Journal of Elasticity manuscript No.
(will be inserted by the editor)

A new proof that the number of linear elastic symmetries in two dimensions is four

Jeremy Trageser* · **Pablo Seleson**

Received: date / Accepted: date

Abstract We present an elementary and self-contained proof that there are exactly four symmetry classes of the elasticity tensor in two dimensions: oblique, rectangular, square, and isotropic. In two dimensions, orthogonal transformations are either reflections or rotations. The proof is based on identification of constraints imposed by reflections and rotations on the elasticity tensor, and it simply employs elementary tools from trigonometry, making the proof accessible to a broad audience. For completeness, we identify the sets of transformations (rotations and reflections) for each symmetry class and report the corresponding equations of motions in classical linear elasticity.

Keywords linear elasticity · anisotropy · two dimensions · elasticity tensor · symmetry transformations

---

This manuscript has been authored in part by UT-Battelle, LLC, under contract DE-AC05-00OR22725 with the US Department of Energy (DOE). The US government retains and the publisher, by accepting the article for publication, acknowledges that the US government retains a nonexclusive, paid-up, irrevocable, worldwide license to publish or reproduce the published form of this manuscript, or allow others to do so, for US government purposes. DOE will provide public access to these results of federally sponsored research in accordance with the DOE Public Access Plan (http://energy.gov/downloads/doe-public-access-plan).

Sandia National Laboratories is a multimission laboratory managed and operated by National Technology & Engineering Solutions of Sandia, LLC, a wholly owned subsidiary of Honeywell International Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA0003525. This paper describes objective technical results and analysis. Any subjective views or opinions that might be expressed in the paper do not necessarily represent the views of the U.S. Department of Energy or the United States Government.

*Corresponding author, E-mail: jtrages@sandia.gov

Jeremy Trageser
Computer Science and Mathematics Division
Oak Ridge National Laboratory
P.O. Box 2008, MS-6211, Oak Ridge, TN 37831-6211
E-mail: jtrages@sandia.gov
Present address:
Engineering Sciences Center
Sandia National Laboratories
P.O. Box 5800, MS-1322, Albuquerque, NM, 87185-1322
https://orcid.org/0000-0003-3219-8448

Pablo Seleson
Computer Science and Mathematics Division
Oak Ridge National Laboratory
P.O. Box 2008, MS-6013, Oak Ridge, TN 37831-6013
https://orcid.org/0000-0003-3279-4231

Declarations

Funding

Research sponsored by the Laboratory Directed Research and Development Program of Oak Ridge National Laboratory, managed by UT-Battelle, LLC, for the U. S. Department of Energy.

Conflicts of interest/Competing interests

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Availability of data and material

Not applicable

Code availability

Not applicable

1 Introduction

Many materials exhibit symmetry in their material response. This can be captured to a certain degree in linear elasticity by imposing symmetry on the elasticity tensor; however, the elasticity tensor is only capable of describing a rather limited number of symmetry classes. In fact, it has been shown that the elasticity tensor has exactly eight symmetry classes in three dimensions [6,4,11,3] and four symmetry classes in two dimensions [1,8]. In addition to being important for describing material response, imposing symmetry on the elasticity tensor is also necessary for dimension-reducing approximations such as plane strain, plane stress, and axisymmetry [10].

In this work, we explore the symmetry classes of the two-dimensional elasticity tensor. In recent years, several works studied various properties of the two-dimensional elasticity tensor, e.g., [1,2,7,8,13,14]. The aim of this paper is to present a new proof that the quantity of material symmetry classes in two-dimensional linear elasticity is exactly four. While a proof was presented in [1,8], an additional focus of the present work is the use of elementary methods in order to facilitate understanding among a broader audience. In fact, by simply employing trigonometry and the definition of a group from abstract algebra, we demonstrate that the elasticity tensor has exactly four symmetry classes[1]: oblique, rectangular, square, and isotropic.

The organization of this paper is as follows. In Section 2, we review the classical theory of linear elasticity to provide a background of material anisotropy, the elasticity tensor, and material symmetry. In Section 3, we look at symmetry classes and present the main result showing there are four symmetry classes of the elasticity tensor in two dimensions. Lastly, in Section 4, we present concluding remarks.

2 The elasticity tensor and the equation of motion in classical linear elasticity

In classical linear elasticity, the stress tensor $\boldsymbol{\sigma}$ and strain tensor $\boldsymbol{\varepsilon}$ are related via the generalized Hooke's law [10]:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \tag{1}$$

where C_{ijkl} are the components of the fourth-order elasticity tensor $\mathbb{C}$ and Einstein summation convention for repeated indices is employed. Due to the symmetries of the stress and strain tensors (i.e., $\sigma_{ij} = \sigma_{ji}$ and $\varepsilon_{kl} = \varepsilon_{lk}$), $\mathbb{C}$ inherits the minor symmetries:

$$C_{ijkl} = C_{jikl} = C_{ijlk}. \tag{2}$$

---

[1] There does not appear to be a consensus in the literature for the naming of the symmetry classes in two dimensions. For instance, in [1] the symmetry classes are labeled monoclinic, orthotropic, tetragonal, and isotropic.

Furthermore, the relation between the strain energy density $W = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$ and $\mathbb{C}$,

$$C_{ijkl} = \frac{\partial^2 W}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}} = \frac{\partial^2 W}{\partial\varepsilon_{kl}\partial\varepsilon_{ij}} = C_{klij},$$

guarantees the major symmetry of $\mathbb{C}$:

$$C_{ijkl} = C_{klij}. \tag{3}$$

Due to the major and minor symmetries, we may utilize Voigt notation to express the fourth-order tensor $\mathbb{C}$ as a symmetric second-order tensor $\mathbf{C}$ and, similarly, represent the second-order tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ as vectors. In this formulation, we express (1) in two dimensions as $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$, which in component form is given by

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1112} \\ \cdot & C_{2222} & C_{2212} \\ \cdot & \cdot & C_{1212} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix}. \tag{4}$$

The inverse relation of (1) can be expressed as

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl}, \tag{5}$$

where S_{ijkl} are the components of the fourth-order compliance tensor $\mathbb{S}$. Similar to $\mathbb{C}$, we may employ Voigt notation to express the fourth-order tensor $\mathbb{S}$ as a symmetric second-order tensor $\mathbf{S}$ [10]. The strain-stress relation (5) may be expressed in two dimensions as $\boldsymbol{\varepsilon} = \mathbf{S}\boldsymbol{\sigma}$, which in component form is given by

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & 2S_{1112} \\ \cdot & S_{2222} & 2S_{2212} \\ \cdot & \cdot & 4S_{1212} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}.$$

In classical linear elasticity, the strain tensor $\boldsymbol{\varepsilon}$ is related to the displacement field $\mathbf{u}$ through the relation

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{6}$$

We are now able to provide the equation of motion in classical linear elasticity [9]. Given a body $\mathcal{B} \subset \mathbb{R}^d$, where d is the dimension, the equation of motion is given by

$$\rho\ddot{\mathbf{u}} = \nabla \cdot \boldsymbol{\sigma} + \mathbf{b}, \tag{7}$$

where ρ is the mass density, $\ddot{\mathbf{u}}$ is the second derivative in time of the displacement field $\mathbf{u}$, and $\mathbf{b}$ is a prescribed body force density field. In component form, (7) may be written as (see (1) and (6))

$$\begin{aligned} \rho\ddot{u}_i =& \frac{\partial\sigma_{ij}}{\partial x_j} + b_i = C_{ijkl}\frac{\partial\varepsilon_{kl}}{\partial x_j} + b_i \\ =& \frac{C_{ijkl}}{2}\left(\frac{\partial^2 u_k}{\partial x_j\partial x_l} + \frac{\partial^2 u_l}{\partial x_j\partial x_k}\right) + b_i \\ =& C_{ijkl}\frac{\partial^2 u_k}{\partial x_j\partial x_l} + b_i, \end{aligned} \tag{8}$$

where we used the minor symmetry $C_{ijkl} = C_{ijlk}$ (see (2)).

3 Material symmetry classes in two-dimensional classical linear elasticity

The fourth-order tensor $\mathbb{C}$ can be expressed in the basis $\{\mathbf{e}_i\}_{i=1,\ldots,d}$, where d is the dimension, as

$$\mathbb{C} = C_{ijkl}\left(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l\right). \tag{9}$$

A transformation between orthonormal bases of $\mathbb{R}^d$, $\{\mathbf{e}_i\}_{i=1,\ldots,d}$ and $\{\mathbf{e}'_i\}_{i=1,\ldots,d}$, where d is the dimension, may be represented by an orthogonal matrix $\mathbf{Q}$, where the components are given by

$$Q_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j. \tag{10}$$

We call $\mathbf{Q}$ a symmetry transformation of $\mathbb{C}$ when the components of $\mathbb{C}$ are invariant under the transformation. In Definition 1 we formalize this concept.

Definition 1 An orthogonal transformation $\mathbf{Q}$ between orthonormal bases $\{\mathbf{e}_i\}_{i=1,\ldots,d}$ and $\{\mathbf{e}'_i\}_{i=1,\ldots,d}$ (see (10)) is a symmetry transformation of $\mathbb{C}$ if

$$C_{ijkl} = Q_{ip}Q_{jq}Q_{kr}Q_{ls}C_{pqrs}, \quad i,j,k,l \in \{1,\ldots,d\}. \tag{11}$$

In view of (11), one may show that if $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are symmetry transformations of $\mathbb{C}$ then $\mathbf{Q}_1^{-1}$ (as well as $\mathbf{Q}_2^{-1}$) and $\mathbf{Q}_1\mathbf{Q}_2$ are also symmetry transformations of $\mathbb{C}$ [10]. Clearly, the identity transformation, $\mathbf{I}$, is a symmetry transformation of $\mathbb{C}$. Therefore, the set of symmetry transformations of $\mathbb{C}$ forms a group (see, e.g., [5]), which we call the *symmetry group* of $\mathbb{C}$ and denote by $\mathcal{G}_{\mathbb{C}}$. We call the set of symmetry groups of $\mathbb{C}$ that are equivalent up to a change in orientation, the *symmetry class* of $\mathbb{C}$. Given a material described by $\mathbb{C}$, its material symmetry class is the corresponding symmetry class of $\mathbb{C}$.

In two dimensions, it is well known that every orthogonal transformation is either a reflection or a rotation. For completeness, this result is presented along with a proof in Proposition 1 in Appendix C. For convenience, we recall the corresponding transformation matrices. For a (counterclockwise) rotation by an angle θ about the origin, the corresponding transformation matrix is given by

$$\mathbf{Rot}(\theta) := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}. \tag{12}$$

Remark 1 The transformation matrix (12) can be viewed as an active or passive transformation. As an active transformation, (12) rotates a vector counterclockwise by θ. Conversely, as a passive transformation, (12) corresponds to rotating the coordinate system and the corresponding basis vectors clockwise by θ.

For a reflection about the line through the origin making an angle of θ with the x-axis, the corresponding transformation matrix is given by

$$\mathbf{Ref}(\theta) := \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}. \tag{13}$$

From the periodicity of the sine and cosine functions, it is clear that any rotation may be represented by a rotation of $\theta \in [0, 2\pi)$, while any reflection may be represented by a reflection through a line making an angle of $\theta \in [0, \pi)$ with the x-axis. We recall a useful identity for later use:

$$\mathbf{Ref}(\phi)\mathbf{Ref}(\theta) = \mathbf{Rot}(2[\phi - \theta]). \tag{14}$$

Since reflections are their own inverses, we additionally have

$$\mathbf{Rot}(2[\phi - \theta])\mathbf{Ref}(\theta) = \mathbf{Ref}(\phi). \tag{15}$$

In the two-dimensional case, there are exactly four symmetry classes of $\mathbb{C}$, as was shown in [1,8]. The proof below of our main result, Theorem 1, provides an alternative proof of this fact. The proof utilizes elementary methods with minimal machinery from abstract algebra.

Theorem 1 *Up to a change in orientation, there are exactly four symmetry groups of the elasticity tensor $\mathbb{C}$ in two dimensions: oblique, rectangular, square, and isotropic (see Figure 1). The corresponding elasticity tensors and group generators for each symmetry group are given by:*

Symmetry Group	**Elasticity Tensor**	**Group Generators**
Oblique	$\begin{bmatrix} C_{1111} & C_{1122} & C_{1112} \\ \cdot & C_{2222} & C_{2212} \\ \cdot & \cdot & C_{1212} \end{bmatrix}$	$\{-\mathbf{I}\}$
Rectangular	$\begin{bmatrix} C_{1111} & C_{1122} & 0 \\ \cdot & C_{2222} & 0 \\ \cdot & \cdot & C_{1212} \end{bmatrix}$	$\{-\mathbf{I}, \mathbf{Ref}(0)\}$
Square	$\begin{bmatrix} C_{1111} & C_{1122} & 0 \\ \cdot & C_{1111} & 0 \\ \cdot & \cdot & C_{1212} \end{bmatrix}$	$\left\{-\mathbf{I}, \mathbf{Ref}(0), \mathbf{Ref}(\frac{\pi}{4})\right\}$
Isotropic	$\begin{bmatrix} C_{1111} & C_{1122} & 0 \\ \cdot & C_{1111} & 0 \\ \cdot & \cdot & \frac{C_{1111}-C_{1122}}{2} \end{bmatrix}$	$\{\mathbf{Ref}(\theta) : \theta \in [0, \pi)\}$

Proof Suppose $\mathcal{G}_{\mathbb{C}}$ is the symmetry group of $\mathbb{C}$. We show that, up to a change in orientation, $\mathcal{G}_{\mathbb{C}}$ is one of the four symmetry groups from Lemma 1 below. By (11), it is clear that $-\mathbf{I} \in \mathcal{G}_{\mathbb{C}}$. Consequently, at least one of the four symmetry groups described in Lemma 1 (i.e., oblique) is a subgroup of $\mathcal{G}_{\mathbb{C}}$. Let $\mathcal{N}$ be the most symmetric (largest quantity of group generators) such subgroup of $\mathcal{G}_{\mathbb{C}}$. By construction, $\mathcal{N} \subseteq \mathcal{G}_{\mathbb{C}}$. We now show that $\mathcal{G}_{\mathbb{C}} \subseteq \mathcal{N}$, which implies $\mathcal{G}_{\mathbb{C}} = \mathcal{N}$.

Up to a change in orientation, Lemma 1 implies every reflection in $\mathcal{G}_{\mathbb{C}}$ is contained in $\mathcal{N}$, as otherwise $\mathcal{N}$ would not be the most symmetric subgroup of $\mathcal{G}_{\mathbb{C}}$ from the groups described in Lemma 1. Similarly, by Lemma 2, we know every rotation in $\mathcal{G}_{\mathbb{C}}$ is contained in $\mathcal{N}$. Since orthogonal transformations in two dimensions are either rotations or reflections, every element of $\mathcal{G}_{\mathbb{C}}$ is contained in $\mathcal{N}$, i.e., $\mathcal{G}_{\mathbb{C}} \subseteq \mathcal{N}$.

Lemma 1 *In two dimensions and up to a change in orientation, there are only four symmetry groups of the elasticity tensor $\mathbb{C}$ generated by reflections and $-\boldsymbol{I}$:*

Symmetry Group	**Lines of Reflection Symmetry**	**Group Generators**
Oblique	*No lines of reflection symmetry*	$\{-\mathbf{I}\}$
Rectangular	*Two lines of reflection symmetry*	$\{-\mathbf{I}, \mathbf{Ref}(0)\}$
Square	*Four lines of reflection symmetry*	$\left\{-\mathbf{I}, \mathbf{Ref}(0), \mathbf{Ref}\left(\frac{\pi}{4}\right)\right\}$
Isotropic	*All lines of reflection symmetry*	$\{\mathbf{Ref}(\theta) : \theta \in [0, \pi)\}$

Proof **No lines of reflection symmetry**
By (11), the group generated by

$$\{-\mathbf{I}\} \tag{16}$$

is a subgroup of any symmetry group of $\mathbb{C}$. This group, $\{\mathbf{I}, -\mathbf{I}\}$, imposes no restrictions on the elasticity tensor, i.e., (11) holds under any transformation in the group. We call the symmetry group generated by (16) the *oblique symmetry group*, and the corresponding elasticity tensor is given by (see (4))

$$\mathbf{C} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1112} \\ \cdot & C_{2222} & C_{2212} \\ \cdot & \cdot & C_{1212} \end{bmatrix}. \tag{17}$$

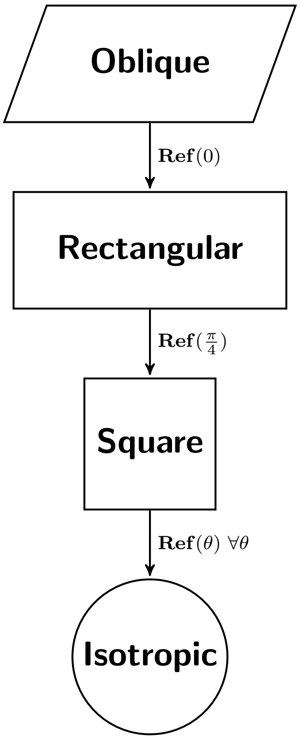

Fig. 1: The four symmetry groups (up to a change in orientation) of the elasticity tensor in two dimensions.

One line of reflection symmetry
We now consider the implications of adding a reflection symmetry transformation to the subgroup $\{\mathbf{I}, -\mathbf{I}\}$. Without loss of generality, we choose an orthonormal basis $\{\mathbf{e}_i\}_{i=1,2}$ so that the line of reflection symmetry coincides with the x-axis. In this basis, we may write the corresponding reflection transformation as (see (13))

$$\mathbf{Ref}(0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{18}$$

We are interested in the restrictions the group generated by

$$\{-\mathbf{I}, \mathbf{Ref}(0)\} \tag{19}$$

imposes on the elasticity tensor $\mathbb{C}$.

In order for (11) to be satisfied for $\mathbf{Q} = \mathbf{Ref}(0)$, any component C_{ijkl} where 2 appears an odd number of times in the indices requires $C_{ijkl} = -C_{ijkl}$ and thus $C_{ijkl} = 0$. Consequently,

$$C_{1112} = C_{2212} = 0. \tag{20}$$

The transformations in (19) impose no additional restrictions on $\mathbb{C}$. We call the symmetry group generated by (19) the *rectangular symmetry group*, and the corresponding elasticity tensor is given by

$$\mathbf{C} = \begin{bmatrix} C_{1111} & C_{1122} & 0 \\ \cdot & C_{2222} & 0 \\ \cdot & \cdot & C_{1212} \end{bmatrix}. \tag{21}$$

Notice $\mathbf{Rot}(\pi)$, a rotation by π, is equivalent to $-\mathbf{I}$ (see (12)). By (15), we have

$$\mathbf{Rot}(\pi)\mathbf{Ref}(\theta) = \mathbf{Ref}\left(\theta + \frac{\pi}{2}\right). \tag{22}$$

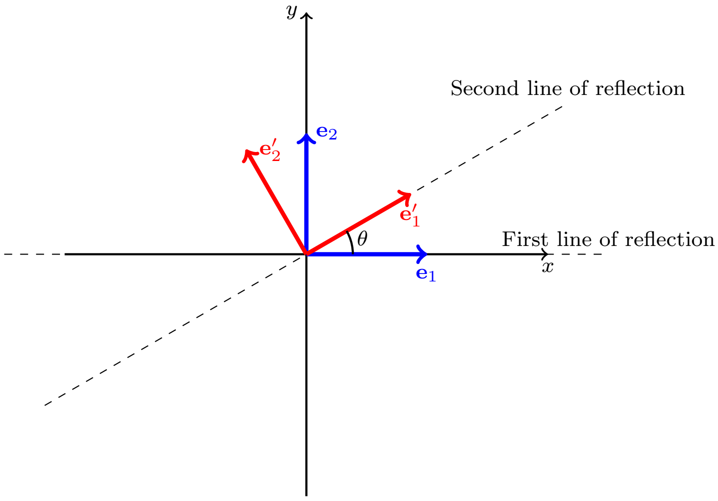

Fig. 2: Basis $\{\mathbf{e}_i\}_{i=1,2}$ and rotated basis $\{\mathbf{e}'_i\}_{i=1,2}$ corresponding to two non-orthogonal lines of reflection symmetry.

Since $-\mathbf{I}$ is in every symmetry group of $\mathbb{C}$, by (22), a line of reflection symmetry automatically induces a second line of reflection symmetry orthogonal to the first line of reflection symmetry. In particular, this implies $\mathbf{Ref}\left(\frac{\pi}{2}\right)$ belongs to the rectangular symmetry group because the reflection transformation $\mathbf{Ref}(0)$ belongs to the group. Thus, to consider a symmetry group distinct from the rectangular symmetry group, we next consider a group containing two symmetry transformations corresponding to reflections across two non-orthogonal lines.

Multiple non-orthogonal lines of reflection symmetry

Consider a group generated by $-\mathbf{I}$ and two symmetry transformations corresponding to reflections across two non-orthogonal lines. Suppose the angle between the two lines of reflection symmetry is given by θ. We further require $\theta \neq \frac{k\pi}{2}$ for $k \in \mathbb{Z}$, so that the lines of reflection symmetry are distinct and not orthogonal to each other.

Similar to [4], one may choose an orthonormal basis $\{\mathbf{e}_i\}_{i=1,2}$ and a rotated basis $\{\mathbf{e}'_i\}_{i=1,2}$ such that

$$\begin{aligned}\mathbf{e}'_1 &= \cos(\theta)\mathbf{e}_1 + \sin(\theta)\mathbf{e}_2,\\ \mathbf{e}'_2 &= -\sin(\theta)\mathbf{e}_1 + \cos(\theta)\mathbf{e}_2,\end{aligned}$$

where $\mathbf{e}_1$ and $\mathbf{e}'_1$ are oriented along the two lines of reflection symmetry (see Figure 2). In the rotated basis, following the argument for one line of reflection symmetry, the condition $C'_{1112} = C'_{2212} = 0$ holds. The orthogonal matrix corresponding to such a basis rotation is $\mathbf{Q} = \mathbf{Rot}(-\theta)$. Using this $\mathbf{Q}$ in (11), in combination with $C'_{1112} = C'_{2212} = 0$, leads to a system of two equations (one for C'_{1112} and one for C'_{2212}) from which the remaining symmetry groups may be identified. Nevertheless, in this work, we instead directly use reflection transformations for the same purpose.

In the basis $\{\mathbf{e}_i\}_{i=1,2}$, the group is generated by

$$\{-\mathbf{I}, \mathbf{Ref}(0), \mathbf{Ref}(\theta)\}\,. \tag{23}$$

We consider (11) with $\mathbf{Q} = \mathbf{Ref}(\theta)$. Since $\mathbf{Ref}(0)$ is already in the set (23), we may assume (20), and the transformation of the components C_{1112} and C_{2212} results in the following equations (after simplification):

$$0 = C'_{1112} = [C_{1111} - C_{2222} + (C_{1111} - 2C_{1122} + C_{2222} - 4C_{1212})\cos(4\theta)]\sin(4\theta), \tag{24a}$$

$$0 = C'_{2212} = [C_{1111} - C_{2222} - (C_{1111} - 2C_{1122} + C_{2222} - 4C_{1212})\cos(4\theta)]\sin(4\theta). \tag{24b}$$

We now consider all solutions of system (24).

First, let us consider the case $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ (recall $\theta \in [0, \pi)$). Recall from (22) that a line of reflection symmetry induces a second line of reflection symmetry perpendicular to the first one. Thus, the choice of $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ is irrelevant as either symmetry transformation induces the other one. Without loss of generality, we suppose $\theta = \frac{\pi}{4}$. Consequently, $\sin(4\theta) = 0$ and no additional restrictions arise from (24). However, in order for $\mathbf{Q}$ to be a symmetry transformation of $\mathbb{C}$, the following remaining equations from (11) need to also be satisfied for the components of the elasticity tensor (we currently only impose (20)):

$$C'_{1111} - C_{1111} = -C_{1111} + C_{1111}\cos^4(2\theta) + C_{2222}\sin^4(2\theta) + \left[\frac{1}{2}C_{1122} + C_{1212}\right]\sin^2(4\theta) = 0, \tag{25a}$$

$$C'_{1122} - C_{1122} = \frac{1}{4}\left[C_{1111} - 2C_{1122} + C_{2222} - 4C_{1212}\right]\sin^2(4\theta) = 0, \tag{25b}$$

$$C'_{2222} - C_{2222} = -C_{2222} + C_{2222}\cos^4(2\theta) + C_{1111}\sin^4(2\theta) + \left[\frac{1}{2}C_{1122} + C_{1212}\right]\sin^2(4\theta) = 0, \tag{25c}$$

$$C'_{1212} - C_{1212} = \frac{1}{4}\left[C_{1111} - 2C_{1122} + C_{2222} - 4C_{1212}\right]\sin^2(4\theta) = 0. \tag{25d}$$

By our assumption that $\theta = \frac{\pi}{4}$, we know $\cos(2\theta) = 0$, $\sin(2\theta) = 1$, and $\sin(4\theta) = 0$. Thus, by (25a) or (25c), we obtain the restriction $C_{1111} = C_{2222}$. No additional restrictions are imposed by (25) on $\mathbb{C}$. We call the symmetry group generated by

$$\left\{-\mathbf{I}, \mathbf{Ref}(0), \mathbf{Ref}\left(\frac{\pi}{4}\right)\right\} \tag{26}$$

the *square symmetry group*, and the corresponding elasticity tensor has the restrictions

$$C_{1112} = C_{2212} = 0 \text{ and } C_{1111} = C_{2222}. \tag{27}$$

The elasticity tensor corresponding to the square symmetry group is given by

$$\mathbf{C} = \begin{bmatrix} C_{1111} & C_{1122} & 0 \\ \cdot & C_{1111} & 0 \\ \cdot & \cdot & C_{1212} \end{bmatrix}. \tag{28}$$

Now, let us consider the case $\theta \neq \frac{\pi}{4}$ or $\frac{3\pi}{4}$. Then, $\sin(4\theta) \neq 0$, and we can simplify (24) by dividing the equations by $\sin(4\theta)$ to obtain:

$$C_{1111} - C_{2222} + (C_{1111} - 2C_{1122} + C_{2222} - 4C_{1212})\cos(4\theta) = 0, \tag{29a}$$

$$C_{1111} - C_{2222} - (C_{1111} - 2C_{1122} + C_{2222} - 4C_{1212})\cos(4\theta) = 0. \tag{29b}$$

In this case, adding (29a) and (29b), we obtain the restriction $C_{1111} = C_{2222}$; then, using this relation in any of the equations in (25), we obtain (after simplification[2]):

$$\left[C_{1111} - C_{1122} - 2C_{1212}\right]\sin^2(4\theta) = 0. \tag{30}$$

As a consequence, we obtain the additional restriction $C_{1212} = (C_{1111} - C_{1122})/2$. These two restrictions ensure (29) is satisfied and thus no additional restrictions are imposed on $\mathbb{C}$. The corresponding elasticity tensor has the restrictions

$$C_{1112} = C_{2212} = 0, C_{1111} = C_{2222}, \text{ and } C_{1212} = \frac{C_{1111} - C_{1122}}{2}, \tag{31}$$

which produce the elasticity tensor

$$\mathbf{C} = \begin{bmatrix} C_{1111} & C_{1122} & 0 \\ \cdot & C_{1111} & 0 \\ \cdot & \cdot & \frac{C_{1111} - C_{1122}}{2} \end{bmatrix}. \tag{32}$$

---

[2] For (25a) and (25c), we employ the trigonometric identity $\sin^4(x) + \cos^4(x) = 1 - \frac{1}{2}\sin^2(2x)$ in combination with $C_{1111} = C_{2222}$ to obtain (30).

Moreover, given (32) one may show (11) holds for $\mathbf{Q} = \mathbf{Ref}(\theta)$ with any choice of θ. Thus, this elasticity tensor remains invariant under reflection for any choice of line of reflection. We call this group the *isotropic symmetry group*. A set of generators of this group is given by

$$\{\mathbf{Ref}(\theta) : \theta \in [0, \pi)\}\,. \tag{33}$$

The last piece to complete the proof is to consider adding an additional reflection symmetry transformation to the square symmetry group. In this case, we have two lines of reflection symmetry which do not intersect at an angle of $\frac{\pi}{4}$ or $\frac{3\pi}{4}$ and, consequently, from the second case above, we arrive at (30). This in turn implies (33) is a set of generators for the group, i.e., we obtain the isotropic symmetry group.

The next lemma shows that adding rotations to the symmetry groups in Lemma 1 does not produce new symmetry groups and therefore those symmetry groups actually describe all symmetry groups of $\mathbb{C}$.

Lemma 2 *The set of symmetry groups described in Lemma 1 is closed under the introduction of rotation symmetry transformations of $\mathbb{C}$. More specifically, if $\mathbb{C}$ is invariant under a rotation transformation $\mathbf{Q}$ and the members of a symmetry group $\mathcal{G}$ from Lemma 1, then the symmetry group generated by $\{\mathbf{Q}\} \cup \mathcal{G}$ is one of the four symmetry groups from Lemma 1.*

Proof Let $\mathbb{C}$ be invariant under a rotation transformation $\mathbf{Q}$ and the members of a symmetry group $\mathcal{G}$ from Lemma 1. If $\mathbf{Q} \in \mathcal{G}$ then $\{\mathbf{Q}\} \cup \mathcal{G} = \mathcal{G}$ and the conclusion of the lemma is immediate. Consequently, we suppose $\mathbf{Q} \notin \mathcal{G}$. We discuss each symmetry group $\mathcal{G}$: isotropic, square, rectangular, and oblique.

Isotropic: In this case, $\mathcal{G}$ is generated by (33) and, consequently, contains all rotation transformations. This is easily seen through the closure property of groups and relation (14). Ergo, $\mathbf{Q} \in \mathcal{G}$, which contradicts the assumption $\mathbf{Q} \notin \mathcal{G}$.

Square: In this case, $\mathcal{G}$ is generated by (26). Recall that in two dimensions the composition of a reflection and a rotation is a reflection (see (15)). Since $\mathbf{Q} \notin \mathcal{G}$, the closure property of groups implies the group generated by $\{\mathbf{Q}\} \cup \mathcal{G}$ contains a reflection symmetry transformation not in $\mathcal{G}$. Consequently, by Lemma 1, the symmetry group generated by $\{\mathbf{Q}\} \cup \mathcal{G}$ is the isotropic symmetry group.

Rectangular: In this case, $\mathcal{G}$ is generated by (19). As in the square case, since $\mathbf{Q} \notin \mathcal{G}$, the group generated by $\{\mathbf{Q}\} \cup \mathcal{G}$ contains a reflection symmetry transformation not in $\mathcal{G}$. Therefore, by Lemma 1, the isotropic symmetry group or the square symmetry group is a subgroup of the group generated by $\{\mathbf{Q}\} \cup \mathcal{G}$. Furthermore, the square and isotropic cases above guarantee the group generated by $\{\mathbf{Q}\} \cup \mathcal{G}$ is either the square symmetry group or the isotropic symmetry group.

Oblique: In this case, $\mathcal{G}$ is generated by (16). We further suppose the rotation transformation $\mathbf{Q}$ is given by (12) with $\theta \neq k\pi$ for $k \in \mathbb{Z}$ since rotations by $k\pi$ are already in the group generated by (16). Because

$\mathbb{C}$ is invariant under the rotation transformation $\mathbf{Q}$, we have by (11) that

$$\begin{aligned}C_{1111} &= C_{1111}\cos^4(\theta) - 4C_{1112}\sin(\theta)\cos^3(\theta) + 2C_{1122}\cos^2(\theta)\sin^2(\theta) + 4C_{1212}\cos^2(\theta)\sin^2(\theta)\\ &\quad - 4C_{2212}\sin^3(\theta)\cos(\theta) + C_{2222}\sin^4(\theta), \end{aligned}\tag{34a}$$

$$\begin{aligned}C_{1112} &= C_{1111}\sin(\theta)\cos^3(\theta) + C_{1112}(4\sin^4(\theta) - 5\sin^2(\theta) + 1)\\ &\quad + C_{1122}(\sin^3(\theta)\cos(\theta) - \sin(\theta)\cos^3(\theta)) + 2C_{1212}(\sin^3(\theta)\cos(\theta) - \sin(\theta)\cos^3(\theta))\\ &\quad + C_{2212}(3\sin^2(\theta) - 4\sin^4(\theta)) - C_{2222}\sin^3(\theta)\cos(\theta), \end{aligned}\tag{34b}$$

$$\begin{aligned}C_{1122} &= C_{1111}\cos^2(\theta)\sin^2(\theta) + 2C_{1112}(\sin(\theta)\cos^3(\theta) - \sin^3(\theta)\cos(\theta))\\ &\quad + C_{1122}(\sin^4(\theta) + \cos^4(\theta)) - 4C_{1212}\cos^2(\theta)\sin^2(\theta)\\ &\quad + 2C_{2212}(\sin^3(\theta)\cos(\theta) - \sin(\theta)\cos^3(\theta)) + C_{2222}\cos^2(\theta)\sin^2(\theta), \end{aligned}\tag{34c}$$

$$\begin{aligned}C_{1212} &= C_{1111}\cos^2(\theta)\sin^2(\theta) + 2C_{1112}(\sin(\theta)\cos^3(\theta) - \sin^3(\theta)\cos(\theta))\\ &\quad - 2C_{1122}\cos^2(\theta)\sin^2(\theta) + C_{1212}(\sin^2(\theta) - \cos^2(\theta))^2\\ &\quad + 2C_{2212}(\sin^3(\theta)\cos(\theta) - \sin(\theta)\cos^3(\theta)) + C_{2222}\cos^2(\theta)\sin^2(\theta), \end{aligned}\tag{34d}$$

$$\begin{aligned}C_{2212} &= C_{1111}\sin^3(\theta)\cos(\theta) + C_{1112}(3\sin^2(\theta) - 4\sin^4(\theta))\\ &\quad + C_{1122}(\sin(\theta)\cos^3(\theta) - \sin^3(\theta)\cos(\theta)) + 2C_{1212}(\sin(\theta)\cos^3(\theta) - \sin^3(\theta)\cos(\theta))\\ &\quad + C_{2212}(4\sin^4(\theta) - 5\sin^2(\theta) + 1) - C_{2222}\sin(\theta)\cos^3(\theta), \end{aligned}\tag{34e}$$

$$\begin{aligned}C_{2222} &= C_{1111}\sin^4(\theta) + 4C_{1112}\sin^3(\theta)\cos(\theta) + 2C_{1122}\cos^2(\theta)\sin^2(\theta) + 4C_{1212}\cos^2(\theta)\sin^2(\theta)\\ &\quad + 4C_{2212}\sin(\theta)\cos^3(\theta) + C_{2222}\cos^4(\theta). \end{aligned}\tag{34f}$$

Take the difference of (34a) and (34f) and simplify to find

$$0 = (C_{1111} - C_{2222})\sin^2(\theta) + 2(C_{1112} + C_{2212})\sin(\theta)\cos(\theta). \tag{35}$$

Similarly, sum the equations for C_{1112} and C_{2212} and then simplify to find

$$0 = (C_{1111} - C_{2222})\sin(\theta)\cos(\theta) - 2(C_{1112} + C_{2212})\sin^2(\theta). \tag{36}$$

Since $\theta \neq k\pi$, we know $\sin(\theta) \neq 0$ and we may divide (35) and (36) by $\sin(\theta)$ to obtain

$$0 = 2(C_{1112} + C_{2212})\cos(\theta) + (C_{1111} - C_{2222})\sin(\theta), \tag{37a}$$

$$0 = (C_{1111} - C_{2222})\cos(\theta) - 2(C_{1112} + C_{2212})\sin(\theta). \tag{37b}$$

Multiplying (37a) by $\cos(\theta)$ and (37b) by $\sin(\theta)$, and taking the difference of the two equations, yields

$$0 = 2(C_{1112} + C_{2212}) \Rightarrow C_{1112} = -C_{2212}. \tag{38}$$

Imposing (38) on (35), recalling $\theta \neq k\pi$ for $k \in \mathbb{Z}$, and then simplifying produces

$$(C_{1111} - C_{2222})\sin^2(\theta) = 0 \Rightarrow C_{1111} = C_{2222}. \tag{39}$$

Substituting (38) and (39) into (34) and simplifying results in just two unique equations:

$$0 = (C_{1111} - C_{1122} - 2C_{1212})\sin^2(2\theta) + 4C_{1112}\cos(2\theta)\sin(2\theta), \tag{40a}$$

$$0 = -4C_{1112}\sin^2(2\theta) + (C_{1111} - C_{1122} - 2C_{1212})\cos(2\theta)\sin(2\theta). \tag{40b}$$

We will consider two cases: $\sin(2\theta) \neq 0$ and $\sin(2\theta) = 0$.

Case 1: Let us consider $\sin(2\theta) \neq 0$. Multiply (40a) by $\cot(2\theta)$ and then subtract the result by (40b) to find (see (38))

$$0 = 4C_{1112}\left(\sin^2(2\theta) + \cos^2(2\theta)\right) \Rightarrow 0 = C_{1112} = -C_{2212}. \tag{41}$$

Imposing (41) on (40a) results in

$$0 = (C_{1111} - C_{1122} - 2C_{1212})\sin^2(2\theta).$$

Since $\sin(2\theta) \neq 0$, we conclude

$$C_{1111} - C_{1122} - 2C_{1212} = 0 \Rightarrow C_{1212} = \frac{C_{1111} - C_{1122}}{2}. \tag{42}$$

Imposing (39), (41), and (42) on the elasticity tensor (17), we obtain (32) and thus the group generated by $\{\mathbf{Q}\} \cup \mathcal{G}$ is the isotropic symmetry group.

Case 2: We suppose $\sin(2\theta) = 0$, i.e., $\theta = \frac{(2k+1)\pi}{2}$ (recall $\theta \neq k\pi$ by assumption) for some $k \in \mathbb{Z}$. We further suppose $C_{1112} \neq 0$ as otherwise (17) reduces to the square elasticity tensor (see (38) and recall (39)), and we have already treated the case where a rotation is added to the square symmetry group. Note that

$$f(x) := 2(\cot(2x) - \tan(2x)) = 2\left(\frac{\cos(2x)}{\sin(2x)} - \frac{\sin(2x)}{\cos(2x)}\right)$$

is continuous on $\left(0, \frac{\pi}{4}\right)$. Moreover,

$$\lim_{x\to 0} f(x) = \infty \quad \text{and} \quad \lim_{x\to \frac{\pi}{4}} f(x) = -\infty,$$

and thus $f(x)$ has range $(-\infty, \infty)$ on the domain $\left(0, \frac{\pi}{4}\right)$ by the Intermediate Value Theorem. Consequently, we may find an $\alpha \in \left(0, \frac{\pi}{4}\right)$ such that

$$\frac{C_{1111} - C_{1122} - 2C_{1212}}{C_{1112}} = 2\left(\cot(2\alpha) - \tan(2\alpha)\right). \tag{43}$$

It turns out $\mathbb{C}$ is invariant with respect to the reflection transformation $\mathbf{Ref}(\alpha)$ (see Lemma 3 in Appendix D). Thus, the group generated by $\{\mathbf{Q}\}\cup\mathcal{G}$ contains the reflection transformation $\mathbf{Ref}(\alpha)$. Noticing $\mathbf{Rot}\left(\frac{\pi}{2}\right)$ is guaranteed to be in $\mathcal{G}$ because $\mathbf{Rot}\left(\frac{(2k+1)\pi}{2}\right)$ and $\mathbf{Rot}\,(\pi)$ are, using (15) we see that

$$\mathbf{Ref}\left(\alpha + \frac{\pi}{4}\right) = \mathbf{Rot}\left(\frac{\pi}{2}\right)\mathbf{Ref}(\alpha) \tag{44}$$

is also in the group generated by $\{\mathbf{Q}\}\cup\mathcal{G}$. Thus, the group generated by $\mathcal{K} := \left\{-\mathbf{I}, \mathbf{Ref}(\alpha), \mathbf{Ref}\left(\alpha + \frac{\pi}{4}\right)\right\}$ is a subgroup of the group generated by $\{\mathbf{Q}\} \cup \mathcal{G}$. On the other hand, by noticing $\mathbf{Ref}(\alpha)$ is its own inverse, from (44) we have

$$\mathbf{Rot}\left(\frac{\pi}{2}\right) = \mathbf{Ref}\left(\alpha + \frac{\pi}{4}\right)\mathbf{Ref}(\alpha), \tag{45}$$

and we deduce $\mathbf{Q}$ is in the group generated by $\mathcal{K}$. Consequently, the groups generated by $\mathcal{K}$ and $\{\mathbf{Q}\}\cup\mathcal{G}$ are the same. By considering a change in orientation, specifically a clockwise rotation by α, we see that the group generated by $\mathcal{K}$ is in the square symmetry class (see (26)).

Remark 2 Notice that adding a rotation to the oblique symmetry group produces either the isotropic symmetry group or the square symmetry group, but not the rectangular symmetry group. In Table 1, we summarize the symmetry transformations and the corresponding restrictions imposed on the elasticity tensor for each symmetry group. Unlike in the three-dimensional case, where symmetry groups in classical linear elasticity can be distinguished by their rotations [6], in two dimensions we cannot distinguish between the symmetry groups solely based on the rotations contained in the group.

4 Conclusions

The objective of this paper was to present a new, elementary, and self-contained proof that there are exactly four symmetry classes of the elasticity tensor in two-dimensional linear elasticity: oblique, rectangular, square, and isotropic. This was accomplished by an exhaustive search for the subgroups generated by reflection symmetry transformations and then showing the set of those groups is closed under the introduction of rotation symmetry transformations. We identified the sets of transformations (rotations and reflections) for each symmetry class. It is interesting to note that unlike for the elasticity tensor in three dimensions, in two dimensions the symmetry groups of the elasticity tensor cannot be characterized solely based on rotation transformations. In addition, for the sake of convenience and completeness, in appendices A and C, respectively, we describe the two-dimensional equations of motion of classical linear elasticity for each of the symmetry classes and prove orthogonal transformations in two dimensions are either rotations or reflections.

Symmetry Group	Reflections	Rotations	$\mathbb{C}$ Restrictions
Oblique	None	$\mathbf{Rot}(\pi)$	None
Rectangular	$\mathbf{Ref}(0), \mathbf{Ref}(\frac{\pi}{2})$	$\mathbf{Rot}(\pi)$	$C_{1112} = C_{2212} = 0$
Square	$\mathbf{Ref}(0), \mathbf{Ref}(\frac{\pi}{4}), \mathbf{Ref}(\frac{\pi}{2}), \mathbf{Ref}(\frac{3\pi}{4})$	$\mathbf{Rot}(\frac{\pi}{2}), \mathbf{Rot}(\pi), \mathbf{Rot}(\frac{3\pi}{2})$	$C_{1112} = C_{2212} = 0$ $C_{1111} = C_{2222}$
Isotropic	$\mathbf{Ref}(\theta), \theta \in [0, \pi)$	$\mathbf{Rot}(\theta), \theta \in [0, 2\pi)$	$C_{1112} = C_{2212} = 0$ $C_{1111} = C_{2222}$ $C_{1212} = \frac{C_{1111}-C_{1122}}{2}$

Table 1: Symmetry transformations and elasticity tensor restrictions by symmetry group in two dimensions.

A Equations of motion in two-dimensional classical linear elasticity

In this section, we consider the equation of motion (8) for representative members of each of the four symmetry classes. The corresponding elasticity tensors are given in Theorem 1.

Oblique: There are no restrictions on the elasticity tensor $\mathbb{C}$ under oblique symmetry and thus the oblique equation of motion (in component form) is given by (8), which we write out explicitly below:

$$\rho \ddot{u}_1 = C_{1111}\frac{\partial^2 u_1}{\partial x^2} + 2C_{1112}\frac{\partial^2 u_1}{\partial x \partial y} + C_{1212}\frac{\partial^2 u_1}{\partial y^2} + C_{1112}\frac{\partial^2 u_2}{\partial x^2} + (C_{1122} + C_{1212})\frac{\partial^2 u_2}{\partial x \partial y} + C_{2212}\frac{\partial^2 u_2}{\partial y^2} + b_1, \tag{46a}$$

$$\rho \ddot{u}_2 = C_{1112}\frac{\partial^2 u_1}{\partial x^2} + (C_{1122} + C_{1212})\frac{\partial^2 u_1}{\partial x \partial y} + C_{2212}\frac{\partial^2 u_1}{\partial y^2} + C_{1212}\frac{\partial^2 u_2}{\partial x^2} + 2C_{2212}\frac{\partial^2 u_2}{\partial x \partial y} + C_{2222}\frac{\partial^2 u_2}{\partial y^2} + b_2. \tag{46b}$$

Rectangular: Imposing the rectangular symmetry restrictions (20) on (46) produces the rectangular equation of motion (in component form):

$$\rho \ddot{u}_1 = C_{1111}\frac{\partial^2 u_1}{\partial x^2} + C_{1212}\frac{\partial^2 u_1}{\partial y^2} + (C_{1122} + C_{1212})\frac{\partial^2 u_2}{\partial x \partial y} + b_1, \tag{47a}$$

$$\rho \ddot{u}_2 = (C_{1122} + C_{1212})\frac{\partial^2 u_1}{\partial x \partial y} + C_{1212}\frac{\partial^2 u_2}{\partial x^2} + C_{2222}\frac{\partial^2 u_2}{\partial y^2} + b_2. \tag{47b}$$

Square: Imposing the square symmetry restrictions (27) on (46) produces the square equation of motion (in component form):

$$\rho \ddot{u}_1 = C_{1111}\frac{\partial^2 u_1}{\partial x^2} + C_{1212}\frac{\partial^2 u_1}{\partial y^2} + (C_{1122} + C_{1212})\frac{\partial^2 u_2}{\partial x \partial y} + b_1, \tag{48a}$$

$$\rho \ddot{u}_2 = (C_{1122} + C_{1212})\frac{\partial^2 u_1}{\partial x \partial y} + C_{1212}\frac{\partial^2 u_2}{\partial x^2} + C_{1111}\frac{\partial^2 u_2}{\partial y^2} + b_2. \tag{48b}$$

Isotropic: Imposing the isotropic symmetry restrictions (31) on (46) produces the isotropic equation of motion (in component form):

$$\rho \ddot{u}_1 = C_{1111}\frac{\partial^2 u_1}{\partial x^2} + \frac{1}{2}(C_{1111} - C_{1122})\frac{\partial^2 u_1}{\partial y^2} + \frac{1}{2}(C_{1111} + C_{1122})\frac{\partial^2 u_2}{\partial x \partial y} + b_1, \tag{49a}$$

$$\rho \ddot{u}_2 = \frac{1}{2}(C_{1111} + C_{1122})\frac{\partial^2 u_1}{\partial x \partial y} + \frac{1}{2}(C_{1111} - C_{1122})\frac{\partial^2 u_2}{\partial x^2} + C_{1111}\frac{\partial^2 u_2}{\partial y^2} + b_2. \tag{49b}$$

B Linearization of the components of the elasticity tensor under reflections and rotations

In this section, we linearize the expressions of the transformed components of the elasticity tensor in order to identify interesting properties of the two-dimensional elasticity tensor. Under the transformation $\mathbf{Q} = \mathbf{Ref}(\alpha)$, the components of the elasticity tensor $\mathbb{C}$ are (see (11))

$$\begin{aligned}
C'_{1111} &= \left(\frac{C_{1111}}{8} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2} + \frac{C_{2222}}{8}\right)\cos(8\alpha) + \left(\frac{C_{1112}}{2} - \frac{C_{2212}}{2}\right)\sin(8\alpha) \\
&\quad + \left(\frac{C_{1111}}{2} - \frac{C_{2222}}{2}\right)\cos(4\alpha) + (C_{1112} + C_{2212})\sin(4\alpha) + \left(\frac{3C_{1111}}{8} + \frac{C_{1122}}{4} + \frac{C_{1212}}{2} + \frac{3C_{2222}}{8}\right), \\
C'_{1112} &= -\left(\frac{C_{1112}}{2} - \frac{C_{2212}}{2}\right)\cos(8\alpha) + \left(\frac{C_{1111}}{8} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2} + \frac{C_{2222}}{8}\right)\sin(8\alpha) \\
&\quad - \left(\frac{C_{1112}}{2} + \frac{C_{2212}}{2}\right)\cos(4\alpha) + \left(\frac{C_{1111}}{4} - \frac{C_{2222}}{4}\right)\sin(4\alpha), \\
C'_{1122} &= -\left(\frac{C_{1111}}{8} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2} + \frac{C_{2222}}{8}\right)\cos(8\alpha) - \left(\frac{C_{1112}}{2} - \frac{C_{2212}}{2}\right)\sin(8\alpha) \\
&\quad + \left(\frac{C_{1111}}{8} + \frac{3C_{1122}}{4} - \frac{C_{1212}}{2} + \frac{C_{2222}}{8}\right), \\
C'_{1212} &= -\left(\frac{C_{1111}}{8} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2} + \frac{C_{2222}}{8}\right)\cos(8\alpha) - \left(\frac{C_{1112}}{2} - \frac{C_{2212}}{2}\right)\sin(8\alpha) \\
&\quad + \left(\frac{C_{1111}}{8} - \frac{C_{1122}}{4} + \frac{C_{1212}}{2} + \frac{C_{2222}}{8}\right), \\
C'_{2212} &= \left(\frac{C_{1112}}{2} - \frac{C_{2212}}{2}\right)\cos(8\alpha) - \left(\frac{C_{1111}}{8} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2} + \frac{C_{2222}}{8}\right)\sin(8\alpha) \\
&\quad - \left(\frac{C_{1112}}{2} + \frac{C_{2212}}{2}\right)\cos(4\alpha) + \left(\frac{C_{1111}}{4} - \frac{C_{2222}}{4}\right)\sin(4\alpha), \\
C'_{2222} &= \left(\frac{C_{1111}}{8} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2} + \frac{C_{2222}}{8}\right)\cos(8\alpha) + \left(\frac{C_{1112}}{2} - \frac{C_{2212}}{2}\right)\sin(8\alpha) \\
&\quad - \left(\frac{C_{1111}}{2} - \frac{C_{2222}}{2}\right)\cos(4\alpha) - (C_{1112} + C_{2212})\sin(4\alpha) + \left(\frac{3C_{1111}}{8} + \frac{C_{1122}}{4} + \frac{C_{1212}}{2} + \frac{3C_{2222}}{8}\right).
\end{aligned} \tag{50}$$

Similarly, under the transformation $\mathbf{Q} = \mathbf{Rot}(\alpha)$, the components of the elasticity tensor $\mathbb{C}$ are

$$\begin{aligned}
C'_{1111} &= \left(\frac{C_{1111}}{8} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2} + \frac{C_{2222}}{8}\right)\cos(4\alpha) - \left(\frac{C_{1112}}{2} - \frac{C_{2212}}{2}\right)\sin(4\alpha) \\
&\quad + \left(\frac{C_{1111}}{2} - \frac{C_{2222}}{2}\right)\cos(2\alpha) - (C_{1112} + C_{2212})\sin(2\alpha) + \left(\frac{3C_{1111}}{8} + \frac{C_{1122}}{4} + \frac{C_{1212}}{2} + \frac{3C_{2222}}{8}\right), \\
C'_{1112} &= \left(\frac{C_{1112}}{2} - \frac{C_{2212}}{2}\right)\cos(4\alpha) + \left(\frac{C_{1111}}{8} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2} + \frac{C_{2222}}{8}\right)\sin(4\alpha) \\
&\quad + \left(\frac{C_{1112}}{2} + \frac{C_{2212}}{2}\right)\cos(2\alpha) + \left(\frac{C_{1111}}{4} - \frac{C_{2222}}{4}\right)\sin(2\alpha), \\
C'_{1122} &= -\left(\frac{C_{1111}}{8} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2} + \frac{C_{2222}}{8}\right)\cos(4\alpha) + \left(\frac{C_{1112}}{2} - \frac{C_{2212}}{2}\right)\sin(4\alpha) \\
&\quad + \left(\frac{C_{1111}}{8} + \frac{3C_{1122}}{4} - \frac{C_{1212}}{2} + \frac{C_{2222}}{8}\right), \\
C'_{1212} &= -\left(\frac{C_{1111}}{8} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2} + \frac{C_{2222}}{8}\right)\cos(4\alpha) + \left(\frac{C_{1112}}{2} - \frac{C_{2212}}{2}\right)\sin(4\alpha) \\
&\quad + \left(\frac{C_{1111}}{8} - \frac{C_{1122}}{4} + \frac{C_{1212}}{2} + \frac{C_{2222}}{8}\right), \\
C'_{2212} &= -\left(\frac{C_{1112}}{2} - \frac{C_{2212}}{2}\right)\cos(4\alpha) - \left(\frac{C_{1111}}{8} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2} + \frac{C_{2222}}{8}\right)\sin(4\alpha) \\
&\quad + \left(\frac{C_{1112}}{2} + \frac{C_{2212}}{2}\right)\cos(2\alpha) + \left(\frac{C_{1111}}{4} - \frac{C_{2222}}{4}\right)\sin(2\alpha), \\
C'_{2222} &= \left(\frac{C_{1111}}{8} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2} + \frac{C_{2222}}{8}\right)\cos(4\alpha) - \left(\frac{C_{1112}}{2} - \frac{C_{2212}}{2}\right)\sin(4\alpha) \\
&\quad - \left(\frac{C_{1111}}{2} - \frac{C_{2222}}{2}\right)\cos(2\alpha) + (C_{1112} + C_{2212})\sin(2\alpha) + \left(\frac{3C_{1111}}{8} + \frac{C_{1122}}{4} + \frac{C_{1212}}{2} + \frac{3C_{2222}}{8}\right).
\end{aligned} \tag{51}$$

One immediate observation from (50) and (51) is that the expressions do not contain sine and cosine of odd multiples of α. This is not entirely unexpected as the transformation (see (11) with $\mathbf{Q}$ given by (12) or (13)) employs terms of the form $(\cos(2\alpha))^n(\sin(2\alpha))^{4-n}$ in (50) and $(\cos(\alpha))^n(\sin(\alpha))^{4-n}$ in (51) for $n \in \{0, 1, 2, 3, 4\}$, which when linearized, produce constant terms and terms of the form $\cos(2m\alpha)$ and $\sin(2m\alpha)$ for $m \in \{1, 2, 3, 4\}$. For rotations, this was observed in [12] wherein the so-called parameters of Tsai and Pagano were introduced. Additionally, there are deeper connections between the harmonic decomposition of the elasticity tensor and this linearized form of the transformed components of $\mathbb{C}$ under rotations which were explored in [7].

C Orthogonal transformations in two dimensions

For the sake of completeness, this section presents a proof that, in two dimensions, orthogonal transformations are either rotations or reflections. The proof presented here is straightforward and we make no claims of originality.

Proposition 1 *In two dimensions, orthogonal transformations are either rotations or reflections.*

Proof Let

$$\mathbf{A} := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{52}$$

be an orthogonal transformation in two dimensions. Consequently, $\mathbf{A}\mathbf{A}^T = \mathbf{I}$ with $\mathbf{I}$ the identity transformation, which implies

$$1 = a^2 + b^2,\ 0 = ac + bd,\ \text{ and } 1 = c^2 + d^2. \tag{53}$$

From (53), we see that the points (a, b) and (c, d) lie on the unit circle and without loss of generality we may introduce angles $\theta, \phi \in [0, 2\pi)$ such that

$$a = \cos(\theta),\ b = \sin(\theta),\ c = \cos(\phi),\ \text{ and } d = \sin(\phi). \tag{54}$$

From (53) and (54), we have

$$0 = ac + bd = \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi) = \cos(\theta - \phi) \Rightarrow \theta - \phi = \frac{\pi}{2} + k\pi, \tag{55}$$

for some $k \in \mathbb{Z}$. If k is even, then from the periodicity of the sine and cosine functions, as well as the fact that the sine function is simply a quarter period shift of the cosine function, we have

$$\mathbf{A} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \cos(\theta - \frac{\pi}{2}) & \sin(\theta - \frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}. \tag{56}$$

From (13), we see that (56) corresponds to a reflection about the line through the origin making an angle of $\frac{\theta}{2}$ with the x-axis. Alternatively, if k is odd, then from the periodicity of the sine and cosine functions, the fact that the sine function is simply a quarter period shift of the cosine function, and the odd and even properties of the sine and cosine functions, respectively, we have

$$\mathbf{A} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \cos(\theta + \frac{\pi}{2}) & \sin(\theta + \frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}. \tag{57}$$

From (12), we see that (57) corresponds to a clockwise rotation by an angle θ about the origin, or equivalently a counter-clockwise rotation by an angle $-\theta$ about the origin.

D Invariance of the elasticity tensor with respect to Ref(α)

This section presents a proof that, under particular conditions on its components, the elasticity tensor $\mathbb{C}$ is invariant with respect to a specific reflection transformation.

Lemma 3 *Let C_{ijkl}, for $i, j, k, l \in \{1, 2\}$, be the components of the elasticity tensor $\mathbb{C}$. If $C_{1111} = C_{2222}$, $C_{1112} = -C_{2212}$, and there exists an $\alpha \in \left(0, \frac{\pi}{4}\right)$ such that*

$$C_{1111} - C_{1122} - 2C_{1212} = 2C_{1112}\left(\cot(2\alpha) - \tan(2\alpha)\right), \tag{58}$$

*then $\mathbb{C}$ is invariant under the transformation **Ref**(α).*

Proof Recall $\mathbb{C}$ is invariant under the transformation $\mathbf{Ref}(\alpha)$ if (11) holds with $\mathbf{Q} = \mathbf{Ref}(\alpha)$. Conveniently, the components of the elasticity tensor under the transformation $\mathbf{Ref}(\alpha)$ are presented in (50). Demonstrating $C'_{ijkl} = C_{ijkl}$ for $i, j, k, l \in \{1, 2\}$ in (50) proves the invariance of the elasticity tensor under the transformation $\mathbf{Ref}(\alpha)$. We start by substituting the assumptions $C_{2222} = C_{1111}$ and $C_{2212} = -C_{1112}$ into (50) to obtain

$$\begin{aligned}
C'_{1111} =& \left(\frac{C_{1111}}{4} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2}\right)\cos(8\alpha) + C_{1112}\sin(8\alpha) + \left(\frac{3C_{1111}}{4} + \frac{C_{1122}}{4} + \frac{C_{1212}}{2}\right),\\
C'_{1112} =& -C_{1112}\cos(8\alpha) + \left(\frac{C_{1111}}{4} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2}\right)\sin(8\alpha),\\
C'_{1122} =& -\left(\frac{C_{1111}}{4} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2}\right)\cos(8\alpha) - C_{1112}\sin(8\alpha) + \left(\frac{C_{1111}}{4} + \frac{3C_{1122}}{4} - \frac{C_{1212}}{2}\right),\\
C'_{1212} =& -\left(\frac{C_{1111}}{4} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2}\right)\cos(8\alpha) - C_{1112}\sin(8\alpha) + \left(\frac{C_{1111}}{4} - \frac{C_{1122}}{4} + \frac{C_{1212}}{2}\right),\\
C'_{2212} =& C_{1112}\cos(8\alpha) - \left(\frac{C_{1111}}{4} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2}\right)\sin(8\alpha),\\
C'_{2222} =& \left(\frac{C_{1111}}{4} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2}\right)\cos(8\alpha) + C_{1112}\sin(8\alpha) + \left(\frac{3C_{1111}}{4} + \frac{C_{1122}}{4} + \frac{C_{1212}}{2}\right).
\end{aligned} \tag{59}$$

Next, we subtract C_{ijkl} from both sides of each equation, where the indices correspond to the indices of C'_{ijkl} in each equation. This is followed by combining like terms to find

$$\begin{aligned}
C'_{1111} - C_{1111} =& -\left(\frac{C_{1111}}{4} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2}\right)(1-\cos(8\alpha)) + C_{1112}\sin(8\alpha),\\
C'_{1112} - C_{1112} =& -C_{1112}(1+\cos(8\alpha)) + \left(\frac{C_{1111}}{4} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2}\right)\sin(8\alpha),\\
C'_{1122} - C_{1122} =& \left(\frac{C_{1111}}{4} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2}\right)(1-\cos(8\alpha)) - C_{1112}\sin(8\alpha),\\
C'_{1212} - C_{1212} =& \left(\frac{C_{1111}}{4} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2}\right)(1-\cos(8\alpha)) - C_{1112}\sin(8\alpha),\\
C'_{2212} - C_{2212} =& C_{1112}(1+\cos(8\alpha)) - \left(\frac{C_{1111}}{4} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2}\right)\sin(8\alpha),\\
C'_{2222} - C_{2222} =& -\left(\frac{C_{1111}}{4} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2}\right)(1-\cos(8\alpha)) + C_{1112}\sin(8\alpha).
\end{aligned} \tag{60}$$

Note that in the equations for C'_{2212} and C_{2222} in (60), we employed the assumptions $C_{2212} = -C_{1112}$ and $C_{2222} = C_{1111}$. To complete the proof, we demonstrate the right-hand side of each equation in (60) is null. Up to a change in sign, there are only two unique expressions on the right-hand sides of the equations in (60):

$$\begin{aligned}
&\left(\frac{C_{1111}}{4} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2}\right)(1-\cos(8\alpha)) - C_{1112}\sin(8\alpha),\\
&C_{1112}(1+\cos(8\alpha)) - \left(\frac{C_{1111}}{4} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2}\right)\sin(8\alpha).
\end{aligned} \tag{61}$$

Based on the final assumption (58), we substitute $\left(\frac{C_{1111}}{4} - \frac{C_{1122}}{4} - \frac{C_{1212}}{2}\right) = \frac{1}{2}C_{1112}(\cot(2\alpha) - \tan(2\alpha))$ into (61) to obtain

$$\begin{aligned}
&\left(\frac{1}{2}(\cot(2\alpha) - \tan(2\alpha))(1-\cos(8\alpha)) - \sin(8\alpha)\right)C_{1112},\\
&\left(1+\cos(8\alpha) - \frac{1}{2}(\cot(2\alpha) - \tan(2\alpha))\sin(8\alpha)\right)C_{1112}.
\end{aligned} \tag{62}$$

Employing the trigonometric identities $\cot(2\alpha) - \tan(2\alpha) = 2\cot(4\alpha), 1-\cos(8\alpha) = 2\sin^2(4\alpha), 1+\cos(8\alpha) = 2\cos^2(4\alpha)$, and $\sin(8\alpha) = 2\sin(4\alpha)\cos(4\alpha)$, we may transform (62) into

$$\begin{aligned}
&\left(2\cot(4\alpha)\sin^2(4\alpha) - 2\sin(4\alpha)\cos(4\alpha)\right)C_{1112} = 0,\\
&\left(2\cos^2(4\alpha) - 2\cot(4\alpha)\sin(4\alpha)\cos(4\alpha)\right)C_{1112} = 0.
\end{aligned} \tag{63}$$

The equalities in (63) follow by recalling the identity $\cot(4\alpha) = \frac{\cos(4\alpha)}{\sin(4\alpha)}$. Consequently, the right-hand sides of (60) are null and thus $C'_{ijkl} = C_{ijkl}$ for $i, j, k, l \in \{1, 2\}$. We conclude $\mathbb{C}$ is invariant under the transformation $\mathbf{Ref}(\alpha)$.

Acknowledgements Research sponsored by the Laboratory Directed Research and Development Program of Oak Ridge National Laboratory, managed by UT-Battelle, LLC, for the U. S. Department of Energy.

References

1. Auffray, N., Kolev, B., Olive, M.: Handbook of bi-dimensional tensors: Part I: Harmonic decomposition and symmetry classes. Mathematics and Mechanics of Solids **22**(9), 1847–1865 (2017)
2. Blinowski, A., Ostrowska-Maciejewska, J., Rychlewski, J.: Two-dimensional Hooke's tensors - isotropic decomposition, effective symmetry criteria. Archives of Mechanics **48**(2), 325–345 (1996)
3. Bóna, A., Bucataru, I., Slawinski, M.A.: Material symmetries of elasticity tensors. The Quarterly Journal of Mechanics and Applied Mathematics **57**(4), 583–598 (2004)
4. Chadwick, P., Vianello, M., Cowin, S.C.: A new proof that the number of linear elastic symmetries is eight. Journal of the Mechanics and Physics of Solids **49**(11), 2471 – 2492 (2001)
5. Dummit, D.S., Foote, R.M.: Abstract Algebra, 3 edn. Wiley, New York (2004)
6. Forte, S., Vianello, M.: Symmetry classes for elasticity tensors. Journal of Elasticity **43**(2), 81–108 (1996)
7. Forte, S., Vianello, M.: A unified approach to invariants of plane elasticity tensors. Meccanica **49**(9), 2001–2012 (2014)
8. He, Q.-C., Zheng, Q.-S.: On the symmetries of 2D elastic and hyperelastic tensors. Journal of Elasticity **43**(3), 203–225 (1996)
9. Timoshenko, S.: Theory of Elasticity. Engineering Societies Monographs. McGraw-Hill (1934)
10. Ting, T.C.T.: Anisotropic Elasticity: Theory and Applications. Oxford University Press (1996)
11. Ting, T.C.T.: Generalized Cowin–Mehrabadi theorems and a direct proof that the number of linear elastic symmetries is eight. International Journal of Solids and Structures **40**(25), 7129–7142 (2003)
12. Tsai, S.W., Pagano, N.J.: Invariant properties of composite materials. Tech. Rep. AFML-TR-67-349, Air Force Materials Laboratory, Wright-Patterson Air Force Base, Ohio (1968)
13. Vannucci, P.: Plane anisotropy by the polar method. Meccanica **40**(4), 437–454 (2005)
14. Vianello, M.: An integrity basis for plane elasticity tensors. Archives of Mechanics **49**(1), 197–208 (1997)