b')] - \frac{\sqrt{(1-\lambda b)(1-\lambda b')}}{b} \right\}$$

where  $b = B/B_0$  and  $b' = B'/B_0$ . Integrating once from  $\lambda = 0$  gives

$$\lambda \left( \langle v_{\parallel} \rangle \frac{\partial h_p^{\omega/\nu}}{\partial \lambda} + \frac{M_i v^2 f_0}{2TB_0} \langle Bu \rangle \right) = \frac{\omega B_0 v^2}{2\Omega_0 \nu Q} \frac{\partial f_0}{\partial \psi} \left\{ \frac{\lambda}{B_0} \langle \frac{\partial^2 F_0}{\partial \alpha^2} \rangle - \langle \frac{\partial^2}{\partial \alpha^2} \int_{\bar{B}}^B \frac{dB'}{\bar{b}' \cdot \nabla B'} \left\{ 1 - \frac{B-B'}{2\sqrt{B B'}} \ln \left[ \frac{2\sqrt{B B'} - B - B'}{B_0} \right] \right\} \right\} \right. \\ \left. + \langle \frac{\partial^2}{\partial \alpha^2} \int_{\bar{B}}^B \frac{dB'}{\bar{b}' \cdot \nabla B'} \left\{ \sqrt{(1-\lambda b)(1-\lambda b')} - \left( \frac{B-B'}{2\sqrt{B B'}} \right) \ln \left\{ 2\sqrt{b b' (1-\lambda b)(1-\lambda b')} + 2b b' \lambda - (b+b') \right\} \right\} \right\} \quad (8.4)$$

Neglecting  $B - B'$  terms as small in  $\delta$  to simplify the analysis, for the moment, gives

$$\langle v_{\parallel} \rangle_p \frac{\partial h_p^{\omega/\nu}}{\partial \lambda} + \frac{M_i v^2 f_0}{2TB_0} \langle Bu \rangle \approx \frac{\omega v^2}{2\Omega_0 \nu Q} \frac{\partial f_0}{\partial \psi} \left\{ \langle \frac{\partial^2}{\partial \alpha^2} [F_0 - \frac{B_0}{\lambda} \int_{\bar{B}}^B \frac{dB'}{\bar{b}' \cdot \nabla B'} (1 - \frac{v_{\parallel} v'_{\parallel}}{v^2})] \rangle \right\} \quad (8.5)$$

where now

$$u = (3T \int d^3 v Q v_{\parallel} h_p^{\omega/\nu}) / (M_j \int d^3 v Q v^2 f_0) = -(3T \int d^3 v Q v_{\parallel} \lambda \partial h_p^{\omega/\nu} / \partial \lambda) / (M_j \int d^3 v Q v^2 f_0). \quad (8.6)$$

A second integration (making  $h_p^{\omega/\nu}$  vanish at the trapped-passing boundary) is not needed. Notice that  $h_p^{\omega/\nu} \lesssim \delta_{no} \omega \rho_{pi} f_0 / v_i a$  exhibits  $\omega/\nu$  behavior when  $\omega \gg v_i$ , but no  $\sqrt{v}$  dependence. For the estimates of this section  $\Omega^{-1} v_{\parallel} \partial F_0 / \partial \alpha \sim \partial \tilde{J} / \partial \alpha$  and  $F_0 \lesssim RB$  are allowed.

To determine the  $h_p^{\omega/\nu}$  modification in the bootstrap current requires evaluating new parallel ion flow velocity contribution  $V_{\parallel i}^{\omega/\nu}$  defined by

$$n_i \langle B V_{\parallel i}^{\omega/\nu} \rangle \equiv \langle B \int d^3 v v_{\parallel} h_p^{\omega/\nu} \rangle_p = - \langle \frac{B_0^2}{B} \int d^3 v v_{\parallel} \lambda \frac{\partial h_p^{\omega/\nu}}{\partial \lambda} \rangle_p = \frac{M_i \langle Bu \rangle}{2T} \langle \frac{B_0}{B} \int d^3 v \frac{v_{\parallel} \lambda v^2}{\langle v_{\parallel} \rangle_p} f_0 \rangle_p \\ - \frac{\omega B_0^2}{2\nu \Omega_0} \langle \int d^3 v \frac{v_{\parallel} \lambda v^2}{\langle v_{\parallel} \rangle_p} \frac{\partial f_0}{\partial \psi} \left\{ \frac{\partial^2}{\partial \alpha^2} [F_0 - \frac{B_0}{\lambda} \int_{\bar{B}}^B \frac{dB'}{\bar{b}' \cdot \nabla B'} (1 - \frac{v_{\parallel} v'_{\parallel}}{v^2})] \right\} \rangle_p. \quad (8.7)$$

Using  $d^3 v \rightarrow 2\pi (B v^3 / B_0 |v_{\parallel}|) dv d\lambda$ , recalling (7.6), and defining the velocity space average

$$\{Q^k\} \equiv (M_j / 3p_j) \int d^3 v v^2 Q^k f_0 = \left( \int_0^{\infty} dx Q^k x^4 e^{-x^2} \right) / \left( \int_0^{\infty} dx x^4 e^{-x^2} \right), \quad (8.8)$$

with  $k = -1, 0$ , and  $+1$ , gives

$$\langle \frac{B_0}{B} \int d^3 v \frac{v_{\parallel} \lambda v^2 f_0}{\langle v_{\parallel} \rangle_p} \rangle_p = 2\pi \int_0^{\infty} dv v^4 f_0 \int_0^{B_0/\bar{B}} \frac{d\lambda \lambda}{\langle \sqrt{1-\lambda B/B_0} \rangle_p} = \frac{2}{3} f_c \int d^3 v v^2 f_0 = \frac{2f_c p_j}{M_j}, \quad (8.9)$$

and

$$\begin{aligned}
\left\langle \frac{B_0}{B} \int d^3v \frac{v_{\parallel} \lambda v^2}{\langle v_{\parallel} \rangle_p Q} \frac{\partial f_0}{\partial \psi} \right\rangle_p &= 2\pi \int_0^{\infty} dv \frac{v^4}{Q} \frac{\partial f_0}{\partial \psi} \int_0^{B_0/\bar{B}} \frac{d\lambda \lambda}{\langle \sqrt{1-\lambda B/B_0} \rangle_p} = \frac{2}{3} f_c \int d^3v \frac{v^2}{Q} \frac{\partial f_0}{\partial \psi} \\
&= \frac{2f_c}{M_j} \left[ \left\{ \frac{1}{Q} \right\} \left( \frac{\partial p_j}{\partial \psi} + Ze n \frac{\partial \Phi}{\partial \psi} \right) + \left\{ \frac{M_j v^2 - 5T}{2TQ} \right\} n \frac{\partial T}{\partial \psi} \right]
\end{aligned} \quad (8.10)$$

To obtain an analytically tractable estimate the last term in (8.7) is simplified further by using the freely passing expansion  $\lambda^{-1} B_0 (1 - v^{-2} v_{\parallel} v'_{\parallel}) \simeq (B + B')/2$  for now. Then the resulting approximation can be added to  $F_0$  and treated in the same way by using

$$G_0 \equiv F_0 - \lambda^{-1} B_0 \int_{\bar{B}}^B \frac{dB'}{\bar{b}' \cdot \nabla B'} \left( 1 - \frac{v_{\parallel} v'_{\parallel}}{v^2} \right) \simeq F_0 - \frac{1}{2} \int_{\bar{B}}^B \frac{dB' (B' + B)}{\bar{b}' \cdot \nabla B'}, \quad (8.11)$$

where the fact that lower limit of the  $F_0$  integral is  $B_0$ , and not  $\bar{B} = \widehat{B}$ , is unimportant since only  $\alpha$  derivatives of  $G_0$  enter.

Also needed is (8.6) which gives

$$\langle Bu \rangle \left[ \int d^3v Q v^2 f_0 - \frac{3}{2} \left\langle \frac{B_0}{B} \int d^3v \frac{v_{\parallel} \lambda v^2}{\langle v_{\parallel} \rangle_p} Q f_0 \right\rangle_p \right] = - \frac{3B_0 \omega T}{2M_i v} \left\langle \int d^3v \frac{\lambda v^2}{\langle v_{\parallel} \rangle_p} \frac{\partial f_0}{\partial \psi} \frac{\partial \Delta_p}{\partial \alpha} \right\rangle_p, \quad (8.12)$$

where

$$\left\langle \frac{B_0}{B} \int d^3v \frac{v_{\parallel} \lambda v^2 Q f_0}{\langle v_{\parallel} \rangle_p} \right\rangle_p = 2\pi \int_0^{\infty} dv v^4 Q f_0 \int_0^{B_0/\bar{B}} \frac{d\lambda \lambda}{\langle \sqrt{1-\lambda B/B_0} \rangle_p} = \frac{2}{3} f_c \int d^3v v^2 Q f_0 \equiv \frac{2f_c p_j}{M_j} \{Q\}. \quad (8.13)$$

Also, noticing that

$$\left\langle \frac{B_0}{B} \int d^3v \frac{v_{\parallel} \lambda v^2}{\langle v_{\parallel} \rangle_p} \frac{\partial f_0}{\partial \psi} \right\rangle_p = 2\pi \int_0^{\infty} dv v^4 \frac{\partial f_0}{\partial \psi} \int_0^{B_0/\bar{B}} \frac{d\lambda \lambda}{\langle \sqrt{1-\lambda B/B_0} \rangle_p} = \frac{2}{3} f_c \int d^3v v^2 \frac{\partial f_0}{\partial \psi} = \frac{2f_c}{M_i} \left( \frac{\partial p}{\partial \psi} + Ze n \frac{\partial \Phi}{\partial \psi} \right). \quad (8.14)$$

gives upon using (8.12)

$$\left\langle \frac{B_0}{B} \int d^3v \frac{v_{\parallel} \lambda v^2}{\langle v_{\parallel} \rangle_p} \frac{\partial f_0}{\partial \psi} \frac{\partial^2 G_0}{\partial \alpha^2} \right\rangle_p \simeq \frac{2f_c}{M_i} \left\langle \frac{\partial^2 G_0}{\partial \alpha^2} \right\rangle \left( \frac{\partial p}{\partial \psi} + Ze n \frac{\partial \Phi}{\partial \psi} \right), \quad (8.15)$$

Substituting in yields

$$\langle Bu \rangle = \frac{-f_c c \omega}{Ze(1-f_c) v n \{Q\}} \left\langle \frac{\partial^2 F_0}{\partial \alpha^2} \right\rangle \left( \frac{\partial p}{\partial \psi} + Ze n \frac{\partial \Phi}{\partial \psi} \right) \lesssim B \frac{\delta_{no} \omega \rho_{pi} v}{v_i a}. \quad (8.16)$$

The collisional bootstrap modification  $\langle BJ_{\parallel}^{\omega/v} \rangle$  to be added to (7.15) is then

$$\langle BJ_{\parallel}^{\omega/v} \rangle \equiv Ze n_i \langle BV_{\parallel}^{\omega/v} \rangle = \frac{c f_c \omega}{(1-f_c) v_i} \left\langle \frac{\partial^2 G_0}{\partial \alpha^2} \right\rangle \left[ \left\{ \frac{1-f_c}{Q} \right\} - \frac{f_c}{\{Q\}} \right] \left( \frac{\partial p_i}{\partial \psi} + Ze n_i \frac{\partial \Phi}{\partial \psi} \right) + (1-f_c) \left\{ \frac{M_i v^2 - 5T_i}{2T_i Q} \right\} n_i \frac{\partial T_i}{\partial \psi}, \quad (8.17)$$

where  $\langle \partial^2 G_0 / \partial \alpha^2 \rangle$  as given by (8.12) only vanishes for an omnigenous flux surface, and  $\{Q\} = 0.4$  and  $\{Q^{-1}\} = 5.4$  (Catto *et al.* 2001). Upon dividing by  $Ze$ ,  $n_i \langle BV_{\parallel}^{\omega/v} \rangle$  is to be added to the parallel ion velocity expression (7.13). Forming the ratio of (8.17) divided by (7.15) gives

$$\frac{\langle BJ_{\parallel}^{\omega/v} \rangle}{\langle BJ_{\parallel} \rangle} \sim \frac{\delta_{no} \omega}{\delta v_i}, \quad (8.18)$$

and therefore a specious modification of the bootstrap current. This modification should be present in simulations that use the radially local form of the drift kinetic equation and does not, in general, vanish on a imperfectly quasi-poloidally symmetric or quasi-isodynamic surfaces.

To remove the approximation (8.11)-(8.12) in (8.17) the substitution

$$G_0 \Rightarrow F_0 - \left\langle \int d^3v \frac{v_{\parallel} v^2}{\langle v_{\parallel} \rangle_p} \frac{\partial f_0}{\partial \psi} \int_{\vec{b}'} \frac{d\vec{B}'}{\vec{b}' \cdot \nabla \vec{B}'} \left(1 - \frac{v_{\parallel} v'_{\parallel}}{v^2}\right) \right\rangle_p / \left\langle \int d^3v \frac{v_{\parallel} \lambda v^2}{B \langle v_{\parallel} \rangle_p} \frac{\partial f_0}{\partial \psi} \right\rangle_p, \quad (8.19)$$

is required. A full generalization can be made similarly by retaining the  $\vec{B} - \vec{B}'$  terms of (8.4).

The collisional, non-omnigenous, tangential  $\vec{E} \times \vec{B}$  drift modification is more important for the ions than the electrons, and dominates for  $\omega \delta_{no}/v_i \gg \delta$ . It introduces unphysical radial electric field dependence through both the tangential drift frequency  $\omega$  and a new  $\partial\Phi/\partial\psi$  force term in (8.17) so that both  $\omega$  and  $\omega^2$  terms enter. It is perhaps responsible for some of the  $\partial\Phi/\partial\psi$  dependence observed in simulations at small collisionalities (Beidler *et al.* 2011). The new term results in no fluxes from  $\langle \int d^3v h_p^{\omega/v} \vec{v}_d \cdot \nabla \psi \rangle$  and  $\langle \int d^3v h_p^{\omega/v} (M_i v^2/2) \vec{v}_d \cdot \nabla \psi \rangle$ , and no frictional particle flux as  $\int d^3v v_{\parallel} C\{h_p^{\omega/v}\} = 0$  for ion-ion collisions; however, it will give rise to a tangential  $\vec{E} \times \vec{B}$  drift dependent, but collision frequency independent heat flux,  $\langle \int d^3v h_p^{\omega/v} (Mv^2/2) \vec{v} \cdot \nabla \psi \rangle = (Mc/Ze) \langle \int d^3v (Mv^2/2) v_{\parallel} C\{h_p^{\omega/v}\} \rangle$ . To see there is such a flux, the solenoidal vector  $\vec{h} = B^{-2} \vec{B} \times \nabla \psi + U \vec{b}$ , is introduced (Simakov & Helander 2009) and the  $\vec{h} \cdot \vec{v} v^2/2$ , moment of the Fokker-Planck equation formed.

## 9. Discussion

The streamlined derivation of the bootstrap current and the parallel ion velocity presented here is possible because the problem is formulated in a way that allows all of the odd terms in  $v_{\parallel}$  to be obtained by evaluating them as part of the leading order corrections to the Maxwellian. As a result, the only other terms that enter in lowest order are trapped terms and they are all even in  $v_{\parallel}$ . The procedure results in a parallel ion flow velocity (7.9) consistent with force balance and continuity, and leads to convenient expressions for the Pfirsch-Schlüter parallel ion velocity (7.14) and parallel current density (7.17). The bootstrap current (7.15) and bootstrap contribution to the parallel ion velocity (7.13) have compact and explicit geometric coefficients that agree with the previous collisionless forms of Helander *et al.* (2017 & 2011) and Landreman & Catto (2012), which are both consistent with Shaing *et al.* (1989).

More importantly, a new, but spurious, collisional modification to the ion parallel flow velocity and bootstrap current given by (8.17) and (8.19) (or its further generalization) is found that can become substantial at low collisionalities whenever there is a tangential  $\vec{E} \times \vec{B}$  drift and the magnetic field is not perfectly omnigenous. In addition to the dependence of (8.17) on the tangential drift frequency,  $\omega = c \partial\Phi/\partial\psi$ , there is a new force term depending on  $\partial\Phi/\partial\psi$ , so terms both linear and quadratic in  $\omega$  occur. This new  $\omega/v_i$  contribution should be present in numerical solutions of the drift kinetic equation and may be a reason why these sometimes do not seem to agree with previous analytical expressions. However, this contribution is an artifact

of the radially local approximation of the drift kinetic equation that is usually employed in analytical theory as well as in neoclassical codes. It disappears when the effect of the full drift velocity on the perturbed distribution function is retained in the kinetic equation. This result is yet another indication that it is not always logically consistent to treat neoclassical transport in stellarators as a radially local process.

For yet lower collisionalities than those treated here, once the population of transitional ions becomes significant and introduces a regime linear in  $v_*$  that arises as  $v_* \rightarrow 0$  (Beidler *et al.* 2011; Catto 2019) that cannot be treated by the procedures herein.

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## Appendix: Lorentz forms

The Lorentz operator form of the results for (8.17) and (7.15) are

$$\langle \mathbf{B} \mathbf{J}_{\parallel}^{\omega/v} \rangle_L \equiv Z n_i \langle \mathbf{B} \mathbf{V}_{\parallel i}^{\omega/v} \rangle_L = \frac{c f_c \omega}{v_i} \left\langle \frac{\partial^2 G_0}{\partial \alpha^2} \right\rangle \left[ \left\{ \frac{1}{Q} \right\} \left\langle \frac{\partial p_i}{\partial \psi} + Z n_i \frac{\partial \Phi}{\partial \psi} \right\rangle + \left\{ \frac{M_i v^2 - 5 T_i}{2 T_i Q} \right\} n_i \frac{\partial T_i}{\partial \psi} \right], \quad (\text{A.1})$$

and

$$\langle \mathbf{B} \mathbf{J}_{\parallel} \rangle_L = c (1 - f_c) \left\langle \frac{\partial p_i}{\partial \psi} + Z n_i \frac{\partial \Phi}{\partial \psi} \right\rangle \left\langle \frac{\vec{\mathbf{B}} \cdot \nabla \psi \times \nabla \mathbf{B}}{\vec{\mathbf{B}} \cdot \nabla \mathbf{B}} \right\rangle + \langle \mathbf{B} \mathbf{J}_{\parallel}^{\omega/v} \rangle_L. \quad (\text{A.2})$$

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