

# Exciton Condensation in Quantum Hall Bilayers at Total Filling $\nu_T = 5$

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We study the coupled quantum Hall bilayers each at half-filled first excited Landau levels with varying the layer distance. Based on numerical exact diagonalization on torus, we identify two distinct phases separated by a critical layer distance  $d_c$ . From  $d_c$  to infinite layer distance, the topological phase is smoothly connected to a direct tensor product of two Moore-Read states, while the interlayer coherence emerges at  $d < d_c$  characterized by the  $xy$  easy-plane ferromagnetic energy spectra, gapless pseudospin excitations, long range current-current correlations and the finite exciton superfluid stiffness, corresponding to the exciton superfluid state. More interestingly, the results of the ground state fidelity, the evolution of energy spectra, and the superfluid stiffness indicate a possible continuous transition. Theoretically it can be interpreted as a topological phase transition which simultaneously changes the topology of ground state and breaks symmetry, providing an interesting example of transitions beyond Landau paradigm.

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*Introduction.*—The quantum Hall bilayers [1, 2], which can be realized in quantum wells [3, 4] or bilayer graphenes [5–7], have triggered substantial interest in pursuing exotic phenomena such as the Bose-Einstein condensation of the excitons [8, 9] and anyonic statistics [10–13]. The possible emerging non-Abelian physics and quantum phase transitions remain not well understood [14–16].

In particular, the bilayers with half filled lowest Landau level (LL) for each layer have attracted great interest from both experimental measurements [9, 17–21] and theoretical investigations [22–47]. The exciton superfluid phase (or Halperin “111 state” [2, 48]) was first established experimentally at a layer distance comparable to the magnetic length [9] based on a zero-bias interlayer tunneling conductance [49] and a vanishing Hall counterflow resistance [20, 50]. Other phases like the composite Fermi liquid (CFL) at larger distance [51] and the novel intermediate phase [26–28] have also been extensively investigated. In addition, the nature of quantum phase transitions among various phases is still controversial. Inspired by the rich physics of these  $\nu_T = 1$  bilayers with half filled lowest LL in each layer, a natural question arises about the quantum phase diagram for the electronic systems with fully filled lowest LL and half-filled first excited LL, corresponding to the bilayers with total filling  $\nu_T = 5$ . Each decoupled layer with filling  $\nu = 5/2$  is believed to be the Moore-Read (MR) state [11, 52, 53]. When the layer distance goes to zero, the interlayer coherent state is theoretically expected though there is no experimental study presented along this line. By tuning the layer distance, the nature of possible intermediate phase and the quantum phase transition remain unclear, which motivate our present work.

Previous theoretical studies have not reached a consistent conclusion on this problem. The calculations based on Hartree-Fock approximation claim a transition from “111 state” to a charge ordered state [38, 54], while the variational and exact diagonalization (ED) calculations on sphere geometry

found a bilayer phase coherent state at small layer distance and two uncoupled  $5/2$  state at large layer separations by Shi *et al* [55]. Nevertheless, unbiased exact simulations for quantum states at intermediate layer distances and the quantum phase transition for torus geometry are still absent. Different from the sphere geometry, there is no orbital number shift on torus and the competing states with the same filling factor can be compared on an equal footing [56].

In the present work, we use ED to calculate systems with up to 18 electrons on torus. Based on the energy spectra, pseudospin gap, exciton superfluid stiffness, current-current correlations, the Berry curvature as well as drag Hall conductance, we identify a direct phase transition at  $d_c$  between the exciton superfluid phase with interlayer coherence and the phase with strong intralayer correlations, the latter can be smoothly connected to the decoupled two copies of MR state. Here the finding of  $d_c$  is consistent with previous variational calculation [55]. Moreover, the calculation of fidelity, the exciton superfluid stiffness, the evolution of energy spectrum, and the ground state energy derivatives indicate the transition is continuous, which is beyond the Landau paradigm [14–16, 57, 58]. Based on the analysis of symmetries and topological orders, we propose the theoretical interpretations of such a transition as exciton condensation which simultaneously breaks  $U(1) \times U(1)$  symmetry and changes the topology. The exciton condensation leads to the  $C = 2$  topologically ordered state in Kitaev’s notation [59], which is consistent with the “111” state.

*Model and Method.*— We consider the  $\nu_T = 5/2 + 5/2$  bilayer electronic systems subject to a perpendicular magnetic field. We neglect the width of these two identical layers and put them on torus spanned by vectors  $\mathbf{L}_x$  and  $\mathbf{L}_y$ . The orbital number (or flux number) in each layer  $N_\phi$  is determined by the area of torus, i.e.,  $|\mathbf{L}_x \times \mathbf{L}_y| = 2\pi N_\phi$ . In the absence of the interlayer tunneling, this system with fully polarized spins can be described by the projected Coulomb interaction, which

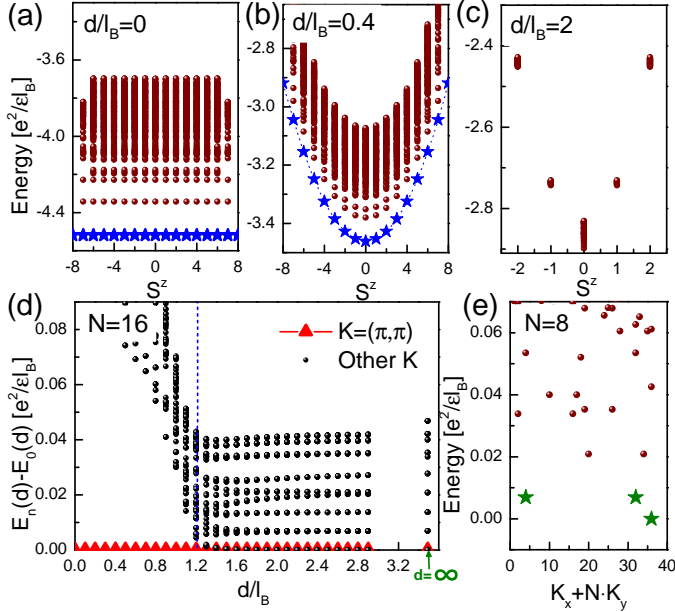


Fig. 1. (Color online) The energy spectra of different pseudospin  $S_z$  sectors at layer distance (a)  $d/l_B = 0$ , (b)  $d/l_B = 0.4$ , (c)  $d/l_B = 2$ . (d) The low-lying energy spectra as a function of layer distance  $d/l_B$ . Here, the total electron number  $N = 16$  and each layer has equal number of electrons. (e) The energy spectrum of single layer  $N = 8$  system at  $n=1$ LL, the green stars highlight the topological sectors of MR state.

reads

$$V = \frac{1}{N_\phi} \sum_{i < j, \alpha, \beta} \sum_{\mathbf{q}, \mathbf{q} \neq 0} V_{\alpha\beta}(\mathbf{q}) e^{-\frac{q^2}{2}} L_n^2\left[\frac{q^2}{2}\right] e^{i\mathbf{q} \cdot (\mathbf{R}_{\alpha,i} - \mathbf{R}_{\beta,j})}. \quad (1)$$

Here,  $\alpha(\beta) = 1, 2$  denote two layers or, equivalently, two components of a pseudospin-1/2.  $q = |\mathbf{q}| = \sqrt{q_x^2 + q_y^2}$ ,  $V_{11}(\mathbf{q}) = V_{22}(\mathbf{q}) = e^2/(\epsilon q)$  and  $V_{12}(\mathbf{q}) = V_{21}(\mathbf{q}) = e^2/(\epsilon q) \cdot e^{-qd}$  are the Fourier transformations of the intralayer and interlayer Coulomb interactions, respectively.  $d$  represents the distance between two layers in the unit of magnetic length  $l_B$ .  $L_n(x)$  is the Laguerre polynomial with Landau level index  $n$  and  $\mathbf{R}_{\alpha,i}$  is the guiding center coordinate of the  $i$ -th electron in layer  $\alpha$ . Here we consider rectangular unit cells with  $L_x = L_y = L$  [60].

**Energy Spectra.**—Without the interlayer tunneling, the bilayer system has separate conservations for the electron number in each layer, which allows us to label eigenstates by pseudospin  $S_z$  defined as  $S_z \equiv (N^\uparrow - N^\downarrow)/2$ , where  $N^\uparrow$  and  $N^\downarrow$  denote the number of electrons for the top and bottom layers, respectively. Then we can study the energy spectra by targeting different pseudospin sectors. Here, the energy shift  $d \cdot S_z^2/N_\phi$  induced by the imbalance of charge in two layers [62] has been considered. When the layer distance goes to zero, as shown in Fig. 1 (a), the lowest energies in each pseudospin  $S_z$  sector are exactly degenerate, indicating that

we have not only conserved  $S_z$  but also full  $SU(2)$  symmetry. This spectrum is consistent with the exciton condensed “111 state”, with spontaneous ferromagnetization which can be seen from the ground state spin degeneracy. However, when the layer distance is finite but small enough, as shown in the Fig. 1 (b), our data shows the nondegenerate ground state located in the  $S_z = 0$  sector, and the low energy excitations are pseudospin excitations among different  $S_z$  sectors, which can be fitted into  $\Delta E = E(S_z) - E(S_z = 0) = \alpha S_z^2$ . These facts indicate that the ground state is an  $xy$  easy-plane ferromagnet instead of Ising ferromagnet, and the interlayer correlations dominate the low energy physics for small  $d$ . Physically, an electron in one layer is bound to a hole in the other layer forming an exciton at  $d = 0$ , then the bilayer system can be mapped into a monolayer at  $\nu = 1$  for the first excited Landau level. When  $d$  is finite but smaller than a critical value, a difference between the interlayer and intralayer Coulomb interaction breaks the pseudospin invariance down to  $U(1)$ , leading to the  $xy$  easy-plane pseudospin ferromagnet as indicated in Fig. 1 (a) and (b). However, for larger layer distance  $d = 2.0l_B$ , the lowest energy excitations exist within the same pseudospin  $S_z$  sector [see Fig. 1 (c)], indicating the low-lying excitations are dominated by the intralayer correlations. These results indicate there are two distinct phases as the layer distance  $d$  is varied.

The flow of low-lying energies with  $d/l_B$  indeed indicates a direct transition at  $d_c/l_B \approx 1.2$  from exciton superfluid phase ( $d < d_c$ ) to a phase with distinct structure of spectra ( $d > d_c$ ) which can be smoothly connected to the decoupled two copies of MR state at  $d/l_B = +\infty$  [see Fig. 1(d)]. Figure 1 (e) shows the energy spectrum of each decoupled layer with 8 electrons, where the threefold degeneracy (in addition to the twofold center of mass degeneracy) in momentum sectors  $(K_x, K_y)/(2\pi/N) = (N/2, N/2), (0, N/2), (N/2, 0)$  occurs supporting that each decoupled layer is indeed in the MR state. When coupling two layers together, we identified a 36-fold near degeneracy of two copies of MR state at  $d > d_c$  side.

**Pseudospin Excitations.**—From the energy spectra we identify a single phase transition at  $d_c/l_B \approx 1.2$  without energy level crossing. Below we characterize the transition from the perspective of low energy excitations. We directly calculate the pseudospin excitation gap, which measures the energy cost when flipping the pseudospin of one particle. The pseudospin gap is defined as  $\Delta_{ps}(d) \equiv E_0(N_\uparrow, N_\downarrow, d) - E_0(N/2, N/2, d) + d \cdot S_z^2/N_\phi$  [61], where  $N_\uparrow = N/2 + S_z$  and  $N_\downarrow = N/2 - S_z$ . As shown in Fig. 2 (a) and (c), the finite size scaling of  $\Delta_{ps}(d)$  for  $S_z = 1$  and  $S_z = 2$  show the excitation gap goes to zero in the thermodynamic limit for  $d/l_B \lesssim 1.2$ . However, for  $d/l_B \gtrsim 1.2$ , the  $S_z = 1$  pseudospin excitation displays significant even-odd effect determined by the electron number in each layer, as shown in Fig. 2 (b). For the systems with even number of electrons in each layer, flipping a single pseudospin costs finite energy [see the inset of Fig. 2 (b)] while the energy cost vanishes when the electron number in each layer is odd. This even-odd effect disappears in

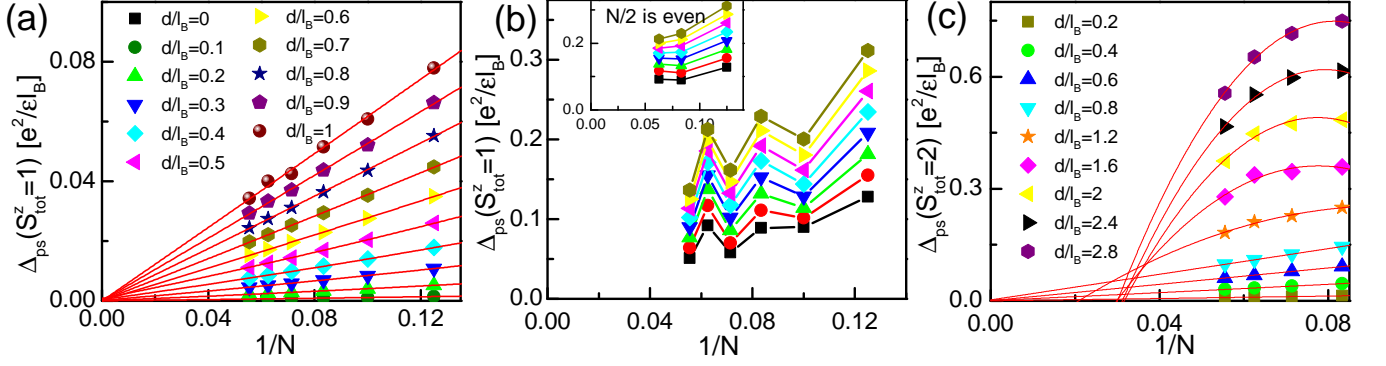


Fig. 2. (Color online) (a) to (c) show the pseudospin excitation gap  $\Delta_{ps}$  for  $S_z = 1$  and  $d < d_c$  (a);  $S_z = 1$  and  $d > d_c$  (b) and  $S_z = 2$  (c). The finite size scaling of  $\Delta_{ps}$  using parabolic function indicates the gapless nature at  $d < d_c$  [(a) and (c)], the inset of (c) shows  $\Delta_{ps}$  for the systems with even number of particles in each layer. The even-odd effect disappears at  $d > d_c$  for  $S_z = 2$  (c).

the  $S_z = 2$  pseudospin gap [see Fig. 2 (c)]. The distinct behavior of  $S_z = 1$  and  $S_z = 2$  pseudo-spin gap is consistent with the picture of the existence of intralayer paired composite fermions. Furthermore, we will show below that the interlayer coherence immediately establishes in the gapless phase at  $d \leq d_c$ , leading to the exciton superfluidity.

*Exciton Superfluidity.*— To study the exciton superfluidity, we calculate both the current-current correlations and superfluid stiffness. We define the interlayer current operator  $J_m \equiv i(c_{m\uparrow}^\dagger c_{m\downarrow} - h.c.)$  and probe the interlayer coherence by studying current-current correlations  $\langle J_m J_n \rangle$ , where  $m, n = 1, \dots, N_\phi$  are orbital indices and the corresponding distance is  $|n - m|L/N_\phi$ . As shown in the inset of Fig. 3 (a),  $\langle J_m J_n \rangle$  decays very slowly and saturates to a finite value when  $d < d_c$  while it becomes vanishingly small when  $d > d_c$ , which directly proves the existence of interlayer coherence in “111 state”. To keep track of such property when tuning layer distance  $d$ , we choose the value of  $\langle J_m J_n \rangle$  at the largest distance  $|n - m| = N_\phi/2$  and study its value versus  $d$ , as shown in Fig. 3 (a), the interlayer coherence is softened with the increase of layer distances and finally disappears after the transition at  $d_c$  smoothly.

In order to get the exciton superfluid stiffness  $\rho_s$ , we add twisted boundary phases  $0 \leq \theta_\lambda^\alpha \leq 2\pi$  along  $\lambda$  direction ( $\lambda = x$  or  $y$ ) in the layer  $\alpha$ , and study the energy evolution. Physically imposing opposite boundary phases for two layers plays a similar role as the counterflow experiments, where the longitudinal counterflow conductivity indicates the superfluidity. Fig. 3 (b) show the energy flow of the lowest two states in the same momentum sector  $(K_x, K_y) = (\pi, \pi)$  with twisted phases. The exciton superfluid stiffness  $\rho_s$ , can be obtained by fitting the energy flows according to [24]

$$E(\theta_t)/|\mathbf{L}_x \times \mathbf{L}_y| = E(\theta_t = 0)/A + \frac{1}{2}\rho_s\theta_t^2 + O(\theta_t^4), \quad (2)$$

where  $E(\theta_t)$  is the ground-state energy with twisted (opposite) boundary phases  $\theta_t$  between two layers. As shown in

Fig. 3 (b), we have finite exciton superfluid stiffness when  $d < d_c$ , while  $\rho_s = 0$  at  $d > d_c$  side due to the totally flat energy curve against twisted phases, indicating the vanish of superfluidity. The quantitative evolution of the superfluid stiffness  $\rho_s > 0$  with the layer distance will be discussed later in Fig. 4 (b) to address the precise nature of the quantum phase transition.

The interlayer correlations can also be detected by the drag Hall conductance, which can be calculated by integrating the Berry curvature  $F(\theta_x^\alpha, \theta_y^\beta)$ . Physically a gapped state has a well-defined Berry curvature and thus well-defined Chern number, while a gapless state has singularities in Berry curvature due to the energy level crossing. As shown in Fig. 3 (b) for the energy gap  $E_1 - E_0$  as a function of twisted phases, one can only get well defined Berry curvature or Chern number at  $d > d_c$  side since there is always a finite gap between the ground state and the first excited state. When  $d < d_c$ , the gap closes near twisted phase point  $(2\pi, 2\pi)$ , indicating singularities in the Berry curvature. At  $d > d_c$  side, we find the Berry curvature is near flat without any singularity (see supplementary materials) and its integral leads to drag Hall conductance  $\sigma_{xy}^d = 0$ , indicating the Hall conductances are equal in both layer symmetric and antisymmetric channels.

*Continuous phase transition.*— Since the exciton superfluid phase and two copies of MR phase have different symmetries and topological orders, a direct continuous transition is beyond the Landau paradigm. From the energy spectra in Fig. 1 (d), the level crossing is absent in the vicinity of the critical distance  $d_c$ , indicating a continuous transition. We further probe the nature of such a transition by calculating the ground state fidelity, superfluid stiffness as well as the ground-state energy derivatives. The fidelity is defined by the wave function overlap between the ground state at  $d - \Delta d$  and  $d$ , i.e.,  $F(d, \Delta d) = |\langle \Psi(d - \Delta d) | \Psi(d) \rangle|$ , which has been shown to be a good indicator to distinguish continuous transition from first-order transition for both symmetry-breaking and topological phase transitions [69, 70]. As shown in Fig. 4 (a), we find the ground-state fidelity displays a single weak dip at the criti-

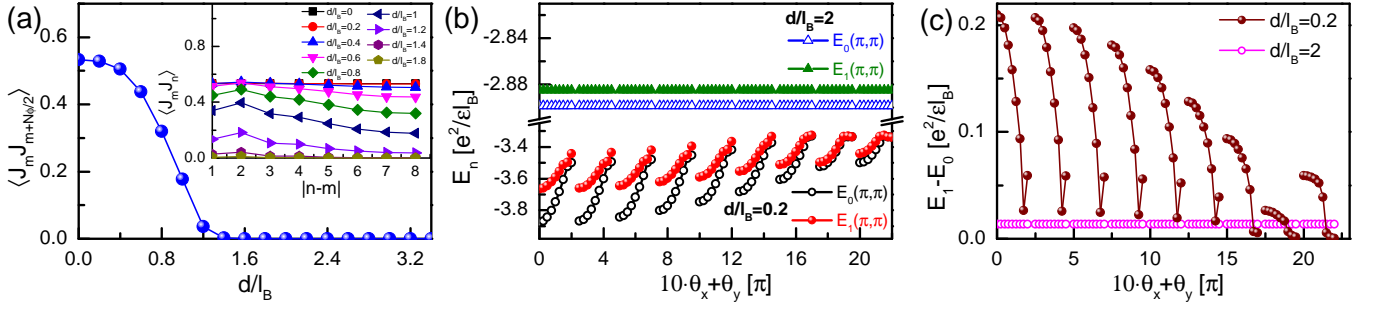


Fig. 3. (Color online) (a) The values of current-current correlations  $\langle J_m J_{m+N_\phi/2} \rangle$  versus layer distances  $d$ , the inset shows  $\langle J_m J_n \rangle$  as a function of orbital distance. (b) The energy flow with twisted boundary phases for  $d/l_B=0.2$  and  $d/l_B=2$ . (c) The energy gap  $E_1 - E_0$  as a function of twisted phases for  $d/l_B=0.2$  and  $d/l_B=2$ . Here, the systems have  $N = 16$  electrons.

cal distance  $d_c$  instead of showing a sudden jump. In addition, as shown in Fig. 4 (b), the exciton superfluid stiffness  $\rho_s$  is finite at  $d < d_c$ , but smoothly decreases with the increase of the layer distance, and becomes vanishingly small after the transition. Figures 4 (c) shows the first-order and second order derivatives of the ground-state energy, which are both smooth functions of layer distances. Thus the numerical evidence indicates the direct transition between these two phases might be continuous.

*Field theory of transition and exciton condensation.*—Here, we provide a possible scenario of the observed transition. We consider the electron to be fractionalized into a boson and a fermion with emergent  $u(1)_i$  gauge field at each layer, i.e.,  $c_i = b_i \psi_i$ , where  $i = \uparrow, \downarrow$  denotes two layers. While the  $\psi_i$  only carries  $u(1)_i$  charge,  $b_i$  carries both emergent  $u(1)_i$  and global  $U(1)_i$  charge (corresponding to the charge conservation at each layer). To obtain MR state at each layer, pairs of fermions form  $p + ip$  superconductor [11, 52], while pairs of bosons form  $\nu = 1/8$  state called  $u(1)_8$  state [71–73]. The effective theory is

$$\mathcal{L} = \sum_{i=\uparrow, \downarrow} \left( \frac{8}{4\pi} \alpha_i d\alpha_i + \frac{2}{2\pi} (eA_i + a_i) d\alpha_i - \frac{2e^2}{4\pi} A_i dA_i + \Psi_i^\dagger [i\partial_0 - a_{i,0} + h_i(\vec{p} + \vec{a}_i)] \Psi_i \right), \quad (3)$$

where  $a_{i,\mu}$  is the emergent gauge field from fractionalization, and  $\alpha_{i,\mu}$  characterizes the  $u(1)_8$  state at  $i$ -th layer.  $ada$  is a short hand notation of the Chern-Simons term  $\epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho$  [74, 75]. The first two terms correspond to  $u(1)_8$  state, and the third term characterizes the Hall response of the filled lowest LL. Integrating out  $\alpha_i$  field gives rise to the quantized Hall conductivity  $\sigma_{xy} = \frac{5}{2} \frac{e^2}{h}$  for each layer. In the second line,  $\Psi_i(p) = (\psi_i(p), \psi_i^\dagger(-p))^T$  is the Nambu spinor,  $h_i(\vec{p}) = \left( \frac{p_x^2 + p_y^2}{2m} - \mu \right) \sigma^z + \Delta_i (p_x \sigma^x + p_y \sigma^y)$  is the Bogoliubov-de-Gennes (BdG) Hamiltonian of  $p + ip$  superconductors (SC) at  $i$ -th layer with Pauli matrix  $\sigma$  acting on Nambu space, and  $\Delta_i$  denotes the pairing condensate.  $m > 0$  and  $\mu$  are the effective mass and chemical potential of frac-

tionalized fermion. When  $\mu > 0$ , the  $p + ip$  SC is in the topological phase with BdG Chern number  $C = 1$  at each layer [52, 59, 76].

The transition to “111 state” is described by interlayer exciton condensation  $\langle c_\uparrow c_\downarrow^\dagger \rangle = \langle b_\uparrow b_\downarrow^\dagger \rangle \langle \psi_\uparrow \psi_\downarrow^\dagger \rangle \neq 0$ , which simultaneously breaks  $S_z$  conservation and leads to  $C = 2$  topological order [59]. It is possible that  $\langle \psi_\uparrow \psi_\downarrow^\dagger \rangle$  becomes nonzero breaking the residue  $Z_2^\uparrow \times Z_2^\downarrow$  of the emergent  $u(1)_\uparrow \times u(1)_\downarrow$  symmetry before exciton condensation, but exciton inducing the condensation of  $\langle \psi_\uparrow \psi_\downarrow^\dagger \rangle$  is not a fine tuned result, and indeed the numerical results show a single transition. In the presence of interlayer coherence, the Hamiltonian of fractionalized fermion is  $H = \begin{pmatrix} h_\uparrow & h_{\uparrow\downarrow} \\ h_{\uparrow\downarrow}^\dagger & h_\downarrow \end{pmatrix}$ , where  $h_{\uparrow\downarrow} = \text{diag}(\Phi, -\Phi^*)$  with  $\Phi = \langle \psi_\uparrow \psi_\downarrow^\dagger \rangle$ . The BdG Chern number is the sum of two layers  $C = 1 + 1 = 2$ . The topological order of the  $C = 2$  state is Abelian, which can be captured by a Chern-Simons term [43, 59]:

$$\mathcal{L} = \sum_i \left[ \frac{8}{4\pi} \alpha_i d\alpha_i + \frac{2}{2\pi} (eA_i + a_i) d\alpha_i + \frac{1}{2\pi} \beta_i da_i \right] - \frac{4}{4\pi} \beta_- d\beta_-, \quad (4)$$

where  $\beta_- = \frac{\beta_\uparrow - \beta_\downarrow}{2}$ ,  $\beta_i$  is the dual theory [73, 77, 78] of Higgs field  $\Delta_i$  that breaks  $u(1)_i$  to  $Z_2^i$ , and gapped fermion part is neglected. The last term is the forecasted Chern-Simons term to capture the quartonic statistics of the  $C = 2$  topological order. Now we can integrate out  $a_i$  since they are linear in the Lagrangian to get  $\mathcal{L} = \sum_{ij} \beta_i K_{ij} d\beta_j + \sum_i \frac{e}{2\pi} A_i d\beta_i$ , where  $K = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$  corresponding to the “111 state” [30]. The  $K$  matrix indicates a gapless gauge field,  $\beta_-$ , dual to Goldstone bosons originated from  $S^z$  symmetry breaking.

*Concluding remarks.*—We have shown a direct continuous transition in  $\nu_T = 5$  quantum Hall bilayers based on both ED calculations on torus, moreover, we proposed an exotic scenario of such transition, where the topology changing and symmetry breaking take place simultaneously. The  $\nu_T = 5$

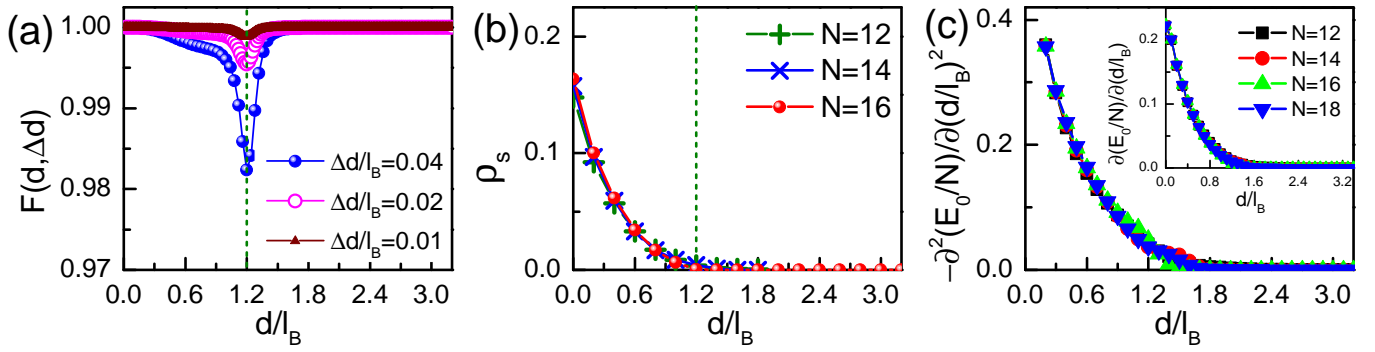


Fig. 4. (Color online) (a) The Fidelity of  $N = 16$  systems with layer distances and different interval of parameters  $\Delta d$ . (b) The exciton superfluid stiffness  $\rho_s$  as a function of layer distance. (c) The first-order (inset) and second-order derivative curves of ground-state energy  $E_0/N$  as a function of layer distance  $d/l_B$ .

bilayers host significantly different physics from  $\nu_T = 1$ , which can be seen more clearly when coupling two independent layers by changing the layer distances. When  $\nu_T = 1$ , each decoupled layer is gapless CFL state, while for  $\nu_T = 5$ , each decoupled layer is fully gapped MR state. The previous studies indicate the  $\nu_T = 1$  system has an intermediate phase when tuning the layer distance [26–28], while it is absent when  $\nu_T = 5$  based on this work. We propose that our findings of the exotic topological quantum transition and exciton superfluid at  $\nu_T = 5$  can be detected in quantum Hall bilayers composed of double well GaAs heterostructures or bilayer graphene, which has been successfully engineered to probe  $\nu_T = 1$  bilayer system.

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