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# Turbulence theories and statistical closure approaches

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**Abstract**

When discussing research in physics and in science more generally, it is common to ascribe equal importance to the three components of the scientific trinity: theoretical, experimental, and computational studies. This review will explore the future of modern turbulence theory by tracing its history, which began in earnest with Kolmogorov's 1941 analysis of turbulence cascade and inertial range [A.N. Kolmogorov, *Dokl. Akad. Nauk SSSR*, **30**, 299, (1941); **32**, 19, (1941)]. The 80th Anniversary of Kolmogorov's landmark study is a welcome opportunity to survey the achievements and evaluate the future of the theoretical approach of turbulence research. Over the years, turbulence theories have been critically important in laying the foundation of our understanding of the nature of turbulent flows. In particular, the Direct Interaction Approximation (DIA) [R.H. Kraichnan, *J. Fluid Mech.*, **5**, 497 (1959)] and its subsequent development, known as the statistical closure approach, can be identified as perhaps the most profound single advancement. The remarkable success of the statistical closure has furnished a platform to study such essential concepts as the energy transfer process and interacting scales, and the roles of the straining and sweeping motions. More recently, the quasi-Lagrangian formulation of V. L'vov & I. Procaccia and Kraichnan's solvable passive scalar model provided powerful ways to explore another fundamental aspect of turbulent flows, the phenomena of intermittency, and the associated anomalous scaling exponents. In the meantime, the theory of fluid equilibria has been developed to describe the large-scale structures that can emerge from turbulent cascades of two-dimensional and geophysical flows at a later time. And yet, despite all these successes, analytical treatments suffer from mathematical complexities. As a result, the utility of theoretical approaches has been limited to relatively idealized flows. On the other hand, in recent decades, computational abilities and experimental facilities have reached an unprecedented scale. Looking beyond the horizon, the imminent deployment of exascale supercomputers will generate complete datasets of the entire flow field of key benchmark flows, allowing researchers to extract additional measurements concerning fully developed, complex turbulent flow fields far beyond those available from the statistical closure theories. Some other potential developments that could influence the future course of turbulence theories include the advancement of machine learning, artificial intelligence, and data science; potential disruptions arising from the advent of quantum computation; and the increasingly prominent role of turbulence research in providing more accurate climate scientific data. Turbulence theorists can leverage these developments by asking the right questions and developing advanced, sophisticated frameworks that will be able to predict and correlate vast amounts of data from the other two components of the trinity.

*Keywords:* Turbulence theories; Statistical closure approaches; Energy transfer; Interacting scales; Straining and sweeping motions; Anisotropy; Intermittency; Anomalous scaling laws; Turbulence experiments; The theory of fluid equilibria; Geophysical turbulence; Plasma and magnetohydrodynamics (MHD) turbulence; Rotating and unstable stratified flows; Rayleigh–Taylor and Richtmyer–Meshkov instabilities; Turbulence modeling; Direct numerical simulation; Large eddy simulation; Exascale computer; Machine learning; Artificial intelligence; Data science; Quantum computing; Climate prediction.

* Dedicated to Dr. Jackson R. Herring and in memory of Dr. Robert H. Kraichnan and Dr. Cecil "Chuck" E. Leith in respectful acknowledgment of their extraordinary contribution and exemplary scientific careers.

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## CONTENTS

## I. INTRODUCTION

### A. Historical context

Although turbulence as a distinct branch of physics is relatively new,[1] turbulence as a physical phenomenon is everywhere in nature and has been observed and documented by scientists, writers, and artists for centuries. In his painting “The Yellow River Breaches its Course,” the famous Chinese artist Ma Yuan (1160-1225) of the Song

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[1] Lev Landau placed hydrodynamics firmly in the physics corpus (Falkovich, 2011) with the publication of his book “Fluid Mechanics,” as Volume 6 of his authoritative “Course of Theoretical Physics” series (Landau and Lifshitz, 1959).

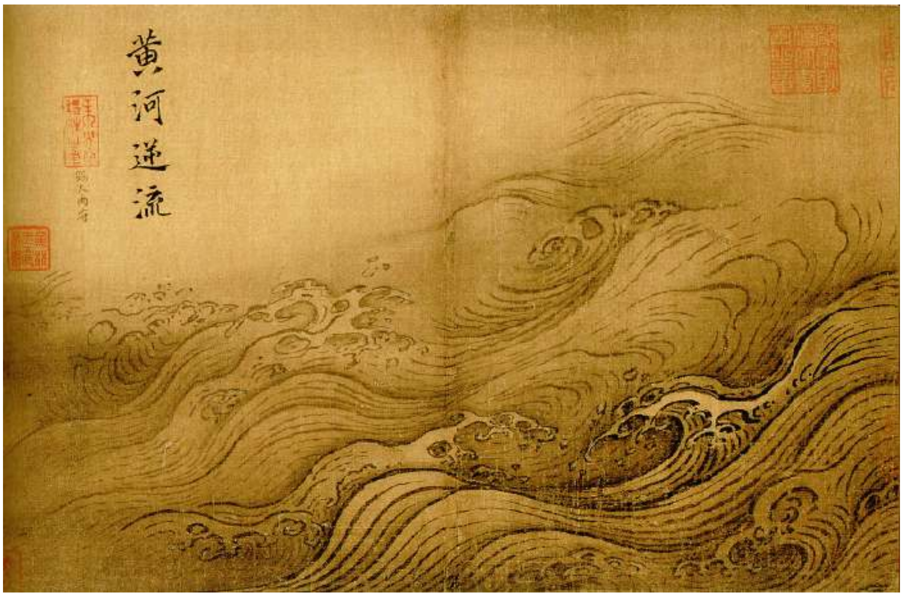


FIG. 1: The Yellow River Breaches its Course, ink on silk, Ma Yuan; circa 1222.

Dynasty showed the devastating flooding of the Yellow River in China by drawing a very turbulent flow (Fig. 1). Several hundred years later, Vincent van Gogh painted the essence of turbulence with a high degree of realism in "Starry Night" and other paintings.[2] The "accidental painting" technique of David Alfaro Siqueiros, a well-known Mexican muralist, has also been shown[3] to relate to the mixing induced by the Rayleigh-Taylor instability (Rayleigh, 1883; Taylor, 1950).

Turbulence presents a significant challenge to researchers because it is both multiscale and highly nonlinear. Analytical solutions for even the simplest turbulent flows do not exist (Moin and Mahesh, 1998) even though the fundamental equations of motion, the Navier–Stokes equations, have been known to scientists for almost two centuries (Navier, 1822, Stokes, 1845).[4] In many problems in fundamental physics, we must simultaneously deal with uncertainty in the underlying equations of motion and with uncertainty in our ability to solve them. In turbulence, we have only the latter (Nelkin, 1994).

Yet, these equations are so challenging that the Clay Mathematics Institute selected them in 2000 as one of seven "Millennium Prize Problems," and offering a $1 million reward to the first individual who could solve them (http://www.claymath.org/millennium-problems/navier-stokes-equation). To date, no one has won this award. For the record, only one of the seven Millennium Prize problems has been solved in 20 years. On March 18, 2010, The Clay Mathematics Institute announces that Dr. Grigoriy Perelman of St. Petersburg branch of Steklov Mathematical Institute, Russia, was the recipient of the Millennium Prize for resolution of the Poincaré conjecture.[5]

Given the ubiquitous nature of turbulence, it is not surprising that the field has attracted some of the most prominent scientific minds of the 19th and 20th centuries. No less than Lord Rayleigh, Werner Heisenberg, Lev Landau, Subrahmanyan Chandrasekhar, Tsung-Dao Lee, Lars Onsager, and Richard Feynman — all Nobel Laureates — have made attempts to address turbulence or associated problems. Heisenberg, for example, wrote his doctoral dissertation on flow instability and turbulence (Eckert, 2015) before leaving fluid mechanics to develop the new field of quantum mechanics and his uncertainty principle. He returned briefly to the study of turbulence, albeit during his months-long detention at Farm Hall, England, following the end of World War II (Bernstein, 2013). Lars Onsager made landmark contributions to several areas of physics and chemistry. However, he only published a short abstract (Onsager, 1945),[6] which was followed a few years later by his only full-length paper (Onsager, 1949) on the subject

[2] "The Starry Night" was painted by van Gogh in 1889 at the asylum in Saint-Remy, France during his periods of prolonged psychotic agitation (Bailey, 2018). Aragón *et al.* (2008) used digital images of these paintings of van Gogh to show that the statistics of luminance contains the characteristic fingerprint of turbulent flow.

[3] David Alfaro Siqueiros (Stein, 1994; White, 2009) employed the method of absorption of two or more superimposed colors which infiltrate one into another. He described the distinctively textured patterns as "the most magical fantasies and forms that the human mind can imagine." Zetina *et al.* (2015) reproduced this so-called "accidental painting" in a controlled manner (Zenit, 2019).

[4] As documented by Darrigol (2002), Navier's original proof of 1822 was not influential, and the equation was rediscovered or re-derived at least four times, by Cauchy in 1823, by Poisson in 1829, by Saint-Venant in 1837, and by Stokes in 1845.

[5] https://www.claymath.org/sites/default/files/millenniumprizefull.pdf.

[6] In this contributed talk at a meeting of the Metropolitan Section of the American Physical Society, Onsager predicted that the energy spectrum for velocity fluctuations should scale as the $-5/3$ power of the wavenumber. Unbeknownst to him, this key result was derived a few years earlier by A. N. Kolmogorov (see Eyink and Sreenivasan (2006) and subsection II.D).

of turbulence in the 1940s. Nevertheless, Eyink and Sreenivasan (2006) documented Onsager's lifelong interest in turbulence and found from historical sources that he made greater inroads into the theory of turbulence than he ever fully made public.

Despite these efforts, even these great minds faced challenges in fully grappling with the complexities of the field. Richard Feynman "struggled for years, in vain, to penetrate the problem of turbulence in gases and liquids" (Gleick, 1993). Indeed, Feynman described turbulence as "the most important unsolved problem of classical physics" (Feynman *et al.*, 1963). Similarly, after authoring a dozen prominent publications on turbulence, Chandrasekhar abruptly ended his work in this area, stating that "[i]t is about the most frustrating in my entire experience" (Sreenivasan, 2019a). T.D. Lee also made a clean break from turbulence after authoring three short but insightful articles on turbulent flows (Lee, 1950, 1951, 1952). Lee most likely did not regret his decision to change paths, however: just a few years later in 1957, he shared the Nobel Prize in Physics[7] with C.N. Yang "for their penetrating investigation of the so-called parity laws which has led to important discoveries regarding the elementary particles."

Professor Lord Julian Hunt and Professor Steven A. Orszag have not been shy in expressing their thoughts about the challenges posed by turbulence over the past century. Orszag (1970) said that "[i]t must be admitted that the principal result of fifty years of turbulence research is the recognition of the profound difficulties of the subject." Lord Hunt added his observation that turbulence is "like certain diseases" (Hunt and Kevlahan, 1993).

"When I meet God, I'm going to ask him two questions: why relativity? And why turbulence? I really believe he'll have an answer for the first." This probably apocryphal quote, attributed to Heisenberg, captures the way many researchers feel about turbulence (Castelvecchi, 2017).

### B. Statistical closure approaches

This review explores the future of modern turbulence theory by tracing its history, summarizing 80 years of activities in the science of fluid turbulence (if we start the clock with Kolmogorov's landmark 1941 papers), with an emphasis on statistical closure models. This anniversary milestone provides a natural opportunity to assess the past achievements of this theory and to speculate on the future outlook.

Andrey Nikolaevich Kolmogorov has been described as the founder of modern probability theory and one of the 20th century's most eminent mathematicians (Gleick, 1987). Regarding his work on turbulence, Kolmogorov (1985) stressed that "[i]t was clear to me from the very beginning that the main mathematical instrument in this study must be the theory of random functions of several variables (random fields) which had only then originated. Moreover, it soon also became clear to me that there was no chance to develop a closed purely mathematical theory. For lack of such a theory it was necessary to use some hypotheses based on the results of treatment of experimental data."

Since its inception, Kolmogorov's phenomenology has formed a foundation in our understanding of the turbulent cascade and inertial range. As pointed out by Yaglom (1994), "[t]wo remarkable short notes (Kolmogorov 1941a,d) undoubtedly marked the greatest achievement in the theory of turbulence since the time of O. Reynolds. They cardinally changed this theory and are basic to all subsequent developments in the field."[8] Thirty years ago, an entire volume of the *Proceedings of the Royal Society of London*, edited by Hunt *et al.* (1991), was devoted to Kolmogorov's ideas about turbulence. Frisch (1995) also entitled his book "Turbulence (The Legacy of A.N. Kolmogorov)," and offered many unique and insightful perspectives.

It is fascinating how Kolmogorov's breakthrough became known internationally during "the darkest hour" of the Second World War.[9] George Keith Batchelor, who later became a revered turbulence researcher himself,[10] made an amazing discovery during his first year in Cambridge — he came upon the landmark papers by Kolmogorov from the English-language editions of the 1941 issues of *Doklady*, the Comptes Rendus of the USSR Academy of Sciences. The bound volumes of *Doklady* and other Soviet journals were used as ballast for supply ships on their return voyage from northern Russian ports having carried armaments for besieged Russian cities on the outward voyage (Barenblatt, 2001; Moffatt, 2002, 2010).

We also identify Kraichnan's Direct Interaction Approximation (DIA) (Kraichnan, 1959) and its related statistical closure theories as perhaps the most profound progress, leading to 60 plus years of extension and refinement. The DIA

[7] https://www.nobelprize.org/prizes/physics/1957/summary/.

[8] Osborne Reynolds (1842-1912), a professor of engineering at the University of Manchester, was one of the greatest scientists in fluid dynamics and turbulence. According to Jackson and Launder (2007), "it is no exaggeration to assert that his two principal papers on turbulent flow essentially provided a marker for the direction of research in Engineering Fluid Mechanics for the next century."

[9] This phrase has long been attributed to Sir Winston Leonard Spencer Churchill, former Prime Minister of the United Kingdom. At that time, Britain's prospects had appeared hopeless in the eyes of most people. A late June 1940 opinion poll found that only one American in three expect the British to win the war (Wheatcroft, 2020). By mid-July of 1941, the Germans armoured divisions were only 200 miles from Moscow, the capital and largest city of the Union of Soviet Socialist Republics (USSR) (e.g., Overy, 1998; Roberts, 2018; Allport, 2020).

[10] See just published "Special Volume in Celebration of the George K. Batchelor Centenary," *J. Fluid Mech.*, Volume 914, (2021). P. Linden, editor. https://www.cambridge.org/core/journals/journal-of-fluid-mechanics/volume/19FFE4E7F64E656981CC8E7D01493B46.

theory was advanced by the formidable physicist Robert H. Kraichnan, one of the last assistants of Albert Einstein (Pearce, 2008), through the deployment of the potent field-theoretic attack. Ironically, according to the acclaimed biography of Albert Einstein, *Subtle is the Lord: The Science and the Life of Albert Einstein* (Pais, 1982), Einstein never had a good word for the quantum field theory. Once, in 1912, he said that the more successful it is, the sillier it looks.

The achievements of the DIA and the related statistical closure theories cannot be overemphasized, especially because they were mostly developed before the rise of powerful supercomputers and advanced experimental diagnostics and facilities. Orszag rightly proclaimed[11] that "DIA was the DNA of turbulence theory." The first few decades of modern turbulence theory have been compared to crossing a desert, with Kraichnan shepherding a small flock of disciples (Eyink and Frisch, 2011). Over the years, revised Lagrangian versions (Kraichnan, 1964c, 1965a, 1966b, 1977) have helped to overcome the deficiencies of the earlier Eulerian system, culminating in the Lagrangian renormalized approximation (LRA) proposed in the early 1980s (Kaneda, 1981).[12] Those formulations have become the anchors of modern turbulence theory.

Lev Landau was the first to recognize the so-called intermittency phenomenon.[13] The energy cascade, i.e. step-by-step transfer of energy that leads to intermittency and multiscaling, is a basic feature of turbulence. The advancement surveyed in this review lays claim to a major breakthrough on theoretical analyses of the anomalous scaling and intermittency.

Kraichnan's last major contribution is a proposed fluid-dynamical framework for studying intermittency. Kraichnan's passive scalar model quickly led the younger generation to identify the mathematical mechanism of intermittency and also to calculate analytically the anomalous scaling exponents (Eyink and Frisch, 2011). Indeed, the application of the methods of statistical mechanics to the description of the motion of fluid particles, i.e., to the Lagrangian dynamics, has led to a new quantitative theory of intermittency in turbulent transport. Many of these studies have been documented in a comprehensive review twenty years ago (Falkovich *et al.*, 2001) and references therein, to which we refer those readers who desire an in-depth tour of the subject. We will discuss several intriguing topics related to the Kraichnan model which have emerged since then.

Around the same time, we have witnessed dramatic progress in the quasi-Lagrangian framework and fusion rules, spearheaded by Professors Victor L'vov and Itamar Procaccia, spurring advancement in our understanding of topics ranging from intermittency and anomalous scaling[14] to anisotropy (Biferale and Procaccia, 2005) and laying the groundwork for further studies on the issues relevant to the locality and multilocality of the energy transfer.

Another significant development is the remarkably beautiful theories of fluid equilibria (Miller-Robert-Sommeria and many follow-ons) which, under various ergodicity assumptions, describe the large scale structures that can emerge from turbulent cascades at late time, mainly in two-dimensional (2D) and geophysical flows.

Nevertheless, the evolution and further growth of the statistical closure theories have suffered because of their mathematical complexity. In reminiscing about the "watershed" turbulence conference held in 1961 in Marseille, France,[15] Professor Henry Keith Moffatt, a renowned expert, remarked that "the new approaches, particularly Kraichnan's (1959) Direct Interaction Approximation, were of such mathematical complexity that it was really difficult to retain that essential link between mathematical description and physical understanding, which is so essential for real progress" (Moffatt, 2002). Indeed, Kraichnan's papers were famous for their dense, opaque text and lengthy, difficult equations even for selected simple flows. Yet, Kraichann observed that "[a]fter initial surprise that turbulence did not succumb rapidly to field-theoretic attack, I have been trapped ever since."[16] In the face of these complexities, scientists have needed to make significant compromises to their statistical closure models in order to address the more practical aspects of flow fields, including the effects of rotation, stratification, shear, magnetic fields, and other external agencies.

In the last several decades, the power of supercomputers has grown exponentially, presenting new methods to address these challenges. The seminal paper of Orszag and Patterson (1972) maybe the first legitimate "numerical

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[11] Remark made during the symposium entitled "Turbulence: Challenges for the 21st Century," a conference held in honor of the 70th birthday of R. H. Kraichnan. Los Alamos National Laboratory, May 18-21, 1998.

[12] The internal consistency of LRA has been further confirmed by an alternative derivation, the so-called sparse direct-interaction perturbation (SDIP) approach; see subsection IV.D.

[13] It has long been known for that small scales of turbulence are intermittent (Batchelor and Townsend, 1949). A signal is called intermittent, if there are relatively calm periods which are irregularly interrupted by strong turbulent bursts either in time or in space (Lohse and Grossmann, 1993). Correspondingly, the probability density function (PDF) develops enhanced tails of large fluctuations and a center peak due to the abundance of calm periods, i.e., the PDF becomes of stretched exponential type instead of being Gaussian (Grossmann and Lohse, 1994). Of particular interest is the intermittency of the rate of turbulent energy dissipation (see for example, Meneveau and Sreenivasan (1987, 1991)). As an aside, it has been noticed that the relative contributions to the dissipation from strain-dominated and enstrophy-dominated structures may be relevant to the strength of intermittency, as well as to the scalings in longitudinal and transversal structure functions (He *et al.*, 1998).

[14] L'vov and Procaccia (1995c, 1996a), see also their follow-up papers (Belinicher *et al.*, 1998a,b; L'vov *et al.*, 1997a; L'vov and Procaccia, 1996b,c,d, 1998, 2000), illustrated that the phenomenon of intermittency is consistent with the Navier-Stokes equation, if one does not used standard closure procedures that forced Kolmogorov (1941) scaling. Instead one has to analyze the diagrammatic series as a whole or to use more general closure procedure which respects "the fundamental rescaling symmetry of the Euler equation" suggested in L'vov and Procaccia (1998).

[15] "Mécanique de la Turbulence," Proceedings of the International Colloquium held in Marseille, August 28th-September 2nd, 1961 (Colloques Internationaux du CNRS, no. 108, 1962).

[16] Correspondence from R.H. Kraichnan to J.R. Herring. See Eyink and Frisch (2011).

experiment.” The authors showed that simulations are feasible and even economical at Reynolds numbers of the order achieved in wind-tunnel turbulence experiments. They also pointed out that the value of the simulations lies not only in the completeness of the data they provide, but also in the opportunity they give for the assessment of the accuracy of turbulence theories under controlled and known conditions. Using direct numerical simulations (DNS) (e.g., Moin and Mahesh (1998)), the Navier-Stokes equations can be solved numerically with physically consistent accuracy in space and time.The entire flow fields of key benchmark flows [homogeneous isotropic turbulence (HIT), for example(Buaria *et al.*, 2019; Cao *et al.*, 1996; Ishihara *et al.*, 2009; Iyer *et al.*, 2018, 2019, 2020; Kaneda *et al.*, 2003; Yeung *et al.*, 2018, 2015)] can be obtained despite spanning a large range of turbulent lengths and time scales.[17] Researchers can extract valuable information about turbulent structures and higher-order statistics by analyzing materials harvested from the flow fields. This information is not available using the DIA and the related statistical closure theories.[18]

### C. Scope and structure of the review

The goal of physics in general, and of the physics of hydrodynamic turbulence in particular, is to describe the basic underlying mechanisms that govern natural phenomenon. The fundamentals undergirding the statitical closure theories of turbulence are reviewed here in order to make it more accessible for interested physicists, astrophysicists, applied mathematicians, and those beginning research in this area.

Given the increasing dominance of computers and experiments in the field of turbulence, to the extent that in what areas can theory offer unique contributions to our understanding of turbulence? This review will seek to answer that question by detailing the past accomplishments of the statistical closure theories and assessing its continued pertinence.

Because of the vastness of the literature in this field, some quite worthy subjects are not covered. As such, we direct the reader to other relevant reviews in these areas:

- Analytical theory of drag reduction by polymers in wall-bounded turbulence has been developed, see e.g. reviews Procaccia *et al.* (2008), Benzi (2010), Benzi and Ching (2018), and references therein.
- Current state of the analytical theory of energy spectra in super fluid $^4$He counter flow turbulence was summarized in L’vov and Pomyalov (2018) and Barenghi, L’vov and Roche (2014).

The outline of the review is as follows: I begin by briefly describing the classic but limited treatment of turbulence prior to the introduction of the statistical closure theories. I present the revolutionary DIA theory in detail and explain how this approach advanced our understanding of the energy transfer mechanism of turbulent flows. In the next few sections, I discuss the Lagrangian versions of the theories that alleviated the defects of the DIA, and the many methods that branched out from it, including test-field model (TFM), eddy damped quasi-normal Markovian (EDQNM), and renormalization group (RG) theory. This is followed by a summary of the promising investigations on the non-perturbative renormalization (NPRG) approach.

Next, I explore how theory, experiment, and computation have been used to probe two different fundamental turbulence processes: local versus non-local interactions and sweeping versus straining motions. I show how closure theories have been utilized to verify the underlying assumptions of those processes, and then conduct the same analysis using databases from large-scale numerical simulations run on supercomputers. It will be clear that the closures could be very helpful in sorting out various issues related to infrared dynamics, large-scale decaying, and other topics.

The following section starts by recounting recent efforts invigorated by Kraichnan’s passive scalar model, the exact solvability of which is rarely found in turbulence research. These are followed by a review of the quasi-Lagrangian representation of Belinicher, L’vov, and Procaccia and the perturbation and nonperturbative theories of L’vov and Procaccia. The so-called fusion rules and balanced equations resulted in significantly improved evaluations of the intermittency and anomalous scaling and set forth early stages of further studies on the issues of multilocality. I then highlight how turbulence closure theories have contributed to the understanding and modeling of selected flows originating in geophysical and plasma physics, astrophysics, and engineering. Along the way, the theories of fluid equilibria are introduced to describe the large scale structures that can emerge from turbulent cascades at a late time.

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[17] We also noted that the fundamental physics-based nature of the direct simulation Monte Carlo (DSMC) algorithm in conjunction with the ever-increasing power of modern computational platforms has allowed DSMC to model a wide range of problems outside its original regime of applicability (Plimpton *et al.*, 2019).

[18] In this review, I have essentially equated “statistical closure” with ”moment closure.” This ignores the very interesting research on PDF closures such as Kraichnan’s “mapping closure” (Chen *et al.*, 1989a; Kraichnan, 1988b). One of the interesting points is that in moment closure the difficulty arises from the nonlinear term, while in mapping closure it is the linear term that causes the problem. See also, Gao (1991); Girimaji (1992); Gotoh and Kraichnan (1993); Kimura and Kraichnan (1993); and Pope (1991).

I conclude this review with a discussion of several factors that could conceivably alter the future course of turbulence theories: the imminent deployment of exascale supercomputers; the advancement of machine learning, artificial intelligence, and data science; the potential for disruptive progress with the the advent of quantum computation; and the need for turbulence research to play a more prominent role in providing more accurate climate scientific data.

## II. FUNDAMENTALS

### A. Governing equations

Fluid flows as varied as steady laminar flows, nonlinear instabilities, and turbulence are all described[19] by the Navier-Stokes equations (Hinze, 1959; Monin and Yaglom, 1971, 1975). Applying momentum conservation to an infinitesimal fluid element gives rise to the well-known relation:

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = \frac{\partial \sigma_{ij}}{\partial x_j}. \tag{1}$$

Here $u_i$ is the velocity vector and $\rho$ is the fluid density. The stress tensor $\sigma_{ij}$ is given by

$$\sigma_{ij} = \mu \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_n}{\partial x_n} \right] - \delta_{ij} p, \tag{2}$$

where $p$ is the pressure, $\mu$ is the dynamic viscosity, and the kinematic viscosity of the fluid is defined as $\nu = \mu/\rho$. Meanwhile, the equation governing the transport of the density, arising from mass conservation, is

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0. \tag{3}$$

In (1)-(3), the Einstein summation convention is applied to repeated indices. When discussing a Navier-Stokes fluid, there is little if any dispute regarding either the dynamical equation (see Eq. 1) or the physical parameters appearing in it. Thus, if one is describing water close to standard conditions, the mass density $\rho$ is nearly 1 gram per cubic centimeter and the kinematic viscosity $\nu$ is roughly 0.01 centimeters squared per second (Bird *et al.*, 1960). The parameters of other fluids, when not calculable from first principles, can be readily obtained from engineering handbooks. For dilute gases, the calculations can, in fact, be carried out from first principles (Chapman-Enskog theory) and have better than qualitative agreement with measurements for monatomic gases (Balescu, 1975; Chapman and Cowling, 1970).

By necessity, experimental investigation was a key vehicle in turbulence research for many years. Observations from the data collected at the University of Manchester's tank resulted in Osborne Reynolds' discovery of a parameter, now bearing his name, for distinguishing a turbulent flow from a laminar one (Reynolds, 1883).

The Reynolds number can be obtained by non-dimensionalizing the Navier-Stokes equation. For a viscous, incompressible fluid[20] with constant properties, Eqs. 1 and 3 reduce to

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i \tag{4}$$

and

$$\frac{\partial u_j}{\partial x_j} = 0. \tag{5}$$

[19] The word "turbulence" was once used for describing the agitation and disorder of crowds or children and was not used by pioneers of fluid dynamics such as Joseph Boussinesq, Osborne Reynolds, Lord Rayleigh, and Horace Lamb, prior to 1887 (Schmitt, 2017). Popularized later by Lord Kelvin (William Thomson) (Thomson, 1887a,b, 1910), the term "turbulence" introduced a new concept, a new understanding of a scientific phenomenon identified as distinct from laminar flow. Turbulent evolution is characterized by the randomness of the flow field in both time and space. A laminar flow, on the other hand, is one that moves as if comprised of layers with different velocities (Landau and Lifshitz, 1959).

[20] The fact that limitations of space do not allow us to discuss compressible turbulence in any detail is practicably irrelevant and immaterial, given how challenge it is to apply the statistical closure theories to such a flow. See remarks in subsection XII.C)

A key mechanism responsible for the formation of the small scales is the process of vortex stretching (Tennekes and Lumley, 1972), which results from the nonlinear coupling between the strain-rate tensor and vorticity, respectively, defined as

$$s_{ij} = \Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \tag{6}$$

and

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}. \tag{7}$$

Therefore, fundamental understanding of the enstrophy (i.e. $\boldsymbol{\omega}\cdot\boldsymbol{\omega}/2$) transport is of key importance for the purpose of characterisation of the energy cascade in turbulent flows (Buaria *et al.*, 2020).

Let us consider a flow with characteristic velocity and length scales $u$ and $L$, respectively. By defining the dimensionless velocity field $\mathbf{u}^* = \mathbf{u}/u$ and gradient $\nabla^* = L\nabla$, Eq. 4 can be recast in the dimensionless form

$$\frac{\partial \mathbf{u}^*}{\partial t} + \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* = -\nabla^* \left(\frac{p}{\rho u^2}\right) + \frac{\nu}{uL}\nabla^{*2}\mathbf{u}^*. \tag{8}$$

In this form it is clear that the pressure scale is the familiar $\rho u^2$, while a single dimensionless parameter, the Reynolds number (Re),

$$Re = \frac{uL}{\nu}, \tag{9}$$

captures the relative importance of the inertial and viscous forces in the flow.[21]

As the title suggests, this review will only focused on fully turbulent flows. It should be noted that transition is a major fluid dynamics field on its own right (see for instance Mullin (2011) and Saric *et al.* (2003)). In particular, hydrodynamic instability induced flows will usually start from potentially infinitesimal spatial perturbations, will eventually transition to a turbulent flow, and then will reach a final state of a true multiscale problem (Zhou *et al.*, 2019) (see subsection XII.B). To qualify as a turbulent flow, it has been determined a minimum value of $Re$ ($1.6\times10^5$) must be met (Zhou, 2007), a requirement which is important to be noted early on — it is seldom sufficiently emphasized and only recently being approached by DNS and experiments. Basically, roughly two orders of magnitude driving and dissipation scale separation are required from each end of the inertial range for it to properly exist.

### B. Trouble with the strong nonlinearity

Only "grudgingly" accepted for publication by two eminent referees, Sir George Stokes and Sir Horace Lamb, Reynolds (1895) published his classical approach to turbulence, in which he began with the Navier-Stokes equation then decomposed a flow variable $\Upsilon$ (e.g., velocity, pressure) into a mean component and a fluctuation about that mean (Launder, 2015),

$$u_i = \overline{u}_i + u_i', \quad p = \overline{p} + p'. \tag{10}$$

For general turbulent flows the mean of any flow variable is taken to be the ensemble average mean

$$\overline{\Upsilon}(\mathbf{x},t) = \langle \Upsilon(\mathbf{x},t)\rangle = \lim_{N\to\infty}\frac{1}{N}\sum_{i=0}^{N}\Upsilon^{(i)}(\mathbf{x},t), \tag{11}$$

where an average is taken over $N$ repeated experiments. For statistically steady turbulence Reynolds assumed the equivalent of an ergodic hypothesis[22] (Panchev, 1971) to conclude that this ensemble average is equal to both the volume average in a homogeneous turbulent flow (Speziale, 1991)

$$\overline{\Upsilon}(t) = \lim_{V\to\infty}\frac{1}{V}\int_V \Upsilon(\mathbf{x},t)dt, \tag{12}$$

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[21] At the low $Re$ limit, the inertial forces are so low that they are negligible. In this case, the velocities of objects are essentially governed by the applied force and viscous drag. An intriguing essay by Purcell (1977) probed small things moving and living in low $Re$ fluids. For intracellular fluid mechanics, Mogilner and Manhart (2018) also remarked that the $Re$ is extremely low at $\sim 10^{-5}$.

[22] According to the ergodic hypothesis time (space) and ensemble averages (moments) are equal to one another. Indeed, it is assumed that all statistical properties of statistically stationary flows of an ensemble are equivalent to those obtained using time series in one very long realization (Galanti and Tsinober, 2004). Higgins *et al.* (2013) have investigated the validity of this hypothesis in the atmospheric surface layer, taking advantages of the advancements in water vapor concentration lidar measurements that simultaneously sample spatial and temporal series. The authors demonstrated that ergodicity is valid in a strict sense above uniform surfaces away from abrupt surface transitions.

as well as the time average

$$\overline{\Upsilon}(\mathbf{x}) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \Upsilon(\mathbf{x},t)dt. \tag{13}$$

When density variations are negligible the Reynolds-averaged-Navier-Stokes (RANS) equation becomes

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \nu \nabla^2 \overline{u}_i - \frac{\partial \tau_{ij}}{\partial x_j} \tag{14}$$

where

$$\tau_{ij} = \overline{u'_i u'_j}. \tag{15}$$

Note that $p$ is now redefined as the modified pressure for the purpose of simplicity (Speziale, 1991). The mean continuity equation is provided by

$$\frac{\partial \overline{u}_j}{\partial x_j} = 0. \tag{16}$$

Meanwhile, the fluctuating velocity equation can be obtained by subtracting Eq. 14 from Eq. 4

$$\frac{\partial u'_i}{\partial t} + \overline{u}_j \frac{\partial u'_i}{\partial x_j} = -u'_j \frac{\partial u'_i}{\partial x_j} - u'_j \frac{\partial \overline{u}_i}{\partial x_j} - \frac{\partial p'}{\partial x_i} + \nu \nabla^2 u'_i + \frac{\partial \tau_{ij}}{\partial x_j}, \tag{17}$$

while the transport equation for the Reynolds stress takes the form (Hinze, 1959)

$$\frac{\partial \tau_{ij}}{\partial t} + \overline{u}_k \frac{\partial \tau_{ij}}{\partial x_k} = -\tau_{ik} \frac{\partial \overline{u}_j}{\partial x_k} - \tau_{jk} \frac{\partial \overline{u}_i}{\partial x_k} + \wp_{ij} - \mathcal{E}_{ij} - \frac{\partial \mathcal{C}_{ijk}}{\partial x_k} + \nu \nabla^2 \tau_{ij}. \tag{18}$$

In Eq. 18, the dissipation-rate correlation, third order diffusion correlation, and pressure-strain correlation are given by

$$\mathcal{E}_{ij} = 2\nu \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}}, \tag{19}$$

$$\mathcal{C}_{ijk} = \overline{u'_i u'_j u'_k}, \tag{20}$$

and

$$\wp_{ij} = \overline{p' \left( \frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i} \right)}. \tag{21}$$

By a contraction of equation 18, the transport equation for turbulent kinetic energy, $K = \frac{1}{2}\tau_{ii}$, is obtained

$$\frac{\partial K}{\partial t} + \overline{u}_j \frac{\partial K}{\partial x_j} = -\tau_{ij} \frac{\partial \overline{u}_i}{\partial x_j} - \mathcal{E} - \frac{\partial}{\partial x_i} \left( \frac{1}{2} \overline{u'_i u'_k u'_k} + \overline{p' u'_i} \right) + \nu \nabla^2 K. \tag{22}$$

The rate of energy transfer $\mathcal{E}$ is obtained from Eq. 19 by noting that $\mathcal{E}_{ij} = \frac{2}{3}\mathcal{E}\delta_{ij}$ (Speziale, 1991). The presence of the first term on the right side of Eq. 22, which couples the mean velocity gradient to the Reynolds stress, revealed a mechanism for transferring energy from the mean flow field to the turbulent fluctuations. This insight was so far ahead of its time, as Jackson and Launder (2007) pointed out, the corresponding equation for the mean square temperature fluctuations of Corrsin (1952) was not published until the 1950s.

Reynolds' work also illustrated the *closure problem* of turbulence. In Eq. 18, the second order velocity correlation is not closed without the third order one. Clearly, the equation for the latter requires expression of a fourth order correlation, and so on. While evolution equations for the correlation functions of the fluctuating velocity field can be formed, each $i$-th order equation so obtained demands the correlation function of $i$+1 order. This closure problem is the main challenge in turbulence and is a direct result of the nonlinearity of the Navier-Stokes equation.

The Reynolds average Navier-Stokes method was further developed for engineering calculations that require quick turn-around times (Chou, 1940, 1945; Launder *et al.*, 1975; Rotta, 1951). However, one challenge to these developments arises from the dependence of the Reynolds-stress tensor, Eq. 15, on the boundary and initial conditions of the fluctuating velocity (Lumley, 1970). If one assumes that the boundary and initial conditions merely set the length and time scales of the flow, eddy viscosity models, which are the simplest of the RANS modeling, take the form

$$\tau_{ij} = \frac{2}{3}K\delta_{ij} - 2\nu_T S_{ij}, \tag{23}$$

where

$$S_{ij} = \frac{1}{2}\left[\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right]. \tag{24}$$

Here the eddy or turbulent viscosity is $\nu_T \propto L_T^2/\tau_T$, where $L_T$ and $\tau_T$ are turbulent length and time scales, respectively. Note that the first term on the right-hand-side of Eq. 23 is needed to obtain the proper trace of $\tau_{ij}$. That is, since $S_{ii} = 0$ for incompressible flow, contracting Eq. 23 yields $\tau_{ij} = 2K$ (e.g. Wilcox, 1993).

The eddy viscosity is clearly inspired by molecular viscosity.[23] Over the years, many RANS models of increasing complexity have been proposed, with varying degrees of success (Wilcox, 1993). In principle, this approach attempts to model all of the fluctuating scales. Nevertheless, even under the best circumstances the RANS models can only provide information about first and second one-point moments (e.g., the mean velocity, turbulence intensity, and mean pressure), which is usually all that is needed for design purposes (Speziale, 1991).

Around 1500, Leonardo da Vinci, a superb artist as well as a good engineer (Taylor, 1974), depicted turbulent flows (see Fig. 2) generated by a jet from a square pipe into a pool. Perhaps the world's first use of visualization as a scientific tool to study a turbulent flow (Gad-el-Hak, 1998), da Vinci observed "the motion of the surface of the water, which resembles that of hair, which has two motions, of which one is caused by the weight of the hair, the other by the direction of the curls; thus the water has eddying motions, one part of which is due to the principal current, the other to the random and reverse motion"[24]. These observations led Lumley (1992) to speculate that da Vinci might have developed his physical understanding of the Reynolds decomposition four centuries before Osborne Reynolds.[25]

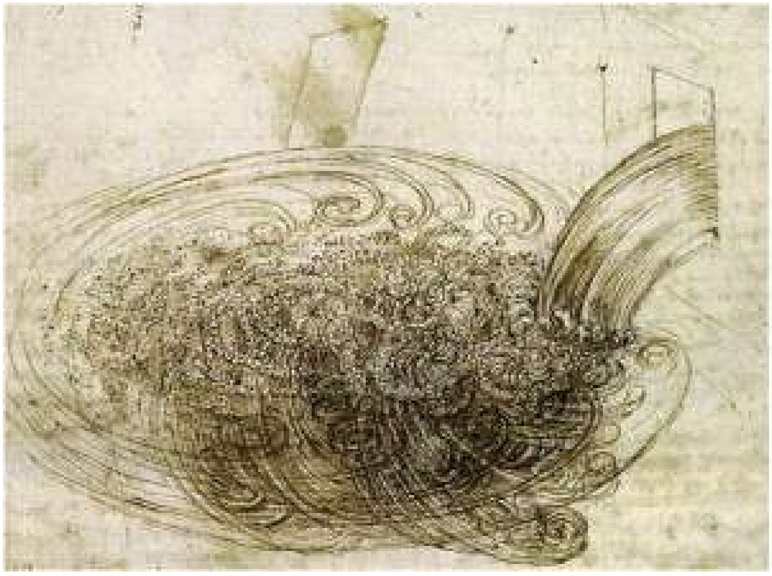

FIG. 2: Illustration of turbulent flows by Leonardo da Vinci; circa 1500.

### C. Quasi-Normal Approximation

The landmark four-part article of Taylor (1935) touched on several issues that will be relevant to our review. Taylor differentiated the turbulence measurements taken in a Lagrangian manner compared to those arising from a Eulerian

[23] Note that various statistical closure theories will be utilized to obtain several analytical eddy viscosities, which will be differentiated among them with either the subscripts or arguments of the function.
[24] Translation credit: Ugo Piomelli (Lumley, 1992).
[25] Leonardo da Vinci also carried out another fascinating study (around 1508-1509 and 1513) in fluid mechanics that focused on cardiac mechanics and the vascular system as unveiled in his more than fifty surviving pages of drawings (Gharib *et al.*, 2002). See also the book entitled *The Heart of Leonardo* (Wells, 2014).

framework. He began with the claim "it is more usual to think in terms of the Eulerian conception of the fluid flow." Next, he introduced a statistical correlation function between velocity fields at two different points in space or two different times. The publication of this work has been hailed as a turning point that distinguished the "statistical" perspectives from the previous "empirical" efforts (Battimelli, 1986). Using a wind tunnel data set measured with a hot wire anemometer, Taylor studied both the curve of correlation between the wind variation and the spectrum of the time variation at two fixed points. In so doing, he showed that the correlation and spectrum curves are simply the Fourier transforms of one another (Taylor, 1938b).

For simultaneously measured velocity statistics of homogeneous turbulence,[26] Millionshtchikov (1941a,b) proposed a hypothesis that the two-point distribution of simultaneous velocity amplitudes in a turbulent fluid is quasi-normal. This hypothesis assumes that the relationship between mean values of quadruple velocity component products and those of double velocity component products, which can be regarded as "known" (Batchelor, 1953), is one appropriate to a jointly normal probability distribution. The third-order covariances are not taken as zero, hence complete joint normality is not postulated (Kraichnan, 1957; O'Brien and Francis, 1962).

The Millionshchikov hypothesis fills the need very well when estimating the covariance of the pressure fluctuations at two points (Batchelor, 1951; Chandrasekhar, 1951; Heisenberg, 1948a; Obukhov, 1949). Heisenberg (1948a) also applied this assumption to the two-time distribution of the Fourier coefficients of the velocity field. Using this extended hypothesis, Chandrasekhar (1955a,b, 1956) proposed a theory of turbulence for fluid and hydromagnetic turbulent flows. Proudman and Reid (1954) and Tatsumi (1957) applied this hypothesis to the turbulence energy decay problem.

For stationary, isotropic,[27] incompressible turbulence, Kraichnan (1957) found that quasi-normality of the two-time velocity amplitude distribution is inconsistent with the equation of motion, and the inconsistency is exhibited as a gross violation of energy conservation. O'Brien and Francis (1962) studied the dynamical consequences of the quasi-normal approximation when applied to the behavior of a scalar field. These authors found that the scalar spectrum developed negative variance after a finite time. Ogura integrated numerically the dynamical equation derived by Tatsumi (1957) in both 2D (Ogura, 1962) and three-dimensional (3D) (Ogura, 1963) homogeneous turbulence. These articles conclusively demonstrated that the dynamical equation based on the quasi-normal approximation resulted in negative energy density at the Reynolds numbers specified.

### D. Kolmogorov phenomenology

Kolmogorov (1941a) took a rather different approach from that of Proudman & Reid and Tatsumi by considering the scaling of a universal equilibrium range phenomenologically. The brevity of our discussion of his landmark phenomenology here is because a detailed description can be found in many excellent turbulence books (Batchelor, 1953; Davidson, 2004; Lesieur, 1987; Monin and Yaglom, 1975; Pope, 2000).

At the onset, a few words should be said regarding notation. The large scales (or so-called energy-containing scales) are usually statistically unsteady and anisotropic, a direct consequence of the external forcing or other mechanisms which produce the turbulence. As introduced already, $L$ represents either the characteristic length scale, defined by the integral scale or the correlation length scale. The Kolmogorov microscale, $\lambda_K$, denotes the size of the smallest eddies beyond which the viscous actions become important and the kinetic energy will be dissipated into heating (Batchelor, 1953).

The structure function of order $m$ constructed from the velocity field, defined over separation $\mathbf{r}$, is given by

$$S_m(\mathbf{r}) = \langle |u_i(\mathbf{x}+\mathbf{r}) - u_i(\mathbf{x})|^m \rangle. \tag{25}$$

The main results of Kolmogorov (1941a) can be stated in plain language as follows (Davidson, 2004):

- Kolmogorov's *First Similarity Hypothesis* states that:

When $Re$ is large enough, and $r \ll L$, the statistical properties of the structure function have a universal form which depends on only $\mathcal{E}$, $r$, and $\nu$. Recall that $\mathcal{E}$ is the fundamental energy transfer rate and has units of energy/time.

---

[26] Here, the unqualified term "homogenous" is applied to fields which are translational invariant in all three mutually perpendicular coordinate directions (McComb, 1990). The implication to the velocity correlation is

$$\langle u_i(\mathbf{x})u_j(\mathbf{x}')\rangle = U_{ij}(\mathbf{x}-\mathbf{x}') = U_{ij}(\mathbf{r})$$

Also, the correlation is not affected by the interchange of $\mathbf{x}$ and $\mathbf{x}'$. Therefore, the correlation function is a symmetic function of $\mathbf{r}$:

$$U_{ij}(\mathbf{r}) = U_{ij}(-\mathbf{r}).$$

[27] The restriction to isotropy implies independence of both direction as well as position in the fluids.

• Kolmogorov's *Second Similarity Hypothesis* states that:

When $Re$ is large, and in the inertial range $\lambda_K \ll r \ll L$, the statistical properties of the structure function have a universal form which is uniquely determined by $\mathcal{E}$ and $r$ alone. Therefore,

$$S_m(r) \propto (\mathcal{E}r)^{m/3} \quad (\lambda_K \ll r \ll L). \tag{26}$$

In particular, one obtains for $m = 2$,

$$S_2(r) \propto \mathcal{E}^{2/3} r^{2/3} \quad (\lambda_K \ll r \ll L). \tag{27}$$

The second-order structure function is the energy contained in all Fourier modes with wavenumbers larger than $1/r$ (Falkovich and Sreenivasan, 2006).

The structure function and the energy spectrum are related in a standard form (Batchelor, 1953; Hou *et al.*, 1998). Reformulating Eq. 27 into the Kolmogorov energy spectrum,[28] we have

$$E(k) = C_K \mathcal{E}^{2/3} k^{-5/3}, \tag{28}$$

where $C_K$ is the Kolmogorov constant with a value around $1.5 - 1.7$ (Donzis and Sreenivasan, 2010; Gotoh and Watanabe, 2005; Kaneda *et al.*, 2003; Sreenivasan, 1995; Yeung and Zhou, 1997).[29] Finally, the energy spectrum of incompressible turbulence has shown a pileup of energy near the dissipation wave number where the viscous actions begin, a phenomenon known as the *bottleneck effect* (Donzis and Sreenivasan, 2010; Falkovich, 1994; Frisch *et al.*, 2008; Katul *et al.*, 2015; Kűchler *et al.*, 2019; Kurien *et al.*, 2004; Saddoughi and Veeravalli, 1994).

While Chang *et al.* (2012) reported that $C_K$ is independent of direction and anisotropy, Chien *et al.* (2013) found that they can make these effects big enough to produce a 20% change in the Kolmogorov constant for the second-order structure function through explicit control of the time dependence of the energy input.[30] Praskovsky and Oncley (1994) experimentally studied high Reynolds number[31] turbulent flows in the atmospheric surface layer and in a large wind tunnel [$Re_\lambda \approx (2.0 - 12.7) \times 10^3$] and reported that the Kolmogorov constant in this law is found to be weakly dependent on $Re_\lambda$. Recently, Ishihara *et al.* (2016) confirmed a $Re_\lambda$ dependence using high-resolution DNS of forced incompressible turbulence.

Kolmogorov's hypothesis essentially states the statistical independence of high and low wavenumbers in large Reynolds number turbulence. Fundamentally, he suggested that the inertial range scales are not influenced by the large scales directly and the large scales evolve very slowly by comparison with the eddies in the inertial range. The $-5/3$ scaling has received excellent confirmations (Chapman, 1979; Grant *et al.*, 1962; Saddoughi and Veeravalli, 1994) from high Reynolds number measurements in geophysical flows and in the world's largest wind tunnels.[32] An asymptotic expansion analysis offers support that Kolmogorov's hypothesis is equivalent to the locality assumption (Branover *et al.*, 1999).

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[28] There are diametrical opinions on how rigorously the Kolmogorov spectrum should be ascribed. The technical aspects of the opposing views below will be given in later sections of the review, and there is no doubt that readers will form their own informed judgment after learning the extensive materials. Indeed, Belinicher and L'vov (1987) showed that line-resummed Wyld's diagrammatic series (Wyld, 1961) for the Navier-Stokes equation in the sweeping-free Belinicher-Lvov-Procaccia representation has an order-by-order solution with Kolmogorov-41 scaling;. Another viewpoint is that it is known only to perhaps 3% accuracy from experiments, and there is no theoretical justication for the exactness of this value beyond dimensional analysis. Moreover, there is no physically realizable model that produces this value, only uncontrolled approximations. DIA is realizable, via $N \to \infty$ multicomponent models (e.g., random coupling) but produces $k^{-3/2}$ — which under ordinary circumstances might have been viewed as remarkably accurate (e.g., to be systematically corrected within some controlled expansion). The K41 result should properly be viewed as being at the level of a mean field result in critical phenomena. It is perhaps unfortunate that its predictions are so close to the experimental value, encouraging decades of fruitless attempts to prove its exactness.

[29] Cheung and Zaki (2014) derived a spectral energy equation where the combination matrix approach is used to characterize the nonlinear triad interactions in wavenumber space in homogeneous isotropic turbulence (HIT). In an appropriate canonical basis and without invoking dimensional scaling arguments, the Kolmogorov energy spectrum in the inertial range is derived from the Navier–Stokes equations. In her survey of the scale interactions of wall turbulence, McKeon (2017) noted that an assembly of component modes (triads) in a self-sustaining system is complementary to the combination matrix.

[30] It would be highly desirable to carry out much higher Reynolds number experiments to entangle the influences between the anisotropy and finite Reynolds number effect (see subsection VII.G).

[31] The Reynolds number is often defined based on the Taylor microscale ($\lambda$). Here,

$$Re_\lambda = \frac{u\lambda}{\nu},$$

where $Re_\lambda$ equals to $(20/3)^{1/2} Re^{1/2}$ for homogeneous and isotropic flow, see Tennekes and Lumley (1972). Here, $Re$ is given by Eq. 9 or by $Re = K^2/\nu\mathcal{E}$ where the turbulent kinetic energy is computed from Eq. 22 . A standard way to compute the Taylor microscale is from the curvature of the velocity autocorrelation (Champagne *et al.*, 1970), $\mathcal{R}(r)$, which is defined as

$$\mathcal{R}_{ii}(r) = \frac{U_{ii}(r)}{U_{ii}(0)}.$$

Note that $\mathcal{R}(r)$ is an even function, so that it is symmetric $\mathcal{R}(r) = \mathcal{R}(-r)$. Performing the Taylor expansion, the leading terms are

$$\mathcal{R}(r) = 1 - \frac{r^2}{\lambda^2},$$

where the Taylor microscale is defined as

$$\lambda = \left[ -\frac{1}{2} \frac{d^2\mathcal{R}(0)}{dr^2} \right]^{-1/2}.$$

Another way of estimating the Taylor microscale from experimental data has been utilized with the variance of the velocity and the variance of the first derivative of velocity (Reese *et al.*, 2018)

$$\lambda = \left[ \frac{2\langle u^2 \rangle}{\langle (\partial u/\partial x)^2 \rangle} \right]^{1/2}.$$

[32] Just to show how robust the Kolmogorov spectrum is, Dűring *et al.* (2019) found the $k^{-5/3}$ scaling in a totally different physical system:

The finding of the $k^{-5/3}$ spectrum has been called “simultaneous discovery” (Battimelli, 1986), because it was also suggested independently by Heisenberg (1948a), von Weizsäcker (1948), Prandtl (Bodenschatz and Eckert, 2011), and Onsager (1945, 1949).

• Kolmogorov's *Four-fifths law* is given by

$$S_3(r) = -\frac{4}{5}\mathcal{E}r. \tag{29}$$

Eq. 29, with coefficent 4/5, does not arise from phenomenology, but as shown by Kolmogorov in 1941, is an exact analytical consequence of the Navier-Stokes equations (under certain, explicitly stated assumptions). Note also that this relation (also credited to von Kármám and Howarth (1938)), along with the exact value of the coefficient, follow rather generally from locality in the energy transfer equation, and do not rely on dimensional analysis. The 4/5 law is not restricted only to HIT, as its extensions have been derived for more complex cases, such as the magnetohydrodynamics (MHD) (Gomez *et al.*, 1999; Politano and Pouquet, 1998a,b) and geophysical (e.g., Augier *et al.* (2012) and Kurien *et al.* (2006)) turbulence.

There is also a “local” version of the 4/5 law (Eyink, 2002) which suggested that the 4/5 law can be recovered in a local subdomain for a locally computed dissipation rate $\mathcal{E}$ at any instant in time if the structure function is averaged over angles. Motivated by that result, Taylor *et al.* (2003) performed the first angle-averaged structure function calculation in order to be able to project the isotropic component of the flow statistics. Many subsequent papers on angle-averaging to extract isotropic or angle-averaged statistics are based on the same methodology (see subsection IX.E). Note also that the $S_3$, inclding three powers of velocity, changes sign under the operation of time reversal. Hence, the nonvanishing of $S_3$ signals a breakdown of time reversibility in the inertial range (Falkovich and Sreenivasan, 2006).

### E. Scalings and intermittency in three- and two-dimensional flows and passive scalar field

It should be noted that significant discrepancies begin to develop between Eq. 26 and measured or simulated data for $n > 3$. The breakdown of scale invariance in the inertial range, so-called anomalous or multifractal scaling, is an important feature of turbulence (Falkovich and Sreenivasan, 2006).[33] Landau is the namesake of a famed remark, appearing as a footnote of Landau and Lifshitz (1959),

> ... averaging these expressions is dependent on the variation of $\mathcal{E}$ over times of large-scale motions (scale $L$), and this variation is different for different specific flows. Therefore, the result of the averaging cannot be universal.

As illustrated in Fig. 3, the values of second-order exponents depart from the Kolmogorov value of 2/3 (Eq. 27) and approach a constant $0.72 \pm 0.004$ at a very higher Reynolds numbers DNS ($Re_\lambda = 1300$). This result indicated the effect of intermittency can be felt even at the level of the energy spectrum, albeit a small one (Iyer *et al.*, 2020).

This point of view is supported by the experimental data of decaying turbulence presented in Kűchler *et al.* (2020). With the measured data obtained from the Max Planck Variable Density Turbulence Tunnel (VDTT), the authors demonstrated that the scaling exponent of the second-order velocity increment statistics to be independent of the Reynolds number and equal to $0.693 \pm 0.003$ for $2000 \le Re_\lambda \le 6000$. The difference from 2/3rds is also attributed to intermittency.[34]

The logarithmic local slopes of velocity difference moments

$$\zeta_m = \frac{d \log S_m(r)}{d \log(r)} \tag{30}$$

are important measurements. Using data from grid turbulence at $R_\lambda = 1030$ (Sinhuber *et al.* (2017)) and DNS at $R_\lambda \sim$1300 (Iyer *et al.* (2020)), Fig. 4a shows that the general trend in experiments (EXP, open circles) and the

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[33] Chen *et al.* (2005) illustrated the relative difference between Kolmogorov's prediction and the measured or simulated scaling exponents.
[34] Gregory P. Bewley, private communication, 2020. The experimentally measured exponent is lower than the DNS value reported in Iyer *et al.* (2020). The reason for the difference between DNS and experiments has not been conclusively identified. However, the instrumentation in the VDTT experiments allow only unidirectional velocity measurements (Kűchler *et al.*, 2021), but scaling laws appear only when projecting onto appropriate symmetry groups when the flow field is not isotropic (see for example, Biferale and Procaccia (2005), Kurien and Sreenivasan (2000), as well as Fig. 4c). It is also interesting that the measured value is essentially the same as that obtained from a study on extended self-similarity (Benzi *et al.*, 1993) in turbulent flows.

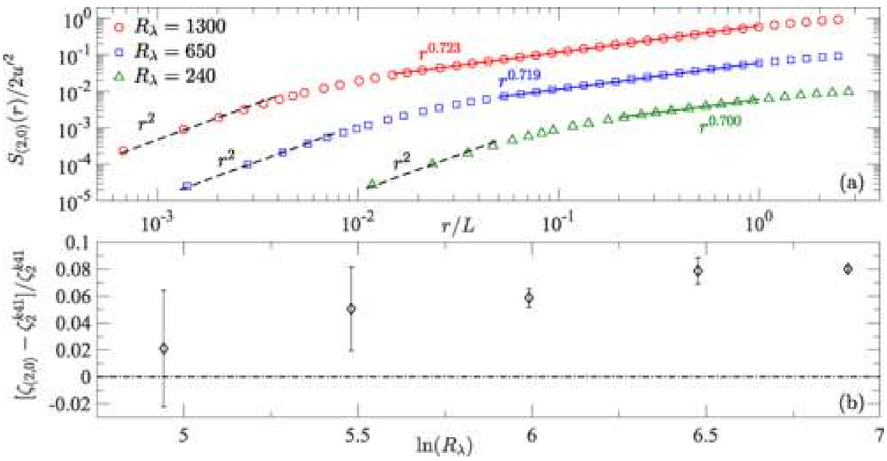


FIG. 3: (a) Scaling of the normalized second-order structure function. (b) Relative deviation of the second-order inertial range exponent from the Kolmogorov value of 2/3. Vertical bars indicate the standard error due to temporal variations in the least-square fits. Iyer *et al.* (2020) *Phys. Rev. Fluids*, ©American Phys. Soc.

isotropic DNS (solid line) data is similar (Iyer *et al.*, 2021) for the first non-trivial ratio between exponents of analytic functions

$$\zeta_{4,2} = \frac{d \log S_4(r)}{d \log S_2(r)}. \tag{31}$$

Both data sets display a crossover "bottleneck" between the viscous and inertial regimes around $r/\lambda_K \approx 80$ and a $R_\lambda$-dependent viscous dip (around $r/\lambda_K \approx 10$) predicated by multifractal models (Frisch and Vergassola, 1991; Meneveau, 1996).[35] The bottleneck, which lies outside the inertial range, decreases in amplitude with increasing $R_\lambda$ (Fig. 4b). The finite (and small) mismatch between EXP and the isotropic sector, as shown in Fig. 4c, from DNS could arise from the fact that the former is not projected onto the isotropic sector (see also, subsection IX.E).

Data from grid turbulence and DNS demonstrate that the flows are intermittent with a departure from the horizontal line $\zeta_{4,2} = 2$, which corresponds to non-intermittent Kolmogorov similarity scaling (Kolmogorov, 1941d). Fig. 4a also shows three different phenomenological models: the *p*-model of Meneveau and Sreenivasan (1987) (dash-dot line), the She and Leveque (1994) model (dashed line) and the Yakhot (2001) model (dotted line).

Lilly (1972a,b) discussed the relevance of idealized 2D turbulence to certain aspects of atmospheric and oceanic dynamics (Pedlosky, 1987; Vallis, 2006). Such 2D turbulent flows are of considerable significance in meteorology, serving as the simplest analog of large-scale planetary motions (Gage, 1979; Herring, 1985; Lilly, 1983; Salmon, 1998). The existence of dual cascades in 2D flows is a key distinction between 2- and 3-dimensional turbulence.[36] More specifically, the expected 2D scalings (Batchelor, 1969; Kraichnan, 1967; Kraichnan and Montgomery, 1980) for the inverse energy cascade follow

$$E(k) = C\mathcal{E}^{2/3} k^{-5/3}, \tag{32}$$

and the direct enstrophy transfer is given by

$$E(k) = C' \beta^{2/3} k^{-3} \left[\ln(k/k_1)\right]^{-1/3}. \tag{33}$$

The coefficient $C$ has been evaluated from the experimental measurement ($6.5 \pm 1$) of Paret and Tabeling (1997) and numerical simulation ($6.0 \pm 0.4$) of Boffetta *et al.* (2000). The requirement of a constant enstrophy flux in the direct cascade led Kraichnan to propose a logarithmic correction to the energy spectrum of the form of Eq. 33 (Boffetta and Ecke, 2012).[37]

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[35] Iyer *et al.* (2021) used $\eta$ to denote the Kolmogorov scale in the caption of Fig. 4, but in this article $\eta$ is reserved for the DIA damping function, defined in Eq. 70.

[36] Yet, Pullin and Saffman (1998) stated: "two-dimensional turbulence, .... is a consequence of the construction of large computers." In any case, in two case studies, inertial confinement fusion (ICF) implosion and supernovae explosion, fully-3D calculations are needed to capture the key physics (Zhou *et al.*, 2019).

[37] Numerical studies of 2D quantum turbulence in superfluids, with quantized pointlike vortices, have identified the formation of an inverse energy cascade to large length scales and demonstrated the development of an enstrophy cascade to small length scales with a $k^{-3}$ spectrum in decaying 2D quantum turbulence. See Forrester *et al.* (2020), and the references therein.

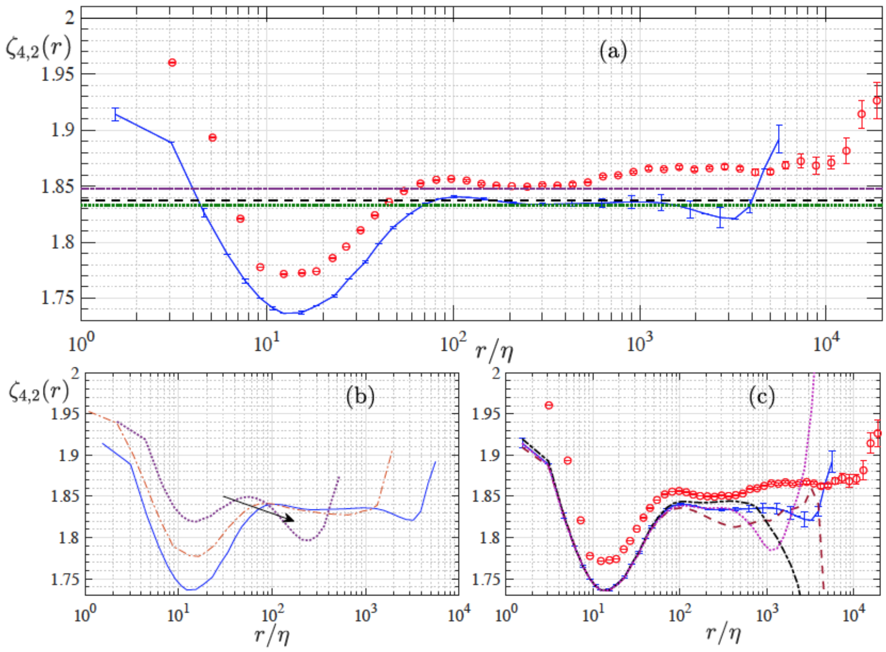

FIG. 4: Ratio of logarithmic local slopes of $S_4$ and $S_2$, $\zeta_{4,2} \equiv d[\log S_4(r)]/d[\log S_2(r)]$, versus spatial separation $r$ normalized by Kolmogorov scale $\lambda_K$ [We should note that this figure uses $\eta$ to denote the Kolmogorov scale, but in this article, $\eta$ is reserved for the DIA damping function, defined in Eq. 70]. (a) Data from EXP at Taylor microscale based Reynolds number, $R_\lambda = 1030$ (open circles) are from Sinhuber *et al.* (2017); the isotropic sector from DNS at $R_\lambda \sim 1300$ (solid line) are compared with the p-model (dash-dot line) of Meneveau and Sreenivasan (1987), She and Leveque (1994) model (dashed line) and that by Yakhot (2001) (dotted line). Horizontal line $\zeta_{4.2} = 2$ corresponds to non-intermittent scaling (Kolmogorov, 1941d). Error bars indicate the standard error obtained from temporal fluctuations of local slopes. (b) Isotropic DNS data at different $R_\lambda$: 240 (dotted line), 650 (dash-dot line) and 1300 (solid line) are plotted to show that the viscous bottleneck around $r/\lambda_K \sim 80$ decreases in amplitude with increasing $R_\lambda$ in the direction shown. (c) Data from EXP (open circles), compared with the DNS data for the isotropic sector (solid line) and the one-dimensional cuts in DNS along the three Cartesian directions: (1, 0, 0) (dotted line), (0, 1, 0) (dashed line) and (0, 0, 1) (dash-dot line). The latter three illustrate that the inertial range value of $\zeta_{4,2}(r)$ can be affected by non-universal large scale effects when not projecting onto the isotropic sector. Fig. 2 of Iyer *et al.* (2021) *Phys. Rev. Lett.*, ©American Phys. Soc.

Falkovich and Lebedev (1994a) predicted logarithmic corrections for higher-order correlators of the vorticity field, $\omega = \nabla \times \mathbf{u}$,

$$\langle [\omega(\mathbf{r})\omega(0)]^m \rangle \simeq \ln^{2m/3}(\ell_f/r), \tag{34}$$

which is in a form independent of the statistics of the forcing. In above equation, $\ell_f$ denotes the length scale of the forcing injection, which is necessary to produce a statistically stationary state when dissipation action is present. Recent experimental and simulation studies of Bruneau and Kellay (2005) and Paret *et al.* (1999) offered support to Eq. 34. Also, Eyink (1996) derived exact bounds for scaling exponents of the vorticity structure functions.

The quasi-Gaussian statistics in the 2D inverse cascade (with no intermittency) were first discussed in Boffetta *et al.* (2000).[38] In a follow-up article, Boffetta and Musacchio (2010) were able to resolve the two cascades simultaneously; again showing consistency with a $-5/3$ energy spectrum without intermittency. Indeed, it appears that the cascade process of 2D turbulence is not subject to intermittency corrections — see, for example, Gkioulekas and Tung (2006) and Tabeling (2002).

The concentration of a substance advected by a turbulent flow exhibits a complex, chaotically evolving structure over a broad range of space and time scales (Shraiman and Siggia, 2000; Sreenivasan, 1991). Passive scalar behavior

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[38] The important point foreseen in Siggia and Aref (1981), as helpfully reminded us by Boffetta *et al.* (2000), is that the smallness of the skewness of velocity difference, $S_3(r)/[S_2(r)]^{3/2}$, would suggest that intermittency might be weak (see also, Paret and Tabeling (1998)).

is important in turbulent mixing, combustion, and pollution and provides impetus for the study of turbulence itself (Warhaft, 2000).[39] For scalar transport by turbulence, $\theta$,

$$\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)\theta = -(\mathbf{u}\cdot\nabla)\theta, \tag{35}$$

where $\kappa$ is the thermal diffusivity.[40]

For passive scalar spectra, there are three distinctive regimes defined by the values of the viscosity and diffusivity (Batchelor, 1959; Batchelor *et al.*, 1959). In the inertial-convective range for which $\nu \sim \kappa \ll 1$,

$$E_\theta = B_{IC}\mathcal{E}^{-1/3}\chi k^{-5/3}, \tag{36}$$

while in the inertial-diffusive range for which $\nu \ll \kappa \ll 1$,

$$E_\theta = B_{ID}\kappa^{-3}\mathcal{E}^{1/3}\chi k^{-17/3}, \tag{37}$$

and finally, in the viscous-convective range for which $\kappa \ll \nu \ll 1$,

$$E_\theta = B_{VC}(\nu/\ \mathcal{E})^{1/2}\chi k^{-1}. \tag{38}$$

Here, $B_{IC}$, $B_{ID}$ and $B_{VC}$ are constants and $\chi$ is the average rate of scalar dissipation per unit mass.

The structure function of order $m$ constructed from the scalar field is provided by

$$S_m^\theta(\mathbf{r}) = \langle|\theta(\mathbf{x}+\mathbf{r}) - \theta(\mathbf{x})|^{\xi_m}\rangle, \tag{39}$$

where $\mathbf{r}$ denotes the separation.[41] Dimensional arguments (Corrsin, 1951; Obukhov, 1949), similar to those of Kolmogorov for the velocity field, resulted in $\xi_m = m/3$ (Odd-order structure functions vanish by symmetry in the isotropic case). Strong evidence, however, indicated that the measured scaling exponents, $\xi_m$, saturate with respect to $m$ for $m > 10$ (Shraiman and Siggia, 2000; Sreenivasan, 1991; Warhaft, 2000).

More recently, the quasi-Lagrangian formulation of V. L'vov & I. Procaccia and Kraichnan's solvable passive scalar model provided new, robust methods for studying the intermittency phenomenon and the associated anomalous scaling exponents. The presentation of these elegant, sophisticated analyses will be postponed to sections VIII and IX.

## III. DIRECT INTERACTION APPROXIMATION

### A. Construction of the DIA models

In this section, the salient points of Kraichnan's theory (Kraichnan, 1959a) will be surveyed. It is a challenge to "justify" the DIA, or any turbulence theoretical approximations in general. As Leslie (1973) pointed out, an approximation of any kind is justified by computing the error which is produced under stated circumstances and showing that it is acceptable. This calculation of the error might be made either by an abstract mathematical process or by comparing some exact solutions with the answers given by the approximation. Unfortunately, no exact statistical solutions of the Navier-Stokes equation are known and direct numerical simulations (DNS) have only recently achieved sufficiently high $Re$ for some limited flow fields.

Kraichnan (1988a) pointed out that if a neutral viewpoint is adopted, the danger is that the effects of characteristic structures of the real flow may be totally lost. With this caveat in mind, the DIA was basically circumscribed by its mathematical complexity and was essentially constructed for the simplistic, idealistic HIT.

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[39] Warhaft (2000) pointed out that a weakly heated flow, such as an air jet, exhibits passive scalar mixing as the cooler air is entrained from the surroundings. Moisture mixing in air and dye in water provides other typical examples. This is labeled as Level-1 mixing by Dimotakis (2005). According to this classification scheme, Level-2 mixing is coupled to the dynamics, such as in the mixing of different density fluids in an acceleration/gravitational field, as in Rayleigh-Taylor (Rayleigh 1883, Taylor 1950) and Richtmyer-Meshkov (Richtmyer 1960, Meshkov 1969) instability flows (Zhou, 2017a,b, Zhou *et al.*, 2019). Level-3 mixing produces changes to the fluid(s), e.g., in composition, density, enthalpy conversion/release, pressure increase, etc., and is in turn coupled to the dynamics. Examples of Level-3 mixing are most combustion phenomena and detonations (Dimotakis, 2005).

[40] Several non-dimensional parameters associated with the scalar field should be noted. The Prandtl number is $Pr = \nu/\kappa$ and the Schmidt number is $Sc = \nu/\mathcal{D}$, where $\nu$ is the kinematic viscosity and $\mathcal{D}$ is mass diffusion coefficient. While Eq. 35 is written with $\kappa$ as the thermal diffusivity, $\mathcal{D}$ may be used instead as appropriate for mass diffusion problems as the relevant coefficient. As a result, the Péclet number can be defined as $\mathcal{P}e = Re \times Pr$ for heat transfer and $\mathcal{P}e = Re \times Sc$ for mass diffusion (Mauri, 2015). The Péclet number signifies the ratio of the strength of convection to the strength of conduction in equation 35.

[41] For simplicity, one considers the inertial-convective region ($Sc = 1$). In the inertial range, the properties of the scalar struction depends only on the energy and scalar flux across the scale. see, e.g. Sreenivasan (2019b).

### 1. Criteria for a valid approximation

In order to optimize the *slim* chances of getting good answers, Kraichnan decided to concoct the DIA by imposing several consistency properties to the approximations (Kraichnan, 1959a, 1966a):

First, the approximations should not destroy the fundamental invariance properties of the exact flow equations. In particular, the basic features of the nonlinear Navier-Stokes interactions requires the conservation of energy and momentum (Kraichnan, 1973; Kraichnan and Chen, 1989), and in two-dimensional flows, the conservation of total vorticity (Kraichnan and Montgomery, 1980).[42]

Second, the existence of formal inviscid equipartition ensembles and the associated fluctuation-dissipation theory (Deker and Haake, 1975; Kraichnan, 1959c) are noble features (Kraichnan, 1958). The statistical studies were initiated by Burgers (1929).[43] Hopf (1952), Kraichnan (1958), and Lee (1952) built on the existence of canonical ensembles based on ideal invariants.

The equilibrium behavior can be nicely described as follows. First, the equation of motion of a uniform incompressible fluid, Eq. 4, may be written in the Fourier transformed spectral space as[44]

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)u_i(\mathbf{k},t) = -iM_{ijm}(\mathbf{k})\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} u_j(\mathbf{p},t)u_m(\mathbf{q},t), \tag{40}$$

where $\mathbf{k}=\mathbf{p}+\mathbf{q}$, $M_{ijk}(\mathbf{k})=\frac{1}{2}\left[k_jP_{ik}(\mathbf{k})+k_kP_{ij}(\mathbf{k})\right]$, $k_iP_{ij}(\mathbf{k})=0$, and $P_{ij}(\mathbf{k})=\delta_{ij}-k_ik_j/k^2$ expresses the action of the pressure. Also, incompressibility requires $k_iu_i(\mathbf{k},t)$ =0. Next, one takes the inviscid limit $\nu\to 0$ and eliminates all terms that contain a wavenumber greater than some cutoff from the equations. The resulting system has a finite number of degrees of freedom per unit volume, obeys a Liouville theorem (Huang, 2001; Tolman, 1938), and conserves kinetic energy (Kraichnan, 1966a).

This is an appropriate point at which to bring up some early theories by Heisenberg (1948a,b), Kovasznay (1948) and Obukhov (1941). Their heuristic approaches relied on dimensional as well as physical analysis to fix the function forms that give energy transfer in terms of energy spectra, and to fix the empirical constant introduced (Herring, 1985). However, Kraichnan and Spiegel (1962) showed these theories lacked an equipartitioning tendency.

### 2. Basic idea of the direct interaction approximation

From Eq. 40, we can construct the energy transfer equation (Kraichnan, 1957)

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)\left[\frac{1}{2}u_i(\mathbf{k},t)u_i^*(\mathbf{k},t)\right] = \mathrm{Im}\sum_{p,q}\left[u_i(\mathbf{k},t)M_{ijm}(k)u_j(\mathbf{p},t)u_m(\mathbf{q},t)\right]. \tag{41}$$

Thus, for the case of homogeneous turbulence, the energy transfer among the Fourier modes is associated with triple correlations among triads of interacting modes.

As articulated by Kraichnan (1959a, 1964c), the DIA assumes that the correlation of any given triad of Fourier modes is induced by continuous direct dynamical interaction of the triad, acting against a relaxation process which destroys the correlation. The relaxation process has two contributions: (1) dynamical relaxation due to interaction of each of the three modes with all the rest of the Fourier modes; (2) viscous decay.

The covariance, $U(k,t,s)$ is defined as

$$(\mathcal{L}/2\pi)^3\langle u_i(\mathbf{k},t)u_j^*(\mathbf{k},s)\rangle = \frac{1}{2}P_{ij}(\mathbf{k})U(k,t,s), \tag{42}$$

where $\mathcal{L}$ is the cyclic box size and $k=|\mathbf{k}|$. $U(k,t,s)$ is related to the energy spectrum, $E(k,t)$, in three dimensions by

$$E(k,t) = 2\pi k^2U(k,t,t). \tag{43}$$

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[42] In a velocity field, the curl of the velocity is equal to the vorticity, $\omega=\nabla\times\mathbf{u}$ (Anderson, 1984).

[43] It is interesting to note that J.M. Burgers (1895–1981), widely regarded as one of the leading scientists in fluid mechanics, moved his research from quantum physics to turbulence and related areas. He contributed many important results, a number of which still bear his name. See Nieuwstadt and Steketee (1995).

[44] Instead of using Cartesian components, a decomposition in terms of helical modes (Cambon and Jacquin, 1989; Waleffe, 1992, 1993) or Craya-Herring frame (Craya, 1958; Herring, 1974) may be more convenient for some applications (see Sagaut and Cambon (2018)).

Analytically, the DIA is devised as a coupled system of integro-differential equations for *two* descriptors of homogeneous turbulence: the *covariance* $U(k,t,s)$ defined in Eq. 42, and the *response function* $G(k,t,s)$ (Kraichnan, 1959a). The covariance is familiar from Taylor's statistical theory of turbulence (Batchelor, 1953; Taylor, 1938b). However, the response function is an entirely new quantity introduced by Kraichnan to analyze perturbations in a turbulent flow.

### 3. The response function

Because the response function will prove to be particularly significant for other closure theories, we would like to begin by reviewing the reasons that led Kraichnan to introduce it (Zhou, 2010).

For our present purposes, all that is really relevant is the quadratic nonlinearity in the Navier-Stokes equations. Thus, we may investigate equation 40 by a simple cartoon problem

$$\dot{u}(t) + \nu k^2 u(t) = M u(t) u(t), \tag{44}$$

ignoring the vector indices and wavevector arguments that would appear in the real problem. The quasinormal theory re-expressed this relation by inverting the viscous operator on the left side as

$$u(t) = \int_0^t ds\ \exp(-\nu(t-s)) M u(s) u(s) \tag{45}$$

We could call $G(t,s) = \exp(-\nu(t-s))H(t-s)$, where $H$ is the Heaviside function, the viscous response function. Of course, equation 45 is essentially "exact," but its use in the perturbative context of analytical theories like the quasinormal theory causes the modeling of time correlations to depend on the viscosity. This dependence is at variance with the Kolmogorov theory (Kraichnan, 1959a). However, this was only one of the problems of the theory, which also proved to predict negative values of the energy spectrum (Ogura, 1963; Orszag, 1970) as discussed in II.C, violating the principle of realizability (Kraichnan, 1959a; Orszag, 1977).

Kraichnan argued that to compute perturbatively, the effect of perturbations should be analyzed using the result of adding a forcing term to the Navier-Stokes equations. For the cartoon problem of Eq. 44, this means solving

$$\dot{u}(t) + \nu k^2 u(t) = M u(t) u(t) + f(t). \tag{46}$$

Of course, this problem is more difficult than the original one, but if we assume the perturbation to be "small," then it is reasonable to replace $f$ by $\delta f$ and ask for the small change in $u$, $\delta u$ that is produced by $\delta f$. This means solving the simpler problem

$$\dot{\delta u}(t) + \nu k^2 \delta u(t) = 2M u(t) \delta u(t) + \delta f(t) \tag{47}$$

The crucial physical distinction between this equation and the quasinormal approximation based on Eq. 45 is that the frequency scale governing the time evolution of $\delta u$ is now $Mu$, which we expect to dominate the viscous frequency scale, $\nu k^2$, in the regime of strong nonlinearity.

To introduce the notion of the infinitesimal response function, which involves two times, we take functional derivatives w.r.t. $f(s)$. The ratio $\delta u(t)/\delta f(s)$ is an exact, linearized response function. DIA achieves closure at the level of second-order statistics by replacing this fluctuating quantity by its average, $G(t,s) = \langle \delta u(t)/\delta f(s) \rangle$, and using this deterministic quantity to compute the response to perturbations. Averaging Eq. 47 leads to the formal equation of motion for $G$,

$$\dot{G}(t,s) + \nu k^2 G(t,s) = 2M \langle u(t) \delta u(t)/\delta f(s) \rangle \tag{48}$$

The correlation on the right side is unclosed, but DIA provides a closure for both this correlation and the third order moments in the equation for $U$.

### 4. A physically motivated, schematic derivation of the DIA equations

Here we provide a physically motivated, schematic derivation of the DIA, following that of Herring (1999). We write down the simplified model equations of motion for the Fourier amplitude $u_i$ of the velocity field twice, once at $\mathbf{x}, t$ and once at $\mathbf{x}', t'$, as

$$(\partial_t + L_0) u = \tilde{f}, \tag{49}$$

$$(\partial_{t'} + L_0)u' = \tilde{f}', \tag{50}$$

where $L_0$ is viscous dissipation and $\tilde{f}_i = \sum_{j,k} C_{ijk}u_j u_k$. Here we are using the indices to represent only wave vectors and do not include the complexity of the vector indices. Note that the model interaction coefficient $C_{ijk}$ is invariant under the interchange of the last two indices. Furthermore, in analogy to the non-interaction of a wavevector mode with itself in the NS equation, we impose $C_{ijj} = 0$. For inviscid energy conservation, the interaction coefficient must also obey

$$C_{ijk} + C_{jki} + C_{kij} = 0. \tag{51}$$

We can construct, from Eqs. 49-50, an exact expression

$$(\partial_t + L_0)(\partial_{t'} + L_0)\langle uu'\rangle = \langle \tilde{f}\tilde{f}'\rangle, \tag{52}$$

which can be rewritten as

$$(\partial_t + L_0)\langle uu'\rangle = G_0'\langle \tilde{f}\tilde{f}'\rangle, \tag{53}$$

where $G_0 = (\partial_t + L_0)^{-1}$.

The DIA may be viewed as first approximating the nonlinear forcing of the velocity field as Gaussian, while simultaneously introducing an eddy viscosity that provides total energy conservation. The avenue for the eddy viscosity is provided by $G(k, t, t')$. Thus, the DIA first makes a crude approximation,

$$(\partial_t + L_0)\langle uu'\rangle = G_0'\langle \tilde{f}\tilde{f}'\rangle_{\mathcal{G}}. \tag{54}$$

where the designation $\mathcal{G}$ is to evaluate the right-hand side of Eq. 54 as if the components of $\tilde{f}$ are from a Gaussian ensemble. However, this assumption results in the loss of inviscid energy conservation in Eq. 54. DIA restores conservation by introducing an analytic eddy viscosity, $\delta L$ as follows:

$$(\partial_t + L_0 + \delta L)\langle uu'\rangle = G'\langle \tilde{f}\tilde{f}'\rangle_{\mathcal{G}}. \tag{55}$$

Where $\delta L$ is to be considered a operator. The notation can be simplified by introducing the full operator $L$ as $L = L_0 + \delta L$, and the nonlinear response function also as an operator: $G = (\partial_t + L)^{-1}$. Therefore, we have

$$(\partial_t + L)\langle uu'\rangle = G'\langle \tilde{f}\tilde{f}'\rangle_{\mathcal{G}}. \tag{56}$$

To determine $\delta L$, we next check inviscid energy conservation.[45] Noting that $\partial_t\langle uu\rangle = 2\partial_t\langle uu'\rangle\big|_{t=t'}$, we see that according to Eq. 56 inviscid energy conservation requires

$$\sum_i \left(\delta L\langle uu'\rangle\right)_i \Big|_{t=t'} = \sum_i \left(G'\langle \tilde{f}\tilde{f}'\rangle_{\mathcal{G}}\right)_i \Big|_{t=t'} \tag{57}$$

We must now write $G'\langle \tilde{f}\tilde{f}'\rangle_{\mathcal{G}}$ in explicit form. We use the non-self-interaction property, the assumption of homogeneity in the form $< u_i u_j > = < u_i u_i > \delta_{ij}$ and Gaussian statistics to obtain

$$\langle \tilde{f}_i\tilde{f}_i'\rangle_{\mathcal{G}} = 2\sum_{jk} C_{ijk}C_{ijk}\langle u_j u_j'\rangle\langle u_k u_k'\rangle. \tag{58}$$

Thus,

$$G'\langle \tilde{f}\tilde{f}'\rangle_{\mathcal{G}} \Big|_{t=t'} = 2\int_0^t ds \sum_{jk} C_{ijk}C_{ijk}G_i(t,s)U_j(t,s)U_k(t,s), \tag{59}$$

where $U_i(t,t') = < u_i(t)u_i(t') >$. Inserting this expression on the right-hand-side of Eq. 57 gives

$$\sum_i \int_0^t ds\, \delta L_i(t,s)U_i(s,t) = 2\int_0^t ds \sum_{ijk} C_{ijk}C_{ijk}G_i(t,s)U_j(t,s)U_k(t,s). \tag{60}$$

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[45] Use of explicit summation on indices for the equations in this subsection is contrary to early use (Einstein summation).

On the rhs, we then substitue the identity $C_{ijk} = -C_{jki} - C_{kij}$ for one of the two factors of $C_{ijk}$ and relabel some of the dummy indices to arrive at the following equation:

$$\sum_i \int_0^t ds\, \delta L_i(t,s) U_i(s,t) = -4 \int_0^t ds \sum_{ijk} C_{ijk} C_{jik} G_j(t,s) U_k(t,s) U_i(t,s). \tag{61}$$

This suggests

$$\delta L_i(t,s) = -4 \sum_{ijk} C_{ijk} C_{jik} G_j(t,s) U_k(t,s). \tag{62}$$

Thus the DIA equation for $U(t,t')$ is

$$(\partial_t + L_{i0}) U_i(t,t') = 2 \int_0^{t'} ds \sum_{jk} C_{ijk} C_{ijk} G_i(t',s) U_j(t,s) U_k(t,s)$$
$$+4 \int_0^t ds \sum_{jk} C_{ijk} C_{jik} G_j(t,s) U_k(t,s) U_i(t',s). \tag{63}$$

To obtain the DIA equation for $G_i(t,t')$, we first note that by definition the operator $G$ satisfies

$$(\partial_t + L) G = 1, \tag{64}$$

which we can now write out explicitly using Eq. 62 as follows

$$(\partial_t + L_{i0}) G_i(t,t') - 4 \int_{t'}^t ds \sum_{jk} C_{ijk} C_{jik} \times G_j(t,s) U_k(t,s) G_i(s,t') = \delta(t-t'), \tag{65}$$

where we have used the property of Green's function that $G(t,t') = 0$ if $t' > t$ to determine the limits on the integral. The exact governing equation for the response function for the Navier-Stokes equation will be given below.

A final remark concerns the uniqueness of the choice for $\delta L$. Looking again at the inviscid energy conservation equation Eq. 60, we see that interchanging the dummy indices $j$ and $k$ on the rhs of the equation brings it into a form that suggests an alternate choice: $\delta L_i(t,s) = 2 \sum_{jk} C_{jik} C_{jik} G_j(t,s) U_k(t,s)$. However, this alternative form leads to a violation of the principle of positive entropy production. This can be shown by assuming the fluctuation-dissipation theorem and slow variation in time of the modal energies $U_i(t,t)$, both assumptions that are valid for small perturbations away from canonical equilibrium. In other words, with this alternate choice for $\delta L$, a Boltzmann-like H-Theorem is not possible. The H-theorem follows only if we choose $\delta L$ in the form given by Eq. 62 (see Carnevale *et al.* (1981); Carnevale and Martin (1982); Salmon (1998, 2018)).

## 5. Renormalized perturbation methods

The Eulerian DIA equations have also been derived Wyld (1961) using methods closely analogous to those of quantum field theory (Peskin and Schroder, 1995), in which the non-perturbative character of turbulence requires certain classes of Feynman diagrams to be summed to infinite order (Abrikosov *et al.*, 1963; Fetter and Walecka, 1971; Kadanoff and Baym, 1962; Kraichnan, 1961).[46] Kraichnan's DIA is the simplest one-loop approximation in Wyld's DT. The procedure was further developed, as we discussed elsewhere in the review. Generalizations of the Wyld (1961) diagrammatic technique (DT) were suggested by Martin, Siggia, Rose (MSR) (Martin *et al.*, 1973) and by Zakharov and L'vov (1975) (L'vov and Procaccia, 1995a). MSR was subsequently reframed in term of path integrals (Phythian, 1977).

Another related procedure, developed by Kraichnan (1970b, 1977, 1988a), will be detailed here as this presentation will also prepare the reader for the next section, as it represents a key pillar of Kaneda's Lagrangian renormalized approximation (LRA) (Kaneda, 1981).

The application of renormalized perturbation methods to the DIA system consists essentially of the following steps (Kraichnan, 1988a):

[46] As Das (2004) pointed out, there is a parallel between the graphical methods of field theory, the DIA and statistical closure theories in the turbulence community, and the self-consistent mode-coupling model (e.g., Götze (2008)) in soft condensed matter field research.

1. Construct a primitive expansion of the scalar covariance and response functions in terms of their zeroth order approximations $U^0(k,t,t')$ and $G^0(k,t,t')$.

2. Invert the primitive expansions to solve for (the unobservable) $U^0$ and $G^0$ in terms of the observable $U(k,t,t')$ and $G(k,t,t')$.

3. Substitute the inverted expansions for $U^0$ and $G^0$ into the primitive expansion for any moment of the exact Fourier amplitudes. This yields a complete line-renormalized expansion for the relevant moment in which only the exact $U$ and $G$ appear.

4. Truncate the expansion at an appropriate order.

The DIA integro-differential equation for the scalar covariance is given by (Kraichnan, 1959a, 1991)

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) U(k,t,t') = S(k,t,t'), \tag{66}$$

where $S(k;t,t')$ is the sum over triple moments,

$$\begin{aligned} S(k,t,t') = \pi k \int\int_\Delta pq\, dp\, dq \Big[ \int_0^{t'} a(k,p,q) G(k,t',s) U(p,t,s) U(q,t,s) ds \\ - \int_0^t b(k,p,q) U(k,t',s) G(p,t,s) U(q,t,s) ds \Big], \end{aligned} \tag{67}$$

$\int\int_\Delta$ denotes integration over all $p, q$ such that $k, p, q$ form a triangle, and

$$\begin{aligned} a(k,p,q) &= \frac{1}{2}(1 - xyz - 2y^2z^2), \\ b(k,p,q) &= \frac{p}{k}(xy + z^3), \end{aligned} \tag{68}$$

where $x, y, z$ are the cosines of the internal angles opposite sides $k, p, q$ (Kraichnan, 1991).

The DIA response function, $G(k,t,t')$, solves the equation

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \nu k^2\right) G(k,t,t') + \int_{t'}^t \eta(k,t,s) G(k,s,t')\, ds = 0, \\ \big[t \geq t', \quad G(k,t',t') = 1\big], \end{aligned} \tag{69}$$

where the dynamical damping, $\eta(k,t,s)$, denotes the sum over triple moments:

$$\eta(k,t,s) = \pi k \int\int_\Delta pq\, dp\, dq\, b(k,p,q) G(p,t,s) U(q,t,s). \tag{70}$$

One important property of $U(k;t,t')$ is that it is the covariance scalar of a realizable probability distribution (Herring and Kraichnan, 1972) for $u_i(\mathbf{k},t)$,

$$\begin{aligned} E(k,t) \geq 0, \\ \big[U(k,t,t')\big]^2 \leq U(k,t,t) U(k,t',t'). \end{aligned} \tag{71}$$

In the language of quantum field theory, the procedure above yields an expansion in terms of the observables $U$ and $G$, rather than the traditional (but unobservable) zeroth order functions $U^0$ and $G^0$ (Kaneda, 2007). The lowest-order terms in the line-renormalized expansions for the quantities $S(k,t,t')$ and $G(k,t,t')$, which appeared in Equations 66 and 69, are precisely the DIA expressions.

It is now clear why DIA proved to be such a breakthrough in understanding the energy transfer process of turbulence: within DIA theory it was finally possible to construct a comprehensive framework for scrutinizing the various energy transfer and interacting scales of turbulence.

Non-Markovian closures such as the DIA are less restrictive in their applications than Markovian closures such as the EDQNM (Orszag, 1970), but this is at the cost of greater computational complexity. This is due to the fact that non-Markovian DIA closures are integro-differential equations with potentially long time-history integrals (e.g., Frederiksen and Davies (2000)). Some efforts have been devoted to tackle the computationally expensive evaluation of the time-history integral of the DIA equations in order to make closure methods more competitive with DNS. Rose

(1985) restarted the integration from time to time, taking into account of the triple correlation that is building during the evolution. In this fashion, the time-history integrals need to be evaluated only along the last time interval, where most of the time history is embodied in the triple correlations (Ottaviani, 1990). Moreover, Frederiksen and Davies (2004) and Frederiksen *et al.* (1994) implemented a generalization of the cumulant update restart procedure used by Rose (1985). On the other hand, Dannevik (1986) reduced the $(t, t')$ dependencies to a single time problem by using Padé tables. Domaradzki and Orszag (1987) also developed a transform method to allow relatively efficient numerical evaluation of the theory's nonlinear transfer terms.

### 6. The random-coupling model

The credibility of the DIA was further enhanced after Kraichnan (1961) and Frisch and Bourret (1970) demonstrated that the DIA is *exact* for the random-coupling model (RCM). The RCM equation, replacing Eq. 40, is a collection of $N$ systems which reads

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_i^n(\mathbf{k}, t) = -i \sum_{r,s=1}^{N} \varphi_{nrs} M_{ijm}(\mathbf{k}) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} u_j^r(\mathbf{p}, t) u_m^s(\mathbf{q}, t). \tag{72}$$

where eventually $N \to \infty$ (Kraichnan, 1961). Here $\varphi_{nrs}$ is a constant coefficient assigned the value 1 or $-1$ at random, subject only to invariance under any permutation of its three indices:

$$\varphi_{nrs} = \varphi_{srn}, \quad \varphi_{nrs} = \varphi_{nsr}. \tag{73}$$

Next,

$$\varphi_{nrs} = \pm N^{-1}. \tag{74}$$

where the plus sign or minus sign is chosen at random subject only to above-imposed symmetries (Herring and Kraichnan, 1972). When a statistical ensemble of realizations of the entire collection of systems is formed, $\varphi_{nrs}$ has precisely the same value in every realization in that ensemble (Kraichnan, 1991).

Kraichnan (1961) pointed out several properties of the RCM in the limit of $N \to \infty$. For brevity, we follow Herring and Kraichnan (1972) by stating three principal results:

1. The ensemble moments may be replaced by averaging over the supersystem:

$$U(k, t, t') = \frac{1}{N} \sum_{n=1}^{N} u_i^n(\mathbf{k}, t) u_i^{n*}(\mathbf{k}, t'), \quad (N \to \infty), \tag{75}$$

   where the right-hand side is evaluated in any typical realization.[47] Here, the initial values of the $u^n$ are chosen in a statistically symmetric fashion, over $n$, of the random coupling.

2. The response for infinitesimal perturbations of the $u^n(\mathbf{k}, t)$ are independent of $n$ and statistically sharp.

3. If perturbation expansions of $U(k, t, t')$ and $G(k, t, t')$ are made in power of $\varphi$'s, it follows from Eqs. 73 and 74 that only those terms survive in which every factor $\varphi_{nrs}$ is paired with another factor $\varphi_{nrs}$ or with a $\varphi$ that differs from $\varphi_{nrs}$ only by an index permutation.

At this point, it is reasonable to consider the question: why close at the level of second-order statistics? It is obvious that such closure has important limitations. Kraichnan (1961) determined that higher order terms in the expansions for the RCM vanish. Therefore, Kraichnan has produced in the RCM a system, related to the original Navier-Stokes, for which DIA is exact but in which the nonlinear interactions are not small (Leslie, 1973). Leslie further argued that this provides some reassurance that it is not unreasonable to apply DIA to Navier-Stokes equations.

Thus, truncation of line-renormalized expansions at the lowest order has a special significance and exhibits assured consistency properties. Closure at the level of third-order correlations, along with the lines of DIA, is indeed possible (Kraichnan, 1985), although the relevant statistical quantities, in particular, the appropriate generalizations of the linear response function, are not at all obvious. As a general proposition, unfortunately, higher order truncations of renormalized expansions lead to unbounded, unphysical results (Kraichnan, 1961).

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[47] The averaging implies an independence of $U$ of the angle $\mathbf{k}$ makes about any direction. Herring and Kraichnan (1972) also assumes isotropy.

#### 7. Generalized Langevin equation

Leith (1971) and Kraichnan (1970a) pointed out that the DIA (Kraichnan, 1964a) constituted exact equations for the covariance of a model system in which the velocity amplitude obeys a generalized Langevin equation (Mori, 1965),

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_i(\mathbf{k}, t) \;\; + \int_0^t \eta(k, t, s) u_i(\mathbf{k}, s)\, ds = \mathfrak{F}_i(\mathbf{k}, t).$$

Here, the random force $\mathfrak{F}_i(\mathbf{k}, t)$ may be interpreted as (Kraichnan, 1991)

$$\mathfrak{F}_i(\mathbf{k}, t) = -iM_{ijm}(\mathbf{k}) \sum_{p,q} \xi_j(\mathbf{p}, t) \xi_m(\mathbf{q}, t), \tag{76}$$

where $\xi_j(\mathbf{k}, t)$ is an isotropic, homogeneous Gaussian velocity that satisfies

$$\langle \xi_i(\mathbf{k}, t) \xi_j^*(\mathbf{k}, t') \rangle = \langle u_i(\mathbf{k}, t) u_j^*(\mathbf{k}, t') \rangle. \tag{77}$$

Thus, Eq. 76 is closed by requiring that the two-time covariance of $\mathfrak{F}_i(\mathbf{k}, t)$ be identical with that of the total nonlinear term in the Navier-Stokes equation for $u_i(\mathbf{k}, t)$ (Kraichnan, 1988a):

$$(\mathcal{L}/2\pi)^3 \langle \mathfrak{F}_i(\mathbf{k}, t) \mathfrak{F}_j^*(\mathbf{k}, t') \rangle = \pi k \int\!\!\int_\Delta pq\, dp\, dq\, a(k, p, q) U(p, t, t') U(q, t, t'). \tag{78}$$

Essentially, the heart of this sort of approximation is that it models nonlinear terms as a random stirrer, plus a compensating eddy damping; an arbitrary coefficient is avoided (Herring, 1990a).

The DIA generalized Langevin is truly "generalized," as a strict Langevin model would have the simpler structure

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_i(\mathbf{k}, t) + \eta(k, t) u_i(\mathbf{k}, t) = \mathfrak{F}_i(\mathbf{k}, t) \tag{79}$$

with Markovian (history-independent) damping and white-noise-in-time random forcing.[48] The Markovianized closure models will be investigated in Section V.

#### 8. The Edwards-Herring approach

Instead of working directly with the random velocity field, Edwards (1964) and Herring (1965, 1966) preferred to calculate the joint probability distribution. Edwards advanced a steady distribution function for homogeneous turbulence starting from Liouville's equation, modified by the introduction of an instantaneously fluctuating, random source of energy. He carried out an expansion of the probability distribution based on a Fokker-Planck depiction of the turbulence dynamics. Herring proposed a self-consistent field (SCF) theory for both stationary and nonstationary turbulence. He found that the SCF statistical equations are identical to those of DIA except that now the time displacement covariance has been replaced by (Herring and Kraichnan, 1972)

$$U(k, t, t') = G(k, t, t') U(k, t') \;\; (t \geq t'). \tag{80}$$

The results from the Edwards and SCF models are also closely related (Herring, 1965) and have been compared to the DIA and other models (Herring and Kraichnan, 1972; Kraichnan, 1964c).

The Edwards-Herring approach using distribution functions can be viewed as a perturbation series in which the "bare" propagator (called $L_0$ as the basis for expansion is replaced by a "dressed" one ($L$, which incorporates an eddy viscosity. This converts a series in powers of the Reynolds number to a series each of it's term is nominally of the same order. One of the valuable contributions of Herring's analysis, as emphasized by Leslie (1973), is its support of DIA's response function.

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[48] For instance, Laval *et al.* (2003) developed a model in which the small-scale velocities are computed using a Langevin model that is derived from a quasi-linear approximation of the Navier-Stokes equations, in the spirit of rapid distortion theory (RDT). See Cambon and Scott (1999) for a more detailed discussion on RDT and other linear/nonlinear models.

## B. Inertial range scaling and the random Galilean transformation

### 1. Kraichnan vs Kolmogorov spectra

While the Kolmogorov (1941a) analysis of turbulence had been publicized by George Batchelor, it had not yet been confirmed in high Reynolds number turbulent flows when Kraichnan's DIA article (Kraichnan, 1959a) was published. Still, the most immediate issue the new theory had to address concerned its discrepant prediction of the inertial range spectrum,

$$E(k) \propto (\mathcal{E}u_0)^{1/2}k^{-3/2}, \tag{81}$$

where $u_0$ is the r.m.s turbulent velocity.[49] Kraichnan (1959a) suggested a counter-consideration to Kolmogorov's original argument by taking into account the intermittency noted by Landau. In the physical space, Kraichnan reasoned, the fine-scale structure of high Reynolds number turbulence consists typically of a complicated tangle of extended vortex filaments and sheets, but not of compact blobs. Thus, an alternative inertial range spectrum in the form of Eq. 81 remained viable. A definite resolution wouldn't come until the conclusive measurement of Grant *et al.* (1962), where the one-dimensional spectra were found to be proportional to $k^{-5/3}$ for several decades, as predicted by Kolmogorov.

The reason for the DIA energy spectrum prediction can be easily seen from Eq. 67. For a wavenumber $k$ in the inertial range, the $a$ and $b$ terms in $S(k,t,t)$ are dominated by the contributions from $p$ or $q$ in the energy-containing range. Because of their opposite signs, there is a strong cancellation between these energy-containing scale contributions. These cancellations are a reflection of the slow modulation that is associated with the advection of small-scale structures by the spatially non-uniform energy-containing range velocity (Kraichnan, 1959a, 1991).

Now let's contrast the above picture with the Kolmogorov phenomenology of the energy cascade. As vividly described by Davidson (2004), the eddies of size $k$ in the inertial range have a complex heritage and are the offspring of larger eddies, which came from yet even larger eddies. It is unlikely that they feel the instantaneous effect of the energy-containing scales. Therefore, a weakened form of Kolmogorov's hypothesis surely is true: zero-wavenumber excitation does not affect the energetics of a finite wavenumber in homogeneous turbulent flows (Kraichnan, 1966a).

The controlling nonlinear transfer time scale provides a useful framework. From Eqs. 66 and 67, the relevant DIA decorrelation time is the sweeping time scale $\tau_{sw} \sim 1/(u_0 k)$. However, in a Kolmogorov inertial range, the characteristic time scale is the so-called intrinsic or straining time scale $\tau_{st} \sim 1/(\mathcal{E}^{1/3}k^{2/3})$. Indeed, Kraichnan and Herring (1978) noted that, when applied to high $Re$ turbulence, the DIA misrepresents energy transfer at a high wavenumber because the Eulerian correlation times entering the triple moments reflect sweeping decorrelation effects that are irrelevant to the straining processes responsible for energy transfer.

### 2. Violation of random Galilean invariance

As for any physical theory, the description of the turbulence must be the same in all inertial frames of reference. Galilean invariance may be expressed by noting that if $\mathbf{u}(\mathbf{x},t)$ is a solution of the Navier-Stokes equation, then $\mathbf{u}(\mathbf{x}-t\mathbf{V}(t),t)+\mathbf{V}(t)$ is also a solution. Here $\mathbf{V}(t)$ is a spatially uniform velocity. Speziale (1985) examined the modeling of the subgrid-scale stresses in the LES of turbulence and showed that one of the more popular models gives rise to equations of motion for the large eddies of turbulence which are not Galilean-invariant. Consequently, this model cannot be of any general applicability. He also investigated whether Reynolds stresses in engineering models satisfy this basic requirement (Speziale, 1991).[50]

The random Galilean invariance should not be confused with the standard one just described above. Suppose that $\mathbf{V}(t)$ is now Gaussianly distributed over the ensemble and is statistically independent of $\mathbf{u}(\mathbf{x},t=0)$. The addition of $\mathbf{V}(t)$ is called a random Galilean transformation of $\mathbf{u}(\mathbf{x},t)$ (Kraichnan, 1991). The various moments of $\mathbf{u}(\mathbf{x},t)$ should be either invariant or else have precisely determined transformation properties (Kraichnan, 1964c).

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[49] The DIA also predicted the dissipation spectrum,

$$E(k) \propto k^3 \exp(-\beta_d k/k_d),$$

where $\beta_d \approx 6.7$. Related studies on this dissipation energy spectrum and energy transfer process can be found in Chen *et al.* (1993); Ishihara *et al.* (2005); Kida *et al.* (1992); and Martinez *et al.* (1997). Using highly resolved DNS with $Re_\lambda$ between 1 and 100, Khurshid *et al.* (2018) confirmed this exponential scaling for $Re_\lambda$ up to about 20, but found that the shape is more complex and may be fitted roughly as a superposition of two exponentials for higher $Re_\lambda$. However, the second exponential seems to be the result of intermittent interactions with the lower part of the spectrum (Khurshid *et al.*, 2018).

[50] Galilean invariance is also an important issue in lattice-based hydrodynamics models. While previous models concentrated on the nonlinear advection term, Qian and Zhou (1998) developed complete Galilean invariant lattice BGK models for the Navier-Stokes equation.

When a random Galilean transformation is performed, the $u_0$ in the DIA energy spectrum, Eq. 81, will be changed as well. This is obviously incompatible with the basic requirement of the Galilean transformation. Thus, when comparing the Kraichnan spectrum with that of Kolmogorov, the existence of $u_0$ promulgated an unphysical feature of the DIA theory.

Mou and Weichman showed that the violation of random Galilean invariance, in some sense, is obvious from the large-$N$ realizable models. These violate Galilean invariance due to the more complex structure of the nonlinear term (reflecting different velocity fields advecting each other). This is discussed at some length, along with possible workarounds, in Mou and Weichman (1995), to which we refer the reader.

Sudan and Pfirsch (1985) demonstrated the equivalence between the DIA (formulated by Kraichnan in the time domain) and the weak-coupling approximation[51] (formulated by Kadomtsev and Pogutse (1973) in the frequency domain) (Krommes, 2015). Of particular significance, Kadomtsev and Pogutse (1973) explained why the weak-coupling approximation (a.k.a, DIA) is unsuitable for the description of the small-scale components of developed turbulence (Monin and Yaglom, 1975).

### 3. The local energy transfer (LET) and wave packet models

Before closing this subsection, we note two Eulerian models that have proclaimed an ability to reproduce the Kolmogorov spectrum.[52] First, Nakano (1988) developed a wave packet representation of the Navier-Stokes equations and applied it to the DIA framework. The author contended that the convection due to large eddies can be eliminated and the Kolmogorov spectrum reproduced without ad hoc approximation.

Second, the local energy transfer (LET) model, first proposed by McComb (1974) and subsequently extended by McComb and his co-workers, was initially based on a reinterpretation of the theory due to Edwards (1964). More recently, McComb and Quinn (2003) and McComb and Yoffe (2017) stated that the additional terms included in the LET response equations canceled the infra-red divergence at $k = 0$ and thus ensured compatibility with the Kolmogorov scaling.[53] We will not discuss them further in this review for brevity, since there is a significant body of literature on this approach (see McComb (1990, 1995), McComb and Quinn (2003), McComb and Yoffe (2017) and references therein).

## C. Triumphs and limitations

The publication of Kraichnan (1959a) marked the inception of the modern era of turbulence theory. Yet, while the DIA has enjoyed significant success, it also suffers a number of significant shortcomings. More specifically, it has

- preserved the fundamental invariance properties of the exact Navier-Stokes equations,
- been shown to be exact when applied to the random coupling model in which the nonlinear interactions are not small,
- provided the integro-differential equations as a starting point to study the energy cascade process,
- identified the key difference between the statistical effects of sweeping of flow structures past fixed points (Heisenberg, 1948a; Kraichnan, 1959a; Tennekes, 1975) and the distortion of flow structures by straining motion,
- produced the DIA $k^{-3/2}$ spectrum and therefore is at variance with that of the confirmed Kolmogorov $k^{-5/3}$ scaling. Crucially, the spectrum resulted from DIA failed to maintain random Galilean invariance, because it depends on the characteristic velocity.

Also, a necessary condition for an approximation to properly describe the energetics of the small scales of turbulence would appear to be the preservation of Galilean invariance. Two approaches, Lagrangian and Markovianized models, have been developed to restore this fundamental basic physics, but with completely different objectives. The former is a sophisticated theoretical approach developed to complete the journey started with the DIA, while the latter is geared toward developing affordable tools for computing relatively simple flows.

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[51] As pointed out by Dubois and Pesme (1985), the Kadomtsev and Pogutse (1973) approach for deriving the DIA equations is essentially that used by Boutros-Ghali and Dupree (1981) who, however, did not apparently recognize the identity of their work and the DIA.

[52] To secure a Kolmogorov $k^{-5/3}$ spectrum, an Eulerian theory needs to suppress the dominating sweeping effects. But the question is whether such procedure was adequately predicated. The concerns can be alleviated, in the view of this writer, by illustrating that the models could obtain the Eulerian time correlations in addition to the desirable $-5/3$ energy spectrum.

[53] Regarding the claim of the compatibility between McComb's LET theory and the Kolmogorov scaling, Herring registered his points of disagreement in a book review (Herring, 1992). In any case, Frederiksen *et al.* (1994) and Kiyani and McComb (2004) stated that the LET model can be shown to differ from the DIA only in how a fluctuation-dissipation theorem is invoked (Frederiksen and O'Kane, 2019).

## IV. LAGRANGIAN FRAMEWORK

### A. The necessity of Lagrangian approaches

Kraichnan (1964c) recognized that the Eulerian analytical framework is intrinsically unsuited to the task of recovering the Kolmogorov inertial range dynamics, and therefore recommended the community pivot to Lagrangian methods. Indeed, knowledge of the Eulerian velocity covariance alone does not permit discrimination between two different time scales (Kraichnan, 1965a). For the reasons set forth in the last section, the controlling time scale in the Eulerian configuration arises from the sweeping motion. Namely, the time dependence is due to the small scale being swept along, almost undistorted, by the large-scale motion. Hence, the random Galilean transformation is violated by the DIA. For the reasons stated above, the DIA formulations (Kraichnan, 1959a) and a related, higher approximation (Kraichnan, 1961) do not retain sufficient information to properly represent the energy transfer process.

Kraichnan (1964c) clarified these issues by considering a modified Navier-Stokes equation in which the convection of any given spatial scale by much larger scales is consistently removed. When applied to the modified equation, the DIA procedure now reproduces the Kolmogorov inertial range scaling.[54] Presumably, this methodology is inspired by the work of Heisenberg (1948a) and Kolmogorov (1941a). As recalled in Kraichnan (1959b), Heisenberg first realized that the convective effects of the large-scale velocity components would give a correlation time of $\tau_{sw}$ in a fixed (Eulerian) framework. Next, Heisenberg determined that the correlation time measured in a quasi-Lagrangian coordinate system moving with the large-scale motion would be of order $\tau_{st}$. Kraichnan (1959b, 1964c) further noted that, in actuality, this modification scheme may be interpreted as an effective transformation of the quasi-Langrangian coordinate system invoked by Kolmogorov (1941a).

Kraichnan identified the over-estimation of sweeping as the reason for the inconsistent predictions of his theory and his next great insight was the idea that reformulating his theory in a Lagrangian representation would remove the effect of sweeping (e.g., Gkioulekas (2016)). Subsequently, Kraichnan moved on to develop his Lagrangian theories (LHDIA, ALHDIA, SBALHDIA).

The generalized velocity field, $\mathbf{u}(\mathbf{x}, t|s)$, is introduced in terms of the Eulerian velocity by the equation

$$\left[\frac{\partial}{\partial t} + \mathbf{u}(\mathbf{x}, t) \cdot \nabla\right]\mathbf{u}(\mathbf{x}, t|s) = 0, \tag{82}$$

with the boundary condition

$$\mathbf{u}(\mathbf{x}, t|t) = \mathbf{u}(\mathbf{x}, t). \tag{83}$$

The vector $\mathbf{u}(\mathbf{x}, t|s)$ is the velocity of a particular fluid element at time $s$, and $(\mathbf{x}, t)$ is a point on the spacetime trajectory of this element. Hence, $s$ is called the *measuring time* and $t$ is named the *labeling time*.

Kraichnan (1965a) offered the visualization for the generalized velocity field. Suppose that, starting from time $t_0$, a non-diffusing ink is continually injected into the fluid at $\mathbf{x}$. Then $\mathbf{u}(\mathbf{x}, t|s)$, for $t_0 \leq t \leq s$, gives the velocities measured at time $s$ at all points on the line of ink.

### B. Velocity-based and strain-based Lagrangian-History approximations

The statistical functions for the Lagrangian-History Direct-Interaction approximation (LHDIA) are the covariance

$$U_{ij}(\mathbf{x}, t|s; \mathbf{x}', t'|s') = \langle u_i(\mathbf{x}, t|s) u_j(\mathbf{x}', t'|s')\rangle \tag{84}$$

and $G_{ij}(\mathbf{x}, t|s; \mathbf{x}', t'|s')$, the Lagrangian response function.

The LHDIA approximation was first heuristically obtained (Kraichnan, 1966b). The DIA formulas for triple moments were altered so that the final equations were integrated over the Lagrangian time-history of the field, measured along the paths of the particle element. Simultaneously, the LHDIA ensured that formal inviscid equipartition exists,

[54] More specifically, Kraichnan (1964c) introduced a localization parameter $\alpha$. While the case $\alpha = \infty$ gives the unmodified Navier-Stokes, he found that when $\alpha$ is finite, the modified equation yields an asymptotic spectrum of the Kolmogorov form rather than Eq. 81. When a calculation imposes a large-scale cutoff, the Kolmogorov $-5/3$ spectrum could in fact be observed because the cutoff will suppress sweeping effects, even if it doesn't eliminate them completely. The problem is that the calculation might be sensitive to this cutoff — so modified equation might give $-5/3$, but the inertial range would depend on the cutoff (Fig. 1 of Kraichnan (1964c)), which it is not supposed to do. More recently, Frederiksen and Davies (2004) found that for 2D HIT the frequent restart allowed $\alpha$ as high as 6 but with longer time integrals 4 is more appropriate. O'kane and Frederiksen (2004) studied the case of 2D inhomogeneous turbulent flow and reported that $\alpha = 4$ is appropriate for their closure model calculations. However, one should be cautious because the sweeping effects are 3D and more numerical simulations are needed to demonstrate this $\alpha$ value.

energy is conserved, and most crucially, the Lagrangian dynamics of homogeneous flows exhibit invariance under random Galilean transformations. Kraichnan (1965a) documented the resulting formulas for the passive scalar field, but only presented the LHDIA equations for the velocity field of homogeneous turbulence. For brevity, we will not reproduce these here but the reader should consult with the original articles for these lengthy expressions. Kraichnan (1968b) treated the scalar spectrum by the LHDIA and obtained the $k^{-5/3}$ inertial-convective range of Obukhov and Corrsin, the $k^{-1}$ viscous-convective range, and the $k^{-17/3}$ inertial-diffusive range of Batchelor *et al.* (1959), but the treatment fails in some respects.

The abridged Lagrangian-History DIA (ALHDIA) was obtained as a further approximation of the LHDIA. More specifically, the ALHDIA is approximated from the LHDIA by invoking the so-called relabeling procedure - by reducing the four-time correlations to a two-time correlation (Kraichnan, 1965a,b, 1966b).

While the essence of the Lagrangian dynamics is in the change of $\mathbf{u}(\mathbf{x}, t|s)$ with respect to the measuring time $s$, the Lagrangian evolution for $\partial\mathbf{u}(\mathbf{x}, t|s)/\partial s$ is extremely complicated (Kaneda, 1986, 2007). The ALHDIA managed this difficulty by working with Eq. 82, the Eulerian velocity, and $\partial\mathbf{u}(\mathbf{x}, t|s)/\partial t$.

Later, Kraichnan (1977) rederived the LHDIA and ALHDIA by applying the renormalized perturbation procedure from Eqs. 4 and 82. The ALHDIA equation sets may be written in the following form (Herring and Kraichnan, 1979):

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\nu k^2\right)U(k,t|t') &= \int_\Delta dpdqC^0_{kq}\int_{t'}^{t}U(k,t|s)U(q,t|t')ds+\int_0^\infty dpK_{kp}U(p,t|t')\\ &+\int_\Delta dpdq\int_{t_0}^{t'}[C^3_{kq}G(k,t'|s)U(p,t|s)-C^4_{kq}G(p,t'|s)U(k,t|s)]U(q,t|s)ds\\ &-\int_\Delta dpdq\int_{t_0}^{t'}[C^3_{kq}G(p,t|s)U(k,t'|s)-C^2_{kq}G(k,t|s)U(p,t'|s)]U(q,t|s)ds\\ &-\int_\Delta dpdq\int_{t_0}^{t}C^7_{kq}G(p,t|s)U(k,t|s)U(q,t'|s)ds,\end{aligned}\tag{85}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\nu k^2\right)G(k,t|t') &= \int_\Delta dpdqC^0_{kq}\int_{t'}^{t}G(k,t|s)U(q,t|t')ds+\int_0^\infty dp\tilde{K}_{kp}G(p,t|t')\\ &-\int_\Delta dpdq\int_{t_0}^{t'}[C^3_{kq}G(p,t|s)G(k,t'|s)-C^2_{kq}G(k,t|s)G(p,s|t')]U(q,t|s)ds.\end{aligned}\tag{86}$$

Here, the velocity field is assumed to be normally distributed at the initial time $t_0$. The coefficients $C_{kq}$ and $K_{kp}$ entering the ALHDIA can be found in Herring and Kraichnan (1979).

Kraichnan and Herring (1978) also considered an alternative version of ALHDIA, the strain-based abridged Lagrangian-History DIA (SBALHDIA). The motivation was that an approach in which straining plays the fundamental role may be more faithful in representing the straining of small scales than one in which velocity is considered fundamental (Herring and Kraichnan, 1979). In this alternative, $U$ and $G$ must be replaced in Eqs. 85 and 86, by $U^B$ and $G^B$, the Lagrangian covariance and response function for the strain field, respectively. Of course, the modified model coefficients must be obtained accordingly.

The LHDIA and ALHDIA have produced a number of physically plausible qualitative predictions: the $k^{-5/3}$ inertial range in 3D (Kraichnan, 1965a, 1966b), the $k^{-5/3}$ inertial range and the $k^{-3}$ logarithmic-corrected enstrophy-transfer range in 2D (Kraichnan, 1967, 1971b), the $k^{-5/3}$, $k^{-1}$ and $k^{-17/3}$ passive scalar spectra in turbulent convection (Kraichnan, 1968b) (subsection IV.C.3), the $k^{-2}$ shock-dominated inertial range[55] (Kraichnan, 1968a), and the $k^{-5/3}$ MHD inertial range (Kraichnan, 1965c) (subsection XI.A).

Numerical simulations of the ALHDIA and SBALHDIA (Herring and Kraichnan, 1979; Kraichnan, 1966b) show an excellent quantitative fit to the measured tidal-channel inertial and dissipation spectra (Grant *et al.*, 1962).[56] The ALHDIA resulted in a somewhat smaller value (1.78) of Kolmogorov's constant (Kraichnan, 1965a) than that from the SBALHDIA (2.0) (Herring and Kraichnan, 1979). In the infinite Reynolds number limit the Kolmogorov

[55] The Burgers equation describes the behavior of weak shocks in compressible turbulence and is also considered a one-dimensional model of the Navier-Stokes equations. Yet, the shockfront structure of evolved solutions of Burgers' equation yields an energy spectrum $E(k) \propto k^{-2}$ (Burgers, 1950). For these reasons it has received a great deal of attention in the literature (e.g., (Girimaji and Zhou, 1995; Gotoh and Kraichnan, 1993, 1998)). Kraichnan (1968a) found that the LHDIA procedure produces the $k^{-2}$ inertial range scaling and is capable of distingushing between Burgers and Navier-Stokes dynamics because it gives a crucial role to Lagrangian correlation times. More recently, Ohkitani and Dowker (2012) compared and contrasted freely decaying evolution of the Navier-Stokes equations with that of the 3D Burgers equations using DNS with the identical incompressible initial data and same kinematic viscosity.

[56] The one dimensional energy spectrum is given by

$$\Phi(k,t) = (1/2)\int_k^\infty dp\big(1-k^2/p^2\big)E(p,t)/p.$$

constant from ALHDIA is 1.77 (Kraichnan, 1966b). At low Reynolds number, SBALHDIA demonstrated agreement with Orszag and Patterson (1972) in all ranges, while ALHDIA displayed excessive energy transfer in the dissipation regime (Herring and Kraichnan, 1979). The ALHDIA and SBALHDIA approximations are both also in reasonable agreement in 2D with the numerical simulations.

Two desirable features of Eulerian DIA are no longer available when working in the Lagrangian framework (Herring and Kraichnan, 1972). First, the exact model representation is lost. Second, the Langevin-model amplitude equation is not known to exist.

## C. Lagrangian renormalized approximation

The improvement of the velocity-based and strain-based Lagrangian-History approximations over the DIA is clearly significant, but is gained at a cost of complication (Kraichnan, 1988a). This is where the Kaneda's Lagrangian renormalized approximation (LRA) (Kaneda, 1981) comes in. The goal of LRA is to alleviate the computational burden of the improved approximations while maintaining consistency with the Kolmogorov -5/3 scaling. The LRA is anchored in three key ideas, one of which, the renormalized expansion, has already been discussed in III.A.5. Therefore, we will now discuss two other subjects: (i) Lagrangian position function and (ii) representatives.

### 1. Lagrangian position function

The essence of the Lagrangian dynamics is in the change of $u_i(\mathbf{x},t|s)$ with respect to $s$. The LRA returned to this description but successfully circumvented the complexities of the LHDIA. In this framework, Kaneda still considered the generalized velocity field, $u_i(\mathbf{x},t|s)$, as fundamental. Yet, it is now separated into $u_i(\mathbf{x},t)$ and the Lagrangian position function $\psi$ (Kaneda, 1981).

For incompressible turbulence, the LRA mapped the Eulerian field $u_i(\mathbf{x},t)$ to the generalized field $u_i(\mathbf{x},t|s)$,

$$\mathbf{u}(\mathbf{x},t|s) = \int \psi(\mathbf{x},t|\mathbf{x}',s)\mathbf{u}(\mathbf{x}',s)d^d\mathbf{x}', \tag{87}$$

where $d$ is the dimensionality of the flow.[57] The Lagrangian position function is

$$\psi(\mathbf{x},t|\mathbf{x}',s) \equiv \delta^d(\mathbf{x}' - \mathbf{z}(\mathbf{x},t|s)), \tag{88}$$

where $\mathbf{z}(\mathbf{x},t|s)$ is the position at time $t$ of the fluid element whose space-time trajectory passes through $(\mathbf{x},t)$. The use of the Lagrangian position function is instrumental for LRA to follow the $s$ dependence of $u_i(\mathbf{x},t|s)$ - the Lagrangian evolution. Indeed, the $\psi$ function obeys the governing equation

$$\frac{\partial}{\partial s}\psi(\mathbf{x},t|\mathbf{x}',s) = -\big[\mathbf{u}(\mathbf{x}',s)\cdot\nabla_{\mathbf{x}'}\big]\psi(\mathbf{x},t|\mathbf{x}',s). \tag{89}$$

As detailed in Kaneda (1981), the statistical functions for the LRA are the covariance

$$U_{ij}(\mathbf{x},t;\mathbf{x}',t') \equiv \langle u_i(\mathbf{x},t'|t)u_j(\mathbf{x}',t')\rangle = \langle\big[\int d^3\mathbf{x}''u_i(\mathbf{x}'',t)\psi(\mathbf{x}'',t|\mathbf{x},t')\big]u_j(\mathbf{x}',t')\rangle \quad (t \geq t'), \tag{90}$$

where the Lagrangian position function $\psi$ has been introduced using Eq. 87. The averaged Lagrangian response function is defined as

$$G_{ij}(\mathbf{x},t;\mathbf{x}',t') \equiv \langle\hat{G}_{ij}(\mathbf{x},t;\mathbf{x}',t')\rangle \quad (t \geq t'). \tag{91}$$

where

$$\hat{G}_{ij}(\mathbf{x},t;\mathbf{x}',t') = \delta u_i(\mathbf{x},t'|t)/\delta f_j(\mathbf{x}',t') = \delta\big[\int d^3\mathbf{x}''u_i(\mathbf{x}'',t)\psi(\mathbf{x}'',t|\mathbf{x},t')\big]/\delta f_j(\mathbf{x}',t') \quad (t \geq t'), \tag{92}$$

[57] Several authors have studied the HIT for space dimensions $d$ (Berera *et al.*, 2020; Fournier and Frisch, 1983; Fournier *et al.*, 1978; Nelkin, 1974, 1975; Yamamoto *et al.*, 2008), looking for possible simplifications. Using the LRA and DNS, Gotoh *et al.* (2008) found that the energy transfer is more efficient in 4D than in 3D.

$$\hat{G}_{ij}(\mathbf{x},t;\mathbf{x}',t') = 0 \quad (t < t'). \tag{93}$$

Recall that $f_i(\mathbf{x},t)$ is an arbitrary source added to the right-hand side of the Navier-Stokes equation (Eq. 47) while $\delta/\delta$ denotes function differentiation (Eq. 48).

### 2. Representatives

A suitable choice of the quantities, prior to carrying out the renormalized perturbation procedure, is important to any statistical closure theory (Kaneda, 1981). The defect in Eulerian DIA has illustrated the importance of this point. These selected quantities are called representatives. If the representatives are judiciously chosen, e.g. ALHDIA, SBALHDIA, or LRA, the approximation will represent the physics accurately.

Kaneda (2007) articulated his considerations in choosing Lagrangian multi-time correlations and response functions as the representatives. The closure is achieved by applying the standard renormalized perturbation method to Eqs. 90 and 91. As summarized in Gotoh *et al.* (1988), the LRA maintained the following basic properties of Navier-Stokes dynamics: (1) invariance under space translation, rotation, and reflection, (2) energy and helicity conservation, (3) existence of absolute equilibrium for an inviscid truncated systems, (4) invariance to random Galilean transformations, and (5) realizability or non-negativity of the energy spectrum for the Modified LRA (MLRA). MLRA slightly modified the fluctuation dissipation relation.

The LRA reproduced the Kolmogorov -5/3 energy spectrum and determined that the Kolmogorov constant is 1.72 (Kaneda, 1986). Kaneda (1981) investigated five alternative representatives and showed that the values of the Kolmogorov constant depend weakly on the choice of the representatives.

Gotoh *et al.* (2000) returned to the issue of picking the optimal representation for computing passive scalar spectrum in the viscous-convective range in 2D steady turbulence. The authors reported that a representative that properly captures the Lagrangian memory decay process of the passive scalar field would produce an improved agreement with the DNS data.

### 3. LRA for two-dimensional, passive scalar, and shear turbulence

Gotoh *et al.* (1988) integrated the LRA and MLRA numerically for decaying HIT for several single-time statistics, including the energy spectrum and skewness factor of velocity derivatives.[58] At a moderate Reynolds number, the results are in good agreement with the DNS of Orszag and Patterson (1972) and Herring *et al.* (1974) in 3D and 2D, respectively.

In 2D, the LRA is consistent with the expected scalings (Batchelor, 1969; Kraichnan, 1967; Kraichnan and Montgomery, 1980) for the inverse energy cascade, while the constants are estimated as $C' = 1.81$ and $C = 7.41$ (Gotoh, 1989; Kaneda, 1987). The $C$ evaluated from the LRA is somewhat higher than the values from the experimental measurement ($6.5 \pm 1$) of Paret and Tabeling (1997) and numerical simulation ($6.0 \pm 0.4$) of Boffetta *et al.* (2000).

For scalar transport by turbulence, $\theta$, The LRA yielded appropriate passive scalar spectra, Eqs. 36- 38 (Gotoh, 1989; Gotoh *et al.*, 2000; Kaneda, 1986)

$$\langle \theta^2(\mathbf{x},t)\rangle = \int_0^\infty E_\theta(k,t)dk. \tag{94}$$

Ariki and Yoshida (2021) argued that there is a shortcoming in previous LRA treatment of the scalar field because its correlation is constant in time. Instead, the authors developed a Hessian-based LRA (HBLRA) closure to overcome this issue and applied it to the inertial-convective regime.[59] Table I compares the values of passive scalar spectra constants computed from the statistical closures (ALHDIA, SBALHDIA, LRA, Hessian-based LRA) against those obtained from experimental measurements and numerical simulations.

Yoshida *et al.* (2003a) have applied the LRA to homogeneous turbulent shear flow, a difficult task because of the complexity of the equations that must be analyzed. The challenge is only mitigated because the authors were aided by an insight from their numerical simulation (Ishihara *et al.*, 2002) - that the analysis can be greatly simplified by taking into the account the symmetry of the problem.

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[58] The skewness factor of velocity derivatives is defined as $S(t) = -\langle(\partial u_1/\partial x_1)^3\rangle/\langle(\partial u_1/\partial x_1)^2\rangle^{3/2}$, which is also known to have a connection to vortex stretching and the energy transfer between different scales (Pope, 2000; Sreenivasan and Antonia, 1997; Sukoriansky *et al.*, 2018).

[59] The HBLRA has also applied to study the scale-similar clustering of heavy particles in the inertial range of turbulence (Ariki *et al.*, 2018)

| | $B_{IC}$ | $B_{ID}$ | $B_{VC}$ |
|---|---|---|---|
| Exp. or DNS | 0.4-1.9 | $0.39 \pm 0.03$ | $3.9 \pm 1.5$ |
| | (Hill, 1978) | (Grant *et al.*, 1968) | (Chasnov *et al.*, 1988) |
| Updated DNS | 0.725 or 0.699 | | 5.7 |
| | (Gotoh and Watanabe, 2015) | | (Gotoh *et al.*, 2014) |
| ALHDIA (Kraichnan, 1966b) | 0.208 | 0.590 | $< 0.9$ |
| SBALHDIA (Herring and Kraichnan, 1979) | | | 2.0 |
| LRA (Kaneda, 1981, 1986) | 0.340 | 0.573 | $\sqrt{10/3} \approx 1.826$ |
| LRA (Ariki and Yoshida, 2021) | 0.754-1.23 | | |

TABLE I: Various constants obtained from the LRA, experiments, simulations for the passive scalar spectrum in three dimensions (Gotoh *et al.*, 2000). Recent DNS data suggested that the values of $B_{IC}$ are 0.725 for the case of random scalar injection and 0.699 for the case of an uniform mean gradient excitation, respectively (Gotoh and Watanabe, 2015; Gotoh *et al.*, 2014). Note that Ariki and Yoshida (2021) used Hessian-based Lagrangian closure theory for passive scalar turbulence.

This analysis found that the anisotropic component of the energy spectra is more sensitive to large-scale turbulence structure than the isotropic component. The scaling of the anisotropic spectrum ($\propto k^{-13/3}$) in the limit $Re \to \infty$ is consistent with that determined by dimensional arguments (Lumley, 1967; Wyngaard and Coté, 1972). The two model constants can be evaluated analytically with LRA. The anisotropic component of the Lagrangian response function, $G^1$, is also derived without any ad hoc adjustable parameters. Moreover, the authors showed that $G^1$ must be treated correctly in order to obtain the correct values of the aforementioned two constants.

### D. Sparse direct-interaction perturbation and other models

Kida and Goto (1997) applied the original Eulerian DIA procedure of Kraichnan (1959a), but took the advantage of the Lagrangian position function, Eq. 89. The so-called Lagrangian DIA is simpler than the LHDIA, ALHDIA, and SBALHDIA, and yields integro-differential equations that are identical to the LRA (Kaneda, 2007). Moreover, Goto and Kida (1999) found that the Lagrangian DIA also reproduced the same passive scalar formulation as that from LRA.

Recall that Kraichnan (1977) showed that the renormalized perturbation method recovered the results of the original DIA (Kraichnan, 1959a). Using a simpler model (Betchov, 1966), Goto and Kida (1998) determined that these two methodologies lead to an identical set of integro-differential equations in the Lagrangian framework. Goto and Kida (2002) also used the Betchov model to highlight the importance of sparse coupling in the DIA approximation. As a result, the name "sparse direct-interaction perturbation" (SDIP) was chosen in place of the term Lagrangian DIA previously proposed by Kida and Goto (1997).

Motivated by the findings of Goto and Kida (1999) that a number of classical scalings for the scalar spectrum can be recovered at different Schmidt numbers ($Sc$), O'Gorman and Pullin (2005) used the SDIP to investigate the effect of $Sc$ on velocity-scalar cospectrum in isotropic turbulence with a mean scalar gradient. Okamura (2018) also proposed a related closure model so that the bottleneck effect (Donzis and Sreenivasan, 2010; Falkovich, 1994; Katul *et al.*, 2015; Kűchler *et al.*, 2019; Saddoughi and Veeravalli, 1994) can be reproduced in the near-dissipation regime.

Although the equations of the SDIP are the same as LRA, Goto and Kida (1998) and Kida and Goto (1997) expressed different views regarding the method of derivation and the range of validity of the LRA equations (Kaneda, 2007).

### E. Triumphs and limitations

The Lagrangian closures reviewed above epitomize the modern and fundamental approach to turbulence. In addition to maintaining a number of desirable features of Eulerian DIA, the statistical closures in the Lagrangian framework have

- predicted the Kolmogorov spectrum and constant,

- simplified the evolution equations for generalized Lagrangian velocity by expressing them in the Lagrangian position function in LRA (Kaneda, 1981),
- retained invariance under random Galilean transformations,
- provided additional confidence from the SDIP (Goto and Kida, 2002) on the robustness and internal consistency of statistial closure in the Lagrangian framework,
- lost the exact model representation in Eulerian DIA (Herring and Kraichnan, 1972),
- failed to show the existence of the Langevin-model amplitude equation,
- precluded a computation of long-time statistics as a result of the two-time character, unless the time- history is truncated at some time difference (Bos and Bertoglio, 2013),
- provided little or no meaningful internal estimates of errors (Herring, 1990a; Kraichnan, 1991),

While the last two items represent the fundamental limitations of statistical closures by construction, it is important to register them again here after documenting the pinnacle of successes of turbulence theories.

## V. MARKOVIANIZED CLOSURE METHODS

It is highly desirable to illustrate the relevance, practicality and utility of the statistical closure theories. To this end we will now turn our attention in this section to more pragmatic models of turbulence. With an eye on the computational cost, Kraichnan (1972) emphasized that such closure models are much faster than the DIA because only single-time statistical functions need be computed. Of course, this advantage comes at the cost of a less rich and faithful representation of the dynamics.

### A. Markovianization

The Markovianized equations are given by (Kraichnan, 1991)

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right)U(k,t,t) = 2S(k,t,t) \tag{95}$$

where

$$S(k,t,t) = \pi k \int\int_{\Delta} pq\, dp\, dq\, \Theta_{kpq}(t)\Big[a(k,p,q)G(k,t',s)U(p,t,s)U(q,t,s) \\ -b(k,p,q)U(k,t',s)G(p,t,s)U(q,t,s)\Big]. \tag{96}$$

Physically, the quantity $\Theta_{kpq}(t)$ is a characteristic memory time for the energetic interactions of $k, p$, and $q$; its presence takes the place of the explicit integration over history. It must be noted that the total freedom in choosing $\Theta_{kpq}(t)$ makes the Markovianized model less than deductive (Kraichnan, 1991).

The form of $\Theta_{kpq}(t)$ model closest to DIA can be obtained by letting (Kraichnan, 1971a)

$$\Theta_{kpq}(t) = \int_0^t G(k,t,s)G(p,t,s)G(q,t,s)ds. \tag{97}$$

Herring (1985) pointed out that an approximate connection between Eqs. (95)-(96) and the DIA is established by taking

$$U(k,t,t') = U(k,t',t'), G(k,t,t') = \exp[-\eta(k)(t-t')], \tag{98}$$

and

$$\Theta_{kpq}(t) \sim 1/\big[\nu(k^2+p^2+q^2)+\eta_{kpq}\big], \tag{99}$$

where

$$\eta_{kpq} = \eta(k) + \eta(p) + \eta(q). \tag{100}$$

In this approximate characterization, the expression for $\eta(k)$ is nearly the same as Edwards (1964):

$$\eta(k,t) \sim k\Big[\int_0^k dp\, E(p,t)\Big]^{1/2}. \tag{101}$$

Carnevale and Frederiksen (1983a) showed that in contrast to those of Kraichnan (1964b) and Leslie (1973), Edwards' prescription resulted from an inconsistent identification of the non-stationary energy-spectrum relaxation rate with the viscosity.

As yet, the above model under Markovianization approximations does not behave properly under random Galilean transformations. The Test-Field Model (TFM) and the Eddy-damped Quasi-Normal-Markovian (EDQNM) remedied the above defect of the DIA's prediction for the inertial range by introducing the relaxation time scales $\Theta_{kpq}(t)$ which are invariant to large-scale convection. Herring (1980a) pointed out that these changes effectively switch the historical integral from an Eulerian to a Lagrangian character. Furthermore, Carnevale *et al.* (1981) showed that for the inviscid flows the second-order Markovian closures satisfy a Boltzmann-type $H$ theorem. From the inviscid closure equations the authors demonstrated that it is because of the Markovian character of closure that the entropy increases monotonically in time to a maximum value at absolute equilibrium.

## B. Test-Field Model

The TFM is based on an insight that the pressure term in the equation of motion is responsible for the Lagrangian decorrelation one would like to estimate (Kraichnan, 1971a). Yet, there are significant complications in working through the theory in a Lagrangian frame. Thus, Kraichnan proposed to work the theory in an Eulerian frame (Orszag, 1977) and the TFM is an attempt to tie the intrinsic distortion of flow structures measured in a Lagrangian frame but with simple advection displacement effects removed. Indeed, the pressure term in the equation of motion is responsible for the Lagrangian decorrelation that we wish to estimate. The pressure term, as articulated by Leith and Kraichnan (1972), preserves the nondivergence of the velocity field, and another way of the measuring its influence is through the tendency of a non-divergent velocity field to become divergent should the pressure term be removed.

Kraichnan (1971a) noted that the rate at which advection couples the solenoidal and compressive components of a "test" velocity field in the absence of pressure can be taken as a measure of the interaction of an eddy with an advecting field. He considered a modified Navier–Stokes system in which the solenoidal component of a "test" velocity field is distorted by advection of the compressive component (and vice versa). That is, in the nonlinearity, the cross-terms that couple the solenoidal and compressive parts of the test field are removed. The TFM also apply the fluctuation-dissipation ansatz and a subsequent Markovianization to the statistical equations for the solenoidal and compressive components (see also, Bowman and Krommes (1997)).

The TFM formulas are

$$d\Theta_{kpq}(t)/dt = 1 - [\nu(k^2+p^2+q^2) + \eta^S(k,t) + \eta^S(p,t) + \eta^S(q,t)]\Theta_{kpq}(t), \tag{102}$$

where

$$\eta^S(k,t) = \pi k \int\int_\Delta pq\, dp\, dq\, b'(k,p,q)\Theta'_{pqk}(t)U(q,t,t). \tag{103}$$

The auxiliary time parameter $\Theta'_{kpq}(t)$ in Eq. 103 is

$$d\Theta'_{kpq}(t)/dt = 1 - [\nu(k^2+p^2+q^2) + \eta^C(k,t) + \eta^S(p,t) + \eta^S(q,t)]\Theta'_{kpq}(t), \tag{104}$$

where

$$\eta^C(k,t) = \pi k \int\int_\Delta pq\, dp\, dq\, b'(k,p,q)\Theta'_{kpq}(t)U(q,t,t). \tag{105}$$

For Eqs. 103 and 105, the coefficient $b'(k,p,q)$ is given by

$$b'(k,p,q) = \frac{1}{2}(1-y^2)(1-z^2). \tag{106}$$

As noted in Herring *et al.* (1982), the TFM effective damping reduces to

$$\eta(k,t) = c_\eta \Big[\int_0^k dp\, p^2 E(p,t)\Big]^{1/2}, \tag{107}$$

while at small $k$ it has a term in the form of Eq. 101. The factor $c_\eta$ is a constant.

An important feature of TFM is its invariance to random Galilean transformations (Kraichnan, 1972). The TFM achieves a $-5/3$ inertial range by tying $\Theta_{kpq}(t)$ to aforesaid distortion effects due to pressure (Kraichnan, 1971b).[60] The characteristic times obtained from the coupling between the solenoidal and compressible parts by advection are at best a measure of the build-up times for triple correlations in the Navier-Stokes system (Kraichnan, 1971a). The model is equally plausible if the characteristic times are scaled by putting a constant factor of order one on the right-hand sides of the coupling equations for $v_i^s(\mathbf{k})$ and $v_i^c(\mathbf{k})$. With an appropriate choice of this adjustable parameter, the TFM yields the inertial range estimates $C_K = 1.40$ (3D) and $C = 6.69$ and $C' = 2.626$ (2D) (Kraichnan, 1971b).

### C. Eddy-damped Quasi-Normal-Markovian

Herring *et al.* (1982) pointed out that the EDQNM (Lesieur, 1987; Orszag, 1970, 1977; Sagaut and Cambon, 2018) may be regarded as a rational approximation to, and simplification of, the TFM. Kraichnan (1972) also alluded to the fact that EDQNM was a simplification of TFM in his implementation of the TFM for inhomogeneous turbulence. The context was that the characteristic memory times for three-mode interactions terms in TFM for such flows were extremely complicated (each being a function of three wave-vectors—e.g., a nine-dimensional space) and a more computable model is achieved by simplifying these, thus yielding EDQNM.

The equation to determined the triad relaxation time $\Theta_{kpq}(t)$ in this model is given by

$$d\Theta_{kpq}(t)/dt = 1 - [\nu(k^2+p^2+q^2) + \eta(k,t) + \eta(p,t) + \eta(q,t)]\Theta_{kpq}(t). \tag{108}$$

If the modes $k, p, q$ are in a steady state, the solution of Eq. 108 is

$$\Theta_{kpq}(t) = \frac{1-\exp[\nu(k^2+p^2+q^2)+\eta_{kpq}]}{[\nu(k^2+p^2+q^2)+\eta_{kpq}]}. \tag{109}$$

The EDQNM model is obtained by choosing $\eta(k,t)$ as Eq. 107 or as the inverse of an eddy circulation time associated with the velocity field in, say, an octave band centered on $k$ (Kraichnan, 1991)

$$\eta(k,t) = c_\eta [k^3 E(k,t)]^{1/2}. \tag{110}$$

This choice of $\eta(k,t)$ thus eliminates the sweeping of inertial range structure by the energy-range velocity field. The factor $c_\eta$ is now adjusted to obtain the desirable value for the Kolmogorov constant (André and Lesieur, 1977).

Orszag (1977) presented a cleaner, but more heuristic route to obtain EDQNM. As summarized by Herring *et al.* (1982), the equation of motion for the third cumulants contains fourth cumulants whose role is to damp the former, thereby providing a mechanism for them to remain properly bounded. The EDQNM may be obtained by simply replacing the fourth cumulants as they occur in the equation of motion for the third cumulants by an appropriate coefficient times the third cumulants (Lesieur, 1987).

In isotropic turbulence, the evolution of the energy spectrum $E(k,t)$ is given by (Lin, 1956)

$$\frac{\partial E(k,t)}{\partial t} + 2\nu k^2 E(k,t) = T(k,t) \tag{111}$$

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[60] The viscous action is negligible in the inertial regime.

where (Chollet and Lesieur, 1981)

$$T(k,t) = \int\int_{\Delta} dp\, dq\, \Theta_{pqk}(t) \frac{xy + z^3}{q} \times \left[k^2 E(p,t)E(q,t) - p^2 E(q,t)E(k,t)\right]. \tag{112}$$

Bos *et al.* (2005) found that the inertial range scaling exponents computed by an EDQNM implementation of Touil *et al.* (2002) agree well with the wind tunnel measurements of Mydlarski and Warhaft (1998). Bos *et al.* (2012a) showed that the EDQNM closure gives power-law corrections to the inertial range scaling of the velocity increment skewness. Moreover, it is shown that this correction is a finite Reynolds number effect, but becomes insignificant at very high Reynolds numbers. Larchevêque (1990) and Lesieur *et al.* (1999) employed the EDQNM to study the pressure spectrum. Lesieur and Herring (1985) investigated the spectral statistics of the fluctuations of a passive scalar convected by a 2D HIT using the EDQNM and TFM.

Herring *et al.* (1982) carried out a comparative assessment of the TFM and EDQNM as applied to passive scalar diffusion. This work recommended that "it appears that the EDQNM, being the simpler procedure, would be the proper tool to employ." This advice was certainly followed in the cases we have discussed since their assessment and in the opinion of the present author, this course is entirely justified.

### D. "Weak" wave turbulence theory

This is an excellent point to mention the connection between statistical closure theories and significant body of literature known as "weak" wave turbulence (WT) theory. Such a relationship is obscured because of the very particular terminology used by the "Russian school" exemplified by Zakharov *et al.* (1992) or Nazarenko (2011). Yet, the connection between asymptotic QNM (Quasi Normal Markovian) techniques and the WT is obvious from seminal paper by Benney and Newell (1969) and more recent works by Cambon and Jacquin (1989), Bellet *et al.* (2006), and Scott (2018).

The astute reader will note that we are referring to the relevant statistical closure theory as QNM. Here, the eddy-damping, the "ED" in the EDQNM, is no longer needed in the asymptotic limit of WT because the energy transfer is dramatically reduced by the dispersivity of interacting waves. This is contrary to the standard EDQNM method applied to the HIT where the dispersive waves are absent (Orszag, 1970). Next, instead of a complex 'Markovian' process in the EDQNM, the "M" in the QNM draws from a different physical process, which has not yet been proven. It refers to the separation between the 'rapid' waves and 'slow' amplitudes, when the focus is on the slow dynamics of the envelopes. Furthermore, the QN is analyzed in Benney and Newell, whereas it is considered self-evident in the "Russian school" as the random phase approximation (RPA).

The WT turbulence theory (in contrast to that of fluid turbulence) is believed to provide an asymptotically exact description — due to the dominance of the underlying wave mode oscillations relative to the much slower evolution of their spectral amplitudes in response to wave-wave scattering.

Before closing, it should be noted that as a common basis for the WT and EDQNM, the spectral equations for the three-point third-order correlations (for Navier-Stokes equations, with quadratic nonlinearity in a rotating frame for example), or four-point fourth-order correlations (for cubic nonlinearity, e.g. in Quantum Gross-Pitaevskii equation[61]), to be closed by the QN approximation, ought to be projected on the eigenmodes of the linear regime (Cambon and Scott, 1999). Bellet *et al.* (2006) and Cambon and Jacquin (1989) have termed this procedure (ED)QNM2 to (ED)QNM3. The reader is referred to Sagaut and Cambon (2018) for further details and the references therein.

### E. Parameterization of small scales of turbulence utilizing spectral closures

Eddy viscosity and diffusivity have been utilized for many years in engineering calculations of turbulent flows at high Reynolds number. It is critical, however, to stress the key conceptual difference between the molecular and eddy viscosities. The former is a valid notion when there is a robust separation of space and time scales between hydrodynamic modes and gas-kinetic collision process. In high *Re* turbulence, however, the latter represents the unresolvable small-scale modes, but there is typically a continuous range of significantly excited modes between the largest and small motions (Kraichnan, 1987a).

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[61] The Gross-Pitaevskii equation (Gross, 1961, 1963; Pitaevskii, 1961), a specific form of self-interacting nonlinear Schrödinger equation, is commonly used to study the evolution of Bose-Einstein condensation (Griffin *et al.*, 1995).

To make contact with closure theories, the analysis below is limited to homogeneous turbulence. One begins by introducing $k_c$ as the cut-off wavenumber separating the resolvable large scales ($k < k_c$) from the unresolvable subgrid scales ($k_c < k < k_d$). We adopted the superscript notation $>$ to denote the unresolvable subgrid fields and $<$ to denote the supergrid fields.

$$u_i(\mathbf{k}, t) = \begin{cases} u_i^< & \text{if } k < k_c, \\ u_i^> & \text{if } k > k_c. \end{cases} \tag{113}$$

The governing equation for the resolvable scale Navier-Stokes equation can be written in the schematic form

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u^< = M(u^<, u^<) + M(u^<, u^>) + M(u^>, u^>) \ \ (k < k_c), \tag{114}$$

where the last two terms must be modeled.[62]

As the net effect of the unresolved subgrid is related to the question of how to account for the energy transfer of these subgrid modes, it is advantageous to work with the statistical closure theories because of their precise definition of the energy transfer. The spectral eddy viscosity is introduced in Kraichnan (1966b, 1971b, 1976) using the TFM.

Using the EDQNM (Chollet and Lesieur, 1981), the subgrid modeling in Eqs 111-112 will consist in expressing the part of $T(k)(k < k_c)$, but with $p$ and/or $q > k_c$ (denoted as $T(k|k_c)$). From Eq. 112, $T(k|k_c)$ has the structure of

$$T(k|k_c) = \mathscr{U}(k) - 2\nu_d(k) k^2 E(k, t) \ \ (k < k_c), \tag{115}$$

where $\mathscr{U}(k)(k < k_c)$ denotes the first term on the right-hand side involving the product of $E(p, t)$ and $E(q, t)$. Both $\mathscr{U}(k)$ and $\nu_d(k)$ will be zero unless $p$ and/or $q$ are in the subgrid scale $p, q > k_c$.

If the analogy with molecular viscosity were perfect the effective eddy viscosity $\nu(k|k_c)$ would be independent of both $k$ and $E(k, t)$ (Kraichnan, 1976). For $k \ll k_c$, there is a clear separation of small- and large-scales and a spectrum with gap at $k_c$ (Chollet and Lesieur, 1981), analogous to the separation between hydrodynamic and thermal agitation scales. Indeed,

$$\nu(k|k_c) \to \nu(k \ll k_c \ll p \sim q) = \frac{1}{15} \int_{k_c}^{\infty} \Theta_{pqk}(t) \left[5E(q) + q\frac{\partial E}{\partial q}\right] dq, \tag{116}$$

and $\nu(k|k_c)$ is in fact independent of both $k$ and $E(k, t)$.

Yet, the eddy viscosity cannot be neatly viewed as the subgrid drain. Indeed, in general the spectral gap assumption breaks down as there is a continuity of scale size between the resolvable and subgrid scales. From the subgrid-scale transfer Kraichnan's expression for the eddy viscosity is given by

$$T(k|k_c) = -2\nu(k|k_c) k^2 E(k, t), \tag{117}$$

where

$$\nu(k|k_c) = \nu_d(k) - \frac{\mathscr{U}(k)}{2k^2 E(k, t)}, \tag{118}$$

and where Kraichnan (1976) termed the $\nu_d(k)$ and $\nu_b(k) = \mathscr{U}(k)/2k^2E(k, t)$ output and input contributions, respectively. Leslie and Quarini (1979) designated these terms, however, the drain and backscatter eddy viscosities.

In terms of resolvable scale Navier-Stokes equation, Eq. 114,

$$\left(\frac{\partial}{\partial t} + [\nu + \nu(k|k_c)]k^2\right) u^< = M(u^<, u^<) \quad (k < k_c). \tag{119}$$

There are a number of EDQNM-based large-eddy simulations (Herring, 1990b; Lesieur and Métais, 1996; Lesieur and Rogallo, 1989; Métais and Lesieur, 1992; Schilling and Zhou, 2002a). Yoshida *et al.* (2003b) noted that the advantage of the modeling based on LRA would be that the model parameters are completely derived from the closure.

Finally, it should be noted that closure-based subgrid model postulate a deterministic eddy viscosity while what is done in practice frequently uses a stochastic one.

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[62] In general, Leonard (1975) and Leslie and Quarini (1979) separate the velocity field into a resolvable large scale component $\bar{u}_i(\mathbf{x}, t)$ and a subgrid component $u_i'(\mathbf{x}, t)$ by a filtering operation, and the resulting resolvable scale evolution equation has been solved routinely in many applicatins. Note that the filter function with a number of popular choices (Leonard, 1975) and the above procedure, Eq. 113, is equivalent to using a sharp cutoff filter. The role of the filters has been discussed, for example, in Buzzicotti *et al.* (2018); Domaradzki and Adams (2002); Langford and Moser (2004); and Leslie and Quarini (1979).

### F. Triumphs and limitations

- Markovianized closures, the TFM and EDQNM, are simplified from the DIA and related models. Such closure models are much faster than the DIA because only single-time statistical functions need be computed (Kraichnan, 1972).
- By construction, the TFM and EDQNM maintain the Kolmogorov scaling, but at the expenses of the introduction of an adjustable constant.
- The EDQNM, being the simpler procedure than the TFM, has gained popularity and become a mainstay of statistical closure applications to various flows (Lesieur, 1987; Sagaut and Cambon, 2018).

## VI. RENORMALIZATION GROUP (RG)

The work of Wilson in developing the RG method for critical phenomena led to his being awarded the Nobel prize in Physics in 1982: "Wilson clarified a natural phenomenon that has puzzled scientists throughout the centuries: the behavior of substances at critical points, or phase transitions. With his renormalization group theory, Wilson divided this seemingly insoluble problem into a number of small ones." While RG is simply an extremely powerful tool for physics calculations when applied to problems for which it is suited, the difficulty is that we cannot yet say with certainty that turbulence is one of those problems (Smith and Woodruff, 1998).

Conceptually, the simplest application of RG to turbulence involves the progressive elimination of high-$k$ modes, with the fixed point corresponding to a renormalized viscosity in the inertial range. The equivalent, in terms of computational fluid dynamics (CFD), would be a subgrid effective viscosity in large-eddy simulations (LES) (Sagaut, 2006). It should be noted, however, that iterated mode elimination leads to two fundamental problems: it generates nonlinearities of a higher order than appear in the Navier-Stokes equations and it requires some simplifying assumptions to permit averaging over the high wavenumber components of the motion.

### A. Perturbative RG

#### 1. RG procedures of mode elimination

This subsection excerpts several salient points from a broad review (Zhou, 2010) on RG for fluid and plasma turbulence. In applying the RG method one partitions the unresolvable subgrid scales, $[k_c, k_0]$, into shells, characterized by a scale factor $\mathfrak{h}$, $0 < \mathfrak{h} < 1$. The portion of the spectral scale to be eliminated, is partitioned by the wave number set $k_c \equiv k_N = \mathfrak{h}^N k_0, k_{N-1} \equiv \mathfrak{h}^{N-1} k_0, ...., k_1 = \mathfrak{h} k_0, k_0$. $k_0$ is typically chosen to be on the order of the Kolmogorov dissipation wave number $(2\pi/\lambda_K)$ while $k_c = k_N$ is the wavenumber separating the actually resolvable scale $k < k_c$ from the unresolvable scales $k_c < k < k_0$.

The RG iterative procedure consists of first eliminating the highest wave number subgrid shell $k_1 < k \leq k_0$ from the Navier-Stokes equation for the remaining supergrid wavenumber $k < k_1$. For $k < k_1$, the resolvable scale Navier-Stokes equation is

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_i^<(\mathbf{k}, t) = \lambda M_{imn}(k) \times \int d^3p \, [u_m^<(\mathbf{p}, t) u_n^<(\mathbf{k} - \mathbf{p}, \mathbf{t}) + \underbrace{2 u_m^>(\mathbf{p}, t) u_n^<(\mathbf{k} - \mathbf{p}, t)}_{\mathbf{A}} + \underbrace{u_m^>(\mathbf{p}, t) u_n^>(\mathbf{k} - \mathbf{p}, t)}_{\mathbf{B}}]. \tag{120}$$

The strength of the nonlinear interaction is denoted by $\lambda$, a formal ordering parameter for perturbation theory, which is eventually set to unity.

The RG multiple-scale elimination schemes can be illustrated by a simple cartoon. The formalism introduces a cutoff wavenumber $k_1$ which is initially around the Kolmogorov scale. Dropping the indices and wavevector arguments for the sake of clarity, we rewrite the Navier-Stokes equation as

$$u^< = G^< M(u^< u^< + u^< u^> + u^> u^< + u^> u^>), \tag{121}$$

$$u^> = G^> M(u^< u^< + u^< u^> + u^> u^< + u^> u^>). \tag{122}$$

The resolvable scale propagator is now denoted by $G^{<} \equiv (\partial/\partial t + \nu k^2)^{-1}$, for $k$ residing in the subgrid shell and the corresponding subgrid scale propagator is given by $G^{>} \equiv (\partial/\partial t + \nu k^2)^{-1}$, when the wavenumber $k$ is in the resolvable scale.

Formally, the $u^{>}$ modes are eliminated from Eq. 121 by solving for $u^{>}$ using Eq. 122. This is found to result in the replacement of the viscosity in Eq. 121 by an enhanced viscosity. Another effect is the generation of the triple product of the resolvable scale velocity fields.

The treatment of averaging over the small scale motion, which appears to proceed as if motions of different scales were independent, has been discussed in Kraichnan's (1959) original presentation of DIA: the same requirement arises in the derivation of the DIA response equation. Heuristically, the DIA closure assumes that the velocity field is only weakly non-Gaussian. Therefore, the motions of different scales, which are uncorrelated because of the kinematic hypothesis of homogeneity, are independent to leading order. This statistical hypothesis also justifies the breakup of fourth order moments into products of second order moments. A theory in which the motions of different scales are not independent to leading order, but in which fourth order moments are treated by the quasi-Gaussian hypothesis, must be carefully formulated to avoid inconsistency.

In the process of removing all subgrid shells, the modification to the Navier-Stokes equation are (a) a renormalized eddy viscosity coefficient and (b) a triple nonlinearity. One then proceeds iteratively, removing at the $i$-the step the subgrid shell $k_i < k < k_{i-1}$ until one reaches the actual resolvable scales at the $N$-th step.

### 2. The $\epsilon$-expansion

Following the pioneering work by Forster *et al.* (1977), the Navier-Stokes equations are assumed driven by a random force. The random force is assumed to be Gaussian with correlation function

$$\langle f_i(\hat{k}) f_j(\hat{k}')\rangle = 2D(2\pi)^{d+1} k^{-y} P_{ij}(\mathbf{k})\delta(\hat{\mathbf{k}} + \hat{\mathbf{k}}'), \tag{123}$$

where the notation, $\hat{k} = (\mathbf{k}, \Omega)$, is introduced for space-time Fourier transforms.

The mode elimination procedure is implemented, without retention of the triple nonlinearity (DeDominicis and Martin, 1979). Fournier and Frisch (1983) studied the relationship between the Forster *et al.* (1977) theory and the TFM and EDQNM closures. Later, Yakhot and Orszag (1986) and others have applied the $\epsilon$-expansion scheme utilizing an additional hypothesis. Iterative mode elimination then generates an equation of the same form with an enhanced viscosity, which is found to satisfy the recurrence relation

$$\frac{d\nu(k)}{dk} = A\frac{D}{\nu^2 k^{\epsilon-1}} \tag{124}$$

where $A$ is computed explicitly by the theory and

$$\epsilon = 4 + y - d. \tag{125}$$

The energy spectrum is found to be of the form

$$E(k) = C_K D^{2/3} k^{-5/3+(4-\epsilon)/3}, \tag{126}$$

where $C_K$ is computed explicitly (Dannevik *et al.*, 1987; Yakhot and Orszag, 1986).

The conclusion of Yakhot and Orszag (1986) is that the nonlinear term in the Navier-Stokes equation is replaced, in the limit of infinite Reynolds number, by the combination of random forcing by $f_i$ and a scale dependent viscosity $\nu(k)$, so that

$$-i\Omega u_i(\hat{k}) + \nu(k)k^2 u_i(\hat{k}) = f_i(\hat{k}). \tag{127}$$

This equation should be compared with the strict DIA Langevin model, Eq. 79, after a Fourier transformation in time. The defining characteristic of RG, at least in the perhaps overly restricted sense of some kind of iterated mode elimination, is the replacement of the integrations in DIA-based analytical closures over a two-dimensional wavenumber slot by much simpler one-dimensional integrations over a single wavenumber argument (Zhou, 2010).

At this point, the scaling exponent of the forcing $y$ in Eq. 123 must be considered. While DeDominicis and Martin (1979) noticed the $\epsilon = 4$ relation to the 5/3 law long before Yakhot and Orszag (1986), the latter popularized it and computed some interesting physical quantities. Substituting $y = d$ in Eqs. 124 and 126 gives the scaling laws

$$\nu(k) \sim D^{1/3} k^{-4/3} \tag{128}$$

$$E(k) \sim D^{2/3} k^{-5/3} \tag{129}$$

which coincide with Kolmogorov scaling if $D$ can be identified with the energy flux $\mathcal{E}$. Note from the definition Eq. 125 that $y = d$ corresponds to $\epsilon = 4$. Although the results are extrapolated from a valid asymptotic expansion, Smith and Woodruff (1998) noted that the only justification for the "$\epsilon$-expansion" at $\epsilon = 4$ is the fact that it leads to good value such as the Kolmogorov constant (Dannevik *et al.*, 1987; Yakhot and Orszag, 1986), and good predictions for turbulence models (Avellaneda and Majda, 1990; Majda and Kramer, 1999; Smith and Woodruff, 1998; Yakhot *et al.*, 1992).

The broad program of Yakhot and Orszag (1986) (hereafter $\epsilon$-RG) at first glance suggested that RG offers a significant theoretical improvement over DIA on one hand, and a major simplification on the other. Indeed, a successful translation of Wilson's idea of expansion about a Gaussian theory [Wilson's $\epsilon$-expansion RG, see Wilson (1975, 1983); and Wilson and Kogut (1974)] to hydrodynamic turbulence would have established RG as a rational approximation, a claim which could never be made for DIA (Zhou, 2010). Yet, the proposal of Yakhot and Orszag (1986) on the basis of the $\epsilon$-expansion that higher order nonlinearities are negligible was contradicted by Eyink (1994), who determined that, on the contrary, higher nonlinearities always scale as $k^0$ rather than $k^{4-\epsilon}$. In this case, these higher nonlinearities cannot be said to be small in any perturbative sense: in the statistical mechanical language of this debate, the higher order nonlinearities are always "marginal," never "irrelevant." The crucial claim is that this conclusion holds, even at $\epsilon = 0$.

Kraichnan (1987b) offered an analysis of the Yakhot–Orszag theory based on comparison with a simple perturbative model. The latter appears to parallel much of the physical and operational content of the lowest order of the Yakhot–Orszag theory, without using RG methods. See also, Kraichnan (1982).

Mou and Weichman (1995) also provided a detailed assessment. The $\epsilon$ expansion describes an infrared fixed point — small scales are sequentially integrated out to obtain an asymptotic description of the large scale ($k \to 0$). However, turbulence is an ultraviolet fixed point — with large scale forcing driving small scale fluctuations. Thus, it is difficult to reason that for a perturbation expansion based on the ramifications of small scale forcing on large scales to successfully describe the effects of large scale forcing on small scales. In RG language, the two fixed points reside in entirely different regimes, and there is no reason why they should continuously exchange stability. As argued in Mou and Weichman (1995) (see especially their Fig. 2, not shown), the radius of convergence of the $\epsilon$ expansion is $\epsilon = 3$ (in particular, not $\epsilon = 4$), and entirely different behavior is observed for $\epsilon > 3$. This is supported by exact solutions to the DIA equations of Mou and Weichman (1995) (their Figs. 11-13, not shown), which perfectly reproduce the RG results for $\epsilon \leq 3$ and demonstrate explicitly how they fail for $\epsilon > 3$ (where, in particular, large scale sweeping effects become important.)[63] Note that the agreement of RG and DIA for small $\epsilon$ is essentially due to the fact that both are based on the same leading order Feynman diagrams (and is likely equivalent to Kraichnan's observation above).

### 3. Eddy damping, backscatter, and subgrid modeling

Rose (1977) applied the recursive RG ($r$-RG) approach to the linear problem of passive scalar convection. Here, the form invariance can be imposed, when both the quadratic and the RG-induced cubic nonlinearities are retained, by truncating the $\lambda$-expansion. Beyond the second iteration, the RG-induced triple products also contribute to the eddy viscosity in the renormalized Navier-Stokes equation. The $r$-RG, therefore, has potential for capturing a variety of dynamical features that depend on the interplay between local and nonlocal triad interactions.

The $r$-RG theory extends Rose's treatment to Navier-Stokes turbulence and eliminates the small scales recursively. The Navier-Stokes equation includes (1) a renormalized eddy viscosity and (2) a triple product in the fluid velocity. The resulting eddy viscosity term exhibits a mild cusp behavior in the renormalized momentum equation (Zhou *et al.*, 1988, 1989).

In fact, the third-order nonlinearities play a major role in the energy-balance equation with the corresponding energy transfer process resulting in an analytic eddy-viscosity formulation (Zhou and Vahala, 1993). Moreover, it is shown that these RG-induced triple nonlinearities give rise to a backscatter in the energy from small scales to large spatial scales, which is in agreement with that from closure theories (Chollet and Lesieur, 1981; Kraichnan, 1976). The region of integration is extended to the so-called near-grid ($k_c, 2k_c$). Leith and Rose (1977) recognized that as a fundamental building block of subgrid modeling, it is important to account for the strong coupling between the resolvable scale eddies and the (near-grid) eddies lying just below the limit of resolution of the grid.[64]

Zhou and Vahala (1993) show that the eddy viscosity associated with the subgrid-subgrid interactions is the only contribution in the limit $k \to 0$. In fact, this portion of the subgrid scale eddy viscosity is essentially a constant for

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[63] See also a discussion in Zhou (2010) on an attempt to eliminating the sweeping with $\epsilon$-RG by Yakhot *et al.* (1989).
[64] Private communication between C.E. Leith and A. Pouquet, U. Frisch, and J.P. Chollet (Pouquet *et al.*, 1983).

the whole range of resolvable scales. On the other hand, the eddy viscosity contributed by the subgrid-resolvable scale interactions is zero in the $k \to 0$ limit but increases rapidly as $k \to k_c$ . Therefore, this portion of the subgrid eddy viscosity is responsible for the cusp in the spectral eddy viscosity. Based on the anlysis of the energy and enstrophy spectra, as well as higher-order statistics of the velocity and velocity derivatives, Dubois *et al.* (1997) found that a faithful subgrid model must take into account the interactions between the subgrid and resolvable scales. Analytically, Sukoriansky *et al.* (2003) have shown that many of these issues can be resolved with proper accounting for this cross-term.

McComb (1990, 1995) has provided a detailed account of his Iterative-RG procedure (a.k.a. Two-field theory). At its core, this method uses the same procedure described in subsection VI.A.3, but with the "resolved-subgrid" terms ignored at each mode-elimination iteration - an additional hypothesis only valid with a spectral gap (Pouquet *et al.*, 1983). McComb *et al.* (2001) eventually arrived at the same conclusion as Zhou and Vahala (1993) and stated that "[i]t is shown analytically that different subgrid eddy viscosities are required to characterize the evolution of the velocity field, the energy spectrum and the dissipation rate, respectively. Also, a subgrid viscosity calculated recursively from the conditionally averaged "subgrid-subgrid" stress can only be used to renormalize the dissipation relation. It is also shown, by direct numerical simulation, that the spectral correlations of subgrid and resolved stresses are dominated by "resolved-subgrid" terms."

## B. Nonperturbative and functional RG

More recently a new non-perturbative renormalization group (NPRG) approach has been concocted and applied to quantum many-particle systems, high-energy physics and quantum gravity (Dupuis *et al.*, 2021). The NPRG is a new implementation of Wilson's concept of the RG (Wilson and Kogut, 1974), previously conceived for critical phenomena for efficiently averaging over fluctuations that have developed for all scales.

The idea of applying NPRG methods to turbulence has shown promising results (Canet *et al.*, 2016, 2017; Tarpin *et al.*, 2018), following early work by Tomassini (1997) and Mejía-Monasterio and Muratore-Ginanneschi (2012). Previously, some confidence-building steps have been carried out first by investigating the stochastic growth of interfaces described by Kardar-Parisi-Zhang (KPZ) equation (Kardar *et al.*, 1986) and statistical description of kinetically roughened surfaces surveyed by Halpin-Healy and Zhang (1995). While the standard perturbative RG failed in perturbation to all orders to find the relevant fixed point and the associated scaling behavior (Wiese, 1998), the NPRG approach captured the strong coupling physics of the KPZ equation at and above one dimension (Canet *et al.*, 2010, 2011; Kloss *et al.*, 2014a, 2012, 2014b).

### 1. Nonperturbative RG approach

The Navier-Stokes equation subject to an external stochastic stirring force is represented by the standard Martin-Siggia-Rose-Janssen-de Dominicis (MSRJD) response functional formalism (DeDominicis, 1976; Janssen, 1976; Martin *et al.*, 1973). MSRJD allows the most systematic way in deriving the generating functional $\mathcal{Z}$ of velocity correlation and response functions under the form of a functional integral. The resulting action $\mathcal{S}_0$ involves not only the original velocity field $u_i$ and pressure $p$,[65] but also their response fields, $\tilde{u}_i$ and $\tilde{p}$

$$\mathcal{Z}[\mathcal{J},\tilde{\mathcal{J}},\mathcal{K},\tilde{\mathcal{K}}] = \int \mathcal{D}\mathbf{u}\mathcal{D}p\mathcal{D}\tilde{\mathbf{u}}\mathcal{D}\tilde{p}\ \exp\big[-\mathcal{S}_0[\mathbf{u},\tilde{\mathbf{u}},p,\tilde{p}] + \mathscr{J}[\mathbf{u},\tilde{\mathbf{u}},p,\tilde{p}]\big]\exp\big[-\Delta S_{0,L^{-1}}[\mathbf{u},\tilde{\mathbf{u}}]\big], \tag{130}$$

where $\mathscr{J}$ contains the source terms $\mathcal{J},\tilde{\mathcal{J}},\mathcal{K},\tilde{\mathcal{K}}$ for the velocity, pressure and response fields

$$\mathscr{J}[\mathbf{u},\tilde{\mathbf{u}},p,\tilde{p}] = \int_{\mathbf{x}} \big[\mathcal{J}\cdot\mathbf{u} + \tilde{\mathcal{J}}\cdot\tilde{\mathbf{u}} + \mathcal{K}p + \tilde{\mathcal{K}}\tilde{p}\big]. \tag{131}$$

The NS action, separated into a local and a nonlocal contribution, reads

$$\mathcal{S}_0[\mathbf{u},\tilde{\mathbf{u}},p,\tilde{p}] = \int_{\mathbf{x},t}\left\{\tilde{p}\partial_i u_i(\mathbf{x}) + \tilde{\mathbf{u}}_i(\mathbf{x})\left[\partial_t u_i(\mathbf{x}) - \nu\nabla^2 u_i(\mathbf{x}) + u_j(\mathbf{x})\partial_j u_i(\mathbf{x}) + \frac{1}{\rho}\partial_i p(\mathbf{x})\right]\right\} \tag{132}$$

[65] Canet *et al.* (2015) have shown that the pressure field, $p$, is very simple to handle since it is not renormalized. Thus, it is kept in the formalism instead of of being eliminated via a Poisson equation.

and

$$\Delta S_{0,L^{-1}}[\mathbf{u},\tilde{\mathbf{u}}] = -\int_{\mathbf{x},\mathbf{x}',t} \tilde{\mathbf{u}}_i(\mathbf{x},t) F_{K^{-1},ij}(|\mathbf{x}-\mathbf{x}'|)\tilde{\mathbf{u}}_i(\mathbf{x}',t), \tag{133}$$

where $\int_{\mathbf{x},t} \equiv \int d^d\mathbf{x}dt$ and $F_{L^{-1}}$ denotes the forcing correlation function, which is local in time in order to preserve Galilean invariance, and is concentrated, in Fourier space, on the inverse of the integral scale. Canet *et al.* (2016) remarked that $F_{L^{-1}}$ is a smooth regular function (both in the IR and in the UV). However, it is not required, as those perturbative RG studies discussed in the previous subsection, to impose a power-law spectra with a purpose to introduce a formal expansion parameter.

The application of the NPRG consists of building a sequence of scale-dependent effective models such that fluctuations are smoothly averaged as the spectral scale, $\varsigma$, is marched from the microscopic UV scale (some large wavenumber $\varsigma \simeq \Lambda$ at which the continuous hydrodynamical description of the fluid in terms of the NS equation starts to be valid) where no fluctuations are yet included, toward the macroscopic IR scale ($\varsigma \to 0$, infinite volume). To achieve the separation of fluctuation modes within the NPRG procedure, Canet *et al.* (2016) included a regulator in $\mathcal{S}$, which is a wave-number and scale-dependent quadratic term,

$$\Delta\mathcal{S}_\varsigma[\mathbf{u},\tilde{\mathbf{u}}] = -\int_{t,\mathbf{x},\mathbf{x}'} \tilde{u}_i(\mathbf{x},t)F_\varsigma(|\mathbf{x}-\mathbf{x}'|)\tilde{u}_i(\mathbf{x}',t) + \int_{t,\mathbf{x},\mathbf{x}'} \tilde{u}_i(\mathbf{x},t)R_\varsigma(|\mathbf{x}-\mathbf{x}'|)u_i(\mathbf{x}',t). \tag{134}$$

Here, the additional term $\mathcal{R}_\varsigma$, which is proportional to the velocity, can be interpreted (Canet *et al.*, 2015) as an Ekman friction term[66] and is essential to properly implement the RG procedure and to correctly regularize the flow — both in the UV and in the IR (Mejía-Monasterio and Muratore-Ginanneschi, 2012). Note that the inverse integral scale in the nonlocal quadratic term is let running and is here incorporated in the regulator $\Delta S_{0,L^{-1}} \to \Delta S_{0,\varsigma}$ (Tomassini, 1997).

The scale-dependent generating functional now becomes (Canet *et al.*, 2016; Mejía-Monasterio and Muratore-Ginanneschi, 2012; Tarpin *et al.*, 2018; Tomassini, 1997),

$$\mathcal{Z}[\mathcal{J},\tilde{\mathcal{J}},\mathcal{K},\tilde{\mathcal{K}}] = \int \mathcal{D}\mathbf{u}\mathcal{D}p\mathcal{D}\tilde{\mathbf{u}}\mathcal{D}\tilde{p}\ \exp\big[-\mathcal{S}_0[\mathbf{u},\tilde{\mathbf{u}},p,\tilde{p}] + \mathscr{J}[\mathbf{u},\tilde{\mathbf{u}},p,\tilde{p}]\big]\exp\big[-\Delta S_\varsigma[\mathbf{u},\tilde{\mathbf{u}}]\big]. \tag{135}$$

The effective average action, $\Gamma_\varsigma$, can be obtained by the Legendre transform of $\mathcal{W}_\varsigma = \ln \mathcal{Z}_\varsigma$ (up to terms proportional to $\mathcal{R}_\varsigma$ and $\mathcal{F}_\varsigma$),

$$\Gamma_\varsigma[\mathbf{u},\tilde{\mathbf{u}},p,\tilde{p}] + \mathcal{W}_\varsigma[\mathcal{J},\tilde{\mathcal{J}},\mathcal{K},\tilde{\mathcal{K}}] = \mathscr{J}[\mathbf{u},\tilde{\mathbf{u}},p,\tilde{p}] + \int_{t,\mathbf{x},\mathbf{x}'} \Big\{\tilde{u}_i(\mathbf{x},t)R_\varsigma u_i(\mathbf{x},t) - \tilde{u}_i(\mathbf{x},t)F_\varsigma\tilde{u}_i(\mathbf{x},t)\Big\}. \tag{136}$$

Thus, evolution of the $\Gamma_\varsigma$ with respect to $\varsigma$, often termed as the flow of the said function, is provided by the Wetterich equation

$$\partial_\varsigma\Gamma_\varsigma[\mathbf{u},\tilde{\mathbf{u}},p,\tilde{p}] = \frac{1}{2}\int_{\mathbf{x},\mathbf{x}'} \partial_\varsigma[\mathcal{R}_\varsigma]_{ij}(|\mathbf{x}-\mathbf{x}'|)\big[\Gamma_\varsigma^{(2)} + \mathcal{R}_\varsigma\big]^{-1}_{ji}, \tag{137}$$

where the regular matrix $[\mathcal{R}_\varsigma]$ is obtained by taking functional derivatives with respect to the fields of $\Delta\mathcal{S}_\varsigma$ in Eq. 134 and $\Gamma_\varsigma^{(2)}$ is the Hessian of $\Gamma_\varsigma$. The RG flow equation is exact and Eq. 137 provides the exact interpolation between these two scales, $\varsigma \approx \Lambda$ and $\varsigma \to 0$. More specifically, its initial condition corresponds to the "microscopic" model, which is the NS equation. At this scale, one can show that $\Gamma_\varsigma$ identifies with the bare action, $\mathcal{S}_0$, since no fluctuation is yet incorporated. When $\varsigma \to 0$, the regulator is removed and one obtains the actual properties of the model, when all fluctuations have been integrated over.

The detailed NPRG treatments of 3D turbulence have been laid out in Canet *et al.* (2016, 2017); and Tarpin *et al.* (2018). This procedure has been extended recently to 2D flows (Tarpin *et al.*, 2019).

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[66] The Ekman friction term is important in 2D to damp energy transfer towards larger and larger scales.

### 2. Time dependence of multi-point correlation functions

The 2-point NPRG correlation is given by

$$[\Gamma_{\varsigma}^{(2)}]_{i_1,i_2}[\mathbf{x}_1,\mathbf{x}_2,\{\varphi_i\}] = \frac{\delta^2\Gamma_{\varsigma}[\{\varphi_i\}]}{\delta\varphi_{i1}(\mathbf{x}_1)\delta\varphi_{i2}(\mathbf{x}_2)}, \tag{138}$$

with analagous definitions for higher order $n$-point vertex functions, $\Gamma_{\varsigma}^{(n)}$. Any physical multi-point correlation function $\langle u_{i1}(\mathbf{x}_1)\cdots u_{i2}(\mathbf{x}_2)\rangle$ can be straightforwardly expressed in terms of the $\Gamma_{\varsigma}^{(n)}$ obtained from the NPRG.

Interestingly, within this framework, rigorous results have been obtained for the space-time dependence of all $n$-point correlation function of Eulerian velocities in the limit of large wave-numbers (Tarpin *et al.*, 2018). Essentially, this possibility is due to the fact that the the regulator renders the exact flow equation of any $\Gamma_{\varsigma}^{(n)}$ (obtained through functional derivatives of Eq. 137) analytic, such that it can be expanded in the limit of large wave-numbers. The leading term of this expansion can be calculated exactly using the Ward identities ~~ensuing~~ arising from the symmetries of the NS field theory, without further assumptions regarding the limit of large wave-numbers. For the 2-point correlation function, the NPRG result shows that it decays as a Gaussian in the variable $tk$ where $k$ is the wave-number and $t$ a small time delay in the stationary state. This corresponds to the sweeping-induced form predicted by Kraichnan's DIA. However, in contrast with the DIA, the the NPRG recovers the correct $k^{-5/3}$ spectrum. Furthermore, the NPRG shows that the Gaussian form is valid at small time delays, and the temporal decay crosses over to a simple exponential at large time delays. Similar explicit expressions have been derived for any arbitrary multi-point function, generalizing the 2-point results (Tarpin *et al.*, 2018). These results demonstrate that a breaking of scale-invariance occurs, due to a non-decoupling of the scales, and they give its exact form for all correlations at large wave-numbers.

A noteworthy consequence to the results summarized above is that the frequency spectrum within NPRG is found to scale as a function of the frequency with a power law exponent $-5/3$ (Canet *et al.*, 2017). While this is a subject which will be discussed in detail in subsections VI.E and VI.F, it suffices for the moment to state that this result corresponds to what is observed in experiments and numerical simulations for velocities measured in a fixed reference frame (Eulerian velocities). Therefore, the NPRG solution has correctly accounted for the sweeping effect, which is imposed by the large-scale motion on the Eulerian velocities at small scales.

### 3. Near-dissipative range scaling

Whilst Kraichnan (1959a) predicted a pure exponential form for the far-dissipation range (see, footnote 49), Canet *et al.* (2017) used the NPRG result for the 2-point correlation at large wave-number, and under an additional assumption, to show that the near-dissipation spectrum is given by

$$E(k) \propto k^{-5/3}\exp\big[-\tilde{\beta}_d(k/k_d)^{2/3}\big], \tag{139}$$

where $\tilde{\beta}_d$ is a constant.

To test the NPRG prediction, Eq. 139, Debue *et al.* (2018) and Gorbunova *et al.* (2020) analyzed the near-dissipative range scaling with high-resolution DNS and extended experimental data of von Kármán swirling turbulent flows and grid turbulence generated in the Modane wind tunnel. All the results, from DNS and from experiments, and from the different determinations, are consistent and strongly supportive. Fig. 5 illustrated the third logarithmic derivative of $\ln E(k)$ in respect to $\ln(k/k_d)$. Clearly, there is a plateau in the near-dissipative range, approaching $2/3$, for all the curves as $Re_\lambda$ increases. Recently, Buaria and Sreenivasan (2020) further confirmed the $\exp[-(k/k_d)^{2/3}]$ scaling for all simulated flow fields they considered (up to $Re_\lambda = 650$). However, the authors argued that the anticipated range of validity is much smaller than that asserted in the NPRG work.

### 4. Closing words on NPRG

Before closing this subsection, it is appropriate to comment on the possibility of determining corrections to the -5/3 scaling exponent of the energy spectrum in the inertial range. Canet *et al.* (2016) remarked that the existence of the fixed point entails the power-law behavior for the correlation functions. However, the non-decoupling of the large wave numbers allows for violations of simple scaling to occur, which means that the exponents can deviate from their dimensional values; therefore this opens the door to multiscaling and intermittency. The value of the corresponding

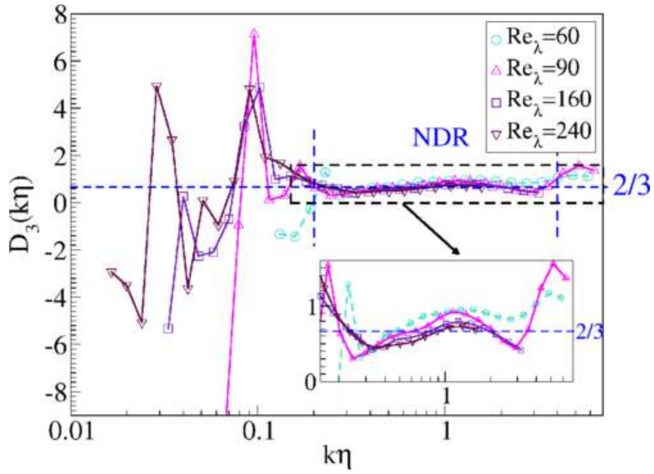


FIG. 5: Third logarithmic derivative of $E(k)$, given in Eq. 139, with respect to $\ln(k/k_d)$. Only the curves for the highest resolution for each $Re_\lambda$. are shown for clarity. They all show a plateau in the near-dissipative range, approaching 2/3, as $Re_\lambda$ increases. Fig. 3 of Gorbunova *et al.* (2020), *Phys. Rev. Fluids*, Reproduced with permission from ©American Phys. Soc.

intermittency exponents may potentially be computed by integrating the exact flow equations obtained in the large wave-number regime.

### C. Triumphs and limitations

- The "$\epsilon$-expansion" RG approach of Yakhot and Orszag (1986) leads to good values of quantities such as the Kolmogorov constant (Dannevik *et al.*, 1987), and good predictions for turbulence models (Yakhot *et al.*, 1992). However, the key hypothesis introduced does not appear to be justified (Eyink, 1994).
- The recursive RG method focused on subgrid modeling and delineated the importance of the interactions between the resolvable and near-grid scales. The eddy viscosity generated from the subgrid-subgrid interactions is analogous to that of the molecular viscosity.
- The non-perturbative RG has shown significant early promise. The most important result that this approach has permitted so far is the time dependence of any $n$-point correlation functions. This result is quite general and most importantly rigorous; it is exact at the limit of large wave-numbers, which makes NPRG quite uncommon among (less controlled) closure methods.

## VII. INFRARED-DYNAMIC, ENERGY TRANSFER, INTERACTING SCALES, AND SWEEPING VS STRAINING MOTIONS

In this section, we segue into an assessment of statistical closure theories as the tools to understanding fundamental aspects of turbulence.

### A. Infrared-dynamic and turbulence free decay

The decay of total energy of turbulence

$$K(t) = \int_0^\infty E(k,t)\, dk, \tag{140}$$

and scalar variance is a classical problem (Byers *et al.*, 2017; Comte-Bellot and Corrsin, 1966, 1971; Mohamed and LaRue, 1990; Skrbek and Stalp, 2000). Based on Loitsianski's invariant (Loitsianski, 1939) the Kolmogorov's total energy decay should follow the $t^{-10/7}$ law (Kolmogorov, 1941c). The decay exponent of $K(t)$ can also be obtained by enforcing a symmetry in the problem (Clark and Zemach, 1998; Clark *et al.*, 2007; Oberlack, 2002).

If $k$ is much smaller than the energy-containing range it is easy to show from the DIA covariance that

$$\partial E(k,t)/\partial t = T(k,t) = k^4 A(E) - \big(\nu + \nu(k)\big) k^2 E(k,t) \tag{141}$$

based on the concepts of equipartitioning and viscosity (Herring, 1985; Herring *et al.*, 1982). Here $A(E)$ and $\nu(k)$ are yet-to-be-determined functionals of $E(k,t)$. The first term on the right hand side can be obtained by taking suitable expansion of the right-hand size of the convariance equation, Eqs. 95-96. The second term on the right hand side, the eddy viscosity term, is necessary to restore energy conservation. Herring (1985) remarked that Eq. 141 alone suffices - in large measure - for analyzing the decaying of total energy.

If the initial energy spectrum behaves as $E(k,0) \sim k^s$ (as $k \to 0$), then a short time later, according to Eq. 141, $E(k,t) \to k^{\tilde{s}}$, where $\tilde{s} = 4$ for $s \geq 4$ and $\tilde{s} = s$, if $1 \leq s < 4$. Note that $s = 1$ is the true self-similar decaying case (Monin and Yaglom, 1975), while the flows associated with $s = 4$ and $s = 2$ are often referred to as Batchelor and Saffman turbulence, respectively.[67]

In an extensive EDQNM study, Lesieur and Schertzer (1978) showed that for the case of $s \geq 4$, the $k^4$ spectral backscatter would dominate the infrared transfer (see also Kida and Goto (1997)).[68] Lesieur and Schertzer (1978) further suggested that the kinetic energy should decay as $t^{-1.38}$ instead of Komogorov's $t^{-10/7}$ law. This proposal has been contradicted, however, by a high-resolution numerical simulation. Ishida *et al.* (2006) concluded that the Kolmogorov's classical decay law still holds true. This issue has been investigated again with the EDQNM in Meldi and Sagaut (2012).

Lesieur and Schertzer (1978) also found that for integer $s = 1, 2$, and 3 no significant backscatter occurs and permanence of large eddies holds.[69] Yoshimatsu and Kaneda (2018, 2019) considered the large-scale structure of decaying homogeneous turbulence with $E(k \to 0, 0) \sim k^2$ and applied an argument similar to that of Saffman (1967). Based on the invariance and a self-similarity assumption for the large-scale evolution, one of the findings suggested that the anisotropy of the velocity and passive scalar fields is persistent at large scales. Under shear-released homogeneous turbulence, the scalar anisotropy is predicted by an EDQNM model to persist during the passive scalar decay (Briard *et al.*, 2016a).

In their investigation of the decay of Burgers turbulence, Gurbatov *et al.* (1997) suggested that $s$ does not have to be an integer. Eyink and Thomson (2000) further argued that for 3D Navier-Stokes turbulence, there is a crossover dimension $s \approx 3.45$, above which a $k^4$ backscatter occurs. Lesieur and Ossia (2000) performed a follow-up EDQNM study, in conjunction with a LES (Ossia and Lesieur, 2000), to evaluate this proposal with simultions for $s \in (3.2, 3.9)$ with an interval of increase of 0.1. While the crossover dimension was not observed, the study found that for $3 \leq s \leq 4$, the backscatter $\propto k^{\tilde{s}}$, with $\tilde{s}$ slightly higher than $s$.

Yet, Meldi and Sagaut (2012) reported that the large-scales close to the peak of the energy spectrum are actually most important in the energy decay. Mons *et al.* (2014a) further utilized a data assimilation method (Gronskis *et al.*, 2013; Le Dimet and Talagrand, 1986) that uses a variational approach for optimal control of the initial energy spectrum shape and minimizing the error committed on some aspects of decay turbulence for a targeted EDQNM calculation. This work found that the finite time decay is not dictated by the initial enegy spectrum $E(k \to 0, t = 0)$, which is a theoretical construct and impossible to measure at the asymptotic large scale. The authors found that the controlling length scale in the energy decay is the one close to the peak of the energy spectrum, $kL(t) = O(1)$, instead of the length scale corresponds to the $k$ with $kL \ll 1$ or $k \to 0$. This finding is significant since detailed features of the energy peak may be intimately related to the turbulence production mechanism, which is not universal (Meldi and Sagaut, 2012).

Briard *et al.* (2015) have investigated the decay of a passive scalar with various Reynolds numbers. They found that the temporal decay exponent from the EDQNM are in very close agreement with those from a dimensional analysis, extending that of Comte-Bellot and Corrsin (1966) for the situation where the permanence of large eddies is broken. The decay exponent of the scalar variance in HIT is not affected by the initial scalar integral length scale or the

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[67] Kolmogorov (1941c) assumed that a self-similar decay of the spectrum could be based on the invariance of the Loitsianski integral (Loitsianski, 1939). This leads to the small wavenumbers energy spectrum $E(k) \sim k^4$ (near $k \to 0$). Proudman and Reid (1954) and Batchelor and Proudman (1956) later determined that the Loitsianski integral actually was not invariant. However, Lesieur (1987) and Chasnov (1993) found that the Loitsianski integral is self-similar, giving credence to Kolmogorov's estimate. Also, Saffman (1967) showed that if the turbulence is created by a non-solenoidal impulsive body force per unit mass applied to the fluid at the initial instant, the small wave number spectrum scales as $E(k) \sim k^2$. This is also the energy equipartition spectrum in three dimensions, with zero flux, a scaling which is often observed.

[68] As an aside, it's probably worth noting that the entire construct of the theories is the Fourier Integral Transform (e.g., a continuous $k$-space), and this is not the general "proper representation" of flow in a box — either a wind-tunnel or a DNS, where discrete Fourier transforms are applicable. Consequently, discussions about what happens near $k \to 0$ are generally tendentious. Likewise, there are no experiments that look at time approaching infinity or probably even the time needed to approach self-similarity of any kind — except maybe astrophysical flows.

[69] The principle of permanence of large-eddies assumes the existence of large-scale invariants, as concisely summarized by Soulard *et al.* (2018), the value of which is set by initial conditions. These invariants can be expressed in terms of large-scale integrals of turbulence, such as thoses from Loitsianski (Kolmogorov, 1941c; Landau and Lifshitz, 1954; Loitsianski, 1939) or Saffman (Saffman, 1967).

Prandtl number. For the case where a mean scalar gradient is present, Briard *et al.* (2017) used the EDQNM model to study the scaling of the scalar variance spectrum at $Pr \ll 1, \sim 1$, or $\gg 1$.

The dependence of the normalized dissipation rate

$$C_\varepsilon = \frac{\mathcal{E}L}{u^3} \tag{142}$$

on the $Re$ number has been investigated since the publication of the classical textbook of Batchelor (1953). Recently, Djenidi *et al.* (2017) provided a discussion from the classical perspectives of Taylor (1935) and Kolmogorov (1941c). When $Re$ is sufficiently large, $C_\varepsilon$ appears to asymptotically approach a constant (Frisch, 1995) and early experimental and simulation results have been collected in Sreenivasan (1984, 1998). Vassilicos (2015) considered numerous decaying turbulence experiments, especially those he and his co-workers performed with a variety of grids: regular, active and fractal. Yoffe and McComb (2018) developed an onset criterion for freely decaying isotropic turbulence and this article is a resource for many representative values of $C_\varepsilon$ from numerical simulations and experiments. Burattini *et al.* (2005) pointed out that the measurement of $\mathcal{E}$ is fraught with uncertainties associated with the electronic noise and probe resolution and involved with errors that are challenging to quantify. In this regard, grid turbulence can sidestep these issues as $\mathcal{E}$ can be estimated indirectly, but more reliably, from the streamwise decay rate of the mean turbulent kinetic energy, a large-scale quantity. Of course, such technical difficulty does not exist for DNS, in which $\mathcal{E}$ can be evaluated accurately (see for example, Kaneda *et al.* (2003)).

From their EDQNM computations, Bos *et al.* (2007) provided support to an observation in Sreenivasan (1998) - originally based on four DNS datasets - that the aymptotic value of $C_\varepsilon$ for decaying turbulence is significantly higher than that from forced turbulence. Extending a previous study (Meldi, 2016), Meldi and Sagaut (2018) investigated the effect of a continuous fractal/multiscale forcing over a range of large scales. The authors also used the initial energy spectrum proposed by Meyers and Meneveau (2008), but with a steady, then a time-evolving forcing function, to probe the sensitivity of $C_\varepsilon$ to the forcing.

### B. Locality of energy transfer

Almost all turbulence theories and models rely on assumptions about the energy transfer process. It is very difficult from experiments to observe the details of the energy transfer processes. On the other hand, analytical theories are not fully satisfactory since they have already involved certain assumptions.

Within the Lagrangian framework, the Kolmogorov assumption is sustained when the *representations* of the models are chosen appropriately. In other words, the controlling time scale must be built in as the time scale of the intrinsic or straining motions, $\tau_{st}$. In Kraichnan's LHDIA, in fact, the Kolmogorov local energy transfer process is ensured by using a modified Navier-Stokes equation in which the convection of any given spatial scale by much larger scales is consistently removed (Kraichnan, 1964c). The controlling time scale is explicitly introduced into the Markovianized closures (TFM and EDQNM). In these models, the triad relaxation time, $\Theta$, and the damping, Eq. 101, are constructed in a fashion so that both are consistent with the straining time scale.

The argument, constructed based on closure theories above, can be viewed from a simple perspective adopted by Kraichnan (1965c); Matthaeus and Zhou (1989); and Zhou and Matthaeus (1990b). The flux rate of energy across a scale of $k$ is the most basic measure of the energy transfer process. In the Kolmogorov theory of the universal equilibrium range, it is the only link between the energetic and dissipative scales of motion (Zhou, 1993b). Let $\tau_s(k)$ denote the time scale for the decay of triple correlations, which may depend on any relevant parameters and are responsible for inducing turbulent spectral transfer from wavenumbers lower than $k$ to higher wavenumber. Following Kraichnan, we assume that $\mathcal{E}$ is proportional to $\tau_s$, depends on the power of the energy spectral density and the wavenumber. From the dimensional argument, one finds that

$$\mathcal{E} = C_3^2 \tau_s(k) E^2(k) k^4. \tag{143}$$

Apart from the introduction of the explicit factor $\tau_s$, the above argument is just a reordering of the usual Kolmogorov phenomenology and $C_3$ is a constant, the value of which depends on the specific problem. For a Kolmogorov turbulence, the triple correlations decay on the order of the strain time (shown in Lagrangian framework), so that $\tau_s = \tau_{st}(k) = [k^{3/2}E^{1/2}(k)]^{-1}$, from which the $k^{-5/3}$ scaling is recovered. By proper formulation of the factor $\tau_s$, Eq. 143 allows us to include external influence such as magnetic field or other external agencies.

At the heart of this predilection for the local energy transfer assumption is the existence of definitive experimental measurements of $-5/3$ Kolmogorov spectrum (Chapman, 1979; Grant *et al.*, 1962; Saddoughi and Veeravalli, 1994).

Still, it is satisfying that spectral numerical simulations, with $Re_\lambda$ up to 1000, provide direct supporting evidence (Domaradzki and Rogallo, 1990; Mininni *et al.*, 2006; Ohkitani and Kida, 1992; Yeung and Brasseur, 1991; Zhou, 1993a,b). Eyink (2005) also provided a rigorous proof of the scale locality property for solutions of the Navier-Stokes equations, pointwise in space and without statistical averaging. Effectively, these scaling assumptions are relatively reasonable and can be heuristically thought of as a velocity spectrum scaling with a power law $k^{-n}$ with $1 < n < 3$. It should be noted that the rigorous proof of Eyink was demonstrated numerically in Eyink and Aluie (2009).

Scrutinizing the correlation between the local energy dissipation rate and energy fluxes both in Eulerian and Lagrangian frames of reference, Wan *et al.* (2010b) provided evidence for the Lagrangian nature of the turbulent energy cascade and the temporal and spatial multiscale properties of turbulent fluid motion. Recently, Cheikh *et al.* (2019) investigated the small-scale energy cascade in homogeneous isotropic turbulence, incorporating small-scale structures and independent degrees of freedom for translational and rotational motions. While both forward and inverse energy cascades exist in HIT at the small-scale, the analysis found that the forward cascade is present globally.

### C. Interacting scales

The issue of interacting scales is a fundamental question regarding the energy transfer process. According to Tennekes and Lumley (1972), "[a]lthough we expect that there will be a net flux of energy from smaller to larger wave number, we do not know which eddy sizes are involved in the spectral energy transfer across a given wave number. For example, does the energy come from eddies that are slightly larger than a given wavelength, or does it come from all larger eddies indiscriminately? In the same way, is the energy absorbed at wave numbers slightly larger than a given value, or is it absorbed by all larger wave numbers?"

The equation for the energy transfer is given by

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) E(k) = T(k), \tag{144}$$

where $T(k)$ is the energy transfer function. The contribution to $T(k)$ resulting from nonlinear interactions between Fourier modes in wave-number band $k$ and those in bands $p$ and $q$ is denoted by $T(k, p, q)$, which follows directly from equation 41 as

$$T(k, p, q) = \mathrm{Im} \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \Big[ u_i(\mathbf{k}, t) M_{ijm}(k) u_j(\mathbf{p}, t) u_m(\mathbf{q}, t) \Big] \tag{145}$$

Here, $\sum$ denotes summation over spherical shells in $\mathbf{k}$, $\mathbf{p}$, $\mathbf{q}$ subject to the triangle constraint $\mathbf{p} + \mathbf{q} = \mathbf{k}$. In turn, the net energy transfer into band $k$ is the result of all contributing interactions:

$$T(k) = \sum_{p,q} T(k, p, q). \tag{146}$$

The raw interaction measurements $T(k, p, q)$, formally the transfer due to the interaction of sharply truncated Fourier bands, form mathematical building blocks for spectral analysis of the energy transfer process.[70] At first blush, it is understandable that some are predisposed to use the raw triadic interaction function $T(k, p, q)$ to assess the degree of locality on interacting scales. Indeed, based on such raw interaction measurements, and particularly the behavior at large $k$, the classical Kolmogorov picture of energy transfer was questioned by Domaradzki and Rogallo (1990), Yeung and Brasseur (1991), and Ohkitani and Kida (1992) who concluded that this local energy transfer resulted from nonlocal interactions, a notion clearly at variance with the classical Kolmogorov picture.[71]

While there is no disagreement with these studies concerning the actual measurement of the raw interaction statistics-the triad nonlinear transfer $T(k, p, q)$, Zhou (1993a,b) stressed that $T(k, p, q)$ is not the appropriate quantity from which to determine whether the nonlinear interactions contributing to the energy transfer are local or not.

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[70] Recently, Dotti *et al.* (2020) endeavored to visualize triad interactions by measuring experimentally the downstream development of a single or a few Fourier components injected into a well-defined flow. In related work, Wu *et al.* (2021) extracted the exact time scale in typical numerical experiments of triad interactions of HIT.

[71] Alexakis *et al.* (2005a), Mininni *et al.* (2006), and Mininni *et al.* (2008) analyzed the transfer functions $T(k, p, q)$ and concluded that the cascade in hydrodynamic turbulence is nonlocal since the most dominant triads are those transferring energy from $q$ to $k = q + k_0$, catalyzed by modes $p = k_0 \sim 1/L$ at the largest, most energetic scales. This is similar to previous studies that had analyzed $T(k, p, q)$ (e.g. Domaradzki and Rogallo (1990); Ohkitani and Kida (1992); and Yeung and Brasseur (1991)), in a process described as "local transfer through nonlocal interactions," which suggested that energy is transferred in spectral space via steps of a fixed, small size $k_0$, driven by large-scale (small wavenumber) advection. However, Aluie and Eyink (2009) showed that such transfer quantified by $T(k, p, q)$ (which does not account for the number of triads transferring energy) is vanishingly small deeper in the inertial range and that the bulk of energy transfer is due to local triads once their number is accounted for (Fig. 8 in Aluie and Eyink 2009). This view is in agreement with that previously discussed in Zhou (1993a,b).

Rather, we follow Kraichnan's argument that these raw interaction statistics should be viewed only as a mathematical building block in the energy transfer process and their physical interpretation requires further summation, during which much additional cancellation occurs. Using the helical wave decomposition and an "instability assumption," Waleffe (1992) argued that the physical process of the straining of small scales by large scales, which results in local transfer by nonlocal interaction, must be represented by at least two triads, resulting in cancellation between their individual triad transfers $T(k, p, q)$.

The main idea from Aluie and Eyink (2009) is that the quantity $T(k, p, q)$ as a measure of triadic energy transfer does not account for the *number* of triads that exists. Therefore, while a single non-local triad transfers more energy compared to a single local triad (which was the main reason many previous studies claimed that the transfer is non-local), there are exponentially many more local triads than nonlocal ones. The bulk of energy transfer is accomplished by local triads. Unlike, $T(k, p, q)$, the definition of the energy flux, $\Pi$, naturally incorporates the number of triads participating and therefore, exhibits local energy transfer. The above holds true even without decorrelation. When decorrelation is accounted for, the nonlocal contribution is even smaller than the predictions of Eyink (2005).

At issue is the appropriate choice of a statistical quantity to indicate the nature of energy transfer across the spectrum, and particularly its dependence on the relative scales involved in the nonlinear interactions (Zhou, 1993b). The work of Kraichnan (1966b, 1971a,b) provided a theoretical criterion by which one can determine the relative importance of local and nonlocal interactions. For a given scale $k$, he argued that all raw interaction statistics must be summed such that physical quantities contain only one parameter which indicates the scale disparity of the interaction. We use the scale disparity parameter

$$s(k, p, q) = \frac{\max(k, p, q)}{\min(k, p, q)} \tag{147}$$

to study the flow of energy to small scales and determine the scaling laws for the contributions of various interactions (Zhou, 1993a,b; Zhou *et al.*, 1996).

The flux rate of energy across a scale $k$ is the most fundamental measure of the energy transfer process. In the Kolmogorov theory of the universal equilibrium range it is the only link between the energetic and dissipative scales of motion. Contributions to the total flux from the various scale interactions can be written as

$$\Pi(k) = \sum_s \Pi(k, s), \tag{148}$$

where

$$\Pi(k, s) = \int_k^\infty T(k', s) dk'. \tag{149}$$

Figs. 8-9 of Zhou (1993b) indicate that the fractional LES energy flux $W(s) = \Pi(k, s)/\Pi(k)$ is dominated by local interactions (small scale disparity $s$) for all scales $k$ (not shown). This closely resembles the classical picture of the energy transfer process described in detail by Tennekes and Lumley (1972). Moreover, the dependence upon the scale disparity parameter is the same for all inertial-range scales, that is beyond the forced scales the normalized individual energy flux contributions are essentially independent of $k$ as would be expected in a scale-similar inertial range. More specifically,

$$W(s) = \Pi(k, s)/\Pi(k) \sim s^{-4/3}. \tag{150}$$

This scaling is consistent with the prediction from the ALHDIA (Kraichnan, 1965a, 1966b), TFM (Kraichnan, 1971a,b), and LRA (Gotoh and Watanabe, 2005).[72]

When $s$ is very large, the LRA asymptotic behavior of the function $W(s)$ is

$$W(s) \sim 2.34 s^{-4/3} \left( \frac{4}{9} \log s + \frac{16}{45} \right), \tag{151}$$

while the LRA fractional flux of the passive scalar scales is given by

$$W_\theta(s) \sim 0.70 \left( \frac{22}{45} s^{-2/3} + \frac{4}{3} s^{-4/3} \right). \tag{152}$$

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[72] Remarkably, this scaling is also found in a recent study of the turbulent bubble breakup cascade (Chan *et al.*, 2021).

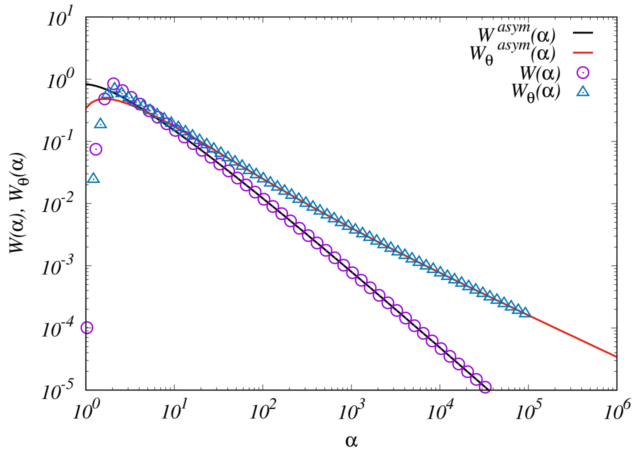


FIG. 6: The $W(s) \sim s^{-4/3}$ was discussed in the statistical closure theories of Kraichnan and confirmed numerically by Zhou. In a more recent numerical investigation, $W(s)$, $W_\theta(s)$ from numerical simulations are compared with their asymptotic expansions $W^{asym}(s)$, $W_\theta^{asym}(s)$ from LRA. Fig. 5 of Gotoh and Watanabe (2005), *J. Turbul.*. Reproduced with permission from Taylor & Francis.

Thus, the scalar transfer of the passive scalar delays more slowly and is less local than that of the velocity field (Gotoh and Watanabe, 2005) (Fig. 6).

Domaradzki and Carati (2007a,b) and Domaradzki *et al.* (2009) revisited the issue of the locality of interacting scale and computed the fractional flux function. The numerical simulations from these authors, as well as those from Gotoh and Watanabe (2005), confirmed the $s^{-4/3}$ scaling predicted by statistical closure theories, in agreement with Zhou (1993a,b).

Alexakis *et al.* (2005a) also found that when considering the energy flux, the relevant interactions are mostly local. In simulations with $Re_\lambda$ up to 1000, the fraction of the flux due to nonlocal interactions is found to be 20% (Mininni *et al.*, 2006). When $Re_\lambda$ is increased to 1300, of the order of 10% of the energy flux is from nonlocal interactions (Mininni *et al.*, 2008). These results are not inconsistent with the Kolmogorov theory, as it is important to stress that the phenomenology is valid only for very high Reynolds number turbulent flows. However,

To ensure the integrity of the physics of the large-scale dynamics of the flows of interest, the corresponding large-scale modes computed or measured in a simulation or an experimental setting should not be contaminated because of their interaction with the dissipation range, which is not universal. This requirement can be satisfied by maintaining a sufficiently broad inertial range (Zhou, 2007; Zhou *et al.*, 2019).

The only question is what is the lowest Reynolds number that will provide a sufficiently wide inertial range to separate the energy-containing and dissipation scale. Studying the energy transfer process and interacting scales, Zhou (2007) determined that the lowest $Re$ satisfing this condition is around $1.6 \times 10^5$ ($Re_\lambda \approx 1000$ in isotropic turbulent flows).

The *"minimum state"* is significant because it implies that one needs not to perform numerical simulations or experimental measurements of extremely high $Re$ flows. All that extra work would simply extend the length of the inertial range, which is well understood via self-similar analysis, and has little impact on the actual physical processes of interest. In other words, once the *"minimum state"* is achieved, the most important physics of the turbulence are already captured.

### D. Energy transfer and interacting scales in physical space

Turbulence energy transfer processes and interacting scales have often been described in Fourier space where their wave numbers become a surrogate for their physical scales. This is especially fortunate for, as indicated previously, the DIA and other related theories have their moment closure in spectral space and are structureless by design.

The alternative strategy, from the physical space perspective, is almost as old as the statistical approach (Davidson, 2004). As noted by Doan *et al.* (2018), since the classical work by Taylor (1938a) and Taylor and Green (1937), it has often been suggested that the mode of energy transfer across the scales is through vortex stretching (e.g., Vela-Martín (2021)), in which vortical structures of a given scale are stretched and intensified by larger vortices, leading to the

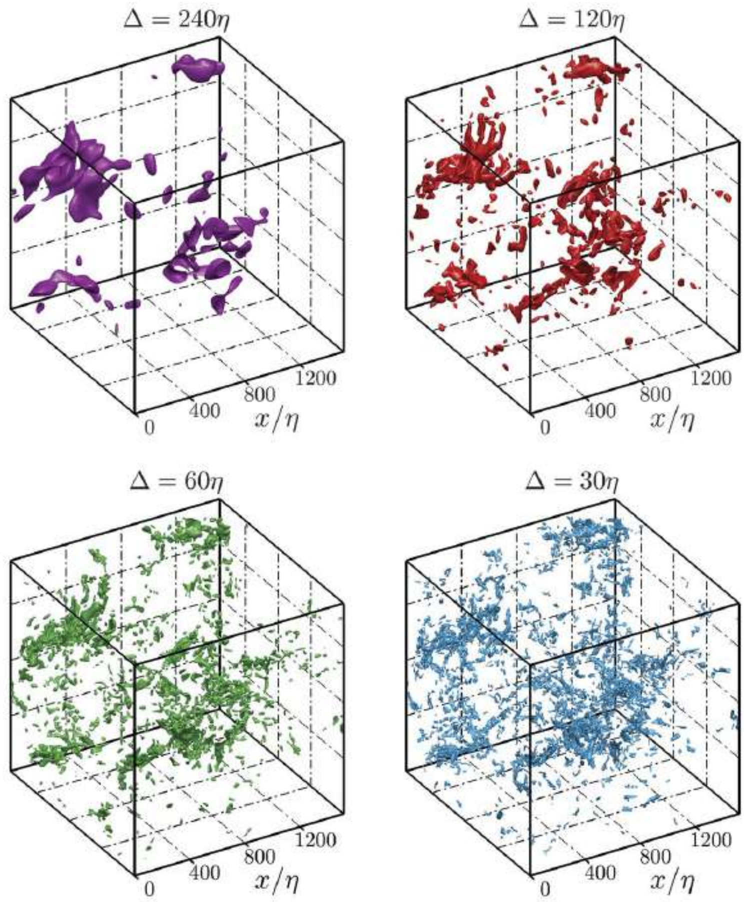


FIG. 7: Energy-eddies at four different scales for the same instant in a numerical simulation of turbulence in a periodic cube. The flow structures observed are the spatially connected regions of the flow where the energy at scale $\Delta$ is above a certain threshold. Fig. 1 of Cardesa *et al.* (2017). Adapted with permission from *Science*, ©American Assoc. for the Adv. Sci.

transfer of energy from the larger to the smaller eddies, with the smallest eddies having a worm-like shape (Burgers, 1948).

The energy cascade in HIT can be caused by tubular vortex pairs stretching smaller-scale vortices (Goto, 2008; Tennekes and Lumley, 1972). More recently, Goto *et al.* (2017) found that the scale twice as large as the stretched vortex is the strongest contributor to the stretching and creation of smaller-scale vortices, but even strain at the scale eight times as large can contribute. With a simulated DNS of $Re_\lambda \approx 750$, the observed scale locality of the vortex stretching is consistent with the locality of energy transfer. Using a decaying HIT with an initial $Re_\lambda \approx 200$, Xiong and Yang (2019) suggested that the stretching and twisting of vortex surfaces/lines are correlated to the energy cascade.

Cardesa *et al.* (2015) showed that the delay required by the energy to cross an octave of scales centred at $r$ is proportional to the local eddy turnover time. Cardesa *et al.* (2017) and Jiménez *et al.* (2019) pointed out that a measure of how related are the location of two flow scales is the volume of the intersection of their intense structures, which has to be compared to the null hypothesis of randomly located point sets with the same overall volume fractions. It is found that structures in energy bands separated by a factor of 2 are more correlated than random, but that those separated by 4 or more are not. Moreover, Figs. 7-8 demonstrated that structures of size $r$ are more strongly correlated with those of size $2r$ at the beginning of their life than at the end. The opposite is true for their correlation with smaller structures of size $r/2$, demonstrating that energy passes from larger to smaller structures at the same physical location.

It is often suggested that high $Re_\lambda$ turbulence presents dynamical features that differ from lower Reynolds number flows (Doan *et al.*, 2018; Ishihara *et al.*, 2013), as seen in Fig. 9. In Fig. 10, Doan *et al.* (2018) investigated the behavior of the normalized energy transfer function for various values of $S/L$, where a large scale $L$ is chosen first for

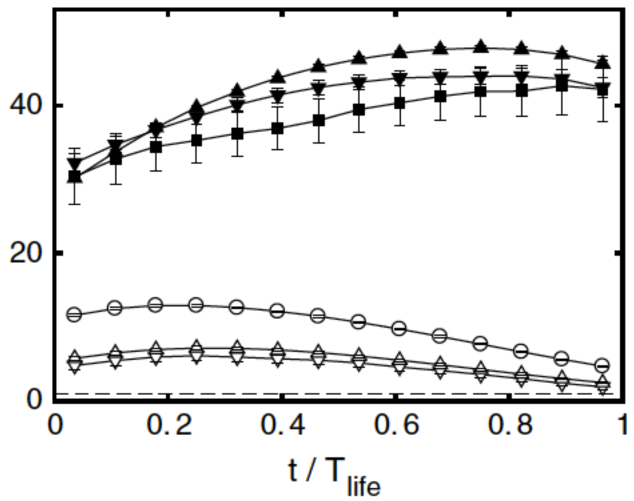


FIG. 8: Intersection ratio of individual structures of size $r_A$ with the set of all structures of size $r_B$, separated from $r_A$ by a factor of two. Normalized to unity for random sets, and plotted as function of the time within the life of each structure. Solid symbols are for $r_B = r_A/2$; open ones are $r_B = 2r_A$. Error bars are two standard deviations. Fig. 2 of Jiménez *et al.* (2019). Adapted with permission from *Turbulent Cascade II*, ©Springer.

each case ($Re_\lambda \approx 97$, 223, and 1131) and the small scale $S$ is varied to determine which small scale receives the most energy from the structures of size $L$. The authors observed that peaks for values of $S/L$ around 0.3 for all values of $L$. This demonstrates that structures at scale $L$ mostly transfer energy to structures having a size of about $0.3L$. This once again supports the energy cascade picture, which dictates that the energy is transferred from one scale to its neighboring smaller scale.[73]

Bassenne *et al.* (2018) also highlighted the effect of the Reynolds number in the turbulent energy cascade, applying the continuous wavelet transform to one-dimensional instantaneous velocity signals obtained from DNS datasets of three-dimensional isotropic turbulence (Cardesa *et al.*, 2017; Ishihara *et al.*, 2016). The algorithm converted one-dimensional physical-space velocity signals into two-dimensional arrays of wavelet coefficients (Grossmann and Morlet, 1984) that represent the local velocity fluctuations at a given scale around a fixed position. Thus, the pitchfork pattern described the distribution of energy across scales around a fixed position and provided visual evidence of the turbulent energy cascade. Fig. 11 showed that while the energy-containing eddies represented in the core of the figures at small-radius values exhibit little sensitivity to the increase in Reynolds number ($Re_\lambda$), the inertial-range scales shown at larger-radius values contain an increasing number of cascading stages (Bassenne *et al.*, 2018).

The so-called "deterministic cartoons" approach[74] seems to produce results that are consistent with those from the classical phenomenology and closure theories. While this methodology is less well developed (Davidson, 2004), it does have potential to incorporate other interesting physics that are fundamentally incapable of being captured by statistical closures.

### E. Space-time covariance and frequency spectra for sweeping and straining motions

The theory of turbulent time correlations can be said to begin with Kraichnan's demonstration that the DIA response equation is infrared divergent when evaluated on an infinite Kolmogorov inertial range (Kraichnan, 1959a). His calculation illustrated the nonlocality of Eulerian time correlations. In later Lagrangian modifications (Kraichnan, 1965a) of DIA, the corresponding integrals converge; this confirmed the intuitive expectation (Kraichnan, 1964c) that Lagrangian time correlations are local in wave number.

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[73] Doan *et al.* (2018) also found that the normalized enstrophy flux exhibits similar behavior as that of its energy flux counterpart (not shown), providing support to the suggestions of Betchov (1956) and Cocke (1969) that the rate of enstrophy generation by vortex stretching could serve as a proxy for the energy transfer process. One view of the connection is that while $\int T(k)\,dk = 0$, $\int k^2 T(k)\,dk$ is not zero and represents production of enstrophy, which is a (single point) measure of energy transfer.

[74] Instead of dissecting the full complexity of turbulence through statistical analysis, idealized simplifications that illuminate signature processes or mechanisms of turbulence could also be extremely useful (e.g, Jumars *et al.* (2009)). The advantages of idealizations are succinctly encapsulated by Davidson (2004, p. 302): "...one might speculate that, in the decades to come, deterministic cartoons will play an increasingly important role, if only because they allow us to tap into our highly developed intuition as to the behaviour of individual vortices. We do not have the same intuitive relationship to the statistical theories, which in any event are plagued by the curse of the closure problem."

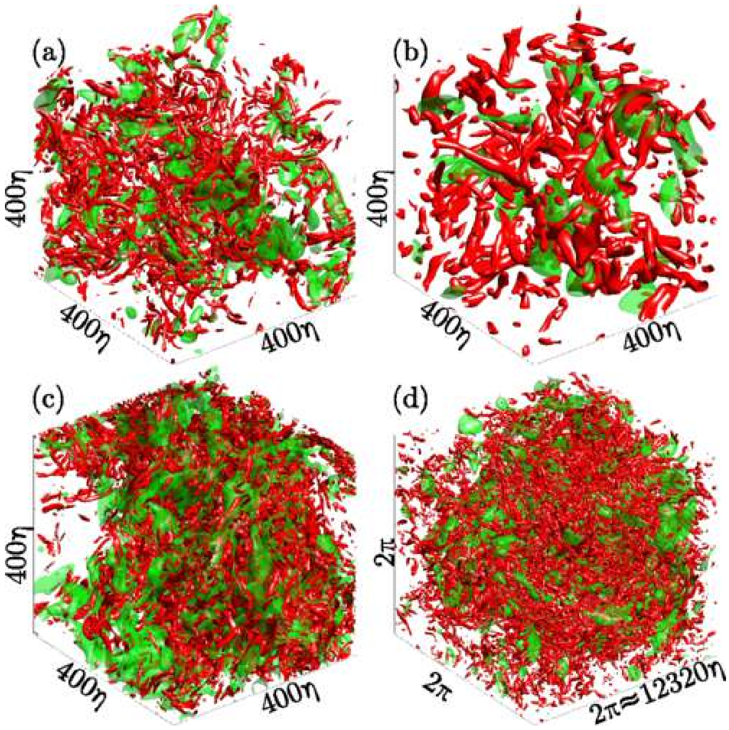


FIG. 9: Isosurfaces of enstrophy (red) and straining (green) structures with a threshold value of the mean plus twice of the rms. Panels (a) and (b) are for $Re_\lambda \approx$140 and panels (c) and (d) are for $Re_\lambda \approx$1131. Note that this figure uses $\eta$ to denote the Kolmogorov length scale, but we have employed $\lambda_K$ instead. Fig. 1 of Doan *et al.* (2018). Adapted with permission from *Phys. Rev. Fluids*, ©American Physical Soc.

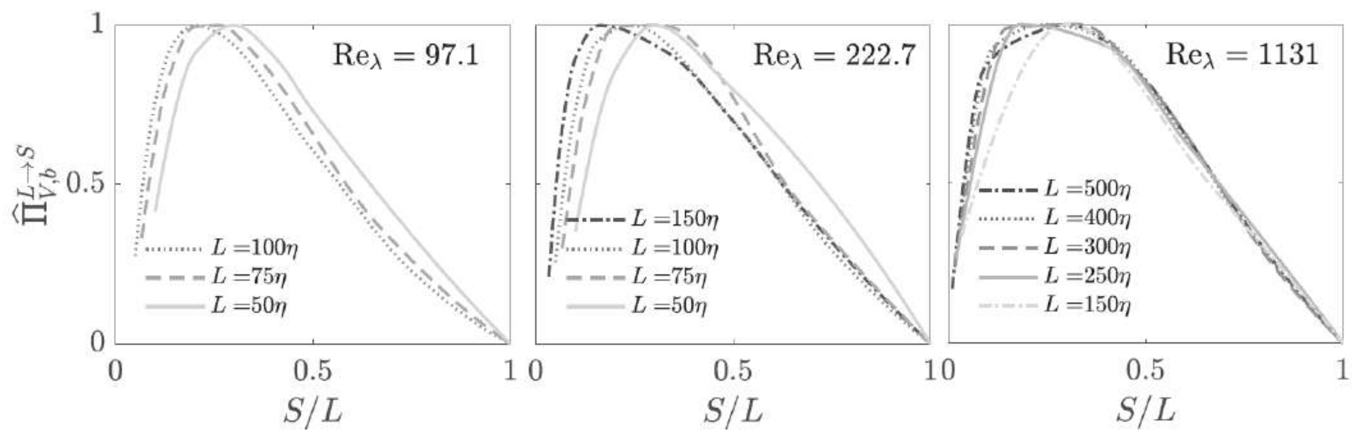


FIG. 10: The normalized energy transfer function, is shown for three different Reynolds numbers: $Re_\lambda \approx$ 97, 223, and 1131. The results of local energy transfer appear to be insensitive for all the Reynolds numbers considered here, suggesting that the cascade picture is robust and does not change with $Re_\lambda$. Fig. 4 of Doan *et al.* (2018). Adapted with permission from *Phys. Rev. Fluids*, ©American Physical Soc.

Given the wavenumber locality of the time correlations, the two-time correlation can be written as

$$Q(k,\tau) = Q(k)r(k,\tau), \tag{153}$$

where $r$ has the similarity form

$$r(k,\tau) = r[\tau\tilde{\theta}(k)]. \tag{154}$$

Whereas there is little doubt that the spatial correlations must follow the Kolmogorov laws $\langle[u(\mathbf{x}+\mathbf{r})-u(\mathbf{x})]^2\rangle \sim r^{2/3}$ or equivalently $E(k) \sim k^{-5/3}$, there are two plausible alternatives for the time correlations (Tennekes, 1975): temporal

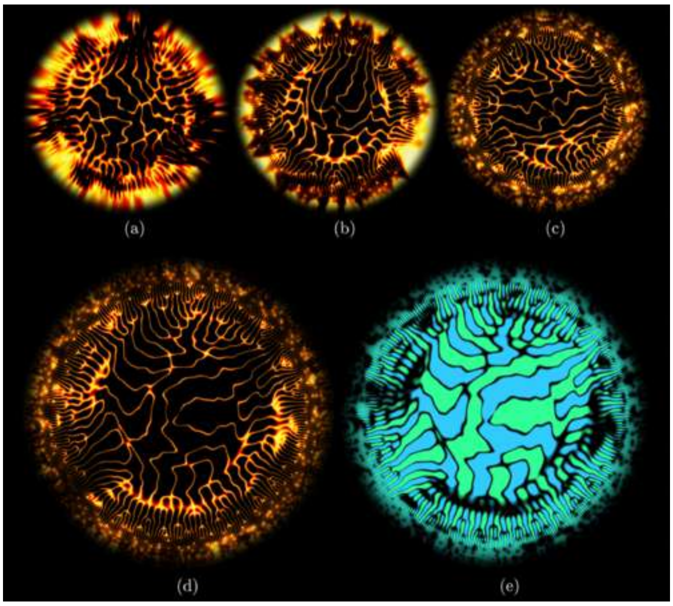


FIG. 11: Polar plots of the wavelet transform of one-dimensional velocity signals extracted from threedimensional isotropic turbulence data at different Taylor-microscale Reynolds number, $Re_\lambda$: (a) 140, (b) 380, (c) 1100, (d) & (e) 2300. Fig. 1 of Bassenne *et al.* (2018), Adapted with permission from *Phys. Rev. Fluids.* Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license.

decorrelation of the Eulerian velocity field can be caused either by sweeping by the largest energetic scales or by local straining.

A definitive discussion of these decorrelation mechanisms and their physical basis is given by Kraichnan (1964c) and in subsection III.B; briefly, under the straining hypothesis, temporal decorrelation is a local property of inertial range scales, but under the sweeping hypothesis, it is a nonlocal property of the energy containing range of scales. The decorrelation time $[\tilde{\theta}(k)]^{-1}$ is given by (Gotoh *et al.*, 1993; Kaneda, 1981; Kraichnan, 1959a; Zhou and Rubinstein, 1996)

$$\tilde{\theta}(k) = \begin{cases} u_0 k, & \text{sweep}, \\ \mathcal{E}^{1/3} k^{2/3}, & \text{strain}. \end{cases} \tag{155}$$

In the context of dynamic critical phenomena, the scaling $\tau k^z$ of time and wavenumber defines the dynamical exponent $z$, with the sweeping hypothesis yielding $z = 1$.

It is well-known that the straining hypothesis is appropriate for Lagrangian time correlations (Chen and Kraichnan, 1989; Kaneda, 1993; Kraichnan, 1964c), and the locality in scale of Lagrangian correlations naturally leads to results consistent with dimensional analysis,

$$\Phi_L(\omega) = \beta_L \mathcal{E} \omega^{-2}, \tag{156}$$

where $\beta_L$ is a constant presumably of order one (Corrsin, 1963; Inoue, 1951; Tennekes and Lumley, 1972). Using kinematic simulation of homogeneous turbulence by unsteady random Fourier modes, Fung *et al.* (1992) found that $\beta_L \approx 0.8$, a value consistent with the atmospheric turbulence measurements by Hanna (1981).

Meanwhile, the Eulerian time correlations are dominated by the sweeping effect of random convection by the largest scales of motion (Tennekes, 1975). The Eulerian frequency spectrum is defined as the kinetic energy per unit frequency:

$$\Phi_E(\omega) = \beta_E \mathcal{E}^{2/3} u_0^{2/3} \omega^{-5/3}, \tag{157}$$

where $u_0$ is the r.m.s turbulent velocity and $\beta_E$ is another unknown constant. For isotropic Gaussian large-scale turbulence at very high Reynolds number, $\beta_E \approx 0.78$, which is close to the computed value of 0.82 (Fung *et al.*, 1992).

It should be emphasized here that the sweeping hypothesis ensures that the frequency spectrum, Eq. 157, reproduces the 1D wavenumber spectrum, hence reproducing here the Kolmogorov 5/3 law. This is also the basis for interpretation of time series from experiments that use fixed position sensors.

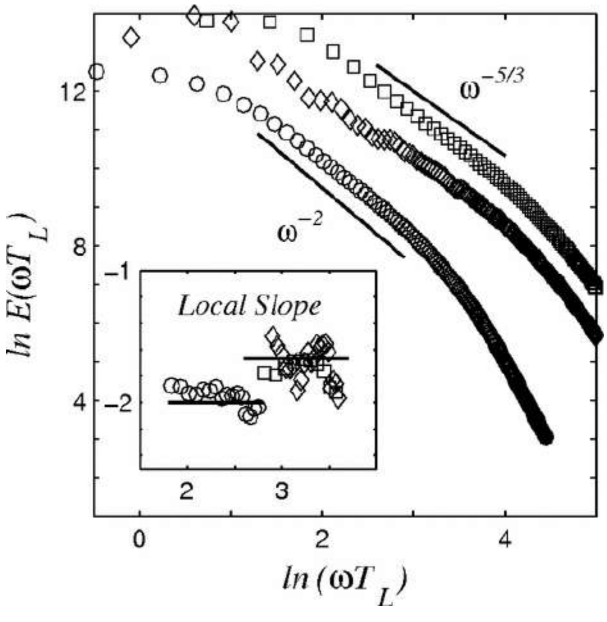


FIG. 12: Power spectral density of one component of the simulated velocity of tracer particles in the dynamic (∘) and static (□) cases, and for the virtual Eulerian probes (⋄). For these three cases, Chevillard *et al.* (2005) first computed the velocity changes of fluid particles that are advected by a frozen 3D Eulerian velocity field; that is, the authors considered a single snapshot of a converged turbulent flow, and used it to advect fluid particles in this frozen Eulerian field— this is called the static case. Then, the authors computed the velocity variations of true Lagrangian particles, a situation in which the Eulerian flow is also evolved in time according to the Navier-Stokes equations—this case is called the dynamic case. Finally, Chevillard recorded the time evolution of the Eulerian velocity at fixed locations of the computation domain, as it would be measured by virtual velocity probes. Fig. 1 of Chevillard *et al.* (2005). Adapted with permission from *Phys. Rev. Lett.*, ©American Phys. Soc.

Chen and Kraichnan (1989) examined the effects of the advection of a passive scalar by a constant and time-varying sweeping and by a multivariate-Gaussian velocity field. Later, Yeung and Sawford (2002) found that the Tennekes relation between wavenumber and frequency spectra resulted in similarly shaped spectra for the scalar.

An alternative version of Kraichnan's calculations was proposed by Kaneda (1993), who recognized that short time Taylor series expansions using the Navier–Stokes equations could produce useful information without explicitly invoking a specific closure scheme. These expansions express two-time correlations in terms of single-time correlations. Later, Kaneda and Gotoh (1991) posed the question of the effect of helicity on Lagrangian and Eulerian time correlations. Rubinstein and Zhou (1999) completed this calculation using Kaneda's method and illustrated a dynamic distinction between Eulerian and Langrangian time correlations. The agreement to second order of the Taylor series expansion of helical and nonhelical Lagrangian time correlation functions suggests that the effect of helicity on Lagrangian time correlations is relatively weak, whereas helicity strongly alters Eulerian correlations.

Malik and Vassilicos (1996) demonstrated the consistency of Tennekes' statistical advection analysis and universal equilibrium arguments with the Eulerian and Lagrangian spectral scaling properties of three different vortex tubes with power-law wavenumber spectra: a Burgers vortex tube, an inviscid Lundgren single spiral vortex sheet, and a vortex tube solution of the Euler equation. O'Gorman and Pullin (2004) considered that the Eulerian two-point, two-time correlations of a turbulent velocity field and those of a passive scalar. The authors showed that stretched-spiral vortex model (Lundgren, 1982) could provide a simple framework, resulting in both of these characteristic time scales in appropriate limits. While a sweeping time scale arises from the movement of the centers of the vortex structures, a straining time scale arises from the radial motion caused by axial stretching of the vortex structures and winding by the vortex cores.

Eulerian time correlations arise in the problems of sound scattering and radiation[75] by turbulence: in the sound radiation problem, random sweeping will cause a random Doppler shift in the radiated sound; consequently, sound

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[75] A unified persective on sound radiation and scattering was given by Kraichnan (1953). The problem of sound radiation by isotropic turbulence has been analyzed from the standpoint of Lighthill's acoustic analogy (Lighthill, 1952) by Proudman (1952): by invoking the assumption of quasi-normality to close the fourth-order velocity moment generated by Lighthill's theory, this analysis leads to a relation

radiation cannot be determined by Lagrangian time correlations. More generally, in the limit of very weak compressibility, the radiated or scattered sound waves do not modify the incompressible turbulence; accordingly, the scattering and radiation problems are determined by the properties of fixed regions of space, and it is the Eulerian rather than the Lagrangian correlations which arise (Rubinstein and Zhou, 1999). The frequency spectrum of the sound radiated by isotropic turbulence is found to scale as $\omega^{-4/3}$ at high frequencies. In contrast, dimensional analysis based on Kolmogorov scaling predicts the dependence $\omega^{-7/2}$ (Rubinstein and Zhou, 2000). The discrepancy is explained by the dependence of Eulerian time correlations on an additional dimensional parameter: the sweeping velocity of the most energetic scales of motion. Rubinstein and Zhou (2002) provided a first step to an analytical theory of the subgrid contribution to radiated sound. It shows how the theory of isotropic turbulence can be applied to derive a theory of subgrid-scale sound radiation.

Wallace (2014) surveyed the conversion between temporal and spatial signals from experimental measurements. Using the random Taylor hypothesis, Brouwers (2004) obtained an analytical expression valid for small times to relate Eulerian spatial and temporal structure functions. He *et al.* (2017) offered a summary on how space-time correlations are used to construct time-accurate turbulence models for the large-eddy simulation of particle-laden turbulence and turbulence-generated noise. As detailed by the authors, the characteristic time scales of space-time correlations are frequently overpredicted by the common class of dissipative subgrid scale (SGS) models because these models are based only on energy balance equations and the time scales are not explicitly considered. Space-time correlations, therefore, can be used not only as a diagnostic tool, but also as an important guidance for constructing time-accurate SGS models. One of the particular issues is the effect of subgrid-scale (SGS) modeling on space-time velocity correlations in decaying isotropic turbulence (He *et al.*, 2004).

### F. Sweeping revisited and elaborated

The validity of the random sweeping hypothesis is one of the important questions of high Reynolds number turbulence theory. *The hypothesis is based on the assumption of complete decorrelation of energy- and inertial-range excitation.* The main consequence of the sweeping hypothesis is that the higher-order spectra $E^m$ of velocity powers $u_i^m$, $m \geq 2$, scale in the inertial subrange as $k^{-5/3}$. The question was further examined by Nelkin and Tabor (1990) who showed that random sweeping dominates if the kinetic energy spectrum scales as $k^{-5/3}$ in the inertial subrange. Based on Van Atta and Wyngaard's (1975) experimental results, Nelkin and Tabor (1990) concluded that the random sweeping hypothesis is valid at high Reynolds numbers.

In addition, Zhou and Vahala (1993) presented the ratio of higher-order to modal energy spectra. The spectra were determined from two velocity time series in high Reynolds number laboratory shear flows: in the return channel $R_\lambda \approx 3.2 \times 10^3$ and in the mixing layer with the $R_\lambda \approx 2.0 \times 10^3$ (Praskovsky *et al.*, 1993). For these flows, the energy-viscous scale separation, i.e. the ratio between the integral and Kolmogorov dissipation scales, $Lk_d$, exceeded 6000 - which was large enough for the existence of a substantial inertial subrange. Zhou and Vahala (1993) showed that the $k^{-5/3}$ scaling for both velocity and kinetic energy spectra,

$$e_{i_i}^{(n)} = \frac{E_n^{u_i}(k)}{n^2 \langle u_i^{2n-2} \rangle E_1^{u_i}(k)}, \tag{158}$$

where $n = 1$ refers to velocity, $n = 2$ to kinetic energy. The spectra have been normalized to remove the influence, if any, of the Taylor hypothesis contamination.

Katul *et al.* (1995) also studied the random sweeping decorrelation hypothesis in thermally stratified turbulent flows over a dry lakebed with a uniform fetch exceeding 10 km in Owens Valley, California. Focusing on two runs that represent near-neutral and unstable atmospheric conditions, the data were collected from highly turbulent flows with $R_\lambda \approx 1.1 \times 10^4$ and $4.9 \times 10^3$, respectively. Once again, the authors found that the higher (up to $4th$) order spectra exhibit a well defined $k^{-5/3}$ power law. More recently, Drivas *et al.* (2017); Favier *et al.* (2010); He and Tong (2011); Poulain *et al.* (2006); Wilczek *et al.* (2015); Yeung (2002); and Yeung and Sawford (2002), among others, also provided support for the sweeping hypothesis.

These results should be contrasted to the earliest attempts, by Dutton and Deaven (1972), to extend Kolmogorov's dimensional analysis (Kolmogorov, 1941a,d) to the higher-order spectra, which resulted in the scaling $k^{-7/3}$ for the

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between the acoustic far-field and the second-order statistical properties of the turbulent velocity. This development has been completed by Lilley (1994), who clarified some unresolved issues pertaining to the ordering of limits in this calculation. The acoustic formulation has received support with the validation of Lighthill's relationship on fourth-order correlations by experimental measurements (Zhou *et al.*, 1995).

kinetic energy spectrum in the inertial subrange. In another development, Yakhot *et al.* (1989) also obtained this scaling from their their $\epsilon$-expansion RG methods.

Praskovsky *et al.* (1993) have attempted to analyze again the random sweeping hypothesis, and some relevant questions, both theoretically and experimentally. It was shown that the higher-order structure function analysis

$$S_{u_i}^{(m)}(r) = \langle [u_i^m(x_r) - u_i^m(x)]^2 \rangle, \quad i = 1,2,3, m \geq 1, \tag{159}$$

has two important advantages over the spectral one.[76] The first advantage is that the derivation of the scaling $S^{(m)} \sim r^{2/3}$ is thoroughly clear and all assumptions are completely transparent. The second is that assumptions, in particular the assumption about the statistical independence of large- and small-scale excitation, can be converted to "measurable" form.

Let us define $u_i(x_1) \equiv u, u_i(x_1 + r) - u_i(x_1) \equiv \Delta u$ and $S_{u_i}^{(m)}(r) \equiv S_m$. To express the sweeping decorrelation hypothesis in terms of higher-order structure function, one must adopt the assumption that *velocity and velocity difference fluctuations are uncorrelated.* Thus,

$$\langle u^k \Delta u^l \rangle = \langle u^k \rangle \langle \Delta u^l \rangle \qquad \text{for all } k \geq 1, \ l \geq 2. \tag{160}$$

Eq. 159 can be rewritten as

$$S_m = m^2 \langle u^{2m-2} \rangle \langle \Delta u^2 \rangle \left[ 1 + (m-1) \frac{\langle u^{2m-3} \rangle \langle \Delta u^3 \rangle}{\langle u^{2m-2} \rangle \langle \Delta u^2 \rangle} + (m-1) \frac{7m-11}{12} \frac{\langle u^{2m-4} \rangle \langle \Delta u^4 \rangle}{\langle u^{2m-2} \rangle \langle \Delta u^2 \rangle} + ... \right] \tag{161}$$

If the terms in the square brackets in Eq. 161 can be omitted, the equation reduces to

$$S_{u_i}^{(m)}(r) = m^2 \langle u_i^{2m-2} \rangle \langle [u_i(x+r) - u_i(x)]^2 \rangle, \quad i = 1,2,3, \quad m \geq 2. \tag{162}$$

More explicitly, Eq. 162 scales as

$$S_{u_i}^{(m)}(r) \sim r^{2/3}, \qquad i = 1,2,3, \quad m \geq 2. \tag{163}$$

Experimental data obtained from the return channel and the mixing layer have shown that higher-order structure functions scale in the inertial subrange as $r^{2/3}$, in agreement with the random hypothesis (Praskovsky *et al.*, 1993). However, strong correlation between large-scale parameters $u_i^n, n = 2,4,6$, and small-scale ones $\Delta u_i^2(r)$ for any distance $r$ within the inertial subrange was established. Thus, the sweeping decorrelation hypothesis cannot be exactly valid, in spite of the scaling prediction agreeing with experiments. It was found that the large-small scale correlation has two relevant consequences. Both of these consequences contradict the sweeping hypothesis but they act in opposite directions and the final result becomes close to the hypothesis's prediction due to the significant cancellation. Using the same methodology but using their own experimental datasets, Katul *et al.* (2016, 1995) reached the same conclusion.

Wilczek and Narita (2012) derived a model spectrum weighted by a Gaussian frequency distribution, which includes the sweeping effects as a Doppler broadening and mean flow effects as a Doppler shift term. Cholemari and Jaywant (2006) advanced a model to relate Eulerian spatial and temporal velocity autocorrelations in homogeneous, isotropic and stationary turbulence. The authors model the decorrelation as the eddies of various scales becoming decorrelated through the "eddy scale."

The concept of the random sweeping has been a very active research area in space and astrophysical plasmas. The major outstanding question is at which scales the Taylor hypothesis can be considered a good approximation (Perri *et al.*, 2017). Recently, this question has been addressed using spacecraft observations and has appeared in the literature, e.g. by Bourouaine and Perez (2018); Carbone *et al.* (2011); Matthaeus *et al.* (2016); Perez *et al.* (2021); Perri *et al.* (2017); Perschke *et al.* (2016); and Servidio *et al.* (2011), to which we refer the reader.

Kraichnan (1965c) noted that the mean magnetic field sweeps the small-scale structures, and during that time nonlinear transfer of energy between length scales occurs since in the Kraichnan picture the "wave packets" suffer brief "collisions" during which energy transfer occurs. This was discussed in some detail and illustrated in Fig. 1(b) of Zhou *et al.* (2004). One can see then that the mean magnetic field induces an inhibition of the nonlinear energy cascade (Chen and Kraichnan, 1989). Also, a mean magnetic field is not removed by a Galilean transform, while a mean velocity can be removed in this way. The mean magnetic field therefore imposes a preferred direction (Lugones *et al.*, 2016).

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[76] Eq. 159 is a distinctly different structure function than the one defined in Eq. 25.

### G. Anisotropy

Indeed, there are many natural questions regarding anisotropy in the absence of external agencies: for example, in steady-state anisotropic turbulence, does the anisotropic part of the correlation decay faster with wavenumber than the energy? Frisch (1995) calls this the "recovery of isotropy" at small scales; it is a cornerstone of modeling because it makes isotropic turbulence an approximately universal state in any turbulent flow.

Herring (1974) and Mons *et al.* (2014b) investigated the approach of axisymmetric turbulence to isotropy using the DIA and EDQNM, respectively. Cambon and his co-workers have focused their attention on the treatment of anisotropy of rotating, stratified and MHD turbulence, and worked directly in Fourier space based on the second-order spectral tensor (e.g., Salhi and Cambon (2007)).[77] For the flows with divergence-free velocity fluctuations, it is possible to treat any anisotropy in reducing the spectral tensor to a set of four scalars, as energy, polarization (in term of a complex-valued scalar) and helicity, and to derive the anisotropic EDQNM and WT models from them. Briard *et al.* (2018) presented their most advanced EDQNM model for homogeneous shear flows. The applications of this method to buoyancy-driven and MHD flows have been detailed in Sagaut and Cambon (2018).[78] To predict in the presence of arbitrary mean-velocity gradients and in a rotating frame, Zhu *et al.* (2019) further advanced statistical treatment for homogeneous turbulence undergoing distortions, which improves and extends the model by Mons *et al.* (2016). Rubinstein and Zhou (2014) additionally presented a preliminary analysis based on a very simple Leith diffusion model suggested by Matthaeus *et al.* (2009).

Recently, Carter and Coletti (2018) investigated the topology and energy transfer properties of homogeneous turbulence over the Reynolds number range $Re_\lambda \approx 300 - 500$ (Carter *et al.*, 2016). High-resolution velocity measurements by particle image velocimetry were performed in a jet-stirred zero-mean-flow facility. The small-scale activity, characterized by the level of enstrophy (or, analogously, dissipation) spatially averaged over each instantaneous realization, is found to be strongly correlated with the large-scale fluctuations; the authors suggested these results as evidence that is contrary to arguments of small-scale universality.

Carter and Coletti (2017) measured two velocity components of their jet-stirred zero-mean-flows (Carter and Coletti, 2018). Yielding statistics up to sixth order, the scaling exponents of the velocity structure functions over the inertial range are found to differ between the longitudinal and transverse components.[79] Esteban *et al.* (2019) experimentally investigated the temporal decay of turbulence without invoking Taylor's hypothesis. Forced turbulence was generated by two facing planes of randomly actuated jet arrays with $Re_\lambda \approx 580$. In their open glass (bottom and walls) and steel-framed tank facility, the authors observed that the large-scale anisotropy that exists at the beginning of the decay progressively decreases and becomes statistically negligible beyond 150 eddy-turn-over time.[80]

The reader is referred to further investigation on the issue of anisotropy in subsection IX.E, using advanced SO(3) group framework.

## VIII. TOPICS RELATED TO KRAICHNAN PASSIVE SCALAR MODEL

### A. Kraichnan passive scalar model

Significant evidence suggested that a good theoretical understanding is needed for anomalous scaling. The conservation laws lead to anomalous scaling of fields advected by turbulence (Falkovich and Sreenivasan, 2006). Since it has not yet been possible to obtain this for the scalar transport equation, Eq. 35, it is helpful to have a nontrivial model (Sreenivasan, 2019b).

Kraichnan's model (Kraichnan, 1968b, 1994) provides one such model, which may shed some light on the phenomenon of anomalous scaling. Kraichnan replaced the velocity $\mathbf{u}$ in the scalar equation by a stochastic Gaussian field which satisfies

$$\langle \mathbf{u}_i(\mathbf{x},t)\mathbf{u}_j(\mathbf{x}',t')\rangle = D_{ij}(\mathbf{x}-\mathbf{x}')\delta(t-t'), \tag{164}$$

[77] See Sections XI and XII for details.

[78] The EDQNM, with its asymptotic WT counterpart, could be viewed as more than the single-time, version of the DIA, as it is the case in the HIT. In the presence of anisotropic and dispersive waves, it is possible to match the EDQNM2-3 to the WT. This is not possible in the DIA and almost all subsequent versions, because the basic response function is only a weak perturbation around an isotropic state, and cannot represent the fully anisotropic wave propagator. Thus, the EDQNM model has applied to the cases of strongly anisotropic quasi-homogeneous flows, either with waves (e.g. rotating, stratified homogeneous turbulence (Sagaut and Cambon, 2018)) or without waves (quasi-static MHD (Favier *et al.*, 2011), unstably stratified homogeneous turbulence; see subsection XII.B).

[79] For superfluid $^4$He turbulence in a three-dimensional counterflow channel geometry, Biferale *et al.* (2019) showed that there is an angular dependence of the components' cross-correlation and the selective suppression of the orthogonal velocity fluctuations has its origin in the strong anisotropy of the energy dissipation by mutual friction. This study suggested the enhanced richness of phenomena made possible by anisotropy.

[80] The eddy-turn-over-time is proportional to the time required for the eddy to be distorted, and, in this distortion process, to generate smaller eddies (Rose and Sulem, 1978). Another important issue stressed by Esteban *et al.* (2019) is that the return to isotropy process has been exacerbated by the experimental facility over the large scales of the flow. The integral scales in the horizontal direction started to be affected by the boundaries of the facility much sooner than their vertical counterpart, driving the fast return to isotropy. This so called saturation effect has also been discussed, among others, qualitatively by Davidson (2004); Gravanis and Akylas (2017); and Thornber (2016) as well as with an EDQNM model (Meldi and Sagaut, 2017).

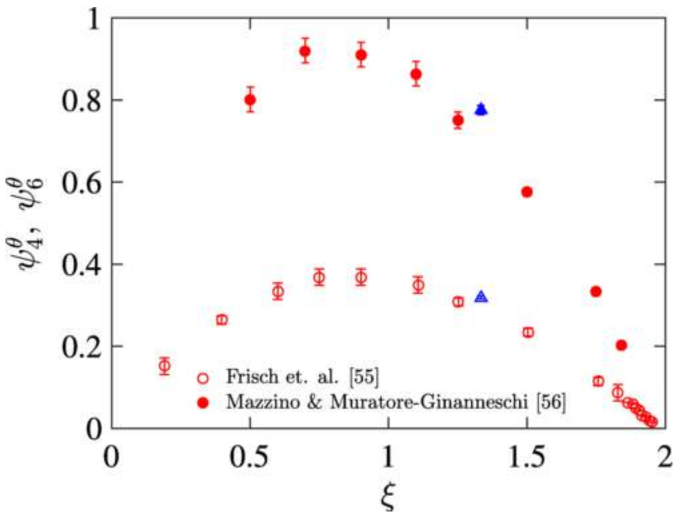


FIG. 13: Flatness anomalyand hyperflatness anomaly) for the scalar versus flow roughness . Circles correspond to 3D Kraichnan model, while triangles are for 3D NS flow with the roughness parameter 4/3. Recall that the roughness parameter of the flow varies between 0 and 2 for the Kraichnan model and is 4/3 for NS turbulence. Fig. 6 of Iyer *et al.* (2018), *Phys. Rev. Lett.*, ©American Phys. Soc.

where a "white in time" correlation is assumed (which decays infinitely rapidly). Here, $D_{ij}$, a diffusivity with the dimension of $L^2T^{-1}$, is given by

$$D_{ij}(\mathbf{r}) = D_0\delta_{ij} - D_1\Big((2+\zeta)\delta_{ij} - \zeta\frac{r_ir_j}{r^2}\Big)r^\zeta, \tag{165}$$

where $\mathbf{r} = \mathbf{x} - \mathbf{x}'$, $r = |\mathbf{r}|$, $i, j = 1, 2, 3$, and $D_0$ and $D_1$ are constants. The spatial correlation has the expected power law with a prescribed scaling exponent $0 < \zeta < 2$, where the cases $\zeta = 2$ and 0 correspond to advection in a smooth flow and very rough flows, respectively. Thus, the power-law scaling of $D_{ij}$ is similar to the Navier-Stokes case, but the temporal scaling is drastically different. For statistical stationarity, a random forcing, $f_\theta$, with the property

$$\langle f_\theta(\mathbf{x}, t)f_\theta(\mathbf{x}', t')\rangle = F(\mathbf{r}/\mathbf{L})\delta(t - t'), \tag{166}$$

is inserted on the right-hand side of the scalar transport equation. Here, the function $F(\mathbf{r}/\mathbf{L})$ is nearly constant for distances $r$ smaller than the integral length scale $L$ and decreases rapidly to zero at small scales

Kraichnan's model maintains the critical elements of the scalar mixing while preserving analytical tractability. In particular, he used this remarkable property and the so-called "linear ansatz" to predict the scaling exponents, $\xi_m$, of the $m$th-order scalar structure function $S_\theta^m$ for all space dimensions $d \geq 2$ and for all velocity scaling exponents $0 \leq \zeta \leq 2$ (Frisch *et al.*, 1998). Indeed, it has been possible to establish anomalous scaling for this model even though the idealized advecting flow itself does not exhibit such an anomaly (Falkovich *et al.*, 2001; Sreenivasan and Schumacher, 2010).

The physical picture that has emerged from the Kraichnan model, as concisely described by Sreenivasan (2019b), is as the following: as particles move in the Kraichnan flow, an $n$-particle cloud grows in size but fluctuations in the cloud shape decrease in magnitude. This happens because the correlation between particles,

$$C_n(\mathbf{r}, t) = \langle\theta(\mathbf{r}_1, t)....\theta(\mathbf{r}_n, t)\rangle, \tag{167}$$

arises because they are contained within the integral scale of the velocity field and weakens with the separation distance. Here, $\mathbf{r}$ denotes the set $\mathbf{r}_1, \ldots, \mathbf{r}_n$. Therefore, one looks for suitable functions of size and shape that have the property of being conserved via the balance between the growth in size and the decrease of shape fluctuations (Celani and Vergassola, 2001; Pumir *et al.*, 2000).[81]

[81] To illustrate this point, it is instructive to consider the case of $n = 3$ (Celani and Vergassola, 2001). The correlation function depends on the size, the orientation, and the shape of the triangle defined by the three points, $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$. Thus, the global size variable can be introduced as $R^2 = (r_{12}^2 + r_{23}^2 + r_{31})^2/3$, where $r_{ij}$ is the distance between the $i$-th and $j$-th particles. Using the scalar field advected by a two-dimensional velocity field generated by an inverse energy cascade process, Celani and Vergassola (2001) demonstrated that $C_3$ scales with $R$ as a power law with the exponent $\zeta_3 = 1.25$. The important result obtained for the Kraichnan model is that the three-point statistics are governed by those trajectories for which the change in the length scale $R$ is compensated by the change in shape of the triangles such that the product $R^{\zeta_3}f(\vartheta_1, \vartheta_2)$ is a constant (Sreenivasan, 2019b). Here, the function $f(\vartheta_1, \vartheta_2)$ is the function of two of the three of the angles of the triangle formed by $r_{12}$, $r_{23}$, and $r_{31}$. All the above arguments are easily generalized to higher-order correlations (Celani and Vergassola, 2001).

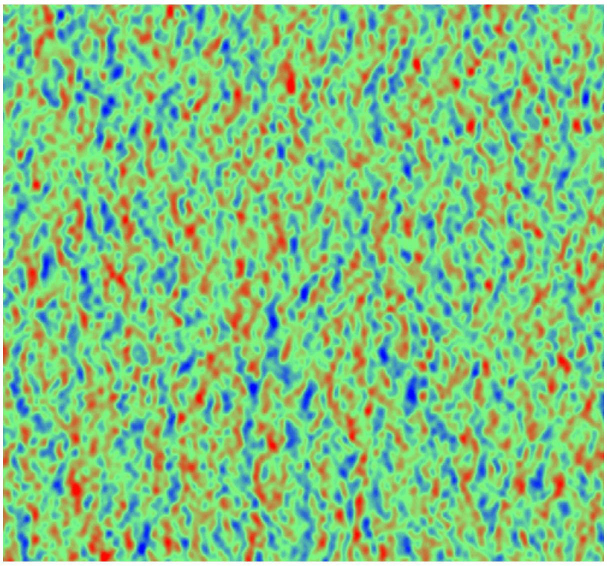

FIG. 14: Image of a scalar field for the Kraichnan model advection by a very rough velocity ($\zeta = 0$) 2D velocity field ($d = 2$). The width of a single scalar patch is $\sim \ell_f$. Forcing is preferentially acting in the vertical direction. Fig. 1 of Celani and Seminara (2006). Adapted with permission from *Phys. Rev. Lett.*, ©American Phys. Soc.

In the so-called zero-mode approach, nontrivial anomalous exponents are related to the zero modes (unforced solutions) of the closed exact differential equations satisfied by the equal-time correlation functions.[82] Their anomalous scaling behavior could be computed numerically (Frisch *et al.*, 1998; Gat *et al.*, 1998) or evaluated perturbatively (Chertkov *et al.*, 1995; Gawedzki and Kupiainen, 1995).

As shown by Bernard *et al.* (1998), zero modes can be interpreted as statistical conservation laws in the dynamics of particle clusters, a concept of statistical conservation laws that has receive support from numerical simulations of a shell model[83] of a passive scalar (Arad *et al.*, 2001) and passive advection in the 2D Navier-Stokes velocity field (Celani and Vergassola, 2001). In contrast to the case of nonzero correlation time, Adzhemyan *et al.* (2005) noted that closed equations are obtained for the equal-time correlations, which are Galilean invariant and therefore unaffected by the sweeping effects.

The reader is referred to Falkovich *et al.* (2001) and references therein for a thorough review pertinent to the major progress relevant to Kraichnan's model. The remainder of this subsection will discuss a few topics previously not covered, such as the application of renormalization group (RG), as well as some notable developments since the publication of that significant survey.

## B. Higher-order anomalous exponents

For the Kraichnan model in high spatial dimensions, Balkovsky and Lebedev (1998) showed the existence of intermittency saturation in the passive scalar field. The authors advanced an "instantonic technique," based on the path-integral representation of the dynamical correlation functions of classical fields (DeDominicis, 1976; Janssen, 1976; Martin *et al.*, 1973) and exploited the saddle-point approximation[84] in the path integral at large $m$ (Falkovich *et al.*, 1996).

Using 2D numerical simulations, Celani *et al.* (2000, 2001a) found that the scalar exponents for the Kraichnan model saturate at different values for different roughness parameters. Celani *et al.* (2001b) and Antonelli *et al.* (2005)

[82] The Lagrangian average of a generic stochastic function $\Upsilon(\mathbf{r})$ of $n$ points $\mathbf{r}_i$ is given by (e.g. Celani and Vergassola (2001))

$$\langle \Upsilon(\mathbf{r}) \rangle_{\mathcal{L}} = \int \Upsilon(\mathbf{r}') \mathcal{P}_n(t, \mathbf{r}'|0, \mathbf{r}) d\mathbf{r}'.$$

The assumptions of the Kraichnan model imply that the $n$-particle propagators $\mathcal{P}_n$ obey closed Fokker-Planck equations (Falkovich *et al.*, 2001; Kraichnan, 1968b). The statistically preserved functions are now identified as zero modes of the $n$-particle Fokker-Planck operator (Chertkov *et al.*, 1995; Gawedzki and Kupiainen, 1995).

[83] The shell models (Biferale, 2003) go back to the pioneering works of the Russian school (Desnyansky and Novikov, 1974).

[84] The saddle-point conditions are integro-differential equations describing an object which has been called an instanton, in analogy to the quantum field theory (Balkovsky and Lebedev, 1998). See also, Balkovsky and Falkovich (1998); Balkovsky *et al.* (1997); Falkovich and Lebedev (1997); and Gurarie and Migdal (1996).

and Mazzitelli and Lanotte (2012) reported that saturation of intermittency with $\xi_\infty = 0.8$ for a 2D Boussinesq model of thermal convection and $\xi_\infty = 0.6$ for a LES of convective boundary layers.

Using a simulated passive scalar field advected by a 3D Navier-Stokes flow at Taylor-scale Reynolds number of 650 on a $4096^3$ grid, Iyer *et al.* (2018) compared and contrasted the anomaly for orders 4 and 6 with that of the 3D Kraichnan model. This comparison is sensible as the small scales evolve with temporal rapidity in a high $Re$ case and the normalized exponents, $\psi^\theta_m = (m/2)\xi^\theta_2 - \xi^\theta_m$, approach the Kraichnan limit of a flow without memory. Figure 13 shows that the anomaly for the fourth and sixth order moments is comparable to that in the Kraichnan model (Frisch *et al.*, 1998; Mazzino and Muratore-Ginanneschi, 2000) for the roughness exponent of 4/3.

### C. Large-scale of a passive scalar and the effects of the shear

Falkovich and Fouxon (2005) analyzed a passive scalar in a random incompressible flow, paying special attention to the scales far exceeding the pumping, or forcing scale $\ell_f$ (quasiequilibrium domain). While it has been shown that the dominant contribution to the statistics of the scalar field at small scale $r \ll \ell_f$ (turbulence domain) is isotropic (Biferale and Procaccia, 2005), *a fortiori* one might anticipate an essentially isotropic concentration field at large-scale $r \gg \ell_f$, since no upscale cascade of scalar fluctuations occurs (Celani and Seminara, 2006). Falkovich and Fouxon (2005) compared and contrasted the multipoint correlation functions of a tracer in an incompressible flow in the quasiequilibrium domain and turbulence regions. The utilization of the Kraichnan model allows the authors to identify the new type of zero modes, which determine an anomalously slow decay of correlations at large scales and break scale invariance.

The findings of Falkovich and Fouxon (2005) — obtained in the context of the Kraichnan model of passive scalar advection where the velocity field is Gaussian, selfsimilar and short-correlated in time — have received support from numerical simulations of Celani and Seminara (2005) and have been further extended to a passive scalar advection by a realistic turbulent flows. Under broad conditions, Celani and Seminara (2006) showed that an arbitrarily small amount of anisotropy propagates to the large scales where it eventually dominates the structure of the concentration field.[85] Figure 14 illustrates that large-scale anisotropy manifests itself in the appearance of "pearl necklaces," which are made of likesign scalar patches of size $\sim \ell_f$, extended for a length $\gg \ell_f$, and aligned along the preferred direction of the forcing.

The Kraichnan model has provided significant assistance in investigating the effects of a large-scale shear on the energy spectrum of a passively advected scalar field. Celani *et al.* (2005) analyzed a new shear regime for a passive scalar transported in shear flows, where the shear with constant intensity $\mathcal{S}$ is superimposed on a turbulent isotropic flow. The authors took advantage of the fact that for Kraichnan's flows the behavior of the second-order correlation function, $C_2$, at the stationary state can be determined exactly and had previously been used to test dimensional arguments. Now, the equation for the second-order correlation function is still closed even in the presence of shear, but the problem is not isotropic anymore. Following Kraichnan (1968b), Celani *et al.* (2005) found that

$$\begin{aligned}\partial_t C_2 + \mathcal{S}(r\sin\varphi\cos\varphi\,\partial_r - \sin^2\varphi\,\partial_\varphi)C_2 = F(r) + (1/r)\,\partial_r(Dr^{(1+\zeta)} + 2\kappa r)\,\partial_r C_2 \\ + (1/r^2)\big[(1+\zeta)Dr^\zeta + 2\kappa\big]\,\partial^2_\varphi C_2,\end{aligned} \tag{168}$$

where $\mathbf{r} = (r, \varphi)$ denotes the relative distance between the two points. The scalar spectrum is shown to scale as $-4/3$ at large scales, where the effects of the shear are felt, but maintains a classical Obukhov–Corrsin scalar spectrum at small scales. The scale of crossover between these two regimes is proportional to the ratio between the turbulent and the shear intensities.

More recently, Attili and Bisetti (2013) performed a direct numerical simulation of a turbulent mixing layer at $Re_\lambda$ of 250 (on a $3072 \times 940 \times 1024$ grid) The measured second-order scaling exponents of passive scalar spectra in both the streamwise and spanwise directions agree with the theoretical analysis by Celani *et al.* Furthermore, the authors found that the scaling exponents of high-order structure functions in the streamwise direction show saturation of intermittency with an asymptotic exponent $\zeta_\infty = 0.4$ at higher-orders. In contrast, intermittency saturation is not observed for the spanwise increments.[86] It would be interesting to determine whether the Kraichnan model could confirm these simulated results.

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[85] The reader may be interested in reading the predictability behavior described in subsection X.D and the related issue that is more popularly known as *butterfly effect.*

[86] The relative scaling exponents of Attili and Bisetti (2013) agree with the results for homogeneous isotropic turbulence with mean scalar gradient (Gotoh *et al.*, 2011), however.

### D. Anomalous scalar dissipation and "Lagrangian spontaneous stochasticity"

Having focused on the large-scale aspects of the passive scalar field, we now pivot to the small scale where anomalous turbulent scalar dissipation occurs. Here, there has been a contestation on the validity of an important result obtained from the Kraichnan model for scalars advected by an actual turbulent flow.

At issue is a phenomenon called spontaneous stochasticity, illustrated by Bernard *et al.* (1998) based on an investigation of the Kraichnan model for a fixed velocity realization and initial particle position (Chaves *et al.*, 2003). At very high Reynolds and Péclet numbers, Lagrangian particle trajectories become non-unique and stochastic for a fixed initial particle position and a fixed velocity realization. When the velocity field is smooth but approximates a 'rough' field over a long range of scales, small stochastic perturbations on Lagrangian trajectories due to molecular diffusivity lead to persistent randomness over any finite time, even as the perturbations vanish (Drivas and Eyink, 2017a). Yet, some researchers, Tsinober (2009) for instance, cast serious doubt on whether "spontaneous stochasticity" occurs in real fluids because the flow fields are smooth.

Any lingering doubt as to whether 'Lagrangian spontaneous stochasticity' is valid for scalars advected by an actual turbulent flow has been dispelled recently. In a neat paper, Drivas and Eyink (2017a) proved that for passive scalars advected by any divergence-free velocity field, including solutions of the incompressible Navier–Stokes equation, and away from walls, that spontaneous stochasticity yields anomalous dissipation for suitable initial scalar fields, so that the two phenomena are completely equivalent. Even for an active scalar, the authors showed that anomalous scalar dissipation requires Lagrangian spontaneous stochasticity.

During this effort, Drivas and Eyink (2017a) advanced an exact relation between the variance of scalar inputs from the initial values, boundary values and internal sources, as those are sampled backward in time by stochastic Lagrangian trajectories and the dissipation rate for either passive or active scalars. This so-called 'Lagrangian fluctuation–dissipation relation' has been further applied to wall-bounded flows (Drivas and Eyink, 2017b) and turbulent Rayleigh–Bénard convection (Eyink and Drivas, 2017).

### E. Perturbative and non-perturbative RG for the Kraichnan and Kazantsev-Kraichnan models

It is clear that the Kraichnan model has revealed many interesting properties and been very productive from an analytical perspective. A significant advantage is that, in contrast to the Navier-Stokes turbulence, the model allows one to construct a controlled expansion (Adzhemyan *et al.*, 1998; Fairhall *et al.*, 1996) for anomalous exponents, which is similar to the $\epsilon$-expansion in the theory of the critical behavior and stochastic dynamics (Vasil'ev, 2004; Zinn-Justin, 1989).

More specifically, Adzhemyan *et al.* (1998, 2005) found that the anomalous scaling emerges as a consequence of the existence in the model of composite fields with negative scaling dimensions ("dangerous composite operators"), which can be identified with the anomalous exponents. This allows the authors to derive the anomalous scaling by a systematic perturbation expansion for the anomalous exponents and to calculate the exponents to the second (Adzhemyan and Antonov, 1998; Adzhemyan *et al.*, 1998, 1999; Antonov and Honkonen, 2001) and third (Adzhemyan *et al.*, 2001a,b; Antonov *et al.*, 2018) (three-loop) orders.

Adzhemyan *et al.* (2005) pointed out that the RG approach complements the zero-mode technique well: the zero-mode technique allows for exact (nonperturbative) solutions for the anomalous exponents related to second-order correlation functions (Chertkov and Falkovich, 1996; Chertkov *et al.*, 1995; Lanotte and Mazzino, 1999; Rogachevskii and Kleeorin, 1997; Vergassola, 1996) while the RG approach forms the basis for systematic perturbative calculations of the higher-order anomalous exponents.

What is more important, as Antonov *et al.* (2018) stressed, is that the RG treatment of the Kraichnan model can be generalized to several more realistic cases: finite correlation time and non-Gaussianity of the advecting velocity field (Adzhemyan *et al.*, 2002; Antonov, 2006; Antonov and Gulitskiy, 2015; Jurčišinová and Jurčišin, 2008), helicity (Hnatič and Zalom, 2016; Jurčišinová *et al.*, 2013, 2014), strong anisotropy (Arponen, 2009; Hnatich *et al.*, 2005; Jurčišinová *et al.*, 2009), and compressibility (Antonov *et al.*, 2017; Menkyna, 2020).

Pagani (2015) applied the NPRG framework to the Kraichnan model, focusing especially on the scaling of the structure functions. It is satisfying that the results of the leading corrections reproduced those obtained via the zero-mode approach in Chertkov and Falkovich (1996) and Gawedzki and Kupiainen (1995) and via the $\epsilon$-expansion by Adzhemyan *et al.* (1998).

The Kazantsev-Kraichnan model (Kazantsev, 1968; Kraichnan, 1968b) was examined independently by Kazantsev for the dynamo instability, instead of the passive scalar field considered by Kraichnan. The model is given by

$$\partial_t b_i = \varkappa \Delta b_i - \partial_j (u_j b_i) + b_j \partial_j u_i + f_i, \tag{169}$$

where $\mathbf{b} \equiv \mathbf{b}(\mathbf{x}, t)$ is the solenoidal magnetic field and $\varkappa$ is the magnetic diffusivity. The forcing function $f$, which maintains the steady state of the system, represents the source of the fluctuations of the magnetic field (Jurčišinová *et al.*, 2017) and takes the Gaussian form with zero mean and a correlation function form given by Eq. 166. Here, the velocity field correlation takes the same structure as that used for the Kraichnan model. Seshasayanan and Alexakis (2016) noted that the problem can be simplified to a one-dimensional eigenvalue problem, the eigenvalue of which gives the growth rate of the magnetic energy. The interested reader may consult, for instance, Vincenzi (2002), for a very neat introduction to the model and important results for the finite $Re$ and $Pr$ cases.

Applying the RG procedure to the Kazantsev-Kraichnan model, Antonov and Gulitskiy (2012); Jurčišinová and Jurčišin (2015); Jurčišinová *et al.* (2017, 2018, 2013) investigated the influence of several factors on anomalous scaling of the magnetic field, including the time correlations, helicity, and compressibility. The interested reader is referred to these articles and references therein.

## IX. QUASI-LAGRANGIAN FORMULATION

### A. Belinicher-L'vov-Procaccia representation

As mentioned already, Kraichnan (1964c) introduced a localization parameter $\alpha$ to remove the sweeping. While the case $\alpha = \infty$ gives the unmodified Navier-Stokes equations, he found that when $\alpha$ is finite, the modified equation yields an asymptotic spectrum of the $-5/3$ Kolmogorov form rather than the erroneous Eulerian DIA scaling, Eq. 81. The LHDIA and LRA further achieved internally consistent theories without invoking the ad hoc $\alpha$ adjustable parameter.

Parallel to these activities, Wyld (1961) showed that the DIA theory is a one-loop line-renormalized diagrammatic theory, derivable from first principles directly from the Navier-Stokes equations. The importance and closely related connections among the Wyld, MSR, and LHDIA have been disucssed by a number of authors (e.g., Gkioulekas (2016)). Although the MSR theory can be used to generate higher-order versions of Kraichnans DIA theory, it is not applicable to the improved LHDIA theory because the Navier-Stokes equations are not local in time when written in the Lagrangian representation. This created a major obstacle to moving forward this line of investigation and resulted in a hiatus that lasted for a couple of decades.

The main breakthrough that made it possible to go beyond the LHDIA theory and onto more exciting developments was the quasi-Lagrangian formulation. The Belinicher-L'vov-Procaccia representation makes it possible to surgically eliminate the sweeping effect directly at the level of the Navier-Stokes equations (Belinicher and L'vov, 1987; Belinicher *et al.*, 1998a,b).

More specifically, the Belinicher-L'vov-Procaccia transformation introduced a field $\mathbf{v}(\mathbf{r}_0, t_0|\mathbf{r}, t)$, which is defined in terms of the Eulerian velocity $\mathbf{u}(\mathbf{r}, t)$,

$$\mathbf{v}(\mathbf{r}_0, t_0|\mathbf{r}, t) \equiv \mathbf{u}[\mathbf{r} + \varrho(\mathbf{r}_0, t), t], \tag{170}$$

where

$$\varrho(\mathbf{r}_0, t) = \int_{t_0}^{t} ds\, \mathbf{u}[\mathbf{r}_0 + \varrho(\mathbf{r}_0, s), s]. \tag{171}$$

Here, the field $\mathbf{v}(\mathbf{r}_0, t_0|\mathbf{r}, t)$ is the Eulerian field in the frame of reference of a single chosen material point, $\varrho(\mathbf{r}_0, t)$, which denotes the Lagrangian trajectory of a fluid particle that is positioned a $\mathbf{r}_0$ at time $t = t_0$.

Belinicher and L'vov (1987) and L'vov and Procaccia (1995b) pointed out that the field of velocity difference, $\mathbf{w}(\mathbf{r}_0, t_0|\mathbf{r}, t)$,

$$\mathbf{w}(\mathbf{r}_0, t_0|\mathbf{r}, t) \equiv \mathbf{v}(\mathbf{r}_0, t_0|\mathbf{r}, t) - \mathbf{v}(\mathbf{r}_0, t_0|\mathbf{r}_0, t), \tag{172}$$

satisfies the Navier-Stokes equation in the limit of incompressible fluid (L'vov and Procaccia, 1998). Crucially, the correlations of $\mathbf{w}$ are directly related to the Eulerian structure functions of $\mathbf{u}$,

$$S_n(|\mathbf{r} - \mathbf{r}_0|) = \langle |\mathbf{w}(\mathbf{r}_0, t_0|\mathbf{r}, t)|^n \rangle, \tag{173}$$

as shown in the Appendix of L'vov and Procaccia (1995b) and with more details in Gkioulekas (2007).

It was demonstrated by Belinicher and L'vov (1987) that line-resummed Wyld's diagrammatic series for the Navier-Stokes equation in the sweeping-free Belinicher-Lvov-Procaccia representation has an order-by-order solution with the enormously successful Kolmogorov (1941a) scaling. In contrast to the Euleraian Navier-Stokes equation, the large scale eddies do not contribute to the quasi-Lagrangian velocity difference.

In a related work, Gkioulekas (2007) determined that using the quasi-Lagrangian formulation of Belinicher, L'vov, and Procaccia to eliminate the sweeping interactions requires a stronger homogeneity assumption which involves many-time correlations instead of one time correlations. This homogeneity assumption is stronger than the homogeneity assumption sufficient for simply ignoring the sweeping terms from the Eulerian formulation of the diagrammstic theory. Consequently, the Belinicher-L'vov-Procaccia representation is an effective approach for removing the sweeping interactions from diagrammatic theories of turbulence, but in itself does not prove that the sweeping interactions are negligible in the Eulerian formulation.

## B. Fusion rules

The Belinicher-L'vov-Procaccia representation serves as the new starting point for computing the anomalous exponents. To tackle the problem, the most informative statistical quantities, the equal-time rank-$n$ tensor correlation functions of velocity differences, are considered

$$\mathcal{F}_n(\mathbf{r}_1, \mathbf{r}'_1; \mathbf{r}_2, \mathbf{r}'_2; \ldots; \mathbf{r}_n, \mathbf{r}'_n) = \langle \mathbf{w}(\mathbf{r}_1, \mathbf{r}'_1)\mathbf{w}(\mathbf{r}_2, \mathbf{r}'_2) \ldots \mathbf{w}(\mathbf{r}_n, \mathbf{r}'_n)\rangle, \tag{174}$$

where all coordinates are distinct and $\langle \cdot \rangle$ represents averaging.

Under the assumptions that the scaling exponents do not depend on the detailed form of forcing, all distances $|\mathbf{r}_i - \mathbf{r}'_i|$ are in the "inertial range," and the correlations are homogeneous functions, Eq. 174 can be rewritten as

$$\mathcal{F}_n(\lambda\mathbf{r}_1, \lambda\mathbf{r}'_1; \lambda\mathbf{r}_2, \lambda\mathbf{r}'_2; \ldots; \lambda\mathbf{r}_n, \lambda\mathbf{r}'_n) = \lambda^{\xi_n}\mathcal{F}_n(\mathbf{r}_1, \mathbf{r}'_1; \mathbf{r}_2, \mathbf{r}'_2; \ldots; \mathbf{r}_n, \mathbf{r}'_n), \tag{175}$$

where $\xi_n$ is the scaling (or homogeneity) exponent.

The aim here is to describe the behavior of the above functions as pairs of coordinates approach one another, or "fuse" (Fairhall *et al.*, 1997). The fusion rules govern the analytical structure of the correlation functions under this coalescence (Eyink, 1993; L'vov and Procaccia, 1996b,d). In mathematical terms, they encapsulate the physical understanding that within the inertial range, turbulence cascades are not affected by the details of random forcing, though they may remember the forcing length scale (Gkioulekas, 2016).

The discussion below follows closely the succinct discussion in Fairhall *et al.* (1997), but a more detailed derivation is given by L'vov and Procaccia (1996a,d), among others. When $p$ pairs of coordinates $\mathbf{r}_1, \mathbf{r}'_1, \ldots, \mathbf{r}_p, \mathbf{r}'_p$, $(p < n)$ of $p$ velocity differences coalesce, with typical separations between coordinates $|\mathbf{r}_i - \mathbf{r}'_i| \sim r$ for $i \leq p$, and all other separations of the order of $R$ in the "inertial" interval of scales, $r \ll R \ll L$, the fusion rules predict

$$\mathcal{F}_n(\mathbf{r}_1, \mathbf{r}'_1; \mathbf{r}_2, \mathbf{r}'_2; \ldots; \mathbf{r}_n, \mathbf{r}'_n) = \tilde{\mathcal{F}}_p(\mathbf{r}_1, \mathbf{r}'_1; \mathbf{r}_2, \mathbf{r}'_2; \ldots; \mathbf{r}_p, \mathbf{r}'_p) \;\; \mathbf{\Psi}_{n,p}(\mathbf{r}_{p+1}, \mathbf{r}'_{p+1}; \ldots; \mathbf{r}_n, \mathbf{r}'_n), \tag{176}$$

where $\tilde{\mathcal{F}}_p$ is a tensor of rank $p$ associated with the first $p$ tensor indices of $\mathcal{F}_n$, and it has a homogeneity exponent $\xi_p$. The $(n-p)$ rank tensor $\mathbf{\Psi}_{n,p}$ is a homogeneous function with a scaling exponent $\xi_n - \xi_p$, and and is associated with the other $n - p$ indices of $\mathcal{F}_n$.

In the special case $p = 1$, the leading order evaluation cancels by symmetry. The next-order result, for a randomly oriented set of pairs with separation $R$, is related to the structure function $S_n(R)$,

$$\mathcal{F}_n(\mathbf{r}_1, \mathbf{r}'_1; \mathbf{r}_2, \mathbf{r}'_2; \ldots; \mathbf{r}_n, \mathbf{r}'_n) \sim \frac{r}{R} S_n(R). \tag{177}$$

## C. Balance equations

Another building block of this infrastructure is the balance equations. The gist of it is, while the details of the derivations are given in L'vov and Procaccia (1996a), the differential equations for the higher-order structure functions can be constructed based on the Belinicher-L'vov-Procaccia representation. Briefly, the procedure involves the following steps:

(i) Rewrite the Navier-Stokes equations in terms of the Belinicher-L'vov-Procaccia velocities $v(\mathbf{r}_0, \mathbf{r}, t)$, Eq. 170;

(ii) Form the transport equation for the velocity difference $w$, Eq. 172 using the equation for Belinicher-L'vov-Procaccia velocities;

(iii) Derive the dynamic equation for the structure function $S_n$, with the Belinicher-L'vov-Procaccia velocities using Eq. 173.

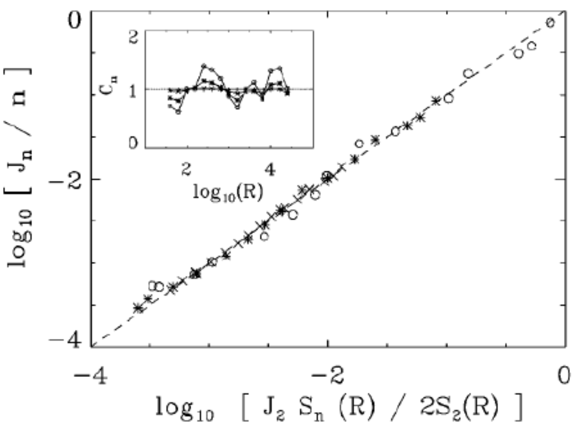


FIG. 15: $\log_{10}[J_n(R)]$ as a function of the fusion rule prediction $\log_{10}[J_2 S_n(R)/S_2(R)]$ for $n = 2, 4, 6$, and 8 denoted by $+$, $\times$, $*$, and $\circ$, respectively. Inset: The coefficient $C_n$ with the same notation. Fig. 5 of from Fairhall *et al.* (1997), *Phys. Rev. Lett.*, ©American Phys. Soc.

L'vov and Procaccia (1996a) derived the balance equations by writing the equations of motion for the structure function $S_n$ and of related quantities. These equations, in the stationary state, exhibit a balance between a convective (interaction) term, $D_n(R)$ and a dissipative term, $J_n(R)$, and $Q_n(R)$, the term that is related to the forcing,

$$D_n(R) = J_n(R) + Q_n(R). \tag{178}$$

Under Kolmogorov scaling, it can be shown that the forcing term can be safely neglected in the inertial range of scales from the balance equation, Eq. 178 (L'vov and Procaccia, 1996a),

$$D_n(R) = J_n(R). \tag{179}$$

The importance of the balance equations is that it allows an opportunity to compare the analytically derived fusion rules against experiments. For comparison with atmospheric turbulence experiments, the dissipative term $J_n(R)$ is expressed in terms of the longitudinal velocity difference, $\hat{\mathbf{r}}$. L'vov and Procaccia (1996c) also noted that the predictions of the fusion rules are sensitive to the crossovers to dissipative behavior, with a prediction for $\hat{\mathbf{r}} \gg \lambda_K$,

$$J_n(R) = nC_n J_2 S_n(R)/2S_2(R), \tag{180}$$

where $C_n$ is $R$-independent dimensionless constants.

Using atmospheric turbulence data obtained using a single hot-wire probe mounted at a height of 35 m on the meteorological tower at the Brookhaven National Laboratory, Fairhall *et al.* (1997) investigated the prediction of Eq. 180 for $R$ in the inertial range and $n = 2, 4, 6$, and 8. The Taylor microscale Reynolds number is 9540 and the Kolmogorov microscale is 0.57 mm. Figure 15 provided a direct confirmation of the fusion rules for the fusion of two points.

Ching *et al.* (1996) also put to the experimental test some nontrivial consequences of the fusion rules for scalar correlations in turbulence by investigating passive turbulent advection as well as convective turbulence. The fusion rules and balance equation can be derived from the scalar transport equation, Eq. 35, as a straightforward extension of those for the velocity fields. The relevant equation, originally obtained for the Kraichnan model in Fairhall *et al.* (1996), reads,

$$J^\theta_{2n}(R) = nC^\theta_{2n} J^\theta_2 S^\theta_{2n}(R)/2S^\theta_2(R), \tag{181}$$

where $C^\theta_{2n}$ is another dimensionless constant and $R$ is in the inertial range. The authors examined the validity of Eq. 181 using temperature data measured in the wake of a heated cylinder (obtained by K.R. Sreenivasan) pertaining to turbulent scalar advection. Once again, Fig. 16 showed that the experimental results supported the theoretical prediction.

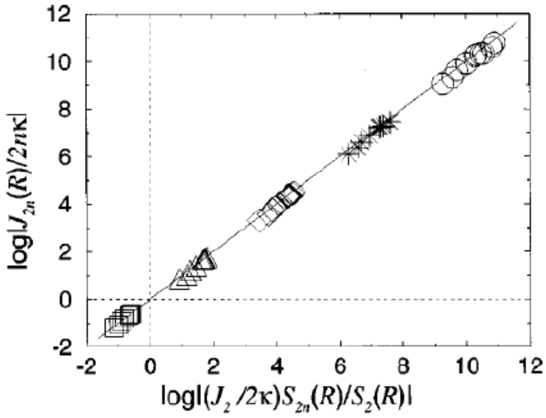


FIG. 16: An analysis of fusion rules for scalar correlations in turbulence, $J_{2n}^{\theta}(R) = nC_{2n}^{\theta}J_2^{\theta}S_{2n}^{\theta}(R)/2S_2^{\theta}(R)$, using experimental data pertaining to turbulent scalar advection. Note that the superscript $\theta$ has been omitted in the labels and $R$ is in the inertial range. The experimental data were obtained from K.R. Sreenivasan. Fig. 1 of Ching *et al.* (1996), *Phys. Rev. E*, ©American Phys. Soc.

### D. Anomalous exponents, locality, and multilocality

The L'vov-Procaccia theory of 3D turbulence, as observed in a recent article by Gkioulekas (2016), is essentially two distinct but connected theories; a perturbative theory (L'vov and Procaccia, 1995a,b,c, 1996a) and a non-perturbative theory (L'vov *et al.*, 1997a; L'vov and Procaccia, 1996b,c,d). The two theories make contact via the fusion rules of generalized structure functions and lead to two separate methods for calculating the structure function scaling exponents: a non-perturbative method (Belinicher *et al.*, 1998a,b; L'vov and Procaccia, 1998) and a perturbative method (L'vov and Procaccia, 2000).

In this series of articles, the Kolmogorov hypothesis of the cascade picture of the energy transfer over scales (locality of the energy transfer) was analytically proven (Belinicher and L'vov, 1987). As a result, the authors demonstrated that the anomalous scaling is consistent with the Navier-Stokes equation, being a non-perturbation effect. A concise summary pertaining to this matter appeared in Gkioulekas (2016): First of all, it is shown that extending Kraichnan's LHDIA theory beyond the 1-loop approximation, into a more general finite order quasi-Lagrangian perturbative theory, represented by the Dyson-Wyld equations governing the second-order velocity difference correlation and the second-order response functions, continues to give Kolmogorov 1941 scaling (L'vov and Procaccia, 1995b). This establishes Kolmogorov 1941 scaling as a baseline initial approximation, and indicates that intermittency corrections cannot be captured by finite-order generalizations of LHDIA. The next step of the argument is to show that without intermittency corrections, universality will be violated (L'vov and Procaccia, 1995c, 1996a).

More recently, Gkioulekas (2016) argued that L'vov-Procaccia theory has much to offer beyond using the fusion rules for analytic calculation of anomalous scaling exponents in turbulence. In particular, it can be used to understand the robustness and stability of the cascades of two-dimensional turbulence (Gkioulekas, 2008) and to predict the behavior of the dissipation scales in two-dimensional turbulence as a consequence of the fusion rules (Gkioulekas, 2010). More generally, it traces out an overarching logical argument in which the perturbative theory is used to show that requiring universal scaling in an extended inertial range implies the fusion rules, which in turn establishes the extent and stability of the inertial range (Gkioulekas, 2016).

"Multilocality" is introduced to represent the generalization of nonperturbative locality to multiple applications of the nonlinear interactions operator (Gkioulekas, 2016). The main result of this work is that the fusion rules hypothesis implies both locality and multilocality in both the IR and UV limits for the downscale energy cascade of 3D turbulence and the downscale enstrophy cascade and inverse energy cascade of 2D turbulence. These claims relate to non-perturbative locality of generalized structure functions on all orders, and not the term by term perturbative locality of diagrammatic theories or closure models that involve only two-point correlation and response functions (Gkioulekas, 2016). Multilocalicality was previously used to establish exact scaling exponent bridge relations (L'vov and Procaccia, 1996c,d), but the argument given at that time for deriving it from the fusion rules, did not consider the scaling of cross-term contributions resulting from multiple applications of the nonlinear interactions operator.

Finally, the fusion rules are placed in juxtaposition to an approach that interprets the turbulent energy cascade in terms of a Markov process of velocity increments in scale (Friedrich *et al.*, 2018). In this perspective, the fusions rules

FIG. 17: Using a technique known as "lucky imaging," researchers have used the Gemini North telescope in Hawaii to capture "ultra-sharp Gemini infrared images" of Jupiter. This image showing the entire disk of the planet in infrared light (4.7 $\mu m$) was compiled from a mosaic of nine separate pointings observed by the international Gemini Observatory, a program of NSF's NOIRLab on 29 May 2019. Photo credit. NASA.

are found to be a direct consequence of the Markov property provided that the structure functions exhibit scaling in the inertial range.

### E. SO(3) symmetry group for anisotropy

L'vov *et al.* (2003) provided an elegant derivation of the scaling exponents in anisotropic hydrodynamic turbulence, where the correlation functions are decomposed in the irreducible representations of the SO(3) symmetry group using the spherical harmonics. The closure analysis here is an extremely difficult one — the authors start with the sophisticated Dyson-Wyld diagrammatic method, but end up invoking the EDQNM approximation.

The "thesis" of the entire SO(3) effort starting from the papers in the 90's is that unless one properly project out the isotropic part of the flow one must assume that objects like structure functions will exhibit some kind of "mixed" scaling due to contamination by anisotropic contributions (Arad *et al.*, 1999, 1998; Biferale and Procaccia, 2005; Clark *et al.*, 2018; Kurien *et al.*, 2001; Kurien and Sreenivasan, 2000, 2001; L'vov *et al.*, 2003; Rubinstein *et al.*, 2015). Those papers introduced for the first time the notion that there is systematic contamination of scaling exponents due to anisotropy and using the SO(3) decomposition is the only way to isolate various contributions. If the Reynolds number is extremely high then the anisotropic contributions have enough "range-of-scales" to fall off yielding to the isotropic contributions in the smallest scales (recovery of local isotropy).

Iyer *et al.* (2017b) studied a random Kolmogorov flow (Biferale and Toschi, 2001) with $Re_\lambda$ up to 450 and found that by going to smaller and smaller scales, isotropy is recovered faster than previously thought. Moreover, the authors decomposed the velocity correlation functions on different sectors of the three-dimensional group of rotations, SO(3), using a high-precision quadrature. Scaling properties of anisotropic components of longitudinal and transverse velocity fluctuations are accurately measured at changing Reynolds numbers. Iyer *et al.* (2019) also remarked that shell-averaged velocity increments (Kurien and Sreenivasan, 2001; L'vov and Procaccia, 1996e) and ball-averaged energy dissipation and enstrophy (Kolmogorov, 1962; Meneveau *et al.*, 1990) all display intermittency and nontrivial anomalous scaling.

Under convincing conditions ($R_\lambda = 1300$), Iyer *et al.* (2020) determined that intermittency increases with increasing order for longitudinal, transverse, and mixed structure functions. However, the transverse exponents[87] differ from

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[87] Iyer *et al.* (2020) also found that the transverse scaling exponents saturate for large moment orders, reminiscent of Burgers turbulence and passive scalars (Celani *et al.*, 2000; Falkovich *et al.*, 2001; Gotoh, 1994; Staicu and van de Water, 2003).

longitudinal ones for orders greater than about 4. This conclusion lends support to a similar result previously obtained from atmospheric turbulence (Dhruva *et al.*, 1997) and numerical simulations measurements (Chen *et al.*, 1997), but which faced recurrent doubts concerning the soundness of it based on symmetry arguments (L'vov *et al.*, 1997b).

## X. GEOPHYSICAL TURBULENCE

### A. Geostrophic and quasi-geostrophic turbulence

Geostrophic turbulence is a key paradigm in the current understanding of large-scale planetary circulations (Galperin *et al.*, 2010; Pedlosky, 1987; Vallis, 2006). It represents the nonlinear advective dynamics of rapidly rotating and strongly stably stratified fluids and it is an important process for large-scale winds and currents on Earth whose Rossby ($Ro$) and Froude ($Fr$) numbers are small (McWilliams *et al.*, 1999).[88] For small $Ro$ and $Fr$, the equations of fluid dynamics asymptotically simplify to the quasi-geostrophic (QG) equations, developed by Charney (1971) utilizing the method of scale analysis on the full governing equations. The QG theory was so named because the theory is valid when fluid motions are close to, but not exactly in geostrophic balance (the theoretical wind that would result from an exact balance between the pressure gradient and the Coriolis forces) (Salmon, 1982). Recently, Cosentino *et al.* (2019) investigated turbulence in Jupiter's atmosphere with archival and ongoing observations from the Hubble Space Telescope and determined that properties of Jupiter's power spectra are consistent with quasi-geostrophic turbulence for shallow fluids.

Herring (1980b) presented a statistical closure study of quasi-geostrophic turbulence, with the goal of discovering how much of the observed dynamics of QG flows can be understood in the context of homogeneous flow. Using a simplified TFM treatment, which is equivalent to the EDQNM approximation, he found that for decaying turbulence at wavenumbers smaller than the energy peak tend toward approximate 2D state, while high wavenumbers tend toward 3D isotropy, as predicted by Charney.

Spatial homogeneity is not geophysically realistic, however, because of boundaries and spatial variations in the Coriolis frequency for the Earth's rotation as well as the buoyancy frequency for the mean density stratification (McWilliams *et al.*, 1999). In quasi-geostrophic three-dimensional turbulence, dual cascades similar to 2D take place. Here, von Hardenberg *et al.* (2000) determined that the properties of same-sign vortex merging determine the flow evolution and they observed the birth of secondary instabilities on the filaments.

Salmon (2018) reminded us that turbulence closure models of the direct-interaction family have nothing to say about the formation of structures in the flow: the closure models predict the evolution of the Fourier amplitudes, whereas the structures clearly depend on the phases of the Fourier coefficients. By examining a method that exposes the closure hypothesis as an assumption about the phases of the Fourier coefficients, he "rederived" the EDQNM model and investigated flow structures that are consistent with the EDQNM model.

The QG equations permit non-trivial linear inviscid solutions, named Rossby waves. On a planetary scale the rotation of the Earth, the Coriolis force, varies with latitude, and this variation gives rise to Rossby waves that interact with the background turbulence. The interactions between turbulence and waves impact both the intensity of the turbulence and the frequency of the Rossby waves (Carnevale and Martin, 1982; Holloway, 1986; Legras, 1980; Rhines, 1975, 1979). Therefore, the frequency shift of Rossby waves provides a measure on the strength of this interactions. Kaneda and Holloway (1994) obtained the LRA estimation of the frequency shift, but it was evaluated numerically using low or moderate Reynolds numbers via numerical simulations of decaying $\beta$-plane turbulence. Subsequently, Ishihara and Kaneda (2001) showed that the the frequency shift of Rossby waves, despite the simplification used in deriving the theoretical estimates, is in fairly good agreement with the simulated values. The authors also suggested that the frequency shifts in the inertial subranges depend in a simple way on the flow conditions at large scales.

Herring (1977) applied the DIA and an extension of the TFM to study 2D rotating turbulent flow above a random topography. Holloway (1978) considered the effect of topography on the overlying motion in the oceans on length and time scales such that the motion is nearly 2D and in geostrophic balance. His TFM may be considered as abridgement of Herring's work. With and without bottom topography and variable rotation rate, Salmon *et al.* (1976) derived the inviscid equilibrium states for one- and two-layer nonlinear QG flows. By applying renormalization methods originally formulated for classical and quantum statistical field theory, Carnevale and Frederiksen (1983b) derived a statistical dynamical closure theory describing the interaction of strongly (and weakly) nonlinear two-dimensional internal waves in the presence of thermal conduction and viscous dissipation. An advantage of this approach is that these methods

[88] The Rossby number, $Ro = u/2\Omega L$, is the ratio of inertial force to Coriolis force, while the Froude number, $Fr = u/\sqrt{gL}$, is a ratio of inertial and gravitational forces, where $g$ is the gravity and $\Omega$ is the rotation rate (Drazin and Reid, 1981; Nieuwstadt *et al.*, 2016).

are applied directly to the strongly nonlinear primitive field equations in Eulerian variables thereby avoiding the small amplitude assumptions inherent in the resonant interaction formalism.

### B. A new look at the Nastrom-Gage spectra

The Nastrom-Gage spectra (Nastrom and Gage, 1985; Nastrom *et al.*, 1984) of horizontal wind and temperature in the atmosphere scales as $k^{-5/3}$ at mesoscales (∼2-500 km) and $k^{-3}$ at synoptic scales (∼500-2000 km), where $k$ is the horizontal wavenumber (see also, Cho *et al.* (1999) and Frehlich and Sharman (2010)). Continued efforts have been devoted toward a determination on whether the energy cascade through the mesoscale range is upscale or downscale.

It has long been theorized that the mesoscale $k^{-5/3}$ spectrum is produced by an upscale energy transfer (Falkovich, 1992; Gage, 1979; Lilly, 1983). This viewpoint received renewed attention upon the publication of an experiment that used a configuration with electromagnetic small-scale forcing. In this work, Xia *et al.* (2011) observed a spectral condensation of turbulence into a box-size vortex and a strong planar vortex suppresses vertical motions, providing a self-organized planar flow that secures an inverse energy transfer.[89] On the other hand, several downscale cascade mechanisms have been identified, including nonlinearly interacting inertia–gravity waves (IGWs) (Dewan, 1979; Van-Zandt, 1982), surface QG turbulence near the tropopause (Tulloch and Smith, 2009), and anisotropic turbulence with strong stable density stratification (Lindborg, 2006).

In what follows, some discussion will be offered on a proposed double cascade in QG turbulence by Tung and Orlando (2003). The mechanism of this process could be understood (Gkioulekas, 2020) in terms of a linear superposition principle, derived from the exact structure of the underlying statistical theory. As such, it should hold for both 2D Navier-Stokes turbulence and for the two-layer QG model (Gkioulekas and Tung, 2005a,b, 2006). More specifically, Tung and Orlando argued that a weak downscale energy cascade is generated from the large-scale forcing, but is shadowed by the downscale cascade of potential enstrophy, which produces a $k^{-3}$ spectrum at synoptic scales, a proposition strongly endorsed by another investigation (Vallgren *et al.*, 2011). There is a minor difference in a basic hypothesis, though — Vallgren *et al.* (2011) assumed that the weak energy cascade is produced in a flow with a finite Rossby number, instead of taking the limit of zero Rossby number (very strong rotation) as in Tung and Orlando (2003). This perspective has received some support from a measurement analysis (Terasaki *et al.*, 2011) and numerical simulations of idealized box (Molemaker and McWilliams, 2010) and more realistic flows (Aluie and Kurien, 2011; Vallgren *et al.*, 2011).

In contrast to 2D Navier-Stokes, a flux inequality, bounding from above the downscale energy flux, does not necessarily persist in two-layer QG models under asymmetric dissipation, and a violation of the flux inequality would correspond to a downscale energy flux strong enough to result in a broken energy spectrum with an observable transition from $k^{-3}$ scaling to $k^{-5/3}$ scaling with increasing wavenumbers $k$ (Gkioulekas, 2012, 2014; Gkioulekas and Tung, 2007). A violation of the flux inequality requires a strong Ekman term to generate sufficient asymmetry between layers, but not so strong that it inhibits the energy injection rate towards small scales (Gkioulekas, 2014). A recent theoretical argument (Gkioulekas, 2020) suggests that the asymmetric placement of the Ekman term at the lower layer allows a regime in which potential enstrophy is not dissipated at large scales, which would prevent a strong Ekman term from interfering with the potential enstrophy cascade. Recent numerical simulations of the two-layer QG model have not been able to reproduce the broken energy spectrum (Iwayama *et al.*, 2019; Sorgentone, 2015), further indicating that it could be a transient effect requiring careful tuning of dissipation at small scales and large scales. However, the broken spectrum has been reproduced more robustly in more complex QG models, beyond the two-layer QG model (Tulloch and Smith, 2009).

In the geophysical regime, it is important for any closures to directly address the conservation laws of the system (namely potential vorticity, potential enstrophy and total (kinetic and potential) energy. Kurien *et al.* (2006) derived a relation analogous to the 4/5 law for potential vorticity correlation in rotating and stratified flows in various limiting cases of the $Ro$ and $Fr$ numbers. Based on the result above Kurien *et al.* (2008) directly showed that the forward cascade of potential enstrophy constrains the spectral distribution of energy.

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[89] For a thin double-layer configuration in the presence of a boundary, a 2D upscale cascade was shown experimentally to form a spectral condensate, or the box-size vortex, which dominated the flow in a steady state (Shats *et al.*, 2007; Xia *et al.*, 2008, 2009). See Musacchio and Boffetta (2019) for a very recent numerical study of the process of formation of large-scale structures in a turbulent flow confined in a thin layer, distribution, showing that the condensate is composed mainly by the 2D mode.

### C. Equilibrium statistical theory and related statistical mechanics approaches

Before attempting to discussion some selected topics of equilibrium statistical theory, it is worth noting at the outset some relevant literature of interest. As noted by Nycander and LaCase (2004), Salmon et al. (1976) adopted a statistical mechanics approach to predict mean flows over topography in the presence of random 2D turbulence while Bretherton and Haidvogel (1976) used a variational approach to examine minimum enstrophy flows over topography. Carnevale and Frederiksen (1987) unified these two methods and further noted the possibility of a more general statistical mechanics based on all the invariants of the continuous system and discussed the difficulty in establishing such a statistical mechanics rigorously. Here, the energy enstrophy theory presented is a rigorous connection between statistical mechanics and nonlinear stability and Fofonoff solutions in the limit of infinite solution (Majda and Wang, 2006). The review article (Les Houches) of Turkington (2009) is also recommended as it is often overlooked because it appears in a book about long-range interacting system (not fluid mechanics).

#### 1. Zonal flows

A general remark is appropriate that applies to this section on geophysical turbulence (section X) and plasma turbulence (section XI), which relates to the extreme importance of zonal flows —azimuthally symmetric bandlike shear flows. Zonal flows are ubiquitous phenomena in the Universe (Busse, 1994; Fujisawa *et al.*, 2004; Marcus *et al.*, 2000; Marcus, 1993); examples include the terrestrial atmospheric jet stream, the rotation profile of the solar tachocline, the super-rotation of the Venusian atmosphere, and Jovian belts and zones. Zonal flows have been expected to be present in magnetically confined toroidal plasmas (Diamond *et al.*, 2005; Hasegawa *et al.*, 1979) as well as inertial confinement fusion implosion (Peterson *et al.*, 2017).

The physical importance of zonal flows is thus indisputable, and their existence has important implications for statistical closures. In both geophysics and plasma physics, a considerable amount of work has been done in the past several decades on statistical closures that incorporate zonal flows. The reader is referred to a recent comprehensive book edited by Galperin and Read (2019), which contains many articles about the physical significance of zonal flows, and it includes articles on appropriate statistical methods by authors from various communities, including plasma physics (Krommes and Parker, 2019).[90]

#### 2. Punctuated Hamiltonian, minimum enstrophy principle, and selective decay hypothesis

Although some indication of serious failures in the closure predictions began to surface with simulations in the 1970's, particularly those by Herring *et al.* (1974) and Fornberg (1977), it was with increasing ease of access to supercomputers in the 1980s and 1990s that new insight into the nature of 2D turbulence crystallized. It became clear that large-scale vortices dominated decaying 2D turbulent flow, making the energy spectrum steeper than $k^{-3}$ in a way that closure theory failed to predict (see Benzi *et al.* (1986); McWilliams (1984, 1990)). This spurred new theoretical research into how these vortices altered the spectral energy range (Benzi *et al.*, 1987, 1988; Santangelo *et al.*, 1989) and the nature of the dynamical interaction between vortices (Melander *et al.*, 1988).[91] The new insight about the importance of coherent vortices and their interactions resulted in a change in emphasis away from the statistical closure theories toward the study of the statistics of the vortices embedded in the flow. The result was the development of a new theoretical model based on Hamiltonian evolution of the interacting vortices punctuated by their irreversible mergers (Benzi *et al.* (1992); Carnevale *et al.* (1991); and Weiss and McWilliams (1993)).[92] This punctuated Hamiltonian model is supplemented by closure theory which is used to explain the statistics of the velocity field in the space between the dominant vortices.

The condensation of vorticity into coherent structures (jets and vortices)[93] occurs regardless of the detailed initial conditions, as Marston (2012) pointed out, suggesting a simple variational interpretation. The minimum enstrophy principle of Bretherton and Haidvogel (1976), or selective decay hypothesis of Matthaeus and Montgomery (1980),

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[90] The reader may also be interested in a special collection of the *Journal of the Atmospheric Sciences* on "Jets and Annular Structures in Geophysical Fluids," which is composed of papers developed from a Chapman Conference (Yoden, conference convener, 2007) and covers diversed interesting subjects, ranging from theoretical and numerical models of widely varying structures and complexity, analyses of the observed variability of Earth's polar vortex, and to laboratory simulations relevant to jets on the gas giant planets.

[91] For more recent studies of the coherent structures in 2D stationary and decaying flows, see Bracco and McWilliams (2010) and Burgess *et al.* (2017), respectively.

[92] From their DNS data, Laval *et al.* (2001) confirmed that a scaling regime can be detected if the scaling laws are expressed in units of mean vorticity and integral scale, as predicted in Carnevale *et al.* (1991). Furthermore, Sire and Chavanis (2000) and Sire *et al.* (2011) considered a kinetic theory of two-dimensional decaying turbulence in the context of two-body and three-body vortex merging processes. In particular, the authors focused their attentions on the role of three-body interactions in the late, dilute, regime of the dynamics involving collisions between vortex dipoles and monopoles.

[93] For an interesting numerical study of 2D flow that exhibits a transition from jets to vortices, see Tobias and Marston (2017).

is such a simple interpretation. In this phenomenological approach, the energy is approximately conserved but the (potential) enstrophy decays for a unforced 2D flow with a small viscosity. In a further generalization, Leith (1984) also incorporated the conservation of angular momentum to take into account isolated vortices.

### 3. Miller-Robert-Sommeria equilibrium statistical method

Much effort has been devoted to understanding the emergence of large-scale coherent structures like jets and vortices, such as the Great Red Spot (GRS) — first reported more than three hundred fifty years ago (Cassini, 1667; Hooke, 1667) – and varieties of other structures (white ovals, the other prominent features, for example) (Bouchet and Sommeria, 2002; Ingersoll, 1990; Porco *et al.*, 2003; Salyk *et al.*, 2006; Wong *et al.*, 2020) in the Jovian atmosphere (see Fig. 17).[94]

Miller (1990); Miller *et al.* (1992); and Robert and Sommeria (1991) (hereafter, MRS) have proposed an exact statistical theory of the late-time steady states of 2D inviscid fluid flows, as the Euler equation reduces to a single scalar equation for the vorticity. Properly accounting for the infinite number of conserved integrals of the motion (Holm *et al.*, 1985; Naso *et al.*, 2010a; Weichman, 2019), the MRS generalized several earlier approximate treatments of Onsager (1949), various formulations of the point vortex problem of Joyce and Montgomery (1973) and Lundgren and Pointin (1977), and the energy–enstrophy theory of Kraichnan (1975) and Salmon *et al.* (1976).[95]

The key idea is to replace the deterministic description of the invisid 2D Euler equation flow vorticity $\omega(\mathbf{r},\omega)$ by a probabilistic description where $\wp(\mathbf{r},\omega)$ gives the density probability of finding the vorticity level $\omega=\sigma$ in $\mathbf{r}$ at time $t$ (Chavanis, 2008). In the physical limit of nonzero but small dissipation, the measured vorticity field of the MRS is given by its coarse-grained mean field[96]

$$\overline{\omega}(\mathbf{r}) = \int \sigma\wp(\mathbf{r},\sigma)\, d\sigma. \tag{182}$$

The mean of the PDF is the expected end-state of the time-evolution. Ergodicity is assumed in statistical mechanics theories such as the MRS.

The equilibrium state is then obtained by maximizing the Boltzmann entropy

$$\mathcal{S}(\wp) = -\int \wp \ln \wp d\mathbf{r}\, d\sigma \tag{183}$$

while conserving $E=(1/2)\int \overline{\omega}\psi d\mathbf{r}$, circulation $\Gamma = \int \overline{\omega} d\mathbf{r}$, and the infinite number of other conserved quantities. Here, the stream-function $\psi$ is related to the vorticity and velocity by $\omega = -\nabla^2\psi$ and $\mathbf{u} = -\hat{\mathbf{z}}\times\nabla\psi$ (note that $\hat{\mathbf{z}}$ denotes the unit vector perpendicular to the 2D plane), respectively. It is assumed that (a) the initial conditions (or equivalently the values of all the Casimirs) are known in detail; and (b) the mixing is efficient and that the evolution is ergodic so that the system will reach, at statistical equilibrium, the most probable state (Chavanis, 2008).[97] In other words, the MRS determines the PDF according to the information-theoretic approach of Jaynes (1957a,b, 2003); and Martyushev and Seleznev (2006) while being constrained by the system's initial conditions. For further details on different variational problems and the relationship between the dynamical and thermodynamical stabilities of 2D flows, the reader is referred to Ellis *et al.* (2002), Bouchet (2008) and Chavanis (2009).

The constraints attached to the 2D Euler equation have been classified into two specifications (e.g., Bouchet and Venaille (2012); Chavanis (2008); and Naso *et al.* (2010a)), (i) the *robust constraints* because they can be expressed in terms of the coarse-grained field $\overline{\omega}$ (neglecting the energy of the fluctuation); and (ii) the *fragile constraints* as they can be represented by the fine-grained moments of the vorticity distribution, or equivalently, the Casimir invariants,

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[94] The Gemini Observatory provided further details on Jupiter, seen with Gemini's Lucky Imaging. From a lucky imaging set of 38 exposures taken at each pointing, the research team selected the sharpest 10%, combining them to image one ninth of Jupiter's disk. Stacks of exposures at the nine pointings were then combined to make one clear, global view of the planet. Even though it only takes a few seconds for Gemini to create each image in a lucky imaging set, completing all 38 exposures in a set can take minutes — long enough for features to rotate noticeably across the disk. In order to compare and combine the images, they are first mapped to their actual latitude and longitude on Jupiter, using the limb, or edge of the disk, as a reference. Once the mosaics are compiled into a full disk, the final images are some of the highest-resolution infrared views of Jupiter ever taken from the ground (see Wong *et al.* (2020) and http://www.gemini.edu/pr/gemini-gets-lucky-and-takes-deep-dive-jupiter-s-clouds).

[95] It is fascinating to note the analogy between MRS and the statistical theory of galaxy dynamics of Lynden-Bell (1967), which was based on the Boltzmann equation and developed for describing the violent relaxation of collisionless stellar systems governed by the Vlasov equation (Chavanis, 2002; Chavanis *et al.*, 1996; Miller *et al.*, 1992). Also, Brands *et al.* (1999); Chavanis and Sommeria (1996, 1998); Naso *et al.* (2010a); and Qi and Marston (2014) discussed the relationship between the MRS and the minimum enstrophy principle. Finally, Chavanis and Sire (2000) and Min *et al.* (1996) studied the statistics of the fluctuations of velocity and acceleration produced by a random distribution of point vortices in two-dimensional turbulence. The authors showed that this case is directly connected to two other interesting problems: the fluctuations of the electric field in a gas composed of simple ions or the fluctuations of the gravitational field produced by a random distribution of stars.

[96] The theory also argued that dissipation acts to smooth out the small-scale motions (Marston, 2012). The language of coarse-grained fields is very similar to the resolvable scales in large-eddy simulations of turbulent flows. Here, the resolved and unresolved (small-scale) parts of the vorticity field are represented by the mean of the PDF and the deviations around the mean, respectively. See related discussion in X.E

[97] In the literature, "most mixed state" has been used by many authors instead of most probable state. This usage should not be confused with the description of the mixing layer in many fluid dynamics applications. See for example, XII.B.

$\int \overline{\omega^m} d\mathbf{r}$, where $\overline{\omega^m} = \int \wp \sigma^m d\sigma$ (Chavanis, 2005). Naso *et al.* (2010a) demonstrated the equivalence between the MRS description that conserves energy, circulation, and fine-grained enstrophy and the phenomenological minimum enstrophy principle, in which there are no other nontrivial conservation laws. Qi and Marston (2014) showed the MRS description that conserves energy, circulation describes a complete condensation of energy to the largest scales, whereas the imposition of the fine-grained quartic Casimir constraint improves agreement with simulations by sharpening the cores and spreading spectral power into smaller-scale modes. The articles of Herbert *et al.* (2012a,b) are good examples of how to apply the MRS to geophysical flows on a rotating sphere. Naso *et al.* (2011) and Venaille and Bouchet (2011) are other examples where the method has been applied to the flows in an oceanic basin with an arbitrary topography (related to Fofonoff flows).

The MRS formalism can be extended straightforwardly to more complex systems, involving multiple interacting fields, but still possessing an infinite number of conserved integrals (Weichman, 2017). For the QG equations, this can be achieved by replacing the vorticity by the potential vorticity (PV) (Bouchet and Sommeria, 2002; Michel and Robert, 1994; Weichman, 2006). The methodology has also been applied to the shallow-water equations (Balk *et al.*, 2011; Chavanis and Sommeria, 2002; Michel and Robert, 1994; Renaud *et al.*, 2016; Weichman, 2017; Weichman and Petrich, 2001).[98] These systems provide more accurate descriptions of geophysical flows than the 2D Euler equations (Pedlosky, 1987).

As articulated by Weichman (2019), 3D axisymmetric flows are interesting because they allow constrained interaction between circulation about the cylinder axis (toroidal flow) and circulation within any given fixed 2D azimuth plane containing that axis (poloidal flow). While the key 2D dynamics occur in the "poloidal plane," the vorticity field in that plane is actually not conserved by the flow. Rather, it is the toroidal field that is passively advected, and is then constrained by an infinite number of conserved integrals. The MRS has been applied to this flow configuration (Leprovost *et al.*, 2006; Naso *et al.*, 2010b,c; Thalabard *et al.*, 2014), with the most recent work by Weichman (2019) further refined and extended the previous treatments. Another flow of significant interest is magnetohydrodynamic (MHD) turbulence.[99] Here, the equilibrium theory of the 2D MHD equations have been derived, accounting for the full infinite hierarchies of conserved integrals (see Jordan and Turkington (1997); Leprovost *et al.* (2005); and Weichman (2012) and references therein).

The MRS can explain not only the annular jet structure of the GRS, but also deduce explicit predictions which can be compared directly against observations. In the QG framework, for instance, Bouchet and Sommeria (2002) showed that the observed eastward and westward jets, much broader than the deformation radius, must be due to a deep zonal flow. This deep shear is reproduced in the atmospheric layer and a systematic westward velocity tends to appear, as a consequence of mixing of the planetary PV. Moreover, the theory went beyond explaining the annular jet structure of the GRS — it resulted in specific predictions on the radius of deformation. As another example, Turkington *et al.* (2001) found that that the structure of the zonal flow in the deep lower layer strongly influences whether or not coherent vortices emerge in the upper layer. When an effective zonal topography is determined in this way, the authors stress, the large-scale zonal jet patterns and monopolar vortices observed in the Jovian weather layer are realized as equilibrium states of the statistical theory.[100]

Chavanis (2005) argued that the MRS statistical prediction presupposes that the (fine-grained) initial conditions are known. Indeed, the equilibrium state depends not only on circulation (the equivalent of mass) and energy as in ordinary statistical mechanics but also on the values of all the vorticity moments.[101]The proposed remedy to sidestep this strong requirement is to treat the constraints associated with the fragile integrals canonically and fix a prior vorticity distribution by small-scale forcing (Ellis *et al.*, 2002; Turkington, 2009). The robust integrals (energy and circulation), of course, are still treated microcanonically.

Several authors have put forward a proposition that the predictions of the MRS may be further improved with the statistical parametrizations of 2D Euler flows in the form of kinetic equations. Here, the kinetic equations are obtained either phenomenologically [e.g, the maximum entropy production principle (MEPP) (Chavanis and Sommeria, 1997; Chavanis *et al.*, 1996; Robert and Rosier, 1997; Robert and Sommeria, 1992)] or systematically [e.g., a quasilinear theory (Chavanis, 2000, 2012)]. In this approach, the evolution of the coarse-grained vorticity is dictated by a competition between a diffusion, and a drift which may be considered as the counterpart of the Chandrasekhar dynamical friction in stellar systems (Chavanis, 2001). These kinetic equations have some additional connections

[98] The QG approximation is related to the more general case of shallow water equations. More specifically, the QG model describes a shallow water system with small Rossby number (a weak vorticity in comparison with the planetary vorticity). As a result, the flow is in geostrophic balance, and the corresponding free-surface deformation is supposed small in comparison with the layer thickness (see, e.g., Bouchet and Sommeria (2002) and Pedlosky (1987)).

[99] For more discussion on aspects of MHD turbulence, see section XI.

[100] A point made Turkington *et al.* (2001) is that the robustness of these observed flows may be theoretically attributed to the stableness of these most probable states with respect to both fine-grained fluctuations and coarse-grained perturbations.

[101] The difficulty on characterinzing the initial conditions is also a very pronounced issue for flows induced by hydrodynamic instabilities, found for instance, during supernovae explosions (see subsection XII.B and Zhou *et al.* (2019)).

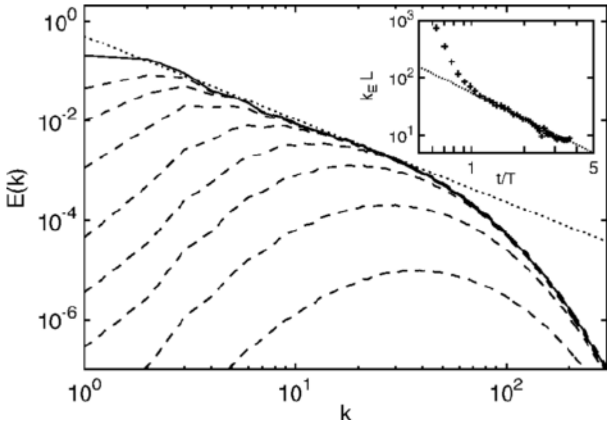


FIG. 18: The spectrum of the error $E_\Delta(k,t)$ at times $t/T$=0.42, 0.56, 0.70, 0.84, 1.1, 1.4, 1.8, 2.1 (dashed lines, from bottom to top) compared with the stationary energy spectrum $E(k)$ (solid line) for simulations at $N$=1024 averaged over 10 independent realizations. The dotted line represents the Kolmogorov scaling. Inset: The error wave number as a function of time (crosses), compared with the dimensional scaling (dotted line). Fig. 3 of Boffetta and Musacchio (2017), *Phys. Rev. Lett.*, ©American Phys. Soc.

to the minimum enstrophy principle (Chavanis, 2014a; Chavanis *et al.*, 2010; Kazantsev *et al.*, 1998) as well as the anticipated vorticity method (Chavanis, 2014b).

The MRS has received strong support from experimental measurements (Monchaux *et al.*, 2008, 2006) and numerical simulations (Jüttner *et al.*, 1995; Thess *et al.*, 1994). Yet, more work should be done to alleviate some issues that have been registered. For example, Marston (2012) cautioned that the treatment of dissipation in the MRS is unsatisfactory, because during the relaxation to equilibrium, viscosity can significantly alter the integrals of motion, especially the high-order moments. Very recently, Modin and Viviani (2020) developed an efficient numerical method especially appropriate for long-time simulations that preserves the geometric features of the exact flow, in particular conservation of Casimirs (eliminating the need for hyperviscosity). Unfortunately, the authors found that the MRS theory incorrectly predicted steadiness and failed to offer insights into the three regimes revealed in the late-time simulated flow fields.

### D. Predictability and growth of uncertainty

Instability is fundamental to turbulent flows (Leith and Kraichnan, 1972) and unpredictability is an essential ingredient for the definition of turbulence.[102] If $\mathbf{u}^1(\mathbf{x})$ and $\mathbf{u}^2(\mathbf{x})$ are two solutions of the Navier-Stokes equation that have identical HIT statistics and whose initial values are imperfectly correlated, the error velocity field can be defined as $\delta\mathbf{u}(\mathbf{x}) = \mathbf{u}^2(\mathbf{x}) - \mathbf{u}^1(\mathbf{x})$ and used to construct statistical measures of the error growth. The spectral covariance, $\Delta(k;t,t')$, is given by

$$\frac{1}{2}P_{ij}(\mathbf{k})\Delta(k;t,t') = (\mathcal{L}/2\pi)^2\langle\delta u_i(\mathbf{k},t)\delta u_j^*(\mathbf{k},t')\rangle \quad (\mathcal{L}\to\infty). \tag{184}$$

Using a closure model for statistically isotropic and homogeneous turbulence, Lorenz (1969) made the original contribution to estimate quantitatively the predictability of fully developed turbulence, motivated mainly by an interest in the problem of predicting atmospheric motions (Robinson, 1967, 1971). Major follow-up studies were performed by Leith (1971), Leith and Kraichnan (1972), Herring *et al.* (1973), and Métais and Lesieur (1992) utilizing statistical closure theories, namely the TFM and EDQNM. Herring *et al.* (1973) found that the DIA results are in reasonably good quantitative agreement with direct numerical simulations of decaying isotropic turbulence at moderate $Re_\lambda \sim 40$. Even when the support of the initial error spectrum is confined to wavenumbers strongly damped by viscosity, there is an error growth that eventually dominant the entire spectrum.

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[102] Frederiksen (2007) surveyed the application of instability theory and chaos theory (Boffetta *et al.*, 2002) for determining their predictability in weather and seasonal climate forecasts.

Boffetta and Musacchio (2001) and Boffetta and Musacchio (2017) performed 2D and 3D DNS of forced incompressible turbulence, respectively. In 2D, Boffetta and Musacchio (2001) focused on the predictability problem in the inverse energy cascade of two-dimensional turbulence, where at stationarity one observes a wide inertial range with a well developed Kolmogorov energy spectrum. The numerical result is in remarkable agreement with the prediction obtained within the test field model closure (Leith and Kraichnan, 1972). Boffetta and Musacchio (2017) computed the 3D spectrum of the error, $E_\Delta(k,t)$. During the initial stage of exponential growth, $E_\Delta(k,t)$ grows exponentially in a self-similar manner. At later times, the error inversely propagates to larger, slower scales, and the error spectrum develops a scaling range $E_\Delta(k,t) \sim k^{-5/3}$. After a transient period, the error wave number $k_E$ also develops a scaling range that is consistent with the dimensional scaling $k_E \sim t^{3/2}$. In this regime, these simulations confirm the predictions based on statistical closure models (Fig. 18). As pointed out by the authors, the two velocity fields can then be assumed to be completely decorrelated at scales smaller than $k_E^{-1}$ and while remaining correlated at larger scales.

Leith and Kraichnan (1972) also determined that the spectrum of the error $E_\Delta(k,t) \propto \Lambda(t)k^4 + o(k^4)$ at small wavenumber $k$, where $\Lambda(t)$ is the only function of time. The $k^4$ behavior of $E_\Delta(k,t)$ has received support from the EDQNM (Métais and Lesieur, 1992), LES (Chollet and Métais, 1989), and more recently, DNS (Yoshimatsu and Ariki, 2019).

Using the TFM, Herring (1984) investigated the homogeneous QG system and compared its predictability results to an equivalent 2D system. Restricted to relatively short integration times, he found that the rate of error growth for the QG system is surprisingly smaller than that for two-dimensional turbulence. Vallis (1983) took a close look at the effects of beta and baroclinicity on the predictability of QG flow.[103] Rotunno and Snyder (2008) advanced a generalized Lorenz model, in order to apply it to the surface QG equations; these equations are very similar to a 2D vorticity equation (with a $-3$ spectrum) but are known to have a $-5/3$ kinetic energy spectrum downscale from a large-scale forcing, producing examples of flows with unlimited and limited predictability, respectively.

Significantly, the results from the statistical closure theories suggest that at least in HIT, predictability behavior is connected to the slope of the flow's kinetic energy spectrum and to error growth as a function of scale. As illustrated by Lorenz (1969) and summarized concisely by Durran and Gingrich (2014), the predictability of certain turbulent systems with $k^{-5/3}$ kinetic energy spectra cannot be extended beyond some finite threshold by reducing the initial-condition errors to any value greater than zero. A key factor limiting the predictability of such systems is the upscale cascade of initial errors, conceivably originating at arbitrarily small scales with arbitrarily rapid eddy turnover times. The possibility that weather forecasting may be limited by perturbations as trivial as those generated by the flying seagull has captured the imagination of the general public as the *butterfly effect*.

Lorenz (1969) and Lilly (1972b) found that flows exhibiting energy spectrum slopes of -3 or steeper would have formally unlimited predictability. As detailed by Tribbia and Baumhefner (2004), a $-3$ range corresponds to a constant eddy turnover time and one might infer that the synoptic- and planetary-scale predictability could be extended indefinitely by sequestering errors to increasingly smaller scales. For a spectrum with a slope steeper than -3, error growth is again scale dependent, but small-scale errors grow more slowly than large-scale errors.[104] Morss *et al.* (2009) also showed that the predictability behavior is linked to the spectral slope of the most energetic model levels, even in the presence of inhomogeneity and anisotropy.

Durran and Gingrich (2014) extended the surface quasigeostrophic dynamics work of Rotunno and Snyder (2008) and stressed an often overlooked finding by Lorenz. Essentially, since the background kinetic energy increases with scale, very small relative errors at long wavelengths have similar impacts on perturbation error growth as large relative errors at short wavelengths. As a result, the authors argued that the influence of initial perturbations generated by butterflies would be swamped by unavoidable tiny relative errors in the large scales, if this model is applied to practical meteorological forecasts.

It remains unclear how predictability theory, developed from simple models for HIT, pertains to the complications of the real atmosphere (Harlim *et al.*, 2005; Ngan and Eperon, 2012).[105] As an example, Ngan *et al.* (2009) found that the predictability loss depends on the Rossby and Froude numbers. Moreover, the loss is significantly slower for subsynoptic flow, $Ro/Fr > 1$, leading to anisotropy in the inverse error cascade and non-self-similar behavior. As a result, the predictability of decaying rotating stratified turbulence differs qualitatively from that of homogeneous isotropic turbulence. Also, Waite and Snyder (2013) investigated the role of moist processes in the development of

[103] The gradient of planetary vorticity is accounted for by a beta effect (Bouchet and Sommeria, 2002). The horizontal variation of buoyancy is referred to as baroclinicity (Smyth and Carpenter, 2019).

[104] As Tribbia and Baumhefner (2004) reminded us, the cascade rates are intimately connected with eddy turnover times in a turbulent fluid, and these are determined by the slope of the energy spectrum.

[105] There has been some developments going from HIT to inhomogeneous turbulence first using the cumulant-discard and quasi-normal hypotheses by Epstein and Pitcher and subsequently the more complete quasi-diagonal DIA for atmospheric barotropic flows where the earlier results are also discussed (O'kane and Frederiksen (2008) and references therein; see also subsection X.E). Note that the first weather forecasts were done with the barotropic equations.

the mesoscale kinetic energy spectrum with numerical simulations of idealized moist baroclinic waves. The authors concluded that the effect of moist forcing of mesoscale kinetic energy presents a challenge for theories of the mesoscale spectrum based on the idealization of a turbulent inertial subrange.

To address the question of whether the atmosphere exhibits an intrinsic limit of predictability (Palmer *et al.*, 2014), Judt (2018) carried out identical twin experiments using a so-called global convection permitting numerical weather prediction model. During the first two days errors grew through moist convection and other mesoscale processes, and the character of the error growth resembled the case of $-5/3$ turbulence. Between 2 and 13 days, errors grew with the background baroclinic instability, and the character of the error growth mirrored the case of $k^{-3}$ turbulence. The existence of an error growth regime with properties similar to $-5/3$ turbulence confirmed Lorenz's idea that the atmosphere has a finite limit of predictability, no matter how small the initial error. More work is needed to further elucidate these complications from the real atmosphere, and how they may act to mitigate or enhance the 'butterfly effect'.

### E. Subgrid and closure models

#### 1. Subgrid modeling with the DIA

Due to the massive disparity between the largest and smallest eddies in the atmosphere and ocean, it is not possible to simulate these flows by explicitly resolving all scales on a computational grid (Kitsios *et al.*, 2016). Two different points of view, a classical ensemble- or time-average point of view and a scale decomposition point of view, have been used to study the parametrizing eddies and the orientation of eddy flux of potential vorticity (PV) in geostrophic turbulence (Nadiga, 2008). Bachman *et al.* (2017) used flow-aware, scale-aware subgrid models based on the Leith (1996) subgrid model to study the quasi-geostrophic dynamics.

Special emphasis is placed on the role played by SGS models in affecting the large-scale flows and energy spectra. It is found that since the very first atmospheric climate simulations (Smagorinsky, 1963) that the accuracy of the large-scale flows is dependent on the appropriate parameterization of the subgrid interactions. In oceanic climate models, adequate modeling of the subgrid processes is perhaps even more important as baroclinic instability occurs at much smaller scales, which are not explicitly resolved in typical climate simulations (Frederiksen *et al.*, 2017). This issue persists today in even the most sophisticated general circulation models (GCMs) and research codes (Koshyk and Boer, 1995; Shutts, 2005, 2013; Tennant *et al.*, 2011; Morrison and Hogg, 2013). A reduction of the resolution dependence will improve the efficiency and accuracy of research and operational activities that require data generated by GCMs (Kitsios *et al.*, 2016).

Frederiksen (1999, 2012) developed the quasi-diagonal DIA (QDIA) closure for inhomogeneous turbulence. The QDIA were derived on the basis that the perturbation fields have, to lowest order, a multivariate Gaussian distribution. This allows the representation of the off-diagonal two-point cumulant and response functions in terms of the diagonal elements. (O'kane and Frederiksen, 2004). The general QDIA closure theory accounts for cross-correlations between field variables (e.g. velocity components or fields at different vertical levels) and between physical space fields, but has the property that the eddy damping and stochastic backscatter terms are diagonal in spectral space (Frederiksen *et al.*, 2013; Kitsios *et al.*, 2016). Frederiksen and Davies (2004) incorporated a regularized form of the QDIA (RQDIA), which acts to localize the eddy–eddy interactions in wavenumber space in order to achieve the consistency with the Kolmogorov hypotheses (Kraichnan, 1964c).

In particular, using the QDIA formalism, but employing stochastic and regression methods, Kitsios and Frederiksen (2019) calculated the 5 subgrid terms for QG inhomogeneous turbulence and were able to reconstruct the flows and spectra at lower resolutions. Looking to the future of closure theory and subgrid modelling this work seem to suggest that future advances in closure theory are much more likely to come from a similar type of analysis.

Nevertheless, the complexity of closure models makes them difficult to formulate and apply to multi-field models such as GCMs. Hence, a direct stochastic modelling approach may be a more tractable methodology (Frederiksen *et al.*, 2013).[106]

Recently, Holloway (2009) presented a review of subgrid modeling of atmospheric and oceanic flows based on entropy approaches. Of particular interest is the utilization of the principle of maximum entropy production for

[106] Two deficiencies of the eddy-viscosity model have been noted: first, it does not provide for the possibility of a back-flow of energy from small-scale turbulence; second, it is not stochastic and therefore does not account for the random forcing of small-scale motions on the large scale motions for the purpose of large-eddy simulations (LES) (e.g. Kraichnan (1976); Leith (1990); Leslie and Quarini (1979); and Rose (1977)). Therefore, Rose (1977) introduced not only the eddy damping but also the eddy noise. Moreover, on the basis of phenomenological modeling, Leith (1990) has found that for a plane shear mixing layer, stochastic backscatter provides the natural seeds from which resolvable turbulence can grow. Zidikheri and Frederiksen (2009) generalized the methodology of Frederiksen and Kepert (2006), which was based on a direct stochastic modeling approach to the subgrid processes observed from the statistics of DNS, to the two-level quasi-geostrophic model.

subgrid parametrizations of the dynamics of geophysical fluid systems (Kazantsev *et al.*, 1998; Verkley *et al.*, 2019). The entropy approaches to subgrid modelling are also discussed in Frederiksen and O'Kane (2008). In particular, the results of Kazantsev *et al.* (1998) and Holloway (1992) are discussed and shown to be inconsistent with the QDIA results.

#### 2. Closures for atmospheric turbulence

The closure theory for atmospheric turbulence based on the budget equations for turbulent kinetic energy, turbulent potential energy and turbulent flux of entropy has been developed during the last 15 years (Kleeorin *et al.*, 2019; Zilitinkevich *et al.*, 2007, 2009, 2013).

## XI. PLASMA TURBULENCE

Plasma physics is the discipline underpinning much of space physics, astrophysics, planetary physics, and fusion research. Such problems are indeed very complex, and we observe that the various other important physics must be incorporated to capture the effects of turbulence. The Sun and stars exhibit a variety of observable phenomena, including turbulent dynamo, sunspots, flares, eruptions, and prominences (e.g., (Mackay *et al.*, 2010)). In the magnetospheres of the Earth, planets and stars, shocks, plasma jets and reconnection are important issues (see, for instance, Collinson *et al.* (2018)). The link between shocks, turbulence, and magnetic reconnection in collisionless plasmas has been addressed, for example, by Karimabadi *et al.* (2014). In the case of tokamaks, as an important example of laboratory magnetized plasmas, various categorizations of the instabilities and turbulence are of value (for example, Goedbloed and Poedts (2004) and Goedbloed *et al.* (2010)).

Orszag and Kraichnan (1967) applied the DIA to Vlasov turbulence (Belmont *et al.*, 2013; Klimontovich, 1967). This comprehensive treatment was largely ignored in the plasma physics community, most likely due to the fact that the theory in this form bears little resemblance to the usual plasma turbulence theories (Dubois and Espedal, 1978). This work emerged from obscurity a decade later, largely attributed (Krommes, 1999) to the work by Montgomery (1977). Because the DIA is quite complicated, Krommes and Kleva (1979) considered a further simplification of Dubois and Espedal (1978), which is basically a renormalized version of the procedure used to extract weak turbulence theory from the DIA. Dubois and Rose (1981), on the other hand, developed a statistical theory of Langmuir turbulence (Rudakov and Tsytovich, 1978; Thornhill and Ter Haar, 1978) in the approximation formally equivalent to the DIA.[107]

For anisotropic turbulence in the presence of linear wave phenomena, Bowman *et al.* (1993) found that EDQNM type closures may be potentially unrealizable, if the EDQNM was derived as a modification of the DIA closure with Markovian response functions and when the two-time covariances were not carefully determined. Thus, Bowman and Krommes (1997) and Bowman *et al.* (1993) developed a realizable Markovian closure (RMC) and a realizable test-field theory (RTFM), respectively.

In the remainder of this subsection we will restrict our attention to statistical closure theories applied to magnetohydrodynamic turbulence and focus on such key threads of the narrative of this review: the sweeping vs. straining time scales, as well as the locality of the energy transfer and interacting scales. The interested reader is referred to the series of survey articles by Krommes (1997, 2002, 2012, 2015); and Krommes and Parker (2019) on other interesting topics regarding statistical descriptions of plasma turbulence.

### A. MHD and sweeping

The magnetohydrodynamic (MHD) approximation has proved to be quite successful in the study of a variety of space and astrophysical flows.[108] In particular, the manifestation of turbulence and other non-linear phenomena in astrophysical plasmas can be explained from an MHD turbulence perspective (Biskamp, 2003; Zhou *et al.*, 2004). The MHD description has also been shown to be an excellent starting point for describing plasma motions when the macroscopic dynamics are well separated from the Coulomb collision/particle gyroscales. The application of MHD turbulence to the solar wind (Goldstein *et al.*, 1995; Matthaeus, 2021) illustrates how the understanding of fundamental plasma physics and of the universe can be improved.

[107] Zhou (2010) documented a number of applications of RG for analyzing the plasma turbulence.

[108] Brown *et al.* (2015) provided an excellent tutorial on the tools of MHD turbulence and illustrated all five temporal statistical tools using a single dataset from the Wind satellite.

The statistical closure theories have played an instrumental role in building our understanding of MHD turbulence, especially because they were advanced before the ascent of modern supercomputers. These MHD equations are

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}=0, \tag{185a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{b}=0, \tag{185b}$$

$$\frac{\partial\boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}=(\nabla\times\boldsymbol{b})\times\boldsymbol{b}+\nu\boldsymbol{\Delta}\boldsymbol{u}-\boldsymbol{\nabla}p, \tag{185c}$$

$$\frac{\partial\boldsymbol{b}}{\partial t}=\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{b})+\varkappa\boldsymbol{\Delta}\boldsymbol{b}, \tag{185d}$$

where $\mathbf{b}$ denotes the magnetic fields and $\varkappa$ is the magnetic diffusivity.[109]

MHD equations are often written in a more symmetric form, in terms of the so-called Elsässer fields (Elsässer, 1950), $\mathbf{z}_\pm=\mathbf{u}\pm\mathbf{b}/\sqrt{4\pi\rho}$,

$$\frac{\partial\mathbf{z}_\pm}{\partial t}\mp\mathbf{v}_A\cdot\nabla\mathbf{z}_\pm=-\mathbf{z}_\mp\nabla\cdot\mathbf{z}_\pm-\frac{1}{\rho}\nabla\tilde{p}-\varkappa\nabla^2\mathbf{z}_\pm, \tag{186}$$

where we have assumed $\nu=\varkappa$ for simplicity and defined the magnetic fluctuation in Alfvén speed units, $\mathbf{v}_A=\mathbf{b}/\sqrt{4\pi\rho}$. The total pressue $\tilde{p}$ acts to enforce the constraints $\nabla\cdot\mathbf{z}_\pm=0$.

Using his Lagrangian framework, Kraichnan (1965c) found that the sweeping time (or the Alfvén wave period), $\tau_A=(k\tilde{b})^{-1}$ must be taken into consideration for MHD turbulence (see also, Iroshnikov (1963).[110]) Here $\tilde{b}$ is the characteristic magnetic field strength in Alfvén speed units. The nonlinear interaction is only between $\mathbf{z}_+$ and $\mathbf{z}_-$ (Chen and Kraichnan, 1989). Thus, small-scale structures suffer brief collisions and there is an inhibition of nonlinear cascade and induction of anisotropy, even when dc magnetic fields are absent. Kraichnan also advanced a rationale surrounding Eq. 143 so that the MHD sweeping can be incorporated. Specifically, Kraichnan argued that the triple correlation in MHD turbulence decay is a time of the order $\tau_s(k)=\tau_A$. A corollary to this development is that $\tau_s$, the time for spectral energy transfer, is increased to a value greater than $\tau_{st}$ (Matthaeus and Zhou, 1989). Dobrowolny *et al.* (1980); Grappin *et al.* (1982, 1983) suggested that the modified spectral transfer time satisfies a "golden rule," $\tau_s(k)\tau_A=\tau_{st}^2$. Here we make use of the viewpoint that the lifetime of triple correlations in MHD turbulence might be more accurately treating by taking into account of the possibility that these correlations decay because of the influence of both wave propagation and convection. Thus the total decay rate of the triples $1/\tau_s$ is simply the sum of the rate due to Alfvén wave propagation and the rate due to nonlinear convection effects (e.g., Pouquet *et al.*, 1976, Kraichnan, 1971b)

$$\frac{1}{\tau_s}=\frac{1}{\tau_A}+\frac{1}{\tau_{st}}. \tag{187}$$

This phenomenological treatment, by substituting Eq. 187 into Eq. 143, leads to a closed-form steady inertial range spectral law that reduces to the Kolmogorov spectrum and Kraichnan's

$$E(k)=C_{Kr}(\tilde{b}\mathcal{E})^{1/2}k^{-3/2}, \tag{188}$$

scaling law in appropriate limits. Note that $C_{Kr}$ is an order one constant of the Iroshnikov-Kraichnan scaling.

Frisch *et al.* (1975) highlighted the importance of three "rugged" invariants of three- dimensional ideal incompressible MHD turbulence theory: the total energy, cross helicity, and magnetic helicity. In particular, an extension of a previous absolute statistical equilibrium analysis of Kraichnan (1967, 1973) suggested that Eqs. 185 may support a dual cascade, in which a direct cascade of energy and cross helicity to high wavenumber might accompany an inverse cascade of magnetic helicity to low wavenumber (Alexakis and Biferale, 2018; Pouquet *et al.*, 2019). Matthaeus and Goldstein (1982) presented the measurements of these "rugged" invariants of the solar wind at 1, 2.8, and 5 AU.

For zero velocity-magnetic field correlation, Pouquet *et al.* (1978, 1976) carried out the EDQNM studies for 2D and 3D helical MHD turbulence, respectively. Of particular significance is that Pouquet *et al.* (1976) recognized that the eddy-damping rate needs to be extended from the fluid to helical MHD turbulence. The new eddy-damping

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[109] In actual astrophysics systems and space physics the nature of the dissipation mechanisms may be quite different from the uniform viscous and resistive dissipation of Eqs. 185. In particular, the dissipation scales are not expected to be universal. Plasma effects, such as damping by waves at ion and/or electron gyroradii or inertial lengths may be important (e.g., Leamon *et al.* (1998); Li *et al.* (2001); Matthaeus *et al.* (2008); and Smith *et al.* (2006)).

[110] Iroshnikov and Kraichnan obtained the $-3/2$ inertial range scaling both with and without a mean magnetic field, respectively. Thus, Kraichnan viewed the magnetic fluctuations of larger scales as a local mean magnetic field for a scale in the inertial range.

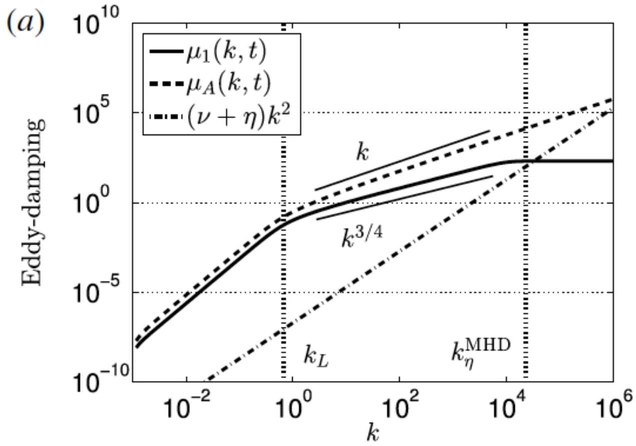


FIG. 19: Different contributions in the MHD EDQNM eddy-damping term. Fig. 1 of (Briard and Gomez, 2018). With permission from *J. Plasma Phys.*, ©Cambridge Univ. Press.

now requires three terms which correspond to the self-distortion (straining) of the flow, the effect of Alfvén waves (sweeping), and the viscous and Joule dissipation. The EDQNM calculations found that the energy spectrum follows the $k^{-3/2}$ energy spectrum, together with the $k^{-2}$ scaling for the inverse cascade of magnetic helicity.

Motivated by solar wind observations Belcher and Davis (1971) and Coleman (1967), Grappin *et al.* (1982, 1983) studied the homogeneous, incompressible MHD turbulence with non-zero velocity-magnetic field correlation (cross helicity). Using the EDQNM closure, the authors found that the power-laws may strongly depart from the $-3/2$ exponent, a direct consequence from the broken symmetry between the two Elsässer fields when the correlation is non-zero (Grappin *et al.*, 1983).

Recently, Briard and Gomez (2018) revisited the EDQNM treatment of the decaying isotropic, homogeneous MHD and considered a number of interesting measurements. In the inertial range, as the Reynolds number increases, the slope of the cross-helical spectrum becomes closer to $k^{-5/3}$ instead of the $k^{-2}$ previously found by Grappin *et al.* (1982) at moderate $Re$. With enough cross-helicity, the Elsässer spectra slowly attain $E_+ \sim k^{-5/3}$ and $E_- \sim k^{-4/3}$. The EDQNM data has also been used to assess several recent proposed 4/3rd laws for isotropic MHD (Politano and Pouquet, 1998a,b; Yousef *et al.*, 2007).[111] extended from that of fluid turbulence in Eq. 29

Fig. 19 shows three contributions to the MHD eddy-damping operator, the inverse of which corresponds to the associated time scales recommended by Pouquet *et al.* (1976). As Matthaeus and Zhou (1989) and Zhou and Matthaeus (1990b) discussed, the inertial range energy spectrum is determined by the smallest time scale, a point that was reiterated more recently by Lee *et al.* (2010). For the entire inertial range, the viscous and Joule dissipation is the smallest. Significantly, however, the term associated with the effect of Alfvén waves is larger than that related to straining motions. Therefore, the sweeping time scale becomes the shortest one of the problem and the $k^{-3/2}$ scaling of Iroshnikov-Kraichnan spectrum is recovered by construction.

### B. MHD, Anisotropy

Anisotropy in solar wind plasma turbulence is an important issue (Galtier, 2018; Oughton *et al.*, 2015). An analysis of 5 years of Wind data at 1 AU revealed that when the relative cross-helicity is smaller than 0.6, the residual energy adopts the scaling $k^{-2}$ (Chen *et al.*, 2013), while the total energy scales as $k^{-5/3}$. In situ observations of the fluctuating solar wind flow also show that the energy of magnetic field fluctuations always exceeds that of the kinetic energy (so-called magnetic excess, see Grappin *et al.* (2016)), and therefore the difference between the kinetic and magnetic energies, known as the residual energy, is always negative.

All major conclusions of 3D MHD simulations with an externally imposed dc magnetic field, $B_0$ (Bigot *et al.*, 2008;

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[111] In related works, Gogoberidze *et al.* (2013) used magnetic field and plasma solar wind data from WIND and Helios 2 spacecrafts while Yoshimatsu (2012) used DNS data on 3D forced incompressible MHD turbulence without a uniformly imposed magnetic field in a periodic box.

Cho and Vishniac, 2000; Oughton *et al.*, 1994), are consistent with the 2D studies of Shebalin *et al.* (1983).[112] With a dc magnetic field, energy transfer to perpendicular modes is enhanced relative to a parallel one (Biskamp, 2003).

For anisotropic 3D MHD turbulence, the quasi-2D motions can be governed by strain and the corresponding time scale is

$$\tau_{st}(k_\perp) = k_\perp^{-3/2} E^{-1/2}(k_\perp), \tag{189}$$

while the quasi-2D modes of MHD turbulence are not strong enough to control the decorrelation of the more wavelike modes. Ng and Bhattacharjee (1997) and Galtier *et al.* (2000) have modified the Kraichnan argument by taking into account the anisotropic feature in the characteristic Alfvén time scales,

$$\tau_A(k_\parallel) = \frac{1}{k_\parallel \tilde{b}}. \tag{190}$$

The sweeping time scale due to propagation effects in the 2D plane is given by

$$\tau_A(k_\perp) = \frac{1}{k_\perp v_A}, \tag{191}$$

where $v_A$ is the Alfvén speed associated with the large-scale quasi-2D magnetic field.

The energy flux in the inertial range for anisotropic 3D MHD turbulence can be obtained (Zhou and Matthaeus, 2005) by extending that of its isotropic counterpart, Eq. 143,

$$\mathcal{E} = C_3^2 \tau_s(k_\perp, k_\parallel) E^2(k_\perp) k_\perp^4, \tag{192}$$

where

$$\frac{1}{\tau_s(k_\perp, k_\parallel)} = \frac{1}{\tau_{st}(k_\perp)} + \frac{1}{\tau_A(k_\perp)} + \frac{1}{\tau_A(k_\parallel)}. \tag{193}$$

Goldreich and Sridhar (1995) adopted the EDQNM closure on the eddy-damping rate and retained two of the effects in Eq. 192, namely $\tau_{st}(k_\perp)$ and $\tau_A(k_\parallel)$. Subsequently, assuming a steady state, Goldreich and Sridhar found the locus of positions in the wavenumber space for which the triple decay rate has equal contribution for nonlocal strain and wave propagating effects, under the assumption of steady perpendicular spectral transfer. In other words, the phenomenology reduces to that of Kolmogorov since there is a balance between the only two available time-scales, $k_\perp^{-5/3}$. This gives rise, in that special case, to the well-known "critical balance" (CB) relation (Goldreich and Sridhar, 1995, 1997; Sridhar and Goldreich, 1994) for a steady anisotropic Kolmogorov cascade,

$$k_\parallel \sim k_\perp^{2/3}. \tag{194}$$

Note that by ignoring $\tau_A(k_\perp)$, Goldreich and Sridhar rule out the Kraichnan spectrum $-3/2$ from the outset. The CB relation has received support from numerical simulations by Beresnyak (2015), who argued that the original CB argument may be sidesteped and made more robust when the anisotropy relation $k_\parallel \sim k_\perp^{2/3}$ is viewed as the correspondence between space domain (Eulerian) and frequency domain (Lagrangian) spectra. In plasma turbulence, Terry (2018) pointed out that these two timescales are only approximately equal because of effects in parallel streaming dynamics and collisions.

The CB relation has received support from numerical simulations and space craft observations. Cho and Lazarian (2004, 2009) provided simulated support to the critical balance theory.[113] Numerically, Mallet *et al.* (2015) found that that in strong Alfvénic turbulence the CB principle is scale invariant, because the probability distribution of the ratio of these times is independent of scale. Analyzing magnetic-field data obtained by the Ulysses spacecraft, Horbury *et al.* (2008) found that the magnetic power spectra in the spacecraft frame that are consistent with the CB assumption. Additional supportive work can be found in Chen *et al.* (2011); Forman *et al.* (2011); Luo and Wu (2010); Podesta (2009); and Wicks *et al.* (2011). As a result, Schekochihin *et al.* (2009) argued that the data has

[112] Oughton *et al.* (1994) found that the anisotropy tends to increase with the strength of $B_0$ the wave number $k$, the mechanical and magnetic $Re$, and decreasing cross correlation. When the turbulence is sufficiently 2D, the sweep time scale due to propagation parallel to $B_0$ becomes comparable to the intrinsic strain interaction times of the fluctuations, and the dynamics of $z_+$ and $z_-$ become similar to those of purely 2D MHD turbulence and are therefore almost independent of $B_0$ (Chen and Kraichnan, 1989; Hossain *et al.*, 1995; Zhou *et al.*, 2004).

[113] While Maron and Goldreich (2001) also stated that the results from their simulations are consistent with the CB prediction, it is important to note that this work has $b/B_0 \sim 1/300$, not the $\sim 1$ that Goldreich and Sridhar (1995) originally used.

approached quantitative corroboration of the CB conjecture and Howes (2015) stated that critical balance is natural for turbulent plasmas.

However, there is still some controversy among researchers, strongly questioning the validity of the critical balance in the solar wind turbulence. Using two 30-day intervals of Ulysses data, Wang *et al.* (2015) presented the scale dependent magnetic-field fluctuation amplitude, which is normalized by the local $B_0$ and evaluated for both parallel and perpendicular sampling directions, The spectral index for the parallel direction is then found to be $-1.75$, whereas the spectral index in the perpendicular direction remains close to $-1.65$. In a very recent paper, Telloni *et al.* (2019) carried out a thorough search in the Wind spacecraft database to extract samples of field-aligned fast solar wind, which allow the precise estimation of the parallel spectral properties of the magnetic fluctuations, and of their intermittency. The authors eliminated the possible role of nonstationarity and large-scale structures by using the Hilbert spectral analysis and concluded that the spectral anisotropy predicted by the critical balance theory is not observed in the selected database.

Critical Balance approaches are perhaps of most interest when there is a well-defined wave timescale to compare with the nonlinear (or strain) timescale. This is the situation in incompressible MHD with a uniform mean magnetic field $B_0$ and its (linearized) Alfvén waves, the original context for critical balance (Goldreich and Sridhar, 1995). A key idea is that wavevector space is partitioned into regions where the strain timescale is faster than the wave timescale, and regions where the converse holds. In Fig. 20 data from a 3D MHD turbulence simulation is employed[114] to plot $\tau_{st}(k)$ and the Alfvén timescales for wavevectors at one chosen angle to $B_0$. There is a wavenumber range where $\tau_{st} \approx k^{-2/3}$, extending a finding from a MHD simulation in the absence of forcing and with no imposed uniform magnetic field (Graham *et al.*, 2011). It is also evident that, for this choise of wavector direction, at larger wavenumbers the wave timescale is faster, but there is a set of smaller magnitude $k$'s (or more oblique wavevectors) where the nonlinear timescale is faster and thus has a strong influence on the physics (Goldreich and Sridhar, 1995; Oughton and Matthaeus, 2020; Terry, 2018).

Gogoberidze *et al.* (2012) and Boldyrev *et al.* (2011) have attempted to incorporate anisotropy into the EDQNM closure. Motivated by the work of Kraichnan and Nagarajan (1967), Veltri *et al.* (1982) considered the feasibility of investigating the cross-helicity effects in anisotropic MHD turbulence with the DIA. Nakayama (1999, 2001) applied the Eulerian DIA and a Lagrangian version of DIA to anisotropic MHD turbulence, respectively. These works were under the simplification of the zero cross-correlation, but this approximation was no longer required in a follow-up work (Nakayama, 2002). The energy spectrum of the $-3/2$ power law emerges in both Lagrangian and Eulerian DIA for anisotropic MHD.

This is an opportunity to succinctly discuss the behavior of MHD turbulence at low magnetic Reynolds number (Knaepen and Moreau, 2008). Léorat *et al.* (1981) applied the EDQNM closure to homogeneous and isotropic helical MHD turbulence. Favier *et al.* (2011) studied the corresponding nonlinear phenomenon Alemany *et al.* (1979) using direct numerical simulation (DNS) of freely decaying turbulence, as well as a two-point statistical spectral closure based on the EDQNM model. The authors found that the transition from the three-dimensional turbulent flow to a 'two-and-a-half-dimensional' flow (Montgomery and Turner, 1982) is a result of the combined effects of short-time linear Joule dissipation and longer-time nonlinear creation of polarization anisotropy. Using the quasi-normal scale elimination (QNSE) method (Sukoriansky and Galperin, 2013; Sukoriansky *et al.*, 2005), Sukoriansky and Zemach (2016) investigated the progressive anisotropization of flow characteristics on increasing spatial scales.

### C. MHD, Locality

For the sake of completeness, we now turn our attention to the locality of interacting scales of MHD turbulence. This is an important subject, but our discussion will be necessarily abbreviated since the closure theories have made rather limited contributions in this area.

The classical MHD turbulence phenomenology of Kraichnan and Iroshnikov assumed scale locality of the nonlinear cascade, as do the more recently proposed models by Goldreich and Sridhar (1995) and Boldyrev *et al.* (2011). On the other hand, Alexakis *et al.* (2005b); Carati *et al.* (2006); Schekochihin *et al.* (2004); and Yousef *et al.* (2007) suggested that the cascades in MHD turbulence are nonlocal processes, allowing a direct exchange of kinetic-magnetic exchange between motions at the largest scales in the system.

While the flux of magnetic helicity may be dominated by nonlocal triads, Aluie and Eyink (2010) demonstrated that fluxes of total energy and cross helicity—or, equivalently, fluxes of Elsässer fields—are dominated by the contributions

[114] Note that $k_\perp \approx k$ because of strong anisotropy.

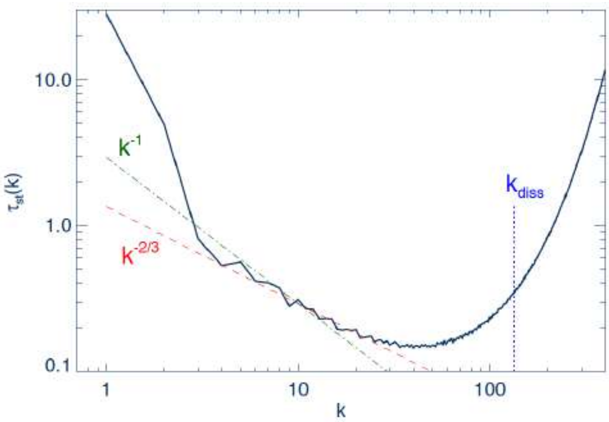


FIG. 20: Wavenumber dependence of the nonlinear timescale $\tau_{NL}(k)$ from a $1024^3$ MHD turbulence simulation with a uniform magnetic field of $B_0 = 1$. Initial conditions were toroidally polarized with $3 \leq |\mathbf{k}| \leq 7$ and low cross helicity. Data is taken from a time just after that of maximum dissipation. Also shown are the $k^{-2/3}$ scaling associated with Kolmogorov theory (red dashed), the Alfvén timescales associated with wavevectors at 70° to $B_0$: $\tau_A = 1/|kB_0 \cos(70°)|$ (green dash dot), and the dissipation wavenumber (blue dotted). Source: Figure courtesy Dr. Sean Oughton, The University of Waikato, New Zealand.

of local interactions. These theoretical predictions are consistent with the numerical results of Domaradzki *et al.* (2010) and Teaca *et al.* (2011), who have carried out MHD simulations at moderate Reynolds numbers. The scalings observed for the normalized versions of the four energy flux functions versus the scale disparity parameter are are

$$W_{uu}(s) \sim W_{ub}(s) \propto s^{-2/3}, \tag{195}$$

and

$$W_{bb}(s) \propto s^{-1/3}. \tag{196}$$

The flux $W_{bb}$ is associated with removing the kinetic energy from the velocity field. It is the least local of these three flux functions.[115] Recall that for fluid turbulence, theory and simulations suggest that $W(s) \sim s^{-4/3}$ (see subsection VII.C), a scaling which is considerably more local than the MHD results.

However, Yoshida and Arimitsu (2007) compared and contrasted the locality of total energy flux between the fluid and MHD turbulent flows using the LRA. Fig. 21 shows that the value of $W(s)$ is slightly larger than that in the fluid turbulence near the peak value of 2. This implies that the triad interactions that contribute to the total energy flux in the MHD turbulence is somewhat more local in spectral space than those in fluid turbulence. Recall that the Navier-Stokes LRA was shown to agree with the DNS on the $s^{-4/3}$ scaling of the interacting scales. It is desirable to clear up why the LRA should produce different results from those summarized in the previous paragraph. A reasonable starting point, perhaps, is to investigate individual contributions $W_{uu}, W_{ub}$ and $W_{bb}$, from the LRA framework.

Finally, Bian and Aluie (2019) have recently showed that scale-locality in MHD results in conservative cascades for magnetic energy and kinetic energy, separately, deep in the inertial range. The kinetic and magnetic energy decouple, which can be viewed as a manifestation of simplicity arising from the ergodicity of turbulence.

### D. MHD modeling

The presence of magnetic fields in a turbulent, electrically conducting fluid flow can dramatically alter the coupling between large and small scales, with potentially profound implications for LES/SGS modeling (Miesch *et al.*, 2015). Historically, subgrid-scale models for MHD turbulence in physical space have been constructed by extending the usual nonmagnetic models to the case of electrically conducting fluids. MHD gradient-diffusion-type subgrid-scale models

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[115] Based on these simulated energy flux functions, Zhou and Oughton (2011) estimated the minimum state kinetic and magnetic Reynolds numbers.

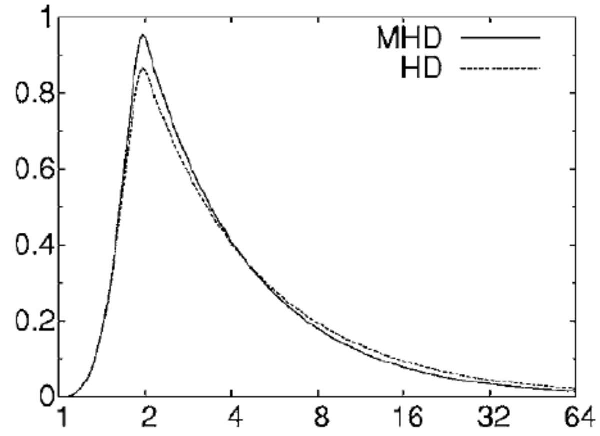



FIG. 21: The LRA locality functions for MHD and hydrodynamic (HD, or fluid) turbulence vs the scale disparity parameter, $s(k,p,q) = \max(k,p,q)/\min(k,p,q)$. Fig. 2 of Yoshida and Arimitsu, (2007). With permission from *Phys. Fluids*, ©American Inst. Phys.

were assessed for isotropic turbulence at magnetic Prandtl number[116] unity (Agullo *et al.*, 2001; Müller and Carati, 2002) and at small magnetic Prandtl number (Knaepen and Moin, 2004).

Yoshizawa (1987) formulated an MHD subgrid model with the aid of his previous statistical result (Yoshizawa, 1985b) from a multi-scale DIA formulation. This result leads to a subgrid model of Smagorinsky type. More recently, Aluie (2017) derived effective equations for the observable velocity and magnetic fields spatially-averaged at an arbitrary scale of resolution using a coarse-graining methodology. These macroscopic effective equations contain both a 'subscale stress' and a 'subscale EMF' generated by the nonlinear interaction of eliminated plasma motions. It is fair to state that this approach shares a similar philosophy to that of the r-RG method (Rose, 1977; Zhou *et al.*, 1988, 1989). There are several interesting RG applications to MHD turbulence, see for example, Fournier *et al.* (1982); Garnier *et al.* (1981); and Sulem *et al.* (1979) and other publications documented in Zhou (2010).

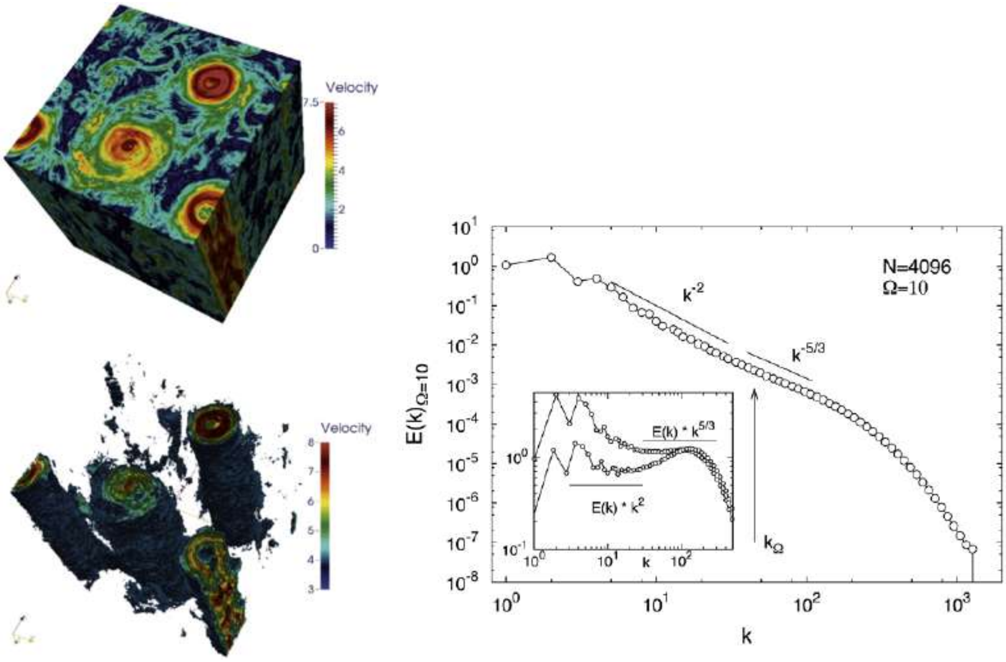
Gomez *et al.* (2007) applied a subgrid model (Zhou *et al.*, 2002) for MHD turbulence from the EDQNM closure in the absence of a mean magnetic field (Schilling and Zhou, 2002b). Baerenzung *et al.* (2008b) extended their large-eddy simulation model, previously derived for fluid turbulence (Baerenzung *et al.*, 2008a), to the MHD equations in its nonhelical version. After a careful examination of the cumulant equations, the author introduced several new characteristic times, distinguishing between magnetic and kinetic energy transfers, as well as between the different quantities entering these transfers. This led to better agreement with direct numerical simulations, but also injected additional degrees of freedom.

### E. Remarks

Kraichnan (1988a) warned that there is a basic dilemma concerning the choice of approaches for researchers. If flow structures are guessed at the start (whether the guess is smoke rings, shear layers, or whatever), there is danger that the subsequent theoretical construction may have its most important prediction.

It is important to stress that the scalings of the EDQNM energy spectra are directly related to the eddy-damping term contructed by the researcher. Definitely the arbitrariness of the triad time in EDQNM is a major drawback, because so much depends on that choice. Built on phenomenological grounds, Pouquet *et al.* (1976) introduced the Alfvén time scale into the eddy-damping term for MHD turbulence in the absence of correlations between the velocity and magnetic field. Baerenzung *et al.* (2008b) remarked that this one change from the usual fluid eddy-damping term, leads to the $k^{-3/2}$ (Iroshnikov-Kraichnan) instead of the $k^{-5/3}$ (Kolmogorov) scaling. In the case that the velocity and magnetic field correlate, Grappin *et al.* (1982, 1983) found that the energy spectra for the Elsässer fields, with the same eddy-damping rate of Pouquet *et al.* (1976), scale as $E_{\pm}(k) \sim k^{-m^{\pm}}$, with $m^{+} + m^{-} = 3$. Thus, the analytical

[116] A dissimilarity of magnetic and kinetic Reynolds numbers is one indication of possible nonequipartition, or possible departure from symmetry, between flow and magnetic field in MHD turbulence (Zhou and Matthaeus, 2005). The magnetic Prandtl number is defined as the ratio of the momentum viscosity $\nu$ and magnetic diffusivity $\varkappa$.




(a) (b)

FIG. 22: Direct numerical simulations (DNS) of rotating turbulent flows at Rossby number of 0.25 and rotation rate of 10. (Left panel). Top: 3D rendering of a turbulent flow. An inverse energy cascade is present in the turbulent dynamics. The stationary behavior is characterized by the formation of three cyclonic coherent columnar vortices emerging from the background of 3D turbulent fluctuations. Bottom: Vortical structures parallel to the rotation axis. Note the turbulent fluctuations exist also inside the core of each vortex. Color scale is based on the velocity amplitude. (Right panel) The $-2$ energy spectrum for rotating flows (Zhou, 1995). Inset: Spectrum compensated with $k^{-2}$ and with $k^{-5/3}$. Figs. 1 and 4 of Biferale *et al.* (2016), *Phys. Rev. X*. Published by the American Physical Society under the terms of the Creative Commons Attribution 3.0 License.

models are not fully satisfactory since they already involved certain assumptions about the energy transfer process. More recently, Grappin *et al.* (2016) carried out an extensive study related to the magnetic excess and resolved some issues in previous EDQNM studies. This work highlighted the limitation on the predictive capabilities of the EDQNM for a flow with various time scales. Of the four possible time scales, their corresponding EDQNM models are compared with both observational data and numerical simulations to identify which eddy-damping model is the correct one.

As for as the spectral exponents for anisotropic MHD turbulence are concerned, on balance it appears that $-5/3$ is better supported (from simulations) than $-3/2$ for the $E(k_\perp)$ spectrum of the total energy spectrum in MHD with a uniform $B_0$ (see, e.g., Beresnyak and Lazarian (2019)). However, as yet the largest 3D simulations are still not large enough to be definitive regarding the value of the slope, even in the zero cross helicity case (Moffatt and Tsinober, 1992). Supposing that at least a decade of inertial range scales is needed to genuinely observe powerlaw behavior (plus at least half a decade at each end of that range Pope (2000), half a decade of dissipation range Wan *et al.* (2010a), and at least a decade of energy-containing and larger scales), one can estimate that simulations resolving at least $10^{3.5} \approx 3000$ scales in each direction will be needed. Thus one should probably be cautious in drawing conclusions about inertial range slopes from simulations with fewer than $8,000^3$ grid points[117]. Such a dataset will be extremely helpful in finally resolving the controversy regarding the degree of locality for high Reynolds number MHD turbulent flows.

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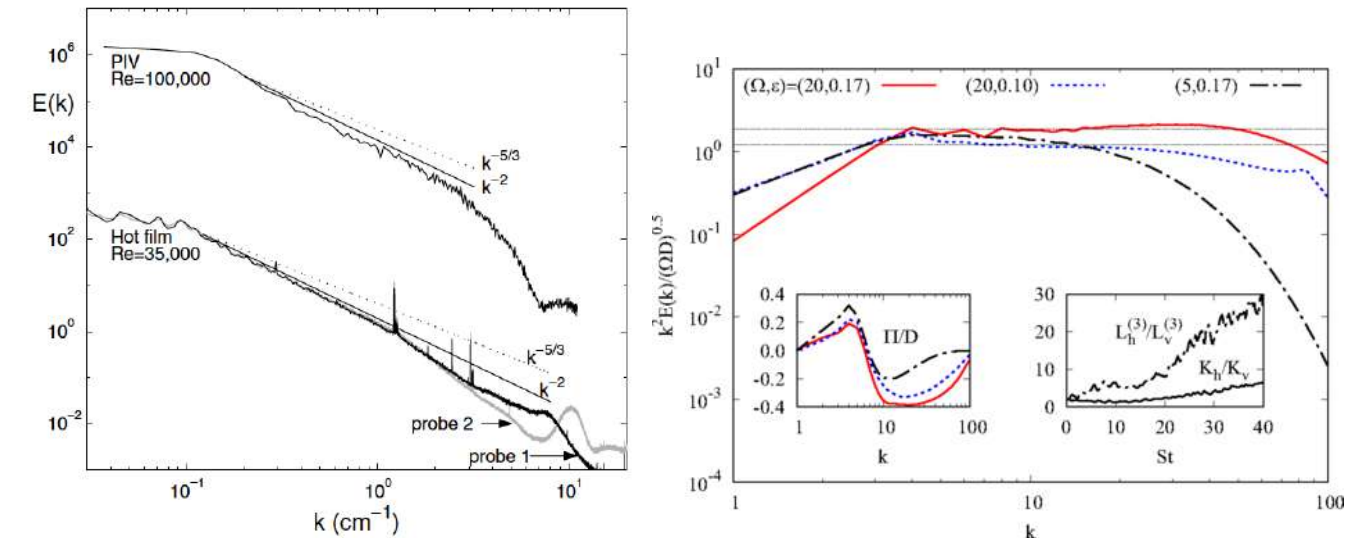
[117] S. Oughton, private communication, 2019




(a) (b)

FIG. 23: Left. The experimental results of $-2$ rotating energy spectrum, as predicted by Zhou (1995). Fig. 2 of Baroud *et al.* (2002), Phys. Rev. Lett. Right. The simulated energy spectrum normalized by rotation scaling by $\sqrt{\Omega}k^{-2}$. Dashed horizontal lines at $C_\Omega \in [1.22, 1.87]$ for reference. See Eq. 198. Fig. 4 of Khlifi *et al.* (2018), Phys. Rev. E
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## XII. ROTATING AND UNSTABLE STRATIFIED FLOWS

### A. Rotating flows

The scientific field of rotating turbulence has been the focus of much effort because of its significant role in stellar and planetary astrophysics (Akiyama *et al.*, 2003, Augustson and Mathis, 2019, Barness, 2001, Cho *et al.*, 2008, Hodgson and Brandenburg, 1998, Tobias, 2009), upper-ocean turbulence (Liu *et al.*, 2018), and enginering applications (Dumitrescu and Vladimir, 2004, Ogawa, 1997). In the astrophysical context, for example, Barker (2016) utilized the model by Mason and Kerswell (2002) and demonstrated that nonlinear interactions saturate the precessional instability and generate turbulence with columnar vortices. As stressed by Khlifi et al. (2018), such structures are a major feature found in turbulence subjected to rapid rotation (e.g., Biferale *et al.*, 2016).

We consider incompressible turbulence subjected to uniform solid-body rotation at rate $\mathbf{\Omega} = (0, 0, \Omega)$, but without mean velocity gradients. In general, rotation gives rise to both centrifugal and Coriolis forces, but when the motion is viewed in a rotating frame of reference only the Coriolis force is relevant. The fluctuating momentum equation in usual Cartesian tensor notation is

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i - 2\epsilon_{ijk}\Omega_j u_k, \tag{197}$$

where $\epsilon_{ijk}$ is the alternating symbol.

It has been recognized that there is a significant similarity between the MHD and strongly rotating turbulence. The controlling time scale associated with the rotating rate is $\tau_\Omega = 1/\Omega$, which is responsible for the energy transfer, $\tau_s = \tau_\Omega$. The rotation modified energy spectrum can be obtained by invoking the same MHD procedure inspired by closure theories. A substitution of $\tau_\Omega$ into Eq. 143 would result

$$E(k) = C_\Omega \sqrt{\Omega \mathcal{E}} k^{-2}, \tag{198}$$

where $C_\Omega = 1.22 - 1.87$ (Zhou, 1995).

At sufficiently low Rossby number (strong rotation), a "split spectrum" model may be appropriate (Zhou, 1995),

$$E(k) \begin{cases} C_0 k^s, & \text{for } k \le k_0, \quad (199a) \\ C_\Omega \sqrt{\Omega \mathcal{E}} k^{-2}, & \text{for } k \le k_\Omega, \quad (199b) \\ C_K \mathcal{E}^{2/3} k^{-5/3}, & \text{for } k > k_\Omega. \quad (199c) \end{cases}$$

Here, $k_0$ denotes the lower edge of the inertial range. The $E(k)$ profile shape for $k \leq k_0$ is, in general, problem dependent. The rotation cutoff scale, $k_\Omega$,

$$k_\Omega = \left(\frac{\Omega^3}{\mathcal{E}}\right)^{1/2}, \tag{200}$$

delimits the region of the spectrum where the rotation effects are important (Zeman, 1994) and is analogous to the Ozmidov scale of the geophysical literature.[118] The study of Delache *et al* (2014) shows that the anisotropy is not monotonic in the range $k < k_0 \sim k_\Omega$, starting with isotropy at smallest wavenumbers until a maximum value is attained, and then rediminishing until the vicinity of the Zeman scale.[119]

Fig. 22a shows the formation of intense and coherent columnar vortical structures, a very striking consequence of the Coriolis force (Biferale *et al.*, 2016). Fig. 22b also demonstrates that the rotation modified energy spectrum scales as $k^{-2}$ predicted in Eq. 198. Additionally, this $k^{-2}$ scaling has also been supported experimentally by Baroud *et al.* (2002) (Fig. 23a), Yarom and Sharon (2014), analytically by Hattori *et al.* (2004), Chakraborty and Bhattacharjee (2007), Kolokolov *et al.* (2020), and numerically by Yeung and Zhou (1998), Müller and Thiele (2007), Mininni *et al.* (2009), Iyer *et al.*, (2015), and Khlifi *et al.*, (2018) (Fig. 23b). For the very latest simulation result of Lin (2021), the interested reader is referred to to Fig. 24.

By ploting the compensated energy spectrum, Iyer *et al.*, (2015) and Khlifi *et al.*, (2018) further determined that $C_\Omega = 1.75$ (not shown) and 1.87 (Fig. 23b), respectively. These values are in good agreement with the prediction of Zhou (1995).

In theoretical studies of non-rotating turbulence, the inviscid truncated system, obtained by imposing a maximum wavenumber and setting viscosity to zero, plays an important role. Since kinetic energy is conserved in the presence of rotation, the inviscid truncated system for rotating turbulence should tend to a state of energy equipartition, with an energy spectrum proportional to $k^2$. This prediction was tested in numerical simulations by (Yamazaki *et al.*, 2002). The results confirmed that equipartition is ultimately achieved for a range of rotation rates and initial conditions, but the effect of rotation is to lengthen the time required to reach equipartition. The effect is significantly enhanced by increasing the rotation rate.

The previous results might be considered largely heuristic, since the energy spectrum, as a spherical average, is by definition insensitive to the anisotropy expected in a rotating flow. An interesting analytical result that reveals at least some of the anisotropy due to rotation was obtained by (Galtier, 2003). This work applies weak turbulence theory to a special limiting case of rotating turbulence. In terms of a 'bihomogeneous' spectrum $E(k_\perp, k_\parallel)$ ($k_\parallel$ is along the rotation axis), defined so that the total energy is given by $\int E(k_\perp, k_\parallel) dk_\perp dk_\parallel$, it is found that $E(k_\perp, k_\parallel) \sim k_\perp^{-5/2} k_\parallel^{-1/2}$. The corresponding mode density is $U \sim k_\perp^{-7/2} k_\parallel^{-1/2}$; the integral over a sphere of radius $k$, the usual energy spectrum, scales as $\oint U(k_\perp, k_\parallel) d\Omega \sim k^{-2}$, consistently with previous results. Restricted to very small, but nonzero, $k_\parallel$, resonant triads were captured numerically in all the spectral domain, resulting in new scaling laws in the presence of finite dissipation (Bellet *et al*, 2006)

For the general case of statistical homogeneity, the information on two-point second-order velocity correlations is given by the tensor $U_{ij}(\mathbf{r})$, and the related spectral tensor $U_{ij}(\mathbf{k})$ obtained by 3D Fourier transform. In this context, the two-point second-order correlations is represented in terms of the set $e, Z$ (Cambon and Jacquin 1989, Cambon *et al.*, 1997, 2017) by

$$U_{ij}(\mathbf{k}, t) = e(\mathbf{k}, t) P_{ij}(\mathbf{k}) + \Re\big[Z(\mathbf{k}, t)\mathbf{N}_i(\mathbf{k})\mathbf{N}_j(\mathbf{k})\big] \tag{201}$$

where the set $(\mathbf{N}, \mathbf{N}^*, \mathbf{k}/|\mathbf{k}|)$ generates an orthonormal frame for projecting the velocity field in Fourier space and is closely related to the so-called Craya-Herring frame (Craya, 1958; Herring, 1974) which is just (vectorial) spherical coordinates and related to the toroidal-poloidal decomposition in physical space.

Note also that helicity can be added as a third term in Eq. 201 for the maximum generality for arbitrary anisotropic homogeneous flows.[120] The helical contribution was not accounted for in the closure theories and DNS of Cambon and Jacquin, (1989), Cambon *et al.* (1997), but was incorporated in Bellet *et al.* (2006) for WT.

It is clear from Eq. 201 that we need only the two terms, $e, Z$, to describe all of the quadratic correlations; but

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[118] The Ozmidov length scale has the physical interpretation of the size of largest eddy unaffected by buoyancy (e.g., Ozmidov, 1965, Mater *et al.* 2013, Li *et al.* 2016).

[119] Delache *et al* (2014) mainly confirmed some aspects of Zhou (1995) (Cambon, private communication, 2021).

[120] This is also connected to the helical mode decomposition of the velocity field alluded to in footnote 44.

note that $Z$ is complex. The starting point for analysis is the following system of equations

$$\left[\frac{\partial}{\partial t} + 2\nu k^2\right] e = T^e,$$
$$\left[\frac{\partial}{\partial t} + 2\nu k^2 + 4i\Omega\frac{k_3}{k}\right] Z = T^Z,$$

where $T^e$ and $T^Z$ are the tranfer terms (see Cambon *et al.*, (1997) for detailed definitions).

For turbulence in the presence of body forces, such as rotation, the Markovianization procedure is less obvious, because of the presence of a nontrivial Green's function in the time integral, and different versions can be proposed (Cambon and Scott, 1999). It is possible to take into account the effect of rotation by incorporating the rotation rate in the eddy damping term

$$\eta(k,t) = \hat{c}_\eta\left[(2\Omega)^2 + \int_0^\infty p^2 E(p,t)dp\right]^{1/2}, \tag{202}$$

which should be compared with that without rotation case Eq. 102. The so-called EDQNM1 model, based on Eq. 202, is the simplest "isotropized model."

Two more complex versions of EDQNM models (Cambon *et al.*, 1997, Sagaut and Cambon, 2018), known as EDQNM2 and EDQNM3, have been introduced to account for the rise of directional anisotropy and the description of the transition from 3D isotropy to 2D structure. Note that EDQNM2 and EDQNM3 only differ in their treatment of $Z$ and the reader is referred to Cambon *et al.* (2004), where EDQNM3 equations are derived from EDQNM2 equations in the case of pure rotation without initial helicity.

Equation 202 suggested that the skewness factor (absolute value) can be written in the following form,

$$S_k = S_k|_{Ro_\omega=\infty}\sqrt{1+\frac{2}{Ro_\omega^2}}, \tag{203}$$

where the instantaneous micro-Rossby number $Ro_\omega = \omega'/(2\Omega)$, with $\omega'$ the rms value of fluctuating vorticity. Fig. 25 shows that the EDQNM1 model compares favorable to the fully anisotropic EDQNM2 model, DNS, and the experimental data of Morize *et al.*, (2005) for some integrated measurements, such as the skewness factor.

Cambon *et al.*, (2004) pointed out EDQNM-based models are not theoretically self-contained, because they rely on an exogenous specification of a relaxation time. The only wholly satisfactory theoretical approach is to compute the nonlinear damping from a comprehensive theory like the DIA or LRA. Since these theories make no assumptions about the relative strength of nonlinear and linear decorrelation, they apply equally to weak and strong rotation and, most importantly, to all modes at any rotation rate. Unfortunately, these more complete models have not been used in practice because of their considerable analytic and computational complexity.

Each year, a large body of publications that focus on either the physics or applications of rotating flows are published. As expected, most of the works are based on numerical and experimental measurements — see a relevant survery by Godeferd and Moisy (2015). Here is a list of limited sample of articles which have appeared very recently to show the significant roles of rotating turbulence on diverse and interesting subject matter: Lagrangian statistics (Maity *et al.*, 2019), thermal convection (Meuel *et al.*, 2018), Rayleigh-Bénard convection (Alards *et al.*, 2018), inertial waves (Le Reun *et al.*, 2017), kinetic-Alfvén wave (Galtier and David, 2020), forcing-dependent dynamics and emergence of helicity (Dallas and Tobias, 2016), spectra and cascade (Rathor *et al.*, 2020, Reshetnyak and Pokhotelov, 2019, Sharma *et al.*, 2018a,b, Pestana and Hickel. 2020, Vallefuoco *et al.*, 2019), stratified flows (Li *et al.*, 2020, Oks *et al.*, 2017), and cross flows in industrial aerodynamics (Su *et al.*, 2020).

### B. Unstable stratified homogeneous turbulence

The Rayleigh-Taylor (RT) instability (Rayleigh, 1883; Taylor, 1950) and Richtmyer-Meshkov (RM) instability (Meshkov, 1969; Richtmyer, 1960) arise in inertial confinement fusion implosions, supernova explosions, and numerous other scientific and engineering applications (Zhou, 2017a,b; Zhou *et al.*, 2021). The RT instability (RTI) is initiated when the lighter fluid is accelerated towards the heavier fluid while the RM instability (RMI) occurs when a shock wave passes across the perturbed interface separating the two fluids. As a result, the interface would roll up into the well-known mushroom structures and eventually lead to turbulent mixing (Zhou *et al.*, 2019). The Kelvin-Helmholtz (KH) instability (Helmholtz, 1868; Kelvin, 1871) occurs because the velocity of the resulting flow is discontinuous

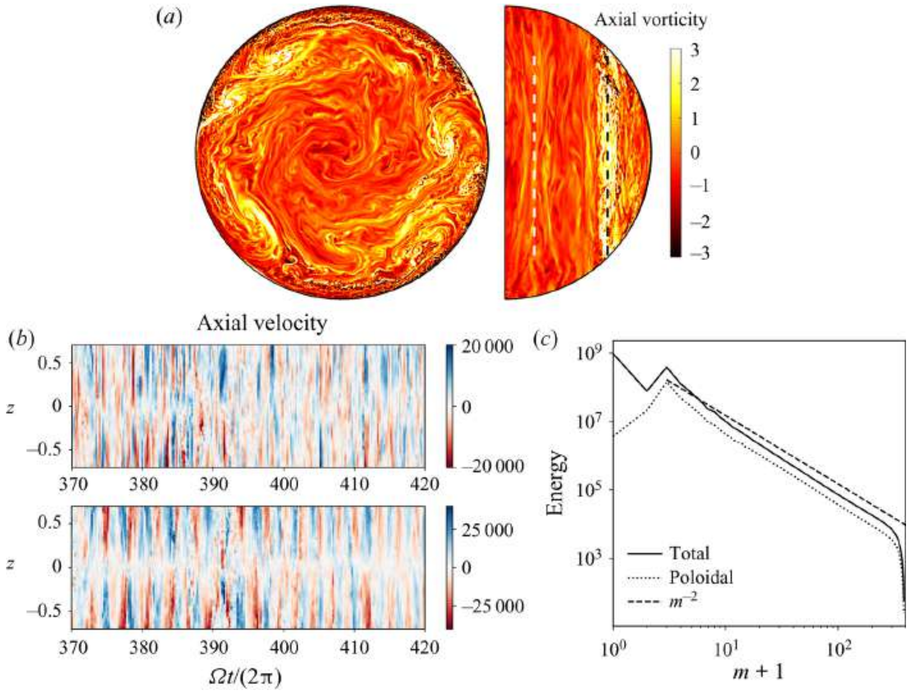


FIG. 24: (a) A snapshot of the axial vorticity in the equatorial plane (left) and in a meridional plane (right). (b) Time evolution of the axial velocity along the white (top panel) and black (bottom panel) dashed lines in panel (a). (c) Energy spectra as a function of the azimuthal wavenumber $m$. Fig. 5 of Lin (2021), *J. Fluid Mech.*. With permission from Cambridge Univ. Press.

at the material interface or contact discontinuity. Therefore, the vorticity sheet develops vortex rolls and eventually breaks in a turbulent fashion.

Naturally, the fluid motions of the instabilities are governed by the Navier-Stokes equations. If the fluids are mutually diffusive and the Mach Number, $Ma$, is small ($Ma \to 0$) the divergence of the velocity field becomes (Cook and Zhou, 2002; Joseph, 1990; Livescu, 2013)

$$\frac{\partial u_n}{\partial x_n} = \frac{\partial}{\partial x_n}\left[\rho \mathcal{D} \frac{\partial}{\partial x_n}\left(\frac{1}{\rho}\right)\right], \tag{204}$$

and the velocity field is only solenoidal in the incompressible, immiscible limit. Here, $\mathcal{D}_i$ is an effective binary species diffusion coefficient (Kays and Crawford, 1993) within an equation for fluid species mass-fractions of the $i^{th}$-species $Y_i$:

$$\frac{\partial \rho Y_i}{\partial t} + \nabla \cdot (\rho Y_i \mathbf{u}) = \nabla \cdot (-\mathbf{J}_i). \tag{205}$$

The diffusional fluxes $\mathbf{J}_i$ (Cook, 2009; Lombardini *et al.*, 1989) are given by

$$\mathbf{J}_i \approx -\rho\Big(\mathcal{D}_i \nabla Y_i - Y_i \sum_{j=1}^{N} \mathcal{D}_j \nabla Y_j\Big). \tag{206}$$

The EDQNM model has only been applied to unstably stratified homogeneous turbulence (USHT) (Gréa *et al.*, 2016; Thoroddsen *et al.*, 1998), thus permitting a better handling of turbulence in a simplified framework that discards inhomogeneous effects (Burlot *et al.*, 2015a; Griffond *et al.*, 2014). The homogeneous assumption can apply to turbulent quantities at the center of the mixing zones, due to a scale separation between the mean density gradient and the integral scale of turbulence (Vladimirova and Chertkov, 2009). Figure 26 shows the idealized profiles of RT mixing zones used to derive the USHT equations solved (see also, Figs. 27-28). However, real atmospheric and other geophysical and engineering flows tend to be inhomogeneous at the larger scales (e.g., Carnevale *et al.* (1995)).

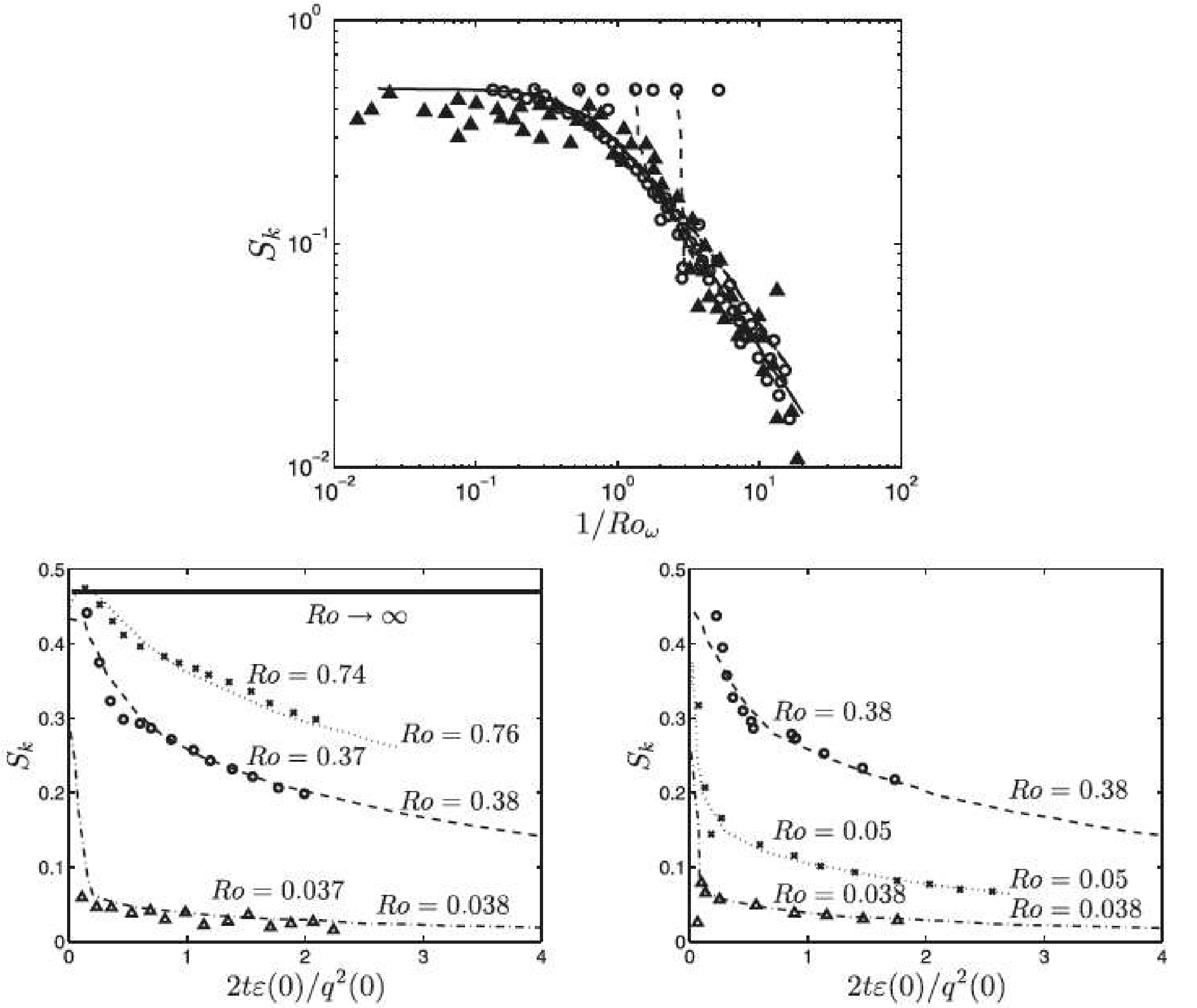


FIG. 25: Top: skewness factor (absolute value) $S_k$ as a function of the instantaneous micro-Rossby number obtained from experimental results in Morize *et al.* (2005) (▲) and DNS (◯). The curve in plain line is defined by Eq. 203 suggested by isotropic EDQNM. Bottom: temporal evolution of $S_k$ for the different indicated macro-Rossby numbers given by DNS (symbols in left figure), fully anisotropic EDQNM2 model (symbols in right figure) and simplified isotropic EDQNM (lines in left and right figures) using Eq. 202 with unsteady temporal variation of $S_k$, started with isotropic initial data; the elapsed time is non-dimensional in terms of the initial turn-over time based on the kinetic energy and its dissipation rate. Fig. 1 of Cambon *et al.* (2017), Computers and Fluids. ©Elsevier.

The equations governing the motion of Boussinesq fluids are similar to those for turbulent convection in astrophysical and geophysical fluid dynamics (e.g., Cabot *et al.* (1990)) and can be written as

$$\frac{\partial u_n}{\partial x_n} = 0, \tag{207a}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i + \mathcal{N}\theta\delta_{i3}, \tag{207b}$$

$$\left(\frac{\partial}{\partial t} - \mathcal{D}\nabla^2\right)\theta = -(\mathbf{u}\cdot\nabla)\theta + \mathcal{N}u_3, \tag{207c}$$

where $\mathcal{D}$ is the diffusivity for buoyancy rescaled as a velocity. The characteristic buoyancy frequency, $\mathcal{N} = |\mathbf{g}\cdot\nabla\rho/\rho|^{1/2}$,

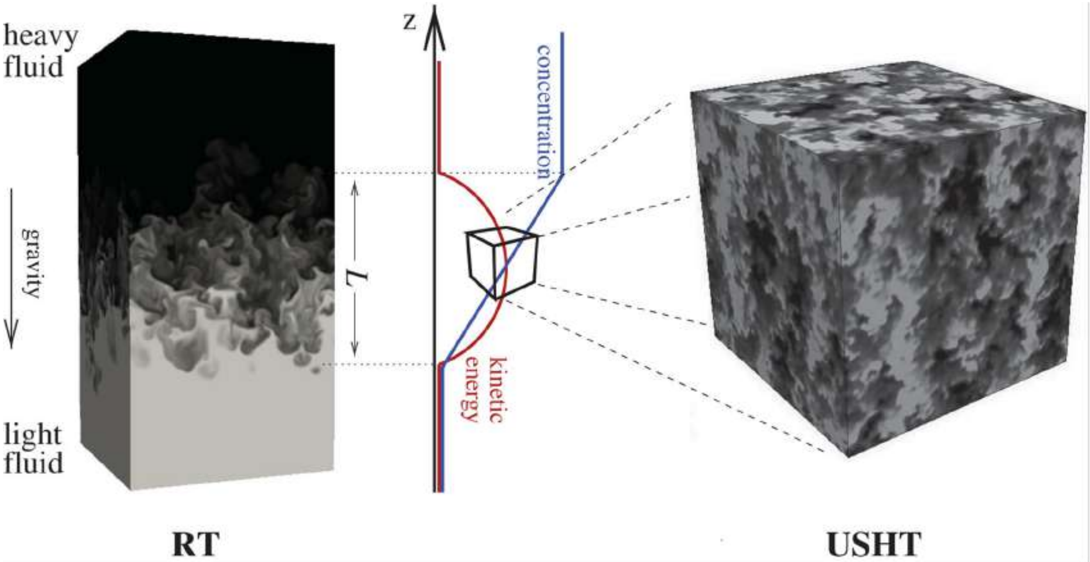


FIG. 26: Sketch of the USHT configuration with the underlying gradient length and reference profiles as schematised from low Atwood number RT configurations. Fig. 1 of Griffond *et al.* (2015). With permission from *J. Turbul.*, Taylor & Francis.

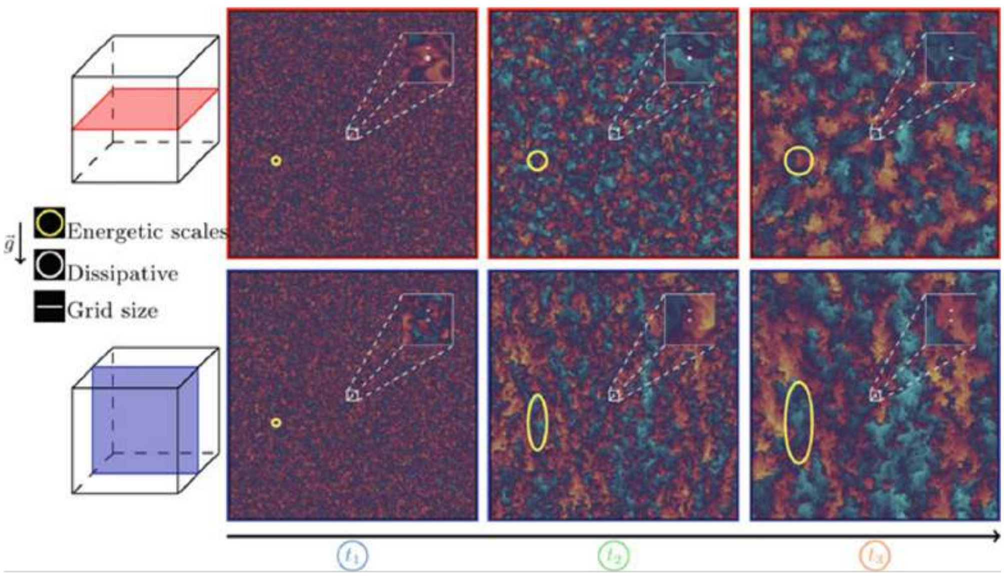


FIG. 27: Fluctuating density field at different times extracted from a DNS of USHT detailed in Gréa *et al.* (2016). The energetic, dissipative and grid scales are also indicated. Top row: Horizontal plane. Bottom row: vertical plane. From the initial isotropic condition, the work of buoyancy forces can be observed by the growth and elongation along the vertical direction of turbulent structures. Fig. 1 of Gréa and Soulard (2019). With permission from *Springer*.

is defined by analogy with the Brunt-Väisälä frequency in the stably stratified case. It is uniform in space and constant in time. The larger $\mathcal{N}$ the stronger the stratification and/or the gravity acceleration.

The buoyancy-driven incompressible flow described by the Boussinesq approximation (Batchelor *et al.*, 1990) is applicable to RTI flows by assuming the fluctuations in density are small relative to the mean density. The flow is now incompressible (see Eq. 207a). Since we use the Boussinesq approximation, we are working in the limit of small Atwood numbers.[121]

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[121] Here, it is important to introduce a non-dimensional parameter to characterize the density difference between two fluids separated by

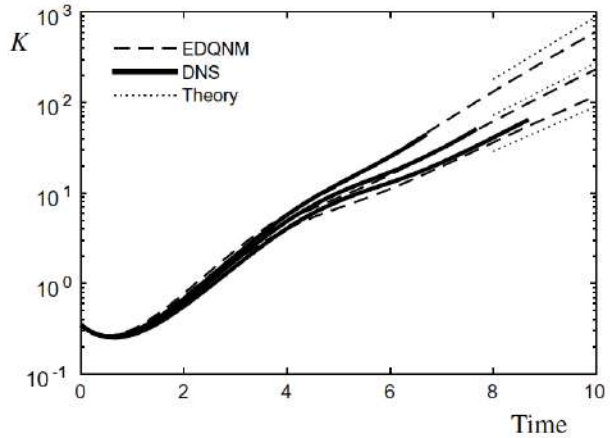



FIG. 28: Time evolution of kinetic energy K for different infra-red slope 2, 3, 4 from Gréa *et al.* (2016). Plain line: DNS. Dashed line: EDQNM. Fig. 3 of Gréa and Soulard (2019). With permission from Springer.

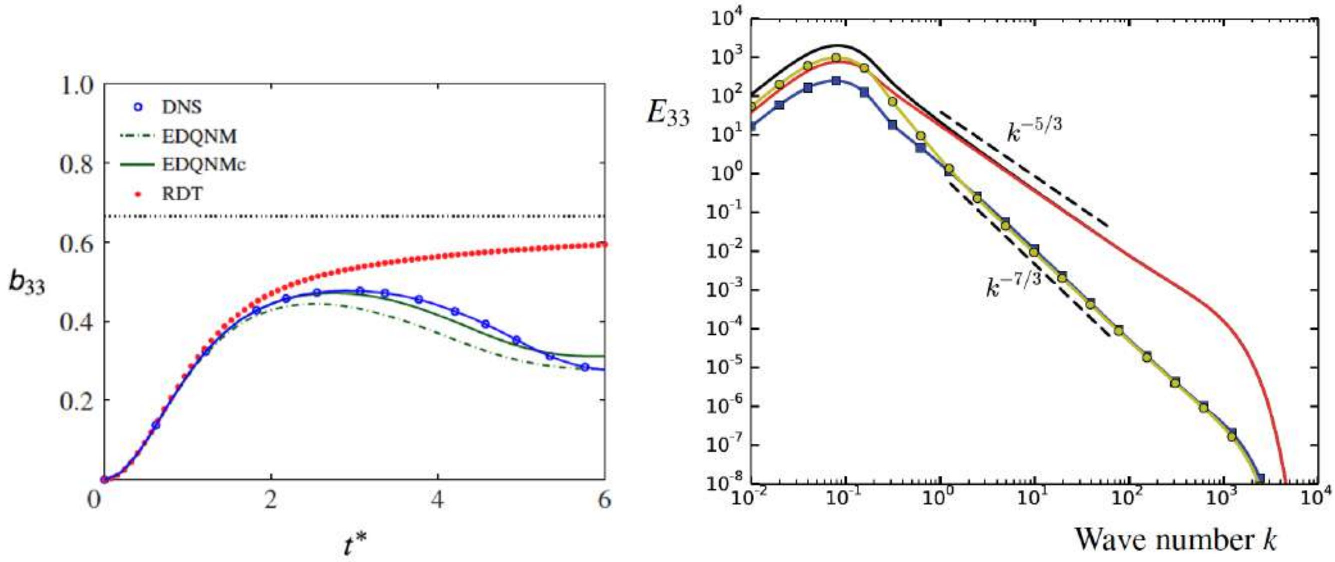
For a stratified turbulent flow with a buoyancy frequency to be applicable as a surrogate for the RTI flow, the ratio between the integral length scale of turbulence and the characteristic scale of the mixing region must be small. This requirement is shown to be valid by Vladimirova and Chertkov (2009) and Zhou and Cabot (2019) from simulated RTI flows. Burlot *et al.* (2015a) advanced a two-point statistical approach using an axisymmetric EDQNM closure that includes buoyancy production. More specifically, the authors transposed the spectral description to a polar-spherical frame of reference, thus taking advantage of the solenoidality of the velocity field and axisymmetric property of the spectra about the axis of gravity. The USHT EDQNM model equations for two-point second-order correlations are derived for a state vector in four components, by virtue of axisymmetry with mirror symmetry: (i)-(ii) the second-order spectral tensor of velocity correlations reduces to toroidal and poloidal energy spectra; (iii) the spectrum of the buoyancy variance reduces to a spectrum of potential energy; and (iv) the cross-correlation between velocity and buoyancy reduces to a cospectrum of poloidal buoyancy flux (Burlot *et al.*, 2015a; Cambon *et al.*, 2017).

The closure of USHT EDQNM model in nontrivial. Burlot *et al.* (2015a) first obtained the analytical solution for the triple correlations equation by assuming the model is Markovian. While the EDQNM model captures the early stages in USHT evolution, a detailed comparison with DNS suggests that EDQNM provided an incorrect nonlinear estimate.

To improve the performance of the EDQNM model, Burlot *et al.* (2015a) propose a simple modification of the eddy-damping term based on the following considerations. In Markovianized closure two-point statistical models, turbulent decorrelation timescales are derived from phenomenological analyses of the interaction between large and small scales and the choice of the eddy-damping timescale must include enough correct physics (Zhou, 2010; Zhou *et al.*, 2004). Two timescales maybe considered, one related to the local straining effect and another to the sweeping of small structures by large ones. Burlot *et al.* (2015a) noted that the vertical buoyant events in USHT flows may trigger sweeping-like phenomena in the turbulent dynamics and and proposed to employ a hybrid timescale, in a same fashion as that of Eq. 187. Briard *et al.* (2017) also modified the eddy-damping constants to slightly increase the global anisotropy of the flow. Fig. 29a displays the deviatoric part of the Reynolds stress, $b_{33} = < u_3 u_3 > / < u_n u_n > -(1/3)$ from DNS, rapid distortion theory (RDT), EDQNM, and modified model EDQNM (marked as EDQNMc). Here, one only needs to retain the component $b_{33}$, since $b_{11} = b_{22} = -b_{33}/2$ from incompressibility. The linear rapid distortion theory (RDT) result evolves from the initial zero value to the asymptotic $2/3$ value, but the linear approximation becomes invalid past $t^* \approx 2$. The result of the modified EDQNM model is very close to the DNS evolution and an improvement over that based on the original version.

As expected, Burlot *et al.* (2015a) determined that the energetic and smaller scales of turbulence are qualitatively well reproduced in USHT, but the larger-scale ones differ because of the inhomogeneous effects. As shown in Gréa *et al.* (2016), the improved EDQNM model can be used to study the influence of initial conditions beyond the reach of DNS (see Figs. 28-29). Here, Burlot *et al.* (2015b) and Soulard *et al.* (2014) demonstrated a direct link between

an initial interface, $\mathcal{A} = (\rho_2 - \rho_1)/(\rho_1 + \rho_2)$, where $\rho_2$ and $\rho_1$ are the "heavy" and "light" fluid densities, respectively and $\mathcal{A}$ is known as the Atwood number of the system.




(a)

(b)

FIG. 29: Left. Statistical characterization of anisotropy. Evolution of component $b_{33} =< u_3 u_3 > / < u_n u_n > -(1/3)$ of the deviatoric part of the Reynolds-stress tensor from DNS, rapid distortion theory (RDT), EDQNM, and modified model EDQNM (marked as EDQNMc). Fig. 9 of Burlot *et al.* (2015), J. Fluid Mech. With permission from Cambridge Univ. Press. Right. Vertical velocity spectrum $E_{33}$ and its isotropic and anisotropic contribution from USHT EDQNM simulation (Soulard and Gréa, 2017). Black line: total. Red line: isotropic part. Yellow and blue line: Anisotropic polarisation and directional part. See also, Fig. 5 of Gréa and Soulard (2019). With permission from Springer.

the late-time anisotropy of the flows and the infrared spectrum, thus demonstrating the long-lasting effect of initial conditions on unstably stratified turbulence. Burlot *et al.* (2015a) and Soulard and Gréa (2017) also reported a strong anisotropy at large scales and isotropy of scales in the inertial range as well as in the dissipative regime. Using data from an EDQNM simulation of a USHT flow, Fig. 29b shows that the isotropic contribution to the vertical velocity spectrum $E_{33}$ is significantly larger than those from anisotropic polarization and directional parts.

Alternatively, Briard *et al.* (2017) developed an anisotropic EDQNM model, taking advantage of an earlier work for passive scalar dynamics (Briard *et al.*, 2016a). This model dramatically reduced the computational cost from that of the axisymmetric EDQNM closure and was capable of incorporating active scalar dynamics, shear and mean scalar gradients, axisymmetric contractions, and distortion (Briard *et al.*, 2016a,b; Mons *et al.*, 2016). Furthermore, interesting results were derived for the case of large Schmidt numbers. Despite some differences between the two approaches, most features of the dynamics studied are in good agreement between the two EDQNM approaches (Briard *et al.*, 2017). Probably due to the modeling of anisotropy in their strategy, the resulting flow field obtained by Briard *et al.* (2017) is less anisotropic than that in (Burlot *et al.*, 2015a,b; Gréa *et al.*, 2016).

The EDQNM models are difficult to apply for an RMI flow because it is initiated after the passage of a shock wave across the perturbed interface separating the two fluids. Soulard *et al.* (2018) sidestepped this problem and supplemented the equations governing the spectra of the velocity and concentration fields with initial conditions stemming from the shock-interface interaction. The authors investigated the properties of large-scale eddies in RMI-induced turbulence with a small Atwood number and found that large scales maintained their initial anisotropy. Of particular interest, they estimated the scaling exponent of a self-similar RMI turbulent mixing zone, which should be compared to values determined by recent work by other researchers documented in Zhou (2017a).

The subject of RT and RM instabilities has been covered extensively in the literature in both computational and experimental investigations. The drawback of experimental studies, howbeit, is that it can be difficult to quantitatively characterize the initial perturbation. Indeed, only three methods can provide precise initial interface perturbations: paramagnetic fluids, numerical simulations, and dedicated high-energy-density physics (HEDP) experiments with solid-state target (Zhou *et al.*, 2019). Recently, Rigon *et al.* (2021) present micron-scale phenomena observed from an HEDP experiment where a laser-produced RT unstable plasma flow evolves towards a possibly turbulent state (Fig. 30). The multi-layer target allows the control over which spatial modes will be favored during RT development. From

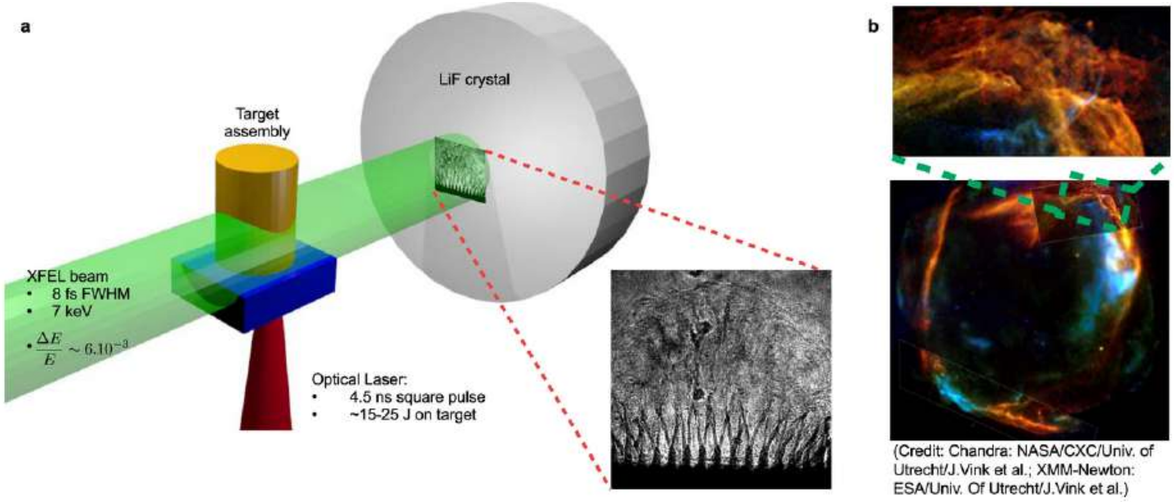



FIG. 30: Diagram of the overall experiment setup (a) and NASA Chandra X-ray Observatory's image of supernovae remnant, RCW 86 (b). The plasma expansion inside the laser-driven target is diagnosed using a XFEL beam and lithium fluorine crystal. This expansion is RT unstable, as in the supernovae remnant. Fig. 1 of Rigon *et al.* (2021), *Nature Comm.*. This is an open access article distributed under the terms of the Creative Commons CC BY license.

the obtained radiographs, the authors extract intensity spatial spectra, which can be associated with velocity spectra.

Analyses for HIT suggest that a full-fledged inertial range (at least an octave in bandwidth) forms for Reynolds numbers exceeding about $1.6{\times}10^5$ (Zhou, 2007). Values of several key measurements using vertical velocity components from the RTI simulation data suggested that much larger RTI simulations are required for well-developed turbulence. In particular, Zhou and Cabot (2019) estimated that a grid size an order of magnitude larger in each direction is needed to generate a fully turbulent state[122] — perhaps in the realm of exascale computing.

### C. Compressible and variable-density flows

The density created insurmountable obstacles for a closure theory of fully compressible turbulence. One can introduce $\rho\mathbf{u}$ as a new unknown, but how to enforce the relation between it and $\rho$ and $\mathbf{u}$? The same is true for the efforts in introducing $\rho^{1/2}\mathbf{u}$ as a new variable where one still cannot eliminate the density as an unknown from the modified Navier-Stokes equation (e.g., Cabot *et al.* (2004) and Kida and Orszag (1986)).

For the DIA, Hartke *et al.* (1988) remarked that the procedure would result in a very large set of coupled integrodifferential equations to be solved for the correlation functions. As the full system of equations would be extremely difficult and time consuming to solve for any real problem, the authors reasoned that a fair strong approximation is justified to obtain a set of equations that are tractable. The equations were derived as part of an extension of this model to compressible stellar turbulent convection (Schilling, 1994) within the approximation of the source function model (Canuto *et al.*, 1991).

Bataille and Zhou (1999) and Bertoglio *et al.* (2001) have performed an EDQNM closure study on the degree of locality of the energy transfer process for a weakly-compressible fluid turbulence (Zank and Matthaeus, 1991, 1993). Two assumptions are introduced in the basic set of equations governing the fluctuating field: barotropicity and linearization of the equations with respect to density fluctuations. These assumptions will preserve some interesting properties of compressibility (sound wave generation, interactions between solenoidal and compressible modes, ... see (Bertoglio *et al.*, 2001)).

It is worth commenting in connection to Eq. 204 that a non-solenoidal velocity field is a substantial complication for a spectral model (where the utility of the $k \cdot u = 0$ is now lost). This is the motivation for examining the so-called Boussinesq limit in subsection XII.B.

[122] Zhou and Cabot (2019) used the rate at which the Reynolds numbers are increasing at late times in their simulations. As an aside, this work generalized Eq. 142, the normalized dissipation rate, to inspect the distinctive features of the flow in the the inhomogeneous direction parallel to gravity and in the homogeneous perpendicular directions.

Homogeneous variable-density turbulence (HVDT) aims to mimic the RT and RM instabilities and reveals new physics that arise from variable-density effects on the turbulent mixing. During late time decay, where most of the flow is well-mixed, Aslangil *et al.* (2020) find that all parameters become independent of the initial composition ratio for both low and high Atwood number.

Petty and Pantano (2019) applied the SDIP to derive the spectra of isotropic variable-density turbulence with an ideal gas thermal state relationship. The flow is assumed to adhere to the zero-Mach variable-density equations (Chassaing *et al.*, 2002). This analysis is based on an asymptotic expansion using a small parameter about the constant-density and resulted the well-known incompressible problem to the leading order. Several interesting results have been obtained. In particular, the scalar spectra indicate significant deviations, and even loss of monotonicity, as determined by the strength and the type of the large-scale source of the mixing. Moreover, this work found that the velocity spectrum owing to variable density is indistinguishable from that of constant-density turbulence, suggested by indirect experimental evidence.

To assess the variable-density effects, Pal *et al.* (2018) applied a two-point spectral closure model originally developed for constant-density turbulent flow (Besnard *et al.*, 1996) and subsequently modified for variable-density flows (Steinkamp *et al.*, 1999a,b). The model does not account for anisotropy explicitly, but it has been shown that the test data can be recovered satisfactorily even with the isotropic assumption. This coupled system is spatially homogeneous, which allows the authors to isolate the variable-density component of the model without the complications introduced by the inhomogeneity, as in the inhomogeneous RT problem. In a related development, Bragg *et al.* (2017) examined a spectral model based on an anisotropic, inhomogeneous generalization of Leith's Diffusion Approximation (Leith, 1968) for isotropic turbulence. The model has been shown to give good results when applied to wind-tunnel experiments of homogeneous, strain flows (Clark and Zemach, 1995). Bragg applied the model to a "shearless mixing layer" (Tordella *et al.*, 2008) and was able to show good agreement with corresponding inhomogeneous DNS results.

Recently, Saenz *et al.* (2021) developed a framework that is consistent with the Navier-Stokes equations in the limit of small length scales, and with RANS formulations in the limit of large length scales. The authors used the formulation to investigate the relationships between scale and dynamics in homogeneous variable density turbulence.

## XIII. ENGINEERING FLOWS

### A. Kraichnan's decimation scheme

A variety of proposals for reduced description have been offered, both to gain physical insight and to make computation feasible for large *Re* (Kraichnan, 1988b). Kraichnan (1985) introduced the so-called constrained decimation scheme (CDS), which can be used to systematically improve models by adjoining higher order statistical symmetry constraints (Kraichnan, 1991) and recover both the DIA and RG theories as special cases.[123] Moreover, Kraichnan (1991) suggested that the successively higher order CDS models are expected to yield statistics that converge to exact Navier-Stokes statistics.

While the theory behind this approach is difficult and subtle (Williams *et al.*, 1987), the CDS can be summarized neatly as follows (Williams *et al.*, 1989a): the large system of evolution equations describing turbulence is replaced - after invoking statistical and dynamical symmetry arguments - by a smaller sample set of evolution equations. This decimated system is driven by stochastic forces which model the couplings to degrees of freedom which have been eliminated. In order to capture the correct statistical behavior of the original full system, these stochastic forces must obey a hierachy of statistical constraints. Indeed, Kraichnan and Chen (1989) further extended the decimation idea to study intermittent phenomena by enforcing more constraints on high-order statistics of fluid turbulence.

### B. Closure approaches for subgrid modeling

The CDS can be naturally translated and recast into the language of subgrid modeling for large-eddy simulations. The influence of subgrid scales on the resolvable large scales is represented by a stochastic forcing. To correctly calculate the mean energy flux, the forcing term must satisfy certain constraints deduced from underlying physics,

[123] Kraichnan's discussion about the link between the CDS and DIA has been supported from a numerical study (Williams *et al.*, 1989b) based on the Betchov model system (Betchov, 1966), which was designed to mimic basic properties of the Fourier mode equations derived from the Navier-Stokes equations.

such as symmetry and conservation (Chen *et al.*, 2012; Shi *et al.*, 2008). Recently, constrained large-eddy simulations have been performed for wall-bounded compressible turbulent flows (Jiang *et al.*, 2013); turbulent flow and heat transfer in a stationary ribbed duct (Jiang *et al.*, 2016), and flow fields past a commercial aircraft at a 14 degree angle of attack (Chen *et al.*, 2013). We will not discuss the results of these LES further but the interesting reader is referred to these original papers for details.

Yoshizawa (1989) used the TSDIA to study the length scale variation in SGS modeling.[124]

## C. Reynolds-Averaging Navier-Stokes modeling

### 1. Nonlinear Reynolds stress models

Two-scale versions of the closure theories have been suggested by Carnevale and Frederiksen (1983b); Carnevale and Martin (1982); and Leslie (1973) and developed including mean fields (Yoshizawa, 1984). This approach takes advanage of length scale separation in many applications,[125] by use of a two-scale asymptotic analysis procedure. In this picture, the small, fast scales represent the turbulent fluctuations and the large, slow scales the mean flow.

Speziale (1991) pointed out that the simplest turbulence closure, the Reynolds stress model, Eq. 23, has two major deficiencies: (i) they are purely dissipative and hence cannot account for Reynolds stress relaxation effects, and (ii) they are oblivious to the presence of rotational strains (e.g., they fail to distinguish between the physically distinct cases of plane shear, plane strain, and rotating plane shear).

In order to overcome these inadequacies, considerable efforts have been directed to the advancement of nonlinear or anisotropic models using statistical closure theories. Within the two-scale framework, one assumes that there exists an asymptotic expansion of the solutions to the governing equations in which the ordering parameter is introduced. The ordering parameter measures the ratio between the small and large length scales. The correlations of fluctuating quantities produced by the perturbation methods are then evaluated using a statistical closure methods. Nonlinear Reynolds stress models result from a perturbation expansion (based on the particular closure approach) that is truncated systematically.

Yoshizawa's nonlinear Reynolds stress models were obtained from a perturbation scheme (Nisizima and Yoshizawa, 1987; Yoshizawa, 1984), the so-called two-scale direction interaction approximation (TSDIA). To obtain a relation between the anisotropy of scalar diffusion and the inhomogeneity of turbulent flows, Yoshizawa (1985a, 1987) evaluated the scalar flux generated by turbulent shear flows up to the second order of a scale parameter. The TSDIA has been applied to various fluid and plasma turbulent flows (Yoshizawa (1998),Yoshizawa *et al.* (2003) and references therein), including the impact of thermal effects in turbulence modeling (Mathelin *et al.*, 2008).

Rubinstein and Barton (1990, 1991) have applied the $\epsilon-$RG to develop a nonlinear algebraic Reynolds stress model of anisotropic turbulence and anisotropic corrections to the scalar eddy diffusivity representation of turbulent diffusion of a passive scalar, respectively. From $\epsilon-$RG, which can only treat nonlocal interactions, it has been shown that the nonlocal contributions to the Reynolds stress give rise to terms that are quadratic in the mean strain rate. Zhou *et al.* (1994) derived an anisotropic turbulence model for the local interaction part of the Reynolds stresses using the r-RG renormalization group theory. It is shown here, by r-RG, that the local interaction effects give rise to a cubic eddy-viscosity model of turbulence.

Based purely on the grounds of continuum mechanics, Speziale (1987) obtained the Reynolds stresses as quadratic functions of the mean velocity gradients. This model was derived by making an analogy between a non-Newtonian fluid and turbulence. He conjectured that a non-Newtonian relationship between stress and strain rate may be an appropriate representation of the effect of turbulence on the mean flow. In fact, approximating the Reynolds stresses with an algebraic connection between the stress and strain has been called algebraic stress models, which can trace its earliest study to Pope (1975) and subsequent refinement by Gatski and Speziale (1993). Indeed, the quadratic and cubic nonlinear algebraic Reynolds stress models advanced by Rubinstein and Barton (1990) and Zhou *et al.* (1994) from the RG methods correspond to those by Speziale (1987) and Craft *et al.* (1996) using more traditional modeling methodologies.

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[124] The reader is referred to Yoshizawa (1998) and Yoshizawa *et al.* (2003) for several other related TSDIA models and references therein.

[125] As an example, observations indicate that smooth average properties of the solar wind vary on length scales of the order of the local heliocentric radial coordinate $\mathcal{R}$ (Hundhausen, 1972). Fluctuations are broadband but generally have correlation scales ($\tilde{\lambda}$) that are much smaller than $\mathcal{R}$, at least for heliocentric distances of the order of 1 AU or more. Much of the MHD turbulence activity takes place at "inertial range" scales that extend roughly from $\tilde{\lambda}$ down to scales 1000 times or so smaller, near the thermal ion gyroscale. Thus the MHD turbulence activity of interest is well separated in length. scale from the large-scale solar wind inhomogeneities (Zhou and Matthaeus, 1990a).

### 2. Two equation models

A variety of two-equations models (Wilcox, 1993) are among the most popular Reynolds stress models for scientific and engineering calculations (Speziale, 1991).

In the standard $K-\mathcal{E}$ model - which is probably the most popular such model - the time and lenghth scales are built up from the turbulent kinetic energy and dissipation rate, $\mathcal{E} = K^{3/2}/L$. As noted by Rubinstein and Clark (2005), two-equation modeling is based on Kolmogorov's theory of the universality of the small scales of motion in turbulent flows. The Kolmogorov spectrum itself depends only on the dissipation rate. Modeling generalizes the Kolmogorov theory in two important ways: (i) it applies the theory to time-dependent problems by replacing the static descriptors $K$ and $\mathcal{E}$ by functions of time $K(t)$ and $\mathcal{E}(t)$, and (ii) it assumes that $K(t)$ and $\mathcal{E}(t)$ satisfy closed equations of motion.

Modeled dissipation rate transport equations are often derived by invoking various hypotheses to close correlations in the corresponding exact equations. The central problem remains to justify the absence in this equation of terms of order $Re^{1/2}$, the possible existence of which is suggested by elementary scaling arguments (Tennekes and Lumley, 1972).[126] More basic difficulties arise when there are coupled fluctuating fields as in buoyant turbulence, or imposed time scales as in rotating turbulence. Leslie (1973) suggested that these models might be derived instead from Kraichnan's wavenumber space integrals (Kraichnan, 1971b) for inertial range transport power. Rubinstein and Zhou (1996) applied these models to the destruction terms in the dissipation rate equations for incompressible turbulence, buoyant turbulence, rotating incompressible turbulence, and rotating buoyant turbulence.

Yakhot and Orszag (1986) derived a dissipation rate equation using the $\epsilon$-RG. This model has been successfully tested for homogeneous shear flows and for flow over a backward facing step (Yakhot *et al.*, 1992). The reader is referred to Smith and Woodruff (1998) for a detailed assessment of such modeling approach.

Yokoi (2006) and Yokoi *et al.* (2008) have also attempted to construct transport equations for MHD turbulence with statistical closure theories.

Statistical closure theories are of significant utility in constructing nonlinear Reynolds stress models and two-equation models. Yet, important limitations remain. For example, the quadratic Reynolds stress model of Yoshizawa (1984) expressed all model constants as integrals of the response functions of the DIA. While these coefficients can be calculated by assuming plausible forms, Nisizima and Yoshizawa (1987) and Shimomura and Yoshizawa (1986) found that empirically adjusted constants provided better results. In another case, Hamba and Kanamoto (2019) used the TSDIA to derived the $\mathcal{E}$ model equation, but the $O(Re^{-1/2})$ corrections to the leading-order terms were not found in the analysis. Only by resorting to their numerical results from the modified LRA equations were the authors able to obtain the $O(Re^{-1/2})$ corrections and the destruction term in the model equation.

Thus, the RANS models derived from statistical closure theories, just as models obtained from other methodologies, may demand expert judgement as well as careful tuning to match a suite of high-quality experiments or simulations. Also, LES based studies inherit many of the limitations of RANS models (Sagaut, 2013). Therefore, there is a need for Quantification of Margins and Uncertainties (QMU) in such closure models to establish these computational approaches as reliable tools for the engineering design process (Mishra and Iaccarino, 2019).

## XIV. CURRENT STATUS

Coinciding with the 80th anniversary of the publication of the Kolmogorov's pioneering work, this review covers the work of the past eight decades focused on the theoretical treatment of turbulence, particularly statistical closure theories. The Direct Interaction Approximation (DIA) (Kraichnan, 1959a) and its Lagrangian versions were revolutionary because these theories provided physicists with a framework to scrutinize energy transfer and interacting scales, and to delineate the straining and sweeping processes, both of which are fundamental to the physics of turbulence. The DIA paved the way for the development of several Markovianized closure methods, all of them retained the key "heritage" from DIA and maintained the multiplicity of dynamical scales, thus allowing a richer dynamical behavior. As a result, these Markovianized closures could be utilized to investigate infrared dynamics, free decaying, and other properties of several basic flows.

More recently, significant progress has been made to understand another fundamental aspect of turbulent flows; namely, the intermittency and the anomalous scaling exponents. This issue has been addressed with an ingenious passive scalar model, which was the subject of Kraichnan's last major contribution. The popularity of this model among researchers continued unabated. Most importantly, the theoretical treatment of Navier-Stokes turbulence

[126] This issue is the subject of active research with statistical closure theories (Hamba and Kanamoto, 2019; Rubinstein and Clark, 2005).

advanced by V. L'vov and I. Procaccia resulted in the establishment of a far-reaching quasi-Lagrangian framework. Renormalization Group (RG) theory, adapted from the original Nobel winning work of Kenneth Wilson, was also contemplated. The application of RG to turbulent flows is extremely challenging, but some promising results have been obtained recently with the newly formulated non-perturbative RG method.

For two-dimensional and geophysical turbulence, the theories of equilibrium statistical theory fluid equilibria (most prominently that of Miller-Robert-Sommeria) have adduced strong evidence that the statistical physics approach was efficacious in addressing aspects of Jupiter's Great Red Spot and other important zonal flows.

Additionally, the statistical closures have been extremely useful in developing the parameterization of unresolvable scales of turbulence for large-eddy simulations and constructing Reynolds-averaging Navier-Stokes models for practical applications in astrophysics, geophysics, engineering, and weather forecasting.

The detailed discussions in this review inevitably lead to the conclusion that turbulence theory has made significant advancements since its early days. Statistical closure theories have harnessed the powerful tools developed in the fields of nuclear, particle, and condensed matter physics, especially those of path integral formulation and renormalization procedures.

However, due in large part to their mathematical complexity, statistical closure theories have only been utilized to study relatively idealized flows. As Pouquet (2019) remarked recently, "[c]losure models can be quite intricate in their development and final formulation as soon as one departs from the homogeneous isotropic neutral fluid case."

Additionally, only limited turbulence statistics can be acquired with statistical closure approaches by construction. This may have been satisfactory in earlier times when computing resources were limited, but now, the cost to capture all scales of motions numerically has become affordable for these idealized cases.[127]

It should be reiterated that, unlike other branches of physics, turbulence researchers do not have to deal with uncertainty in the underlying governing equations (Nelkin, 1994). The Navier-Stokes equations (Navier, 1822; Stokes, 1845) are partial differential equations that likely contain all the physics (Frisch, 1995) necessary to describe fluid motion.

## XV. OUTLOOK FOR THE FUTURE

Like any other branch of physics, turbulence has traditionally relied on the three components of the scientific trinity: theoretical, experimental, and computational studies. Yet, in recent decades, the latter two components have become increasingly poised to dominate the field as computational abilities and experimental facilities have achieved unprecedented advancement.

This review has sought to address the questions that have naturally arisen from these developments. Namely, where do we stand, and what next? More specifically, how much have theoretical approaches actually achieved? Were these contributions only useful in the absence of advanced computational and experimental studies, or will theory remain relevant despite the continued development of the other two legs of the trinity? And what are the factors that could conceivably shape the future direction of theoretical turbulence research?

### A. Machine learning, artificial intelligence, and data science

Massive amounts of data are today widespread across scientific disciplines and gaining insight and actionable information from them has become a new mode of scientific inquiry (Brunton *et al.*, 2020). While artificial intelligence (AI) mimics natural human-like intelligence to solve complex problems and enables decision making, machine learning (ML), a subset of AI, is about improving and maximizing performance through self-learning algorithms (Mehta *et al.*, 2019; Pandey *et al.*, 2020).[128] Both of them require large databases from which to learn: the more high-quality data that becomes available, the better the results – hence the close connection of AI and ML to data science.

Brenner *et al.* (2019) pointed out that although they are often constructed based on heuristics, ML algorithms (Carleo *et al.*, 2019) are themselves well-defined mathematically and they also have a massive infrastructure supporting their broad use. In particular, the concurrent enhancements in statistical inference algorithms, ML and uncertainty

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[127] For the record, the highest resolutions of the recent DNS for fluid and MHD turbulence, as of this writing (2021), are as the follows: (1) HIT turbulence at $18432^3$ (Yeung and Ravikumar, 2020), (2) MHD with a strong magnetic field at $2048^3$ (Bandyopadhyay *et al.*, 2018), (3) Reduced MHD (Kadomtsev and Pogutse, 1973; Strauss, 1976) simulations at $4096^3$ (Beresnyak, 2014), (4) MHD with a large $Re$, but a low magnetic $Re$ number on cubic domains at $2048^3$ or on elongated periodic domains at $16384 \times 2048^2$ (Zhai *et al.*, 2019; Zhai and Yeung, 2018).

[128] Putting the matter more broadly, ML refers to the ability of machines to learn from data, improve at tasks with experience, and make predictions. Indeed, there is an expanding body of literature on AI and ML in other scientific and medical fields, especially cardiovascular medicine (Kilic, 2020).

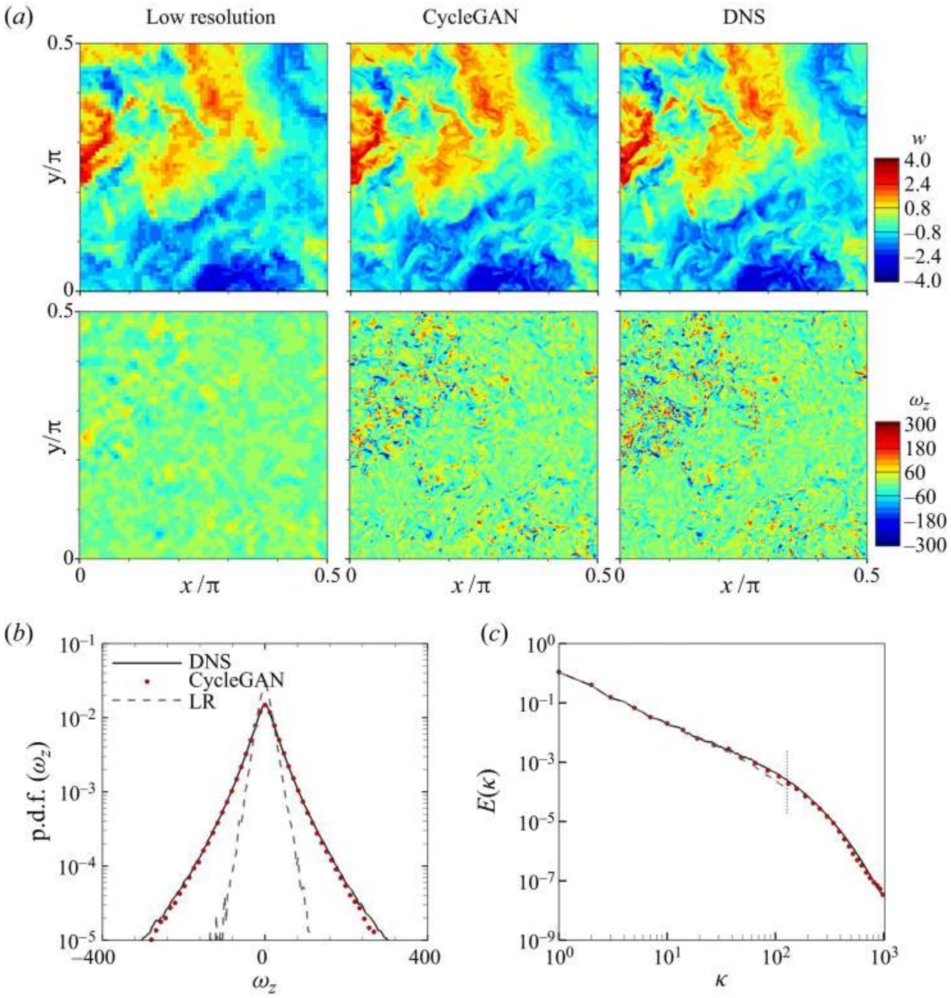


FIG. 31: Test of cycle-consistent generative adversarial network (CycleGAN) for a higher $Re_\lambda \sim 611$ than the trained value for a HIT flow. The network was trained at $Re_\lambda \sim 418$. The resolution ratio for both training and testing was 8. Panel (a) shows the recovered velocity fields (top panels) from the filtered low-resolution field and the vorticity fields (bottom panels) calculated from the recovered velocity field. Panel (b) shows the p.d.f. of the vorticity. (c) Transverse energy spectrum. The vertical dashed line denotes the cutoff wavenumber used for the low-resolution field. Fig. 7 of Kim *et al.* (2021), *J. Fluid Mech.*, with permission.

quantification approaches combined with the growth in available data have sparked renewed interest in turbulence research (Beck and Kurz, 2021; Duraisamy *et al.*, 2019).[129]

Within very recent years there has been substantial works in RANS modeling (Ling *et al.*, 2016a,b; Ling and Templeton, 2015; Matai and Durbin, 2019; Singh *et al.*, 2017), subgrid modeling for LES (Beck *et al.*, 2019; MacArt *et al.*, 2021; Maulik *et al.*, 2018, 2019a,b; Pawar and San, 2021; Wang *et al.*, 2015; Xie *et al.*, 2019; Yang *et al.*, 2019), quantification of uncertainty (Barmparousis and Drikakis, 2017; Duraisamy, 2021; Huang *et al.*, 2021; Meldi *et al.*, 2011a,b; Mishra and Iaccarino, 2019; Poroseva *et al.*, 2016, 2015), and fluid property characterization (Jiménez, 2018). There are also several fascinating articles concerns with the utilization of symbolic regression/genetic programming tools to find closures for RANS and LES (Schoepplein *et al.*, 2018; Vaddireddy *et al.*, 2020; Weatheritt and Sandberg, 2016, 2017).

Of particular interest is the use of ML to recover high-resolution turbulent flows from grossly coarse flow data in space and time. Several authors (e.g., Fukami *et al.* (2019, 2021); Kim *et al.* (2021); and Liu *et al.* (2020)) recently

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[129] A collection just appeared in *Phys. Rev. Fluids* contains seven papers invited by the editors which apply machine learning to fluid mechanics. Three of the papers are cited above in the text, and the others can be found at the following link: https://journals.aps.org/prfluids/collections/machine-learning.

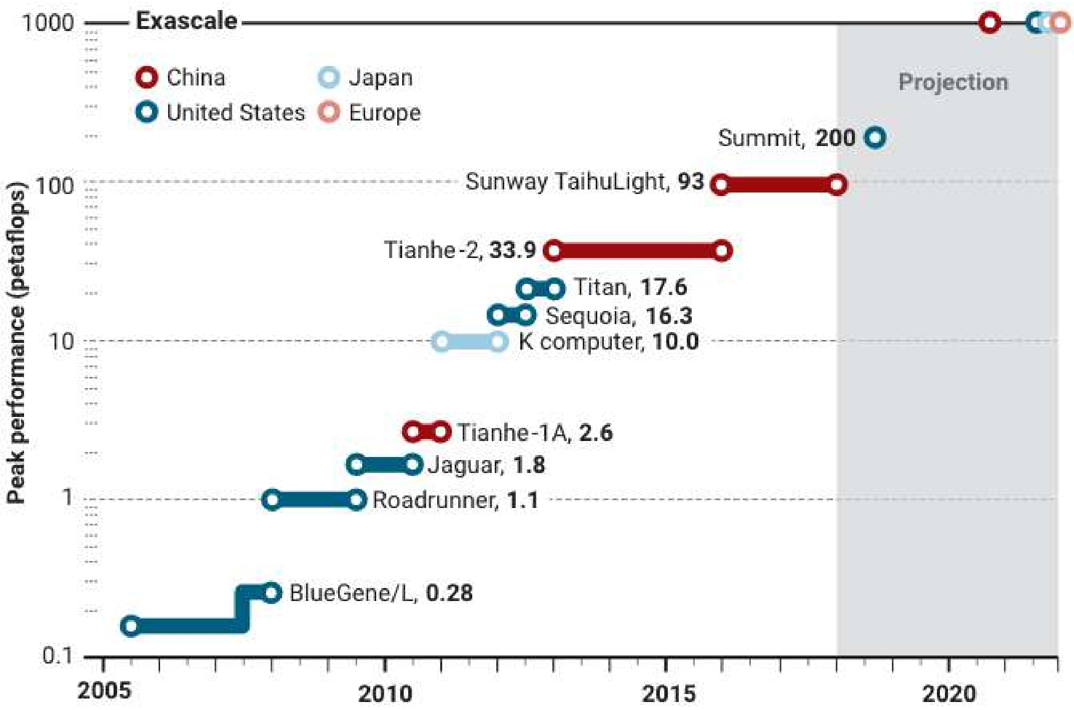


FIG. 32: Performance development of supercomputers. Data compiled from TOP500 (https://www.top500.org). Graphic credits: K. Sutliff. Reprinted from Service (2018), *Science*, with permission.

proposed super-resolution reconstruction methods for fluid flows, which have been tested for channel flows, two-dimensional laminar cylinder wake, and HIT (Fig. 31). These encouraging results suggest that the ML methods have the potential to reconstruct a range of flow fields in support of experimental or computational activities.

As summarized by Xiao and Cinnella (2019), much progress on quantifying and reducing applied model uncertainties has been fostered by: (i) the considerable increase in computer resources, (ii) the ever-increasing mass of high fidelity experimental and numerical data, and (iii) the development of statistical sampling and inference methods guided by physical constraints and prior knowledge in turbulence modeling. Based on rapid advancement in these areas, it is reasonable to anticipate that the ML and related data science may play a critical enabling role in the future of modeling complex flows.

Some have argued that emerging machine learning has demonstrated efficacy for developing and refining engineering models. Though any new techniques would have to follow well-established procedures, they will likely play a major role in the near future in determining and deploying suitable models for a given practical flow simulation.

### B. Exascale computers

In 2021, the first "exascale" computer, Frontier, is scheduled to be delivered to Oak Ridge National Laboratory in Tennessee and this may well be the tipping point. Exascale computer is a machine capable of one billion billion (quintillion, $10^{18}$) floating-point operations per second, or 1 exaflop. With a vast increase over existing systems in memory, storage, and, most importantly, compute power, exascale computing will furnish scientists with a much-needed tool to peer more deeply into the inner workings of complex turbulent flows.

Argonne's Aurora system was originally expected to be the first exascale supercomputer, but a technology delay at Intel makes it close to impossible that the system will be installed on schedule (Clark, 2020). The third planned U.S.-based exascale system, El Capitan, will be located at Lawrence Livermore National Laboratory in California. As shown in Fig. 32, China, Japan, and European Union could cross the exascale threshold around the same time.

The increased computational affordability will open a range of fresh possibilities for investigators because simulation and experimental approaches will allow them to obtain measurements for turbulent structures and any other important statistics that are unavailable from closure theories. Indeed, the exponential growth in computing power (Fig. 32) and the improvements to experimental facilities and diagnostics will increasingly provide scientists with a platform to investigate realistic complex flows beyond the reach of analytical theories. As mentioned already, the recently rapid advancements in ML, AI, Big Data, and generally data-driven methods have opened new doors for their applicability towards turbulence research.

### C. Quantum computing

Quantum computers are quantum systems that can be initialized, measured, and sufficiently controlled to solve a computational task (Ladd *et al.*, 2010; Nielsen and Chuang, 2000). In the 1980s, Feynman (1982) and, independently, Manin (1980) proposed this paradigm shift that fermented the development of quantum computers. These systems may soon become a key component of the future computing landscape as the community moves beyond cutting-edge exascale computers.

At the fundamental, physical computational level, quantum processors (at lest now), are slower than their classical bit counterparts. Quantum processors operate in a fundamentally different way, achieving unprecedented speedups by effectively exploiting the enormous potential parallelism in manipulating coherent, entangled quantum state superpositions (e.g., Outeiral *et al.* (2021)). That is, in quantum circuits, the logic gates modify not the usual binary 1 or 0 bits, but qubits, the fundamental unit of quantum computing (Matthews, 2021).

Qubits come from carefully chosen few state quantum systems, such photon polarizations, atomic hyperfine states, geometrically constrained superconducting Josephson junctions, semiconductor "dots," and basic electronic and nuclear spin systems. Unlike binary bits, qubits are built out of quantum superpositions of these basic state systems and use entanglement to create the key operating computational units. More specifically, quantum computers exploit a phenomenon called quantum entanglement (Einstein *et al.*, 1935), where qubits can also be inextricably linked together — even if they are separated by very long distances. Einstein famously described the entanglement as "spooky action at a distance."[130]

While quantum computing was originally contemplated as a means for solving NP hard combinatorial problems, and quantum mechanical physics systems, it is also tantalizing to consider its potential for solving hard classical physics problems like the simulation of turbulent flows. An interesting study sets out to explore the application of a quantum algorithm (Kacewicz, 1987, 2004, 2005, 2006) for solving the Navier–Stokes equations (Gaitan, 2020). This algorithm is based on an accelerated ordinary differential equation solver that relies on the key "quantum amplitude estimation algorithm," that in turn builds on a generalization of the quantum Grover search algorithm. The work suggests that the quantum algorithm run on a scalable quantum computer may provide a quadratic quantum speed up for direct numerical simulation of fully-developed turbulence at large Reynolds number. It is very important to realize, however, that all the simulations in this study were performed by a conventional, instead of a real quantum computer.

Although still in their infancy, there are good reasons to be optimistic about the future of quantum computers.[131] While not mature enough to solve "hard" problems for at least another decade, they held promise for an exponential speed-up to some scientific and engineering systems. Researchers still need to home in on whether turbulence is such a problem that is hard for a classical computer but easy for a quantum computer.[132] It would be even better if turbulence could be served as an ultimate testbed of difficult problems in physics and chemistry to demonstrate its *quantum supremacy*.[133]

One opinion piece cautions that "building a universal, fault-tolerant quantum computer—one that can correct errors deriving from imperfections in its operations or from environmental disturbances—is a daunting task. Fault-tolerant quantum computers that can be scaled up to solve meaningful problems are a decade away (plus or minus infinity, given the incertitude inherent in such predictions)" (Lloyd, 2021). This wide margin of error for the estimated crossover dates, to a large extent, is associated with various issues related to the maturation of quantum device technology. As an example, even minute charged impulses, such as those from cosmic rays absorbed by the system, can create a blast of relatively high-energy electrons that could heat the quantum device's substrate just long enough to disrupt the qubits and disturb their quantum states (Wilen *et al.*, 2021).

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[130] Quantum entanglement is at the center of a new mathematical proof manuscript (Ji *et al.*, 2020), but the confluence of complexity theory, quantum information and mathematics means that there are very few researchers in the world who say that they can grasp all the facets of the paper. Fields medalist Alain Connes himself told *Nature* that he was not qualified to comment (Castelvecchi, 2020).

[131] The cutting edge of the race for a quantum computer is a quantum spin model with tunable interactions for system sizes ranging from 64 to 256 qubits. According to Ebadi *et al.* (2021), this programmable quantum simulator is based on deterministically prepared two-dimensional arrays of neutral atoms, featuring strong interactions controlled by coherent atomic excitation into Rydberg states (Browaeys and Lahaye, 2020). Offering a new lens into the study of complex quantum matter, the simulator has already allowed physicists to observe several exotic quantum states of matter that had never before been realized experimentally.

[132] For instance, a task of computing properties of physical problems that are inherently quantum mechanical would be a more natural candidate to a quantum computer, for obvious reasons.

[133] "Quantum supremacy," a phrase coined by Caltech particle physicist John Preskill, refers to the ability of a quantum computer to surpass the fastest conventional machine. A team from Google, University of California at Santa Barbara, and NASA Ames Research Center reports that its quantum computer carried out a specific calculation that is beyond the practical capabilities of conventional machines (Arute *et al.*, 2019). The same calculation would take even the best classical supercomputer 10,000 years to complete, according to Google.

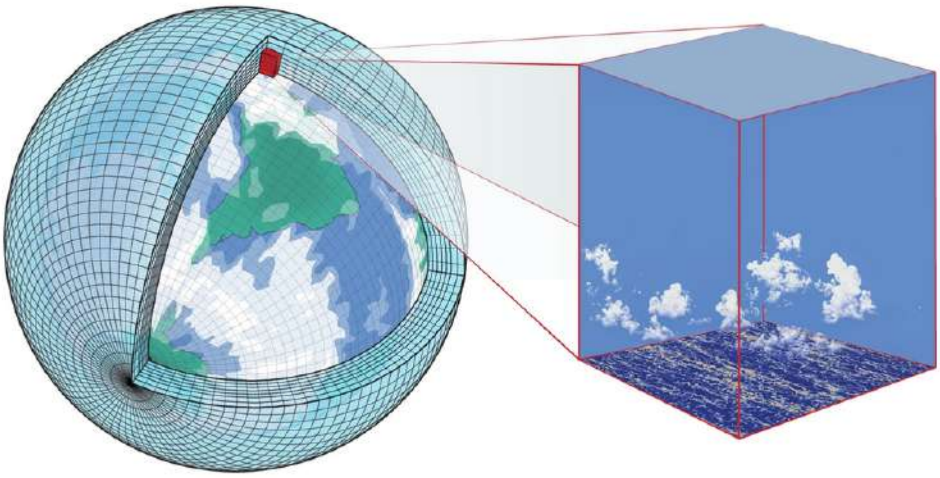

FIG. 33: The colours at the bottom of the LES domain indicate the buoyancy of near-surface air. The horizontal grid spacing of global climate models is approaching the feasible domain size in limited-area cloud simulations. This creates fresh opportunities for developing parameterizations of the turbulent dynamics of low clouds, such as the shallow cumulus clouds here, for climate models with grid spacings of order kilometers and larger (Pressel *et al.*, 2015). Reprinted with permission from Schneider *et al.* (2017), *Nature Climate Change*, with permission.

### D. Climate predictions

Now, by the use of some illuminating examples, a case is made that turbulence research may soon play a more prominent role in providing more accurate climate scientific data.

Modern climate modeling studies indicate that the amount of clear air turbulence on transatlantic flight routes (Williams, 2017; Williams and Joshi, 2013) and over the North Pacific (Watanabe *et al.*, 2019) will increase significantly in the future as the climate changes. Clear air turbulence is already one of the largest causes of weather-related aviation incidents. Often, pilots cannot avoid it, because it is invisible to the naked eye and undetectable by onboard sensors (Storer *et al.*, 2017).

How clouds respond to warming remains one of the greatest sources of uncertainty in climate projections (Schneider *et al.*, 2017). Since fast-changing air motions are primarily responsible for cloud lifetimes, turbulent effects influence the cloud dynamics, and turbulent closure schemes affect the representation of a variety of cloud types (e.g., Grabowski (2007), Ohno *et al.* (2020) and references therein). The physics of clouds remains challenging to represent in climate models since their controlling processes occur on such small, subgrid scales. Even with a 1 km grid, key processes such as subgrid turbulence, the effects of aerosols, and cloud microphysics need to be parameterized (Palmer, 2019). Care must be taken to consider the interaction of numerical error with the SGS closures used in atmospheric LES (Fig. 33), especially where sharp transitions in mixing strength occur (Pressel *et al.*, 2017).

High clouds are chiefly composed of ice-phase hydrometeors, and are effective at trapping longwave radiation (Liou, 2002). Ohno *et al.* (2020) show that a more sophisticated treatment of SGS ice clouds physics in the turbulence, such as incorporating a realistic ice condensation time scale, could result in an improved projection of changes of high clouds due to global warming.

Stratocumulus clouds cover $\sim$20% of the low-latitude oceans and are especially prevalent in the subtropics (Schneider *et al.*, 2017). They cool the Earth by shading large portions of its surface from sunlight and reflect more solar radiation than the surface of the ocean (Hartmann *et al.*, 1992; Wood, 2012). Drizzle is constantly present below and within stratocumulus clouds. Using radar, lidar, and radiometer data, a new study found that drizzle decreases turbulence below stratocumulus clouds, an observation, which, in turn, provided a more realistic representation of drizzle-turbulence interactions for climate models (Zheng *et al.*, 2020).

The Arctic region is experiencing the most rapid environmental changes on Earth, with unparalleled air temperature increases, a warming ocean, and melting permafrost, snow, and ice (Timmermans and Marshall, 2020). In contrast to their subtropical counterparts, Arctic clouds warm the surface in all seasons except summer because their long-wave greenhouse effect predominates over short-wave effects, (Intrieri *et al.*, 2002; Kay and L'Ecuyer, 2013). The presence of snow and sea ice allows the Arctic surface temperature to respond more quickly to cloud changes than over subtropical oceans (Zhang *et al.*, 2020). The low-cloud response is sensitive to parametrizations of clouds, turbulence,

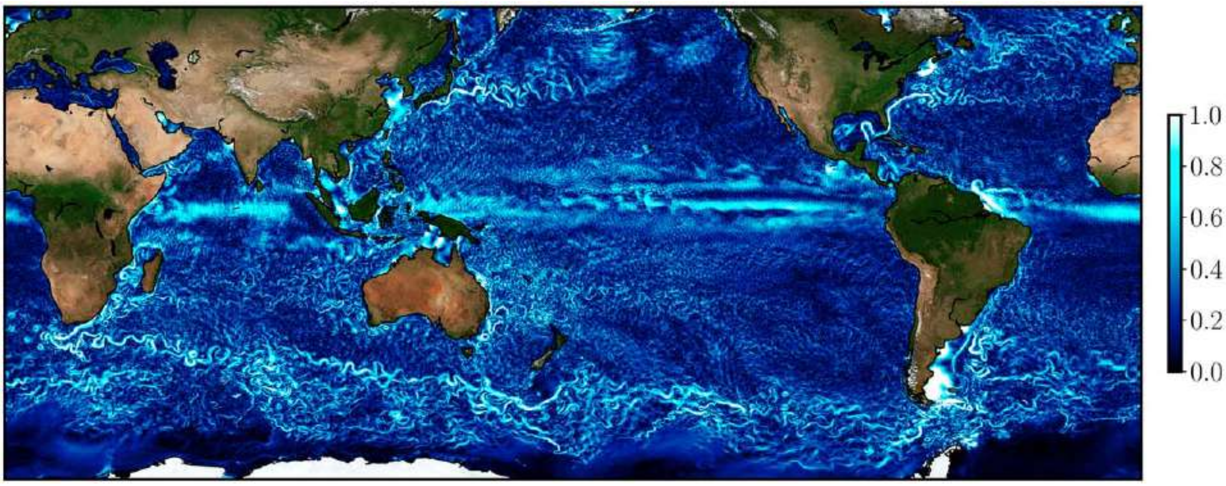


FIG. 34: Instantaneous snapshot of the kinetic energy in the global ocean, from a global Estimating the Circulation and Climate of the Ocean (ECCO) numerical simulation ((1/48)° degree resolution in the horizontal and 90 vertical levels). Fig. 3 of Klein et al. (2019). Credit: NASA (See https://science.jpl.nasa.gov/projects/ECCO–ICES/).

and convection (Yoshimori *et al.*, 2014). Understanding this issue is a first step toward understanding how Arctic low clouds respond to climate change (Zhang *et al.*, 2020).

Mesoscale turbulence (on scales from 50 to 200 km) in the ocean is characterized by an inverse energy cascade (Aluie *et al.*, 2018) to larger scales in the oceanic circulation and it strongly affects the circulation, water mass formation, and transport of tracers (Busecke and Abernathey, 2019). This ocean mesoscale turbulence is characterized by cyclonic and anticyclonic eddies (with a 100- to 500-km size and depth scales of 500–1,000 m) that capture approximatively 60% to 80% (Buzzicotti *et al.*, 2021) of the total kinetic energy and is now known to significantly impact the Earth's climate as a whole (Fig. 34) (Klein *et al.*, 2019). Although energetic eddies are present everywhere, they are intensified in hot spots associated with major oceanic currents such as the Gulf Stream (Fig. 35).

It was recently shown that this mesoscale turbulence is damped by wind, despite the latter being the primary driver of the oceanic general circulation (Rai *et al.*, 2021). This study, based on satellite observations and a method to disentangle multiscale processes on the sphere, showed that "eddy-killing" is a major seasonal sink for the oceanic eddies, removes a substantial fraction (up to 90%) of the wind power input in western boundary currents such as the Gulf Stream and Kuroshio. Eddy killing, often overlooked in analyses and models, seems to be a major dissipation pathway for mesoscales, reaching its peak in winter. Note also that the temporal mean-eddy decomposition (i.e., Reynolds averaging) typically used in oceanography fails to unravel eddy-killing.

Until lately, climate modelers simply didn't worry too much about how turbulent processes in the ocean (known as ocean mixing) affected the outputs of their models (Schiermeier, 2007). Turbulent mixing plays a key role in determining the ventilation of the ocean as well as the ocean uptake and storage of heat and carbon (e.g., Ferrari and Wunsch (2009); Melet *et al.* (2016); and Wunsch and Ferrari (2004)). In the deep ocean, tides supply much of the mechanical energy required to sustain mixing via the generation of internal waves, known as internal tides, whose fate—the relative importance of their local versus remote breaking into turbulence—remains uncertain. Vic *et al.* (2019) show that from an energetic viewpoint, small-scale internal tides, hitherto overlooked, account for the bulk (>50%) of global internal tide generation, breaking, and mixing. Moreover, the authors found that the pronounced geographical variations of their energy proportion, ignored by current parameterizations of mixing in climate-scale models.

In the last decade, marine heatwaves and climate change have led to mass bleaching events and coral mortality, significantly contributing to their worldwide decline (Hughes*et al.*, 2017, 2018). An improved understanding of turbulent processes on coral reefs is crucial for the prediction of momentum, energy, and scalar transport. Furthermore, Davis *et al.* (2021) pointed out that a defining physical characteristic of coral reefs, their extreme hydrodynamic roughness, makes them an ideal environment for the study of turbulence. By way of illustration, we introduce an example drawn from recent research into the role of turbulent hydrodynamics and surface morphology on heat and mass transfer in corals. Stocking *et al.* (2018) found that for rough corals in wave-dominated flows, novel trade-off dynamics for heat and mass transfer exist between narrowly spaced roughness that produces greater surface area versus broadly

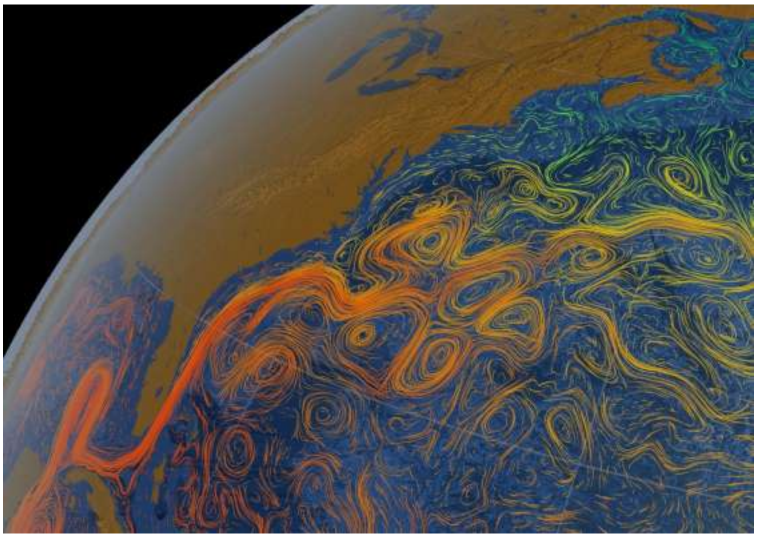

FIG. 35: Eddies such as those seen in the Gulf Stream may play a part in mixing the oceans' water. Image: NASA/Goddard Space Flight Center Scientific Visualization Studio.

spaced roughness that enhances turbulence production.[134] These findings have important implications since those corals best able to mitigate thermal stress through more efficient heat transfer should be expected to demonstrate preferential survivorship and faster recovery following bleaching events, which have become increasingly common with global climate change.

More broadly speaking, adequate treatments of issues intimately related to turbulence are important for the design of next-generation climate models (see a current perspective by Palmer (2019)). The interested reader is also referred to Ghil and Lucarini (2020) for the detailed considerations on the physics of climate variability and climate change.[135]

### E. Concluding Remarks

One of the major issues in turbulence research is that it does not have a "standard model" and that statistical closure theories are currently the closest paradigm available. A "standard model" is a theoretical framework built from the observation that predicts and correlates new data (Gaillard *et al.*, 1999). It also sets the agenda by asking prudent and penetrating questions, which is of course the most difficult part of making significant advancement.[136] As Sreenivasan (1999) noted in his centenary essay: "What is clear is that progress will depend on controlled measurements and computer simulations at high Reynolds numbers, and the ability to see in them the answers to the right theoretical questions."

Some may argue that such a standard model is unnecessary, but when looking at other fields in physics, it becomes apparent that this development would add tremendous value. Professor James Peebles was awarded the 2019 Nobel Prize in Physics for establishing the theoretical framework for the cosmic microwave background and for developing the current standard model for the universe's evolution (Gibney and Castelvecchi, 2019). As Mats Larsson, the chair of the Nobel committee stated, if it was "not for the theoretical discoveries of James Peebles, the wonderful high-precision measurements of this radiation over the last 20 years would have told us almost nothing."

Peter Bradshaw, a well-known turbulence expert, once humorously opined that turbulence might be the "invention of the Devil on the seventh day of creation" (Bradshaw, 1994). The rapid advancement of supercomputers and experimental facilities will undoubtedly help researchers tame this Devil moving forward, but scientists must work vigorously to prevent turbulence from becoming the first branch of physics in which the theoretical leg of the trinity is overshadowed.

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[134] As noted by Davis *et al.* (2021), additional calculations at a much higher Schmidt number, along with a refined near-wall treatment, are needed to make the simulations more realistic.

[135] As shown in this subsection, it is imperative to encourage turbulence researchers to broaden their scope and methodology to tackle the challenge of global climate change. This viewpoint is particularly pertinent as there are currently just a few articles where attempts have been made to undeviatingly use the type of statistical closures that we have described in earlier sections for climate models (e.g., Frederiksen *et al.* (2003, 2015)).

[136] V.I. Arnold, a leading Russian scientist, wrote in a remembrance of his mentor A.N. Kolmogorov: "[h]e never explained anything, just posed problems" (Gleick, 1987).

## XVI. ACKNOWLEDGMENTS

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