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ORIGINAL PAPER

An efficient algorithm for the projection of a point on the intersection of two hyperplanes and a box in $\mathbb{R}^n$

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Abstract
In this work, we present an efficient strongly polynomial algorithm for the projection of a point on the intersection of two hyperplanes and a box in $\mathbb{R}^n$. Interior point methods are the most efficient algorithm in the literature to solve this problem. While efficient in practice, the complexity of interior-point methods is bounded by a polynomial in the dimension of the problem and in the accuracy of the solution. Moreover, their efficiency is highly dependent on a series of parameters depending on the specific method chosen (especially for nonlinear problems), such as step size, barrier parameter, accuracy, among others. We propose a new method based on the KKT optimality conditions. In this method, we write the problem as a function of the Lagrangian multipliers of the hyperplanes and seek to find the pair of multipliers that corresponds to the optimal solution. We prove that the algorithm has complexity $O(n^2 \log n)$.

Keywords Orthogonal projections · Convex programming · Separable quadratic programming · Interior-point methods · Strongly polynomial algorithm

Mathematics Subject Classification 90-08 Computational methods

1 Introduction

Interior-point methods are the most efficient algorithm in the literature to solve projection problems with more than one constraint [see Byrd et al. (1995), Lin and Moré (1998) and Morales and Nocedal (2011)]. While very efficient in practice, their complexity is bounded by a polynomial in the dimension of the problem and in the accuracy of the solution [see Ben-Tal and Nemirovski (2001) and Wright (1987)]. In addition,

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their efficiency is highly dependent on a series of parameters depending on the specific method chosen, such as step size, barrier parameter, accuracy.

In this document, we present an efficient strongly polynomial algorithm for the intersection of two hyperplanes and a box in $\mathbb{R}^n$. We extend a previous result from Maculan et al. (2003). In Maculan and Paula 1989, the problem of finding the projection of a vector on the simplex of $\mathbb{R}^n$ was defined as a special quadratic programming problem. The authors presented an $O(n)$ time algorithm. In Maculan et al. (1997), a new linear-time algorithm was presented for projecting a vector on the intersection of a hyperplane and $\mathbb{R}^n_+$. Different approaches were proposed in Brucker (1984), Michelot (1986), and Pardalos and Kovoor (1990). Kiwiel Kiwiel (2008) also provides approaches to solve quadratic problems with one linear constraint.

The approach taken to solve the problem with two hyperplanes is the same as the one used in Maculan et al. (2003); i.e., we search a pair of Lagrangian multipliers that will solve the problem. However, since $\mathbb{R}^2$ is not endowed with the total order relation, we need to study the combinations of signs of the coefficients of both hyperplanes and how that affects our search for the desired pair of multipliers.

This document is organized as follows: In Sect. 2, we present some definitions and basic results. In Sect. 3, we use the results on the previous section to present the algorithm as well as its complexity. In Sect. 4, we present an example to solve a projection problem on $\mathbb{R}^3$ in order to illustrate the algorithm mechanics. In Sect. 5, we present an application of this work to solve an optimal control problem. In Sect. 6, we show some numerical experiments arising from the comparison of the method proposed on this document and the CPLEX solver for different precisions. In Sect. 7, we present the concluding remarks and the future work.

Throughout this document, let $J = \{1, \ldots, n\}$, where n is the dimension of the problem.

2 The problem

In this section, we define the problem and present some basic results.

Let $\overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)^T \in \mathbb{R}^n$ be given. Also, let $a, c \in \mathbb{R}^n$, $b_1, b_2 \in \mathbb{R}$ and $\mu, d \in \mathbb{R}^n$ such that $\mu_i < d_i$, $\forall i \in J$. We want to solve the following problem:

$$
\begin{aligned}
(P) := \min \quad & \tfrac{1}{2} \sum_{i \in J} (x_i - \overline{x}_i)^2 \\
\text{s.t.} \quad & \\
& \sum_{i \in J} a_i x_i = b_1 \qquad (1) \\
& \sum_{i \in J} c_i x_i = b_2 \\
& \mu_i \leq x_i \leq d_i, \ \forall i \in J. \qquad (2)
\end{aligned}
$$

The strict inequality $\mu_i < d_i$, $\forall i \in J$, required does not lead to loss of generality, since if $\mu_k = d_k$, for some k, then the corresponding variable can be eliminated from the problem.

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We also suppose that the vectors a, c are linearly independent.

In order to simplify our presentation, without loss of generality, we will suppose that $\mu_i = 0,\ \forall i \in J$.

The linearity of the constraints guarantees that any minimizer satisfies KKT conditions. Therefore, the KKT conditions are necessary and sufficient. Thus, there exist Lagrangian multipliers $\alpha, \beta, v_i, w_j \in \mathbb{R},\ \forall i \in J$, such that if $p_i = \overline{x}_i - \alpha a_i - \beta c_i,\ i \in J$, then, for each $i \in J$, we have the following system of inequalities defined:

$$x_i - p_i - v_i + w_i = 0 \tag{3}$$
$$v_i x_i = 0 \tag{4}$$
$$w_i(x_i - d_i) = 0 \tag{5}$$
$$0 \leq x_i \leq d_i \tag{6}$$
$$v_i \geq 0 \tag{7}$$
$$w_i \geq 0, \tag{8}$$

where $x_i, v_i, w_i \in \mathbb{R}$ are variables and $p_i, d_i \in \mathbb{R}$, such that $d_i > 0$, are parameters, $\forall i \in J$.

Lemma 1 *Let $p \in \mathbb{R}$ and $d > 0$. Then, for each $i \in J$, the system composed of inequalities (3)–(8) has a unique solution:*

$$w_i = w(p_i, d_i) = \max\{0, p_i - d_i\}, \tag{9}$$
$$v_i = v(p_i, d_i) = \max\{0, -p_i\}, \tag{10}$$
$$x_i = x(p_i, d_i) = \max\{0, \min\{p_i, d_i\}\}. \tag{11}$$

Proof A proof for this lemma can be found in Maculan et al. (2003). □

Therefore, we need to compute $\alpha^*, \beta^* \in \mathbb{R}$ such that constraints (1) and (2) are satisfied, i.e., such that $\sum_{i \in J} a_i x_i(\alpha^*, \beta^*) = b_1$ and $\sum_{i \in J} c_i x_i(\alpha^*, \beta^*) = b_2$. Then, $x(\alpha^*, \beta^*)$ will be the solution of problem (P).

3 An $O(n^2 \log n)$ algorithm

In this section, we present the algorithm, along with the proof of its complexity and correctness.

Throughout this section, we assume that $a_j, c_j \neq 0,\ j \in J$. If this is not the case, we can always produce a linear combination with the desired property, as long as either $a_j \neq 0$ or $c_j \neq 0$ for each $j \in J$ (if for some $k \in J$ we have $a_k = c_k = 0$, this variable can be removed from the problem). We assume also that $d_j > 0,\ \forall j$. Otherwise, since $0 \leq x_j \leq d_j$, we have $x_j = 0$ and this variable can be removed from the problem.

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3.1 Basic results

Let $\varphi_1 : \mathbb{R}^2 \to \mathbb{R}$ and $\varphi_2 : \mathbb{R}^2 \to \mathbb{R}$ be:

$$\varphi_1(\alpha, \beta) := \sum_{i \in J} a_i x_i(\alpha, \beta) - b_1,$$
$$\varphi_2(\alpha, \beta) := \sum_{i \in J} c_i x_i(\alpha, \beta) - b_2.$$

Therefore, we see that φ_1 and φ_2 are the residuals of the two hyperplane constraints. The approach presented in this paper consists of computing the level sets of $\varphi_1(\cdot, \cdot)$ and $\varphi_2(\cdot, \cdot)$ through the study of the properties of the functions $x_i(\cdot, \cdot)$. Firstly, we consider $x_i(\cdot, \cdot)$ by fixing one of the arguments so that it becomes a function of one variable which has properties that can be exploited. Next, we show that if $x_i(\cdot, \cdot)$ is positive, then we can obtain α as a function of β, this leads to an easy way of studying the properties of the original functions whose level sets we are interested in (i.e., the level sets of $\varphi_1(\cdot, \cdot)$ and $\varphi_2(\cdot, \cdot)$), since these functions will be linear combinations of the $x_i(\cdot, \cdot)$ functions, whenever these functions are positive. In order for us to identify where such functions are positive, a crucial step of our approach is to compute the breakpoints of $x_i(\cdot, \cdot)$, which, in turn, correspond to breakpoints of $\varphi_1(\cdot, \cdot)$ and $\varphi_2(\cdot, \cdot)$. Therefore, this section consists of proving properties of $x_i(\cdot, \cdot)$, $\varphi_1(\alpha, \beta)$ (and $\varphi_2(\alpha, \beta)$) that allows us to compute the level sets of $\varphi_1(\alpha, \beta)$ (and $\varphi_2(\alpha, \beta)$) using only one parameter (β), thereby reducing a two-dimensional problem to a series of one-dimensional problems that arise from computing the breakpoints of $x_i(\cdot, \cdot)$.

From (1) and (2), we need to compute α^*, β^* such that:

$$\varphi_1(\alpha^*, \beta^*) = 0, \tag{12}$$
$$\varphi_2(\alpha^*, \beta^*) = 0. \tag{13}$$

Let $\hat{\alpha}, \hat{\beta} \in \mathbb{R}$. Let us define the following level sets:

$$S_1(\hat{\beta}) := \{(\alpha, \hat{\beta}) \in \mathbb{R}^2 \mid \varphi_1(\alpha, \hat{\beta}) = 0\}, \tag{14}$$
$$S_2(\hat{\alpha}) := \{(\hat{\alpha}, \beta) \in \mathbb{R}^2 \mid \varphi_2(\hat{\alpha}, \beta) = 0\}. \tag{15}$$

Using the definitions of level sets (14) and (15), let us define two new sets:

$$S_1 := \bigcup_{\beta \in \mathbb{R}} S_1(\beta), \tag{16}$$
$$S_2 := \bigcup_{\alpha \in \mathbb{R}} S_2(\alpha). \tag{17}$$

Thus, S_1 is the set that contains all pairs (α^*, β^*) such that $\varphi_1(\alpha^*, \beta^*) = 0$. Analogously, S_2 is the set that contains all pairs (α^*, β^*) such that $\varphi_2(\alpha^*, \beta^*) = 0$.

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3.1.1 Basic properties of φ_1 and φ_2

We will study now basic properties of the constraint functions φ_1 and φ_2 by analyzing their behavior when one of their coordinates is fixed. Since both φ_1 and φ_2 are in $\mathbb{R}^2$, the properties proved here will help us reduce the problem of searching for solutions for our original level set problem in $\mathbb{R}^2$ to the easier task of searching for solutions for the level sets in $\mathbb{R}$.

We introduce the following notation that will be useful throughout this section:

$$z_i(\alpha, \beta) := \overline{x}_i - \alpha a_i - \beta c_i.$$

For $\hat{\alpha}, \hat{\beta} \in \mathbb{R}$, we define the following functions, which are equal to the LHS of the two hyperplane constraints:

$$\phi_{1,\hat{\beta}}(\alpha) := \varphi_1(\alpha, \hat{\beta}) + b_1, \tag{18}$$

$$\phi_{2,\hat{\alpha}}(\beta) := \varphi_2(\hat{\alpha}, \beta) + b_2. \tag{19}$$

Lemma 2 *$\phi_{1,\hat{\beta}}(\cdot)$ and $\phi_{2,\hat{\alpha}}(\cdot)$ are nonincreasing functions, $\forall \hat{\beta} \in \mathbb{R}$.*

Proof We will prove the result for $\phi_{1,\hat{\beta}}(\cdot)$. The proof for $\phi_{2,\hat{\alpha}}(\cdot)$ is analogous.

Since adding a constant will not change our result, we will show that $\phi_{1,\hat{\beta}}(\alpha)$ is nonincreasing. From the previous section, we have:

$$\phi_{1,\hat{\beta}}(\alpha) = \sum_{i \in J} a_i x_i(\alpha, \hat{\beta}) = \sum_{i \in J} a_i \max\{\min\{z_i(\alpha, \hat{\beta}), d_i\}, 0\}.$$

Let $q_i = \overline{x}_i - \hat{\beta} c_i$, $\forall i \in J$. If $\hat{\beta}$ is fixed, then q_i is constant, $\forall i \in J$. Thus, we have:

$$\phi_{1,\hat{\beta}}(\alpha) = \sum_{i=1}^{n} a_i \max\{\min\{q_i - \alpha a_i, d_i\}, 0\}. \tag{20}$$

Now, for every $i \in J$, let $\lambda_i(\alpha) = \max\{0, \min\{q_i - \alpha a_i, d_i\}\}$. Clearly, we have the following characterization of $\lambda_i(\cdot)$:

$$\begin{aligned} &\lambda_i(\cdot) \quad \text{is nonincreasing, } \forall \alpha \text{ if } a_i > 0, \\ &\lambda_i(\cdot) \quad \text{is nondecreasing, } \forall \alpha \text{ if } a_i < 0. \end{aligned}$$

Thus,

$$a_i \lambda_i(\cdot) \quad \text{is nonincreasing, } \forall \alpha, \ \forall i \in J. \tag{21}$$

Therefore, from (20) and (21), we can conclude that $\phi_{1,\hat{\beta}}(\alpha)$ is a nonincreasing function of α, since it is the sum of nonincreasing functions of α. □

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Corollary 3 *The level sets of the functions $\phi_{1,\hat{\beta}}(\cdot)$ and $\phi_{2,\hat{\alpha}}(\cdot)$ defined in (18) and (19), respectively, are convex subsets of the real line.*

Lemma 4 *The following assertions are true:*

1. *If $S_1(\hat{\beta}) \neq \emptyset$ for some $\hat{\beta} \in \mathbb{R}$, then $S_1(\beta) \neq \emptyset$, $\forall \beta \in \mathbb{R}$;*
1. *If $S_2(\hat{\alpha}) \neq \emptyset$ for some $\hat{\alpha} \in \mathbb{R}$, then $S_2(\hat{\alpha}) \neq \emptyset$, $\forall \alpha \in \mathbb{R}$.*

Proof We will prove only Assertion 1. Assertion 2 can be proven analogously.

Since the feasibility of the projection problem does not depend on the point to be projected, i.e., either it will have a solution for any point or it will be infeasible, i.e., the box does not intersect the hyperplane. □

3.1.2 Properties of $\varphi_1|_{S_1}$ and $\varphi_2|_{S_2}$

In this part, we will examine the properties of the restriction of φ_1 and φ_2 to the sets S_1 and S_2, since the solution of the projection problem boils down to finding points on these sets.

Let us define the following index sets:

$$\begin{aligned} J^{a-} &:= \{i : a_j < 0,\ i \in J\}, \\ J^{a+} &:= \{i : a_j > 0,\ i \in J\}, \\ J^{c-} &:= \{i : c_j < 0,\ i \in J\}, \\ J^{c+} &:= \{i : c_j > 0,\ i \in J\}. \end{aligned}$$

The following quantities will be useful in the rest of the section:

$$\underline{b}_1 = \sum_{j \in J^{a-}} a_j d_j, \tag{22}$$

$$\overline{b}_1 = \sum_{j \in J^{a+}} a_j d_j, \tag{23}$$

$$\underline{b}_2 = \sum_{i \in J^{c-}} c_j d_j, \tag{24}$$

$$\overline{b}_2 = \sum_{j \in J^{c+}} c_j d_j. \tag{25}$$

Lemma 5 *Assume (P) is feasible. Then, the following assertions are true:*

1. *$b_1 \in [\underline{b}_1, \overline{b}_1]$;*
2. *$b_2 \in [\underline{b}_2, \overline{b}_2]$.*

Moreover, since $a_i \neq 0, \forall i \in J$ (and $c_i \neq 0, \forall i \in J$), the convex subsets from Corollary 3 will be strict.

Proof We will prove only Assertion 1. Assertion 2 can be proven analogously.

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If (P) is feasible, then $\exists \alpha, \beta \in \mathbb{R}$ such that $\varphi_1(\alpha, \beta) = 0$. Since

$$\underline{b}_1 \leq \varphi_1(\alpha, \beta) + b_1 \leq \overline{b}_1 \Rightarrow \underline{b}_1 \leq b_1 \leq \overline{b}_1, \tag{26}$$

the first part of the lemma follows.

We will now prove the second part of the lemma. First, we will prove the following implication:

$$\underline{b}_1 \neq \overline{b}_1 \Rightarrow \text{ the convex subsets from Corollary 3 will be strict.} \tag{27}$$

Let $L_b(\phi_{1,\hat{\beta}})$ be the level set of the function $\phi_{1,\hat{\beta}}$ on b, i.e., $L_b(\phi_{1,\hat{\beta}}) = \{\alpha : \phi_{1,\hat{\beta}}(\alpha) = b\}$. If $\underline{b}_1 \neq \overline{b}_1$, then $\underline{b}_1 < \overline{b}_1$. Let $\underline{\alpha}, \overline{\alpha} \in \mathbb{R}$ be such that $\phi_{1,\hat{\beta}}(\underline{\alpha}) = \underline{b}_1$ and $\phi_{1,\hat{\beta}}(\overline{\alpha}) = \overline{b}_1$. Therefore, $\underline{\alpha}$ and $\overline{\alpha}$ cannot both belong to $L_b(\phi_{1,\hat{\beta}})$, $\forall b \in \mathbb{R}$. Thus, $L_b(\phi_{1,\hat{\beta}}) \subsetneq \mathbb{R}$, thereby proving (27).

We will prove now the following equivalence:

$$\underline{b}_1 = \overline{b}_1 \Leftrightarrow a_i = 0, \ \forall i \in J. \tag{28}$$

As noted before, we assume $d_j > 0, \ j \in J$. If $\underline{b}_1 = \overline{b}_1$, then $\sum_{j \in J^{a-}} a_j d_j = \overline{b}_1 = \sum_{j \in J^{a+}} a_j d_j$. Thus, $\overline{b}_1 = \sum_{j \in J^{a+}} a_j d_j + \sum_{j \in J^{a-}} (-a_j) d_j = 0$. Since $d_j > 0$, and $a_j \geq 0$ for $i \in J^{a+}$ and $-a_j \geq 0$ for $i \in J^{a-}$, then the above expression is true if and only if $a_j = 0, \forall j$.

Therefore, the result follows from equivalence (28) and implication (27). □

Lemma 6 *Suppose that x^* is a solution to problem (P). Then, the following assertions are true:*

1. *If $b_1 = \underline{b}_1$ then $x_j^* = d_j, \ j \in J^{a-}$ and $x_j^* = 0, \ j \in J - J^{a-}$;*
2. *If $b_1 = \overline{b}_1$ then $x_j^* = d_j, \ j \in J^{a+}$ and $x_j^* = 0, \ j \in J - J^{a+}$;*
3. *If $b_2 = \underline{b}_2$ then $x_j^* = d_j, \ j \in J^{c-}$ and $x_j^* = 0, \ j \in J - J^{c-}$;*
4. *If $b_2 = \overline{b}_2$ then $x_j^* = d_j, \ j \in J^{c+}$ and $x_j^* = 0, \ j \in J - J^{c+}$.*

Proof We will prove Assertion 1. Assertions 2–4 can be proven analogously.

Let $y_j(\alpha, \beta) = \max\{z_j(\alpha, \beta), 0\}$. Let α^* and β^* be the corresponding dual variables that solve problem (P). Then, from the previous results, $x_j^* = \min\{y_j(\alpha^*, \beta^*), d_j\}$, $\forall j \in J$. Thus,

$$\sum_{j \in J} \min\{y_j(\alpha^*, \beta^*), d_j\} a_j = b_1 \, .$$

Since $b_1 = \underline{b}_1$:

$$\sum_{j \in J} \min\{y_j(\alpha^*, \beta^*), d_j\} a_j = \underline{b}_1 = \sum_{j \in J^{a-}} a_j d_j$$

$$\sum_{j \in J} \min\{y_j(\alpha^*, \beta^*), d_j\} a_j - \sum_{j \in J^{a-}} a_j d_j = 0$$

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The last equality can be arranged as:

$$\sum_{j \in J - J^{a-}} \min\{y_j(\alpha^*, \beta^*), d_j\} a_j + \sum_{j \in J^{a-}} (\min\{y_j(\alpha^*, \beta^*), d_j\} - d_j) a_j = 0 \quad (29)$$

Since $\min\{y_j(\alpha^*, \beta^*), d_j\} \geq 0$ and $a_j > 0, j \in J - J^{a-}$, then

$$\sum_{j \in J - J^{a-}} \min\{y_j(\alpha^*, \beta^*), d_j\} a_j \geq 0 \,. \quad (30)$$

Since $\min\{y_j(\alpha^*, \beta^*), d_j\} - d_j \leq 0$ and $a_j < 0, j \in J^{a-}$, then

$$\sum_{j \in J^{a-}} (\min\{y_j(\alpha^*, \beta^*), d_j\} - d_j) a_j \geq 0 \,. \quad (31)$$

Inequalities (30) and (31) imply that both must be zero in order for equality (29) to hold. Since each term of (30) is nonnegative, then the following equivalence holds:

$$\sum_{j \in J - J^{a-}} \min\{y_j(\alpha^*, \beta^*), d_j\} a_j \geq 0$$
$$\iff \min\{y_j(\alpha^*, \beta^*), d_j\} a_j = 0 \,,\ j \in J - J^{a-}$$

The same observation holds for (31):

$$\sum_{j \in J^{a-}} (\min\{y_j(\alpha^*, \beta^*), d_j\} - d_j) a_j$$
$$\iff (\min\{y_j(\alpha^*, \beta^*), d_j\} - d_j) a_j = 0 \,,\ j \in J^{a-}$$

Since $a_j \neq 0 \,, \forall j \in J$, then (29) holds only if

$$\min\{y_j(\alpha^*, \beta^*), d_j\} = 0 \,,\ j \in J - J^{a-} \quad (32)$$
$$\min\{y_j(\alpha^*, \beta^*), d_j\} - d_j = 0 \,,\ j \in J^{a-} \quad (33)$$

From (32) and (33), respectively, we have that

$$y_j(\alpha^*, \beta^*) = 0 \Rightarrow x_j^* = 0 \,,\ j \in J - J^{a-}$$
$$y_j(\alpha^*, \beta^*) = d_j \Rightarrow x_j^* = d_j \,,\ j \in J^{a-} \,. \quad (34)$$

□

In light of Lemma 6, we will assume both $b_1 \in (\underline{b}_1, \overline{b}_1)$ and $b_2 \in (\underline{b}_2, \overline{b}_2)$. We will now bound the feasible region defined by the hyperplane constraints.

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Let us define the following functions $\underline{\psi}_1 : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, $\overline{\psi}_1 : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, $\underline{\psi}_2 : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, and $\overline{\psi}_2 : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$:

$$\underline{\psi}_1(\beta) := \inf\{\alpha : (\alpha, \beta) \in S_1(\beta)\}, \tag{35}$$

$$\overline{\psi}_1(\beta) := \sup\{\alpha : (\alpha, \beta) \in S_1(\beta)\}. \tag{36}$$

$$\underline{\psi}_2(\alpha) := \inf\{\beta : (\alpha, \beta) \in S_2(\alpha)\}, \tag{37}$$

$$\overline{\psi}_2(\alpha) := \sup\{\beta : (\alpha, \beta) \in S_2(\alpha)\}. \tag{38}$$

Lemma 7 *The following assertions are true:*

1. *Functions $\underline{\psi}_1, \overline{\psi}_1$ bound the feasible region of $\varphi_1(\alpha, \beta) = 0$;*
2. *Functions $\underline{\psi}_2, \overline{\psi}_2$ bound the feasible region of $\varphi_2(\alpha, \beta) = 0$.*

Proof We provide a proof for Assertion 1, and the proof of Assertion 2 is similar.

For a given β, we have that

$$\hat{\alpha} \in \{\alpha : (\alpha, \beta) \in S_1(\beta)\} \Leftrightarrow \varphi_1(\hat{\alpha}, \beta) = 0. \tag{39}$$

It is clear that

$$\inf\{\alpha : (\alpha, \beta) \in S_1(\beta)\} \leq \hat{\alpha} \in \{\alpha : (\alpha, \beta) \in S_1(\beta)\}. \tag{40}$$

$$\sup\{\alpha : (\alpha, \beta) \in S_1(\beta)\} \geq \hat{\alpha} \in \{\alpha : (\alpha, \beta) \in S_1(\beta)\}. \tag{41}$$

From (39) and (40)–(41) and using (35)–(36), it immediately follows that for any $\hat{\alpha}$ such that $\varphi_1(\hat{\alpha}, \beta) = 0$, we have:

$$\underline{\psi}_1(\beta) \leq \hat{\alpha} \leq \overline{\psi}_1(\beta).$$

□

Let us define the following index sets that will be used in the proof of lemmas to follow:

$$\begin{aligned}
I^*(\beta) &:= \{i : 0 < z_j(\alpha, \beta) < d_j,\ i \in J\},\\
I^+(\beta) &:= \{i : z_j(\alpha, \beta) > d_j,\ i \in J\},\\
I^{+=}(\beta) &:= \{i : z_j(\alpha, \beta) = d_j,\ i \in J\},\\
I^-(\beta) &:= \{i : z_j(\alpha, \beta) < 0,\ i \in J\},\\
I^{-=}(\beta) &:= \{i : z_j(\alpha, \beta) = 0,\ i \in J\}.
\end{aligned}$$

We will define the set of binding constraints

$$\overline{I}^{=}(\beta) := I^{+=}(\beta) \cup I^{-=}(\beta).$$

Lemma 8 *If $\hat{\beta} \in \mathbb{R}$ is such that $\underline{\psi}_1(\hat{\beta}) > -\infty$, then $\underline{\psi}_1(\beta) > -\infty$, $\forall \beta \in \mathbb{R}$.*

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Proof Let $\hat{\beta} \in \mathbb{R}$. If $\underline{\psi}_1(\hat{\beta}) > -\infty$, then, from the definition of $\underline{\psi}_1$, $\exists \hat{\alpha}$ such that:

$$\hat{\alpha} = \inf\{\alpha : (\alpha, \hat{\beta}) \in S_1(\hat{\beta})\}, \tag{42}$$

then $\varphi_1(\hat{\alpha}, \hat{\beta}) = 0$. We know that $b_1 \in [\underline{b}_1, \overline{b}_1]$ (see Lemma 5).

We also know that for any β, $\phi_{1,\beta}(\cdot)$ is a nonincreasing function (see Lemma 2) and that $\exists \underline{\alpha}, \overline{\alpha} \in \mathbb{R}$ such that $\phi_{1,\beta}(\underline{\alpha}) = \underline{b}_1$ and $\phi_{1,\beta}(\overline{\alpha}) = \overline{b}_1$.

Thus, $\forall \beta$, $\exists \overline{\alpha}$ such that $\phi_{1,\beta}(\overline{\alpha}) = \overline{b}_1$. Since $b_1 < \overline{b}_1$, then if $\varphi_1(\alpha, \beta) = b_1, \alpha > \overline{\alpha}$. Thus:

$$\inf\{\alpha : (\alpha, \hat{\beta}) \in S_1(\hat{\beta})\} > \overline{\alpha}. \tag{43}$$

□

Lemma 9 *If $\hat{\alpha}, \hat{\beta} \in \mathbb{R}$ are such that $\varphi_1(\hat{\alpha}, \hat{\beta}) = 0$, then*

$$\hat{\alpha} = -\hat{\beta}\frac{\sum_{i \in I^*(\hat{\beta})} c_i a_i}{\sum_{i \in I^*(\hat{\beta})} a_i a_i} + \frac{\sum_{i \in I^*(\hat{\beta})} \overline{x}_i a_i + \sum_{i \in I^+(\hat{\beta}) \cup I^{+=}(\hat{\beta})} d_i a_i - b_1}{\sum_{i \in I^*(\hat{\beta})} a_i a_i} \tag{44}$$

Proof

$$\varphi_1(\hat{\alpha}, \hat{\beta}) = \sum_{i \in J} \max\{\min\{z_i(\hat{\alpha}, \hat{\beta}), d_i\}, 0\} a_i - b_1$$

$$\varphi_1(\hat{\alpha}, \hat{\beta}) = \sum_{i \in I^*(\hat{\beta})} z_i(\hat{\alpha}, \hat{\beta}) a_i + \sum_{i \in I^+(\hat{\beta}) \cup I^{+=}(\hat{\beta})} d_i a_i - b_1$$

$$\varphi_1(\hat{\alpha}, \hat{\beta}) = -\hat{\beta} \sum_{i \in I^*(\hat{\beta})} c_i a_i - \hat{\alpha} \sum_{i \in I^*(\hat{\beta})} a_i a_i + \sum_{i \in I^*(\hat{\beta})} \overline{x}_i a_i + \sum_{i \in I^+(\hat{\beta}) \cup I^{+=}(\hat{\beta})} d_i a_i - b_1$$

Since $\varphi_1(\hat{\alpha}, \hat{\beta}) = 0$,

$$\hat{\alpha} = -\hat{\beta}\frac{\sum_{i \in I^*(\hat{\beta})} c_i a_i}{\sum_{i \in I^*(\hat{\beta})} a_i a_i} + \frac{\sum_{i \in I^*(\hat{\beta})} \overline{x}_i a_i + \sum_{i \in I^+(\hat{\beta}) \cup I^{+=}(\hat{\beta})} d_i a_i - b_1}{\sum_{i \in I^*(\hat{\beta})} a_i a_i}$$

□

Lemma 10 *Let $V^- \subseteq I^{-=}(\hat{\beta})$ and $V^+ \subseteq I^{+=}(\hat{\beta})$. If $\hat{\alpha}, \hat{\beta} \in \mathbb{R}$, such that $\varphi_1(\hat{\alpha}, \hat{\beta}) = 0$, then*

$$\hat{\alpha} = -\hat{\beta}\frac{\sum_{i \in I^*(\hat{\beta}) \cup V^- \cup V^+} c_i a_i}{\sum_{i \in I^*(\hat{\beta}) \cup V^- \cup V^+} a_i a_i} + \frac{\sum_{i \in I^*(\hat{\beta}) \cup V^- \cup V^+} \overline{x}_i a_i + \sum_{i \in I^+(\hat{\beta}) \cup (I^{+=}(\hat{\beta}) - V^+)} d_i a_i - b_1}{\sum_{i \in \cup V^- \cup V^+} a_i a_i}.$$

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Proof By the definition of the sets $I^{+=}(\hat{\beta})$ and $I^{-=}(\hat{\beta})$, we have the following equalities:

$$-\hat{\beta}\sum_{i\in I^{+=}(\hat{\beta})} c_i a_i - \hat{\alpha}\sum_{i\in I^{+=}(\hat{\beta})} a_i a_i + \sum_{i\in I^{+=}(\hat{\beta})} \overline{x}_i a_i = \sum_{i\in I^{1+=}(\hat{\beta})} d_i a_i \tag{45}$$

$$-\hat{\beta}\sum_{i\in I^{-=}(\hat{\beta})} c_i a_i - \hat{\alpha}\sum_{i\in I^{-=}(\hat{\beta})} a_i a_i + \sum_{i\in I^{-=}(\hat{\beta})} \overline{x}_i a_i = 0 \tag{46}$$

From (45) and (46), we can see that

$$\begin{aligned}&\sum_{i\in I^*(\hat{\beta})\cup V^-\cup V^+} (\overline{x}_i - \hat{\alpha}a_i - \hat{\beta}c_i)a_i + \sum_{i\in I^+(\hat{\beta})\cup(I^{+=}(\hat{\beta})-V^+)} d_i a_i\\ &= \sum_{i\in I^*(\hat{\beta})} (\overline{x}_i - \hat{\alpha}a_i - \hat{\beta}c_i)a_i + \sum_{i\in I^+(\hat{\beta})} d_i a_i.\end{aligned} \tag{47}$$

The result can be reached from (47) by a reasoning similar to the proof of Lemma 9. □

In order to simplify the formulae, we will use the following notation:

$$\overline{I}^*(\hat{\beta}) = I^*(\hat{\beta}) \cup I^{-=}(\hat{\beta}) \cup I^{+=}(\hat{\beta}). \tag{48}$$

Corollary 11 *If $\hat{\alpha}, \hat{\beta} \in \mathbb{R}$, such that $\varphi_1(\hat{\alpha}, \hat{\beta}) = 0$, then*

$$\hat{\alpha} = -\hat{\beta}\frac{\sum_{i\in\overline{I}^*(\hat{\beta})} c_i a_i}{\sum_{i\in\overline{I}^*(\hat{\beta})} a_i a_i} + \frac{\sum_{i\in\overline{I}^*(\hat{\beta})} \overline{x}_i a_i + \sum_{i\in I^+(\hat{\beta})} d_i a_i - b_1}{\sum_{i\in\overline{I}^*(\hat{\beta})} a_i a_i}.$$

Proof The proof is analogous to Lemma 10. □

The following lemma proves a crucial property of $\underline{\psi}_1$.

Lemma 12 *Then, the following assertions are true:*

1. *Function $\underline{\psi}_1(\cdot)$ is piecewise affine, continuous in its effective domain, i.e., $\forall\beta$ such that $\underline{\psi}_1(\beta) > -\infty$;*
2. *Function $\overline{\psi}_1(\cdot)$ is piecewise affine, continuous in its effective domain, i.e., $\forall\beta$ such that $\overline{\psi}_1(\beta) < +\infty$;*
3. *Function $\underline{\psi}_2(\cdot)$ is piecewise affine, continuous in its effective domain, i.e., $\forall\alpha$ such that $\underline{\psi}_2(\alpha) > -\infty$;*
4. *Function $\overline{\psi}_2(\cdot)$ is piecewise affine, continuous in its effective domain, i.e., $\forall\alpha$ such that $\overline{\psi}_2(\alpha) < +\infty$.*

Proof We will prove only Assertion 1. Assertions 2–4 can be proven analogously. Let $\hat{\beta}$ be such that $\underline{\psi}_1(\hat{\beta}) > -\infty$.

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Since $\underline{\psi}_1(\hat{\beta}) > -\infty$, $\exists \hat{\alpha}$ such that:

$$\varphi_1(\hat{\alpha}, \hat{\beta}) = 0 \, , \, \underline{\psi}_1(\hat{\beta}) = \hat{\alpha}. \tag{49}$$

From Lemma 11, we have that:

$$\underline{\psi}_1(\hat{\beta}) = \hat{\alpha} = -\hat{\beta} \frac{\sum_{i \in \overline{I}^*(\hat{\beta})} c_i a_i}{\sum_{i \in \overline{I}^*(\hat{\beta})} a_i a_i} + \frac{\sum_{i \in \overline{I}^*(\hat{\beta})} \overline{x}_i a_i + \sum_{i \in I^+(\hat{\beta})} d_i a_i - b_1}{\sum_{i \in \overline{I}^*(\hat{\beta})} a_i a_i}. \tag{50}$$

Since the inequalities defining sets $I^*(\hat{\beta})$, $I^+(\hat{\beta})$ and $I^-(\hat{\beta})$ are strict, $\exists \epsilon$ such that:

$$I^*(\hat{\beta}) \subseteq I^*(\hat{\beta} + \epsilon) \, , \, I^+(\hat{\beta}) \subseteq I^+(\hat{\beta} + \epsilon) \, , \, I^-(\hat{\beta}) \subseteq I^-(\hat{\beta} + \epsilon). \tag{51}$$

From Lemma 8, $\underline{\psi}_1(\hat{\beta} + \epsilon) > -\infty$, $\exists \alpha_\epsilon$ such that:

$$\varphi_1(\hat{\alpha}_\epsilon, \hat{\beta} + \epsilon) = 0 \, , \, \underline{\psi}_1(\hat{\beta} + \epsilon) = \hat{\alpha}_\epsilon. \tag{52}$$

Let us define the following sets:

$$\tilde{I}^{*-}(\epsilon) = \{k : z_k(\hat{\alpha}, \hat{\beta}) = 0, z_k(\hat{\alpha}, \hat{\beta} + \epsilon) > 0, |z_k(\hat{\alpha}, \hat{\beta} + \epsilon)| < \epsilon\}, \tag{53}$$

$$\tilde{I}^{-}(\epsilon) = \{k : z_k(\hat{\alpha}, \hat{\beta}) = 0, z_k(\hat{\alpha}, \hat{\beta} + \epsilon) < 0, |z_k(\hat{\alpha}, \hat{\beta} + \epsilon)| < \epsilon\}, \tag{54}$$

$$\tilde{I}^{*+}(\epsilon) = \{k : z_k(\hat{\alpha}, \hat{\beta}) = d_k, z_k(\hat{\alpha}, \hat{\beta} + \epsilon) < d_k, |z_k(\hat{\alpha}, \hat{\beta} + \epsilon) - d_k| < \epsilon\}, \tag{55}$$

$$\tilde{I}^{+}(\epsilon) = \{k : z_k(\hat{\alpha}, \hat{\beta}) = d_k, z_k(\hat{\alpha}, \hat{\beta} + \epsilon) > d_k, |z_k(\hat{\alpha}, \hat{\beta} + \epsilon) - d_k| < \epsilon\}. \tag{56}$$

Thus, we have that:

$$\begin{aligned}
I^-(\hat{\beta} + \epsilon) &= I^-(\hat{\beta}) \cup \tilde{I}^-(\epsilon) - \tilde{I}^{-*}(\epsilon), \\
I^{-=}(\hat{\beta} + \epsilon) &= I^{-=}(\hat{\beta}) - \tilde{I}^- - \tilde{I}^{-*}(\epsilon), \\
I^+(\hat{\beta} + \epsilon) &= I^+(\hat{\beta}) \cup \tilde{I}^+(\epsilon) - \tilde{I}^{+*}(\epsilon), \\
I^{+=}(\hat{\beta} + \epsilon) &= I^{+=}(\hat{\beta}) - \tilde{I}^+(\epsilon) - \tilde{I}^{+*}(\epsilon), \\
I^*(\hat{\beta} + \epsilon) &= I^*(\hat{\beta}) \cup \tilde{I}^{-*}(\epsilon) \cup \tilde{I}^{+*}(\epsilon).
\end{aligned} \tag{57}$$

Now let $\tilde{I}^*(\epsilon) = \tilde{I}^{-*}(\epsilon) \cup \tilde{I}^{+*}(\epsilon)$. Therefore, the following identity holds:

$$\begin{aligned}
&I^*(\hat{\beta} + \epsilon) \cup I^{-=}(\hat{\beta} + \epsilon) \cup I^{+=}(\hat{\beta} + \epsilon) \\
&\quad = I^*(\hat{\beta}) \cup \tilde{I}^*(\epsilon) \cup (I^{-=}(\hat{\beta}) - \tilde{I}^-)(\epsilon) \cup (I^{+=}(\hat{\beta}) - \tilde{I}^+)(\epsilon)
\end{aligned}$$

Since $\tilde{I}^*(\epsilon) \subseteq I^*(\hat{\beta})$,

$$\begin{aligned}
&I^*(\hat{\beta} + \epsilon) \cup I^{-=}(\hat{\beta} + \epsilon) \cup I^{+=}(\hat{\beta} + \epsilon) \\
&\quad = I^*(\hat{\beta}) \cup (I^{-=}(\hat{\beta}) - \tilde{I}^-(\epsilon)) \cup (I^{+=}(\hat{\beta}) - \tilde{I}^+)
\end{aligned}$$

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Since $\tilde{I}^{-}(\epsilon) \subseteq I^{-=}(\hat{\beta})$ and $\tilde{I}^{+}(\epsilon) \subseteq I^{+=}(\hat{\beta})$,

$$\overline{I}^{*}(\hat{\beta}+\epsilon) = \overline{I}^{*}(\hat{\beta}) - (\tilde{I}^{-} \cup \tilde{I}^{+}) \tag{58}$$

From Lemma 10, we can write $\hat{\alpha}_{\epsilon}$ as:

$$\underline{\psi}_1(\hat{\beta}+\epsilon) = \hat{\alpha}_{\epsilon} = -(\hat{\beta}+\epsilon)\frac{\sum_{i\in\overline{I}^{*}(\hat{\beta}+\epsilon)} c_i a_i}{\sum_{i\in\overline{I}^{*}(\hat{\beta}+\epsilon)} a_i a_i} + \frac{\sum_{i\in\overline{I}^{*}(\hat{\beta}+\epsilon)} \overline{x}_i a_i + \sum_{i\in I^{+}(\hat{\beta}+\epsilon)} d_i a_i - b_1}{\sum_{i\in\overline{I}^{*}(\hat{\beta}+\epsilon)} a_i a_i}. \tag{59}$$

Substituting (57) and (58) in (59):

$$\underline{\psi}_1(\hat{\beta}+\epsilon) = -(\hat{\beta}+\epsilon)\frac{\sum_{i\in\overline{I}^{*}(\hat{\beta})-(\tilde{I}^{-}(\epsilon)\cup\tilde{I}^{+}(\epsilon))} c_i a_i}{\sum_{i\in\overline{I}^{*}(\hat{\beta})-(\tilde{I}^{-}(\epsilon)\cup\tilde{I}^{+}(\epsilon))} a_i a_i} + \frac{\sum_{i\in\overline{I}^{*}(\hat{\beta})-(\tilde{I}^{-}(\epsilon)\cup\tilde{I}^{+}(\epsilon))} \overline{x}_i a_i + \sum_{i\in I^{+}(\hat{\beta})\cup\tilde{I}^{+}(\epsilon)-\tilde{I}^{+*}(\epsilon)} d_i a_i - b_1}{\sum_{i\in\overline{I}^{*}(\hat{\beta})-(\tilde{I}^{-}(\epsilon)\cup\tilde{I}^{+}(\epsilon))} a_i a_i}. \tag{60}$$

Equation 60 is valid $\forall\tilde{\epsilon} \in (0, \epsilon]$. In order to prove continuity of $\underline{\psi}_1(\cdot)$ over $[\hat{\beta}, \hat{\beta}+\epsilon]$, it remains only to show that $\underline{\psi}_1(\cdot)$ is continuous from the right at $\hat{\beta}$, i.e.

$$\lim_{\epsilon\to 0^{+}} \underline{\psi}_1(\hat{\beta}+\epsilon) = \underline{\psi}_1(\hat{\beta}). \tag{61}$$

In order to prove (61), lets rewrite (60):

$$\underline{\psi}_1(\hat{\beta}+\epsilon) \sum_{i\in\overline{I}^{*}(\hat{\beta})-(\tilde{I}^{-}(\epsilon)\cup\tilde{I}^{+}(\epsilon))} a_i a_i = -(\hat{\beta}+\epsilon) \sum_{i\in\overline{I}^{*}(\hat{\beta})-(\tilde{I}^{-}(\epsilon)\cup\tilde{I}^{+}(\epsilon))} c_i a_i + \sum_{i\in\overline{I}^{*}(\hat{\beta})-(\tilde{I}^{-}(\epsilon)\cup\tilde{I}^{+}(\epsilon))} \overline{x}_i a_i + \sum_{i\in I^{+}(\hat{\beta})\cup\tilde{I}^{+}(\epsilon)-\tilde{I}^{+*}(\epsilon)} d_i a_i - b_1. \tag{62}$$

Since $\tilde{I}^{-} \cup \tilde{I}^{+}(\epsilon) \subseteq \overline{I}^{*}(\hat{\beta})$ and using (52),

$$\underline{\psi}_1(\hat{\beta}+\epsilon) \sum_{i\in\overline{I}^{*}(\hat{\beta})} a_i a_i = -(\hat{\beta}+\epsilon) \sum_{i\in\overline{I}^{*}(\hat{\beta})} c_i a_i + \sum_{i\in\overline{I}^{*}(\hat{\beta})} \overline{x}_i a_i + \sum_{i\in I^{+}(\hat{\beta})} d_i a_i - b_1 - \hat{\alpha}_{\epsilon} \sum_{i\in\tilde{I}^{-}(\epsilon)\cup\tilde{I}^{+}(\epsilon)} a_i a_i$$

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$$-(\hat{\beta}+\epsilon)\sum_{i\in\tilde{I}^-(\epsilon)\cup\tilde{I}^+(\epsilon)}c_i a_i+\sum_{i\in\tilde{I}^-(\epsilon)\cup\tilde{I}^+(\epsilon)}\overline{x}_i a_i+\sum_{i\in\tilde{I}^+(\epsilon)}d_i a_i. \tag{63}$$

We can also rewrite (59) as:

$$\underline{\psi}_1(\hat{\beta})\sum_{i\in\overline{I}^*(\hat{\beta})}a_i a_i=-(\hat{\beta})\sum_{i\in\overline{I}^*(\hat{\beta})}c_i a_i+\sum_{i\in\overline{I}^*(\hat{\beta})}\overline{x}_i a_i+\sum_{i\in I^+(\hat{\beta})}d_i a_i-b_1. \tag{64}$$

Subtracting (64) from (63):

$$(\underline{\psi}_1(\hat{\beta}+\epsilon)-\underline{\psi}_1(\hat{\beta}))\sum_{i\in\overline{I}^*(\hat{\beta})}a_i a_i=\epsilon\sum_{i\in\overline{I}^*(\hat{\beta})}c_i a_i-\hat{\alpha}_\epsilon\sum_{i\in\tilde{I}^-(\epsilon)\cup\tilde{I}^+(\epsilon)}a_i a_i$$
$$-(\hat{\beta}+\epsilon)\sum_{i\in\tilde{I}^-(\epsilon)\cup\tilde{I}^+(\epsilon)}c_i a_i+\sum_{i\in\tilde{I}^-(\epsilon)\cup\tilde{I}^+(\epsilon)}\overline{x}_i a_i+\sum_{i\in\tilde{I}^+(\epsilon)}d_i a_i. \tag{65}$$

From (53)–(56), we can derive that

$$\left|-\hat{\alpha}_\epsilon\sum_{i\in\tilde{I}^-(\epsilon)\cup\tilde{I}^+(\epsilon)}a_i a_i-(\hat{\beta}+\epsilon)\sum_{i\in\tilde{I}^-\cup\tilde{I}^+(\epsilon)}c_i a_i+\sum_{i\in\tilde{I}^-(\epsilon)\cup\tilde{I}^+(\epsilon)}\overline{x}_i a_i\right.$$
$$\left.+\sum_{i\in\tilde{I}^+(\epsilon)}d_i a_i\right|<|\tilde{I}^-(\epsilon)\cup\tilde{I}^+(\epsilon)|\epsilon \tag{66}$$

Using (66) in (65),

$$(\underline{\psi}_1(\hat{\beta}+\epsilon)-\underline{\psi}_1(\hat{\beta}))\sum_{i\in\overline{I}^*(\hat{\beta})}a_i a_i<\epsilon\sum_{i\in\overline{I}^*(\hat{\beta})}c_i a_i+|\tilde{I}^-(\epsilon)\cup\tilde{I}^+(\epsilon)|\epsilon,$$
$$\underline{\psi}_1(\hat{\beta}+\epsilon)-\underline{\psi}_1(\hat{\beta})<\frac{\sum_{i\in\overline{I}^*(\hat{\beta})}c_i a_i+|\tilde{I}^-(\epsilon)\cup\tilde{I}^+|}{\sum_{i\in\overline{I}^*(\hat{\beta})}a_i a_i}\epsilon<\frac{\|c\|\|a\|+|J|}{\min_{i\in J}a_i^2}\epsilon. \tag{67}$$

Inequality (67) proves that $\lim_{\epsilon\to 0^+}\underline{\psi}_1(\hat{\beta}+\epsilon)=\underline{\psi}_1(\hat{\beta})$. Thus, we showed that $\underline{\psi}_1$ is affine on the interval $[\hat{\beta},\hat{\beta}+\epsilon]$.

Similarly, we can prove that $\underline{\psi}_1$ is affine on the interval $[\hat{\beta}-\epsilon,\hat{\beta}]$.

Therefore, $\forall\hat{\beta}$, $\underline{\psi}_1$ is piecewise affine over $[\hat{\beta}-\epsilon,\hat{\beta}+\epsilon]$. □

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The following definitions will be useful in the proof of the next lemma as well as in the algorithm description:

$$\varphi_1^k(\beta) := \varphi_1\left(\frac{\overline{x}_k - \beta c_k}{a_k}, \beta\right), \tag{68}$$

$$\tilde{\varphi}_1^k(\beta) := \varphi_1\left(\frac{\overline{x}_k - \beta c_k - d_k}{a_k}, \beta\right), \tag{69}$$

$$\mu_i^k := \overline{x}_i - \frac{\overline{x}_k a_i}{a_k},\ i \in J - \{k\}, \tag{70}$$

$$\eta_i^k := c_i - \frac{c_k a_i}{a_k},\ i \in J - \{k\}. \tag{71}$$

Lemma 13 *The functions $\underline{\psi}_1(\cdot)$ and $\overline{\psi}_1(\cdot)$ have at most $4n^2 - 4n$ breakpoints.*

Proof The points of nondifferentiability of $\underline{\psi}_1(\cdot)$ and $\overline{\psi}_1(\cdot)$ are reached when this function intercepts either the level set $x_i(\cdot,\cdot) = 0$ or $x_i(\cdot,\cdot) = d_i$. If this happens at a point (α, β) , then it must satisfy at least one of the following systems:

$$\varphi_1(\alpha, \beta) = 0 \tag{72}$$

$$z_i(\alpha, \beta) = 0,\ i \in J. \tag{73}$$

$$\varphi_1(\alpha, \beta) = 0 \tag{74}$$

$$z_i(\alpha, \beta) = d_i,\ i \in J. \tag{75}$$

Let us examine the breakpoints due to system (72)–(73). A similar analysis can be done for breakpoints due to system (74)–(75).

Since from our assumptions, $a_i \neq 0, \forall i \in J$, from equality (73) we have for any k:

$$\alpha = \frac{\overline{x}_k - \beta c_k}{a_k}. \tag{76}$$

Substituting (76) in (72),

$$\varphi_1\left(\frac{\overline{x}_k - \beta c_k}{a_k}, \beta\right) = 0.$$

Using definition of $\varphi_1^k(.)$ in 68,

$$\begin{aligned}
\varphi_1^k(\beta) &= \sum_{i \in J} \min\left\{\max\left\{\overline{x}_i - \frac{\overline{x}_k - \beta c_k}{a_k} a_i - \beta c_i, 0\right\}, d_i\right\} a_i \\
&= \sum_{i \in J} \min\left\{\max\left\{\left(\overline{x}_i - \frac{\overline{x}_k a_i}{a_k}\right) - \left(c_i - \frac{c_k a_i}{a_k}\right)\beta, 0\right\}, d_i\right\} a_i \\
&= \sum_{i \in J - \{k\}} \min\left\{\max\left\{\left(\overline{x}_i - \frac{\overline{x}_k a_i}{a_k}\right) - \left(c_i - \frac{c_k a_i}{a_k}\right)\beta, 0\right\}, d_i\right\} a_i\ .
\end{aligned}$$

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Rewriting (77) using μ_j^k and η_j^k,

$$\varphi_1^k(\beta) = \sum_{i \in J - \{k\}} \min\{\max\{\mu_i^k - \eta_i^k \beta, 0\}, d_i\} a_i \ .$$

Let us consider the following sets:

$$U_k = \left\{ \frac{\mu_i^k}{\eta_i^k} \ : i \in J - \{k\} \right\}, \text{corresponding to equality } \mu_i^k - \eta_i^k \beta = 0;$$

$$V_k = \left\{ \frac{\mu_i^k - d_i}{\eta_i^k} \ : i \in J - \{k\} \right\}, \text{corresponding to equality } \mu_i^k - \eta_i^k \beta = d_i .$$

□

Claim 14 *Each $\hat{\beta} \in (U_k \cup V_k)$ is a breakpoint of $\varphi_1^k(\cdot)$.*

Proof Let $\hat{\beta} \in \mathbb{R}$. Let us define the following sets:

$$J^*(\hat{\beta}) = \left\{ i \in J - \{k\} : \frac{\mu_i^k - d_i}{\eta_i^k} < \hat{\beta} < \frac{\mu_i^k}{\eta_i^k} \right\}, \tag{77}$$

$$J^+(\hat{\beta}) = \left\{ i \in J - \{k\} : \hat{\beta} < \frac{\mu_i^k - d_i}{\eta_i^k} \right\}, \tag{78}$$

$$J^{+=}(\hat{\beta}) = \left\{ i \in J - \{k\} : \hat{\beta} = \frac{\mu_i^k - d_i}{\eta_i^k} \right\}, \tag{79}$$

$$J^-(\hat{\beta}) = \left\{ i \in J - \{k\} : \hat{\beta} > \frac{\mu_i^k}{\eta_i^k} \right\}, \tag{80}$$

$$J^{-=}(\hat{\beta}) = \left\{ i \in J - \{k\} : \frac{\mu_i^k}{\eta_i^k} = \hat{\beta} \right\}. \tag{81}$$

We can use sets (77), (81) to express $\varphi_1^k(\hat{\beta})$:

$$\varphi_1^k(\hat{\beta}) = \sum_{i \in J^*(\hat{\beta})} (\mu_i^k - \eta_i^k \hat{\beta}) a_i + \sum_{i \in J^+(\hat{\beta}) \cup J^{+=}(\hat{\beta})} d_i a_i . \tag{82}$$

Let $\epsilon \in \mathbb{R}$, $\epsilon > 0$, such that the following holds:

$$\epsilon < \min\{\beta - \hat{\beta} : \beta > \hat{\beta}, \beta \in U_k \cup V_k\}. \tag{83}$$

If $\{\beta - \hat{\beta} : \beta > \hat{\beta}, \beta \in U_k \cup V_k\} = \emptyset$, then we can set $\epsilon = e$, $\forall e > 0$.

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Condition (83) implies that

$$J^*(\hat{\beta}) \subseteq J^*(\hat{\beta}+\epsilon),\ J^+(\hat{\beta}) \subseteq J^+(\hat{\beta}+\epsilon),\ J^-(\hat{\beta}) \subseteq J^-(\hat{\beta}+\epsilon),$$
$$J^{+=}(\hat{\beta}+\epsilon) = \emptyset, \quad J^{-=}(\hat{\beta}+\epsilon) = \emptyset. \tag{84}$$

We can express $\varphi_1^k(\hat{\beta}+\epsilon)$ as:

$$\varphi_1^k(\hat{\beta}+\epsilon) = \sum_{i\in J^*(\hat{\beta})} (\mu_i^k - \eta_i^k\hat{\beta})a_i + \sum_{i\in J^*(\hat{\beta}+\epsilon)-J^*(\hat{\beta})} (\mu_i^k - \eta_i^k\hat{\beta})a_i + \sum_{i\in J^+(\hat{\beta}+\epsilon)} d_i a_i.$$

Clearly, since $\hat{\beta} < \hat{\beta}+\epsilon$,

$$J^*(\hat{\beta}+\epsilon) = J^*(\hat{\beta}) \cup J^{+=}(\hat{\beta}).$$
$$\varphi_1^k(\hat{\beta}+\epsilon) = \sum_{i\in J^*(\hat{\beta})} (\mu_i^k - \eta_i^k\hat{\beta})a_i + \sum_{i\in J^{+=}(\hat{\beta})} (\mu_i^k - \eta_i^k\hat{\beta})a_i - \sum_{i\in J^{-=}(\hat{\beta})} (\mu_i^k - \eta_i^k\hat{\beta})a_i + \sum_{i\in J^+(\hat{\beta})} d_i a_i.$$

Thus,

$$\varphi_1^k(\hat{\beta}+\epsilon) = -\hat{\beta}\left(\sum_{i\in J^*(\hat{\beta})\cup J^{+=}(\hat{\beta})} \eta_i^k a_i - \sum_{i\in J^{-=}(\hat{\beta})} \eta_i^k a_i\right) + \sum_{i\in J^*(\hat{\beta})\cup J^{+=}(\hat{\beta})} \mu_i^k a_i - \sum_{i\in J^{-=}(\hat{\beta})} \mu_i^k a_i + \sum_{i\in J^+(\hat{\beta})} d_i a_i. \tag{85}$$

Using the same reasoning, we can deduce that

$$\varphi_1^k(\hat{\beta}-\epsilon) = -\hat{\beta}\left(\sum_{i\in J^*(\hat{\beta})\cup J^{-=}(\hat{\beta})} \eta_i^k a_i - \sum_{i\in J^{+=}(\hat{\beta})} \eta_i^k a_i\right) + \sum_{i\in J^*(\hat{\beta})\cup J^{-=}(\hat{\beta})} \mu_i^k a_i - \sum_{i\in J^{+=}(\hat{\beta})} \mu_i^k a_i + \sum_{i\in J^+(\hat{\beta})\cup J^{+=}(\hat{\beta})} d_i a_i. \tag{86}$$

Computing the right and left derivatives of $\varphi_1^k(\cdot)$ at $\hat{\beta}$ using expressions (85) and (86), respectively,

$$\partial_+\varphi_1^k(\hat{\beta}) = -\sum_{i\in J^*(\hat{\beta})\cup J^{+=}(\hat{\beta})} \eta_i^k a_i + \sum_{i\in J^{-=}(\hat{\beta})} \eta_i^k a_i \tag{87}$$

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$$\partial_- \varphi_1^k(\hat{\beta}) = -\sum_{i \in J^*(\hat{\beta}) \cup J^{-=}(\hat{\beta})} \eta_i^k a_i + \sum_{i \in J^{+=}(\hat{\beta})} \eta_i^k a_i \tag{88}$$

$\varphi_1^k(\cdot)$ has derivative at $\hat{\beta}$, if and only if $\partial_+ \varphi_1^k(\hat{\beta}) = \partial_- \varphi_1^k(\hat{\beta})$. Therefore,

$$\begin{aligned} &-\sum_{i \in J^*(\hat{\beta}) \cup J^{+=}(\hat{\beta})} \eta_i^k a_i + \sum_{i \in J^{-=}(\hat{\beta})} \eta_i^k a_i = -\sum_{i \in J^*(\hat{\beta}) \cup J^{-=}(\hat{\beta})} \eta_i^k a_i + \sum_{i \in J^{+=}(\hat{\beta})} \eta_i^k a_i \\ &\Leftrightarrow -\sum_{i \in J^{+=}(\hat{\beta})} \eta_i^k a_i + \sum_{i \in J^{-=}(\hat{\beta})} \eta_i^k a_i = -\sum_{i \in J^{-=}(\hat{\beta})} \eta_i^k a_i + \sum_{i \in J^{+=}(\hat{\beta})} \eta_i^k a_i \\ &\Leftrightarrow \sum_{i \in J^{+=}(\hat{\beta})} \eta_i^k a_i = \sum_{i \in J^{-=}(\hat{\beta})} \eta_i^k a_i . \end{aligned} \tag{89}$$

Thus, if condition (89) is satisfied, $\varphi_1^k(\cdot)$ has derivative at $\hat{\beta}$. Clearly,

$$\hat{\beta} \notin U_k \cup V_k \Rightarrow J^{+=}(\hat{\beta}) = \emptyset,\ J^{-=}(\hat{\beta}) = \emptyset. \tag{90}$$

From (90), if $\hat{\beta} \notin U_k \cup V_k$, since $J^{+=}(\hat{\beta}) = J^{-=}(\hat{\beta})$, condition (89) is satisfied and $\varphi_1^k(\cdot)$ has derivative at $\hat{\beta}$. So, if $\varphi_1^k(\cdot)$ does not have derivative at $\hat{\beta}$, then $\hat{\beta} \in U_k \cup V_k$. Thus, $U_k \cup V_k$ contains the breakpoints of $\varphi_1^k(\cdot)$. □

Clearly, $|U_k| + |V_k| = 2(|J| - |\{k\}|) = 2(n-1)$. Therefore, there are at most $2(n-1)$ breakpoints associated with the system composed of equalities (72)–(73). Analogously, it can be checked that there are at most $2(n-1)$ breakpoints associated with system (74)–(75).
Thus, for every $i \in J$, we have at most $2 \times 2(n-1)$ breakpoints.
Since $|J| = n$, the maximum number of breakpoints of $\psi_1(\cdot)$ is $n \times 4(n-1) = 4n^2 - 4n$. □

Let $k \in J$ and $\hat{\beta} \in \mathbb{R}$; let us define the following quantities that will be used to compute $\varphi_1^k(\cdot)$:

$$\nu_1^k(\hat{\beta}) := -\sum_{i \in J^*(\hat{\beta})} \eta_i^k a_i, \tag{91}$$

$$\zeta_1^k(\hat{\beta}) := \sum_{i \in J^*(\hat{\beta})} \mu_i^k a_i + \sum_{i \in J^+(\hat{\beta}) \cup J^{+=}(\hat{\beta})} d_i a_i . \tag{92}$$

Thus,

$$\varphi_1^k(\hat{\beta}) = \nu_1^k(\hat{\beta})\hat{\beta} + \zeta_1^k(\hat{\beta}). \tag{93}$$

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Let us define the following quantities that will be used to compute $\varphi_2^k(\cdot)$:

$$\nu_2^k(\hat{\beta}) := -\sum_{i \in J^*(\hat{\beta})} \eta_i^k c_i, \tag{94}$$

$$\zeta_2^k(\hat{\beta}) := \sum_{i \in J^*(\hat{\beta})} \mu_i^k c_i + \sum_{i \in J^+(\hat{\beta}) \cup J^{+=}(\hat{\beta})} d_i c_i. \tag{95}$$

Thus,

$$\varphi_2^k(\hat{\beta}) = \nu_2^k(\hat{\beta})\hat{\beta} + \zeta_2^k(\hat{\beta}). \tag{96}$$

The following result will be used in the next section to solve $\varphi_1^k(\alpha) = 0$.

Corollary 15 *Let $k \in J$ and $\hat{\beta} \in \mathbb{R}$ and $\epsilon > 0$ such that $(\hat{\beta}, \hat{\beta} + \epsilon) \cap (U_k \cup V_k) = \emptyset$. Then, the following assertions are true:*

1. *$\varphi_1^k(\hat{\beta})$ can be computed in $|J^{+=}(\hat{\beta})| + |J^{-=}(\hat{\beta})|$ steps;*
2. *$\varphi_2^k(\hat{\beta})$ can be computed in $|J^{+=}(\hat{\beta})| + |J^{-=}(\hat{\beta})|$ steps.*

Proof We will prove only Assertion 1. Assertion 2 can be proven analogously.

From equation (93), we have that:

$$\varphi_1^k(\hat{\beta} + \epsilon) = \nu_1^k(\hat{\beta} + \epsilon)(\hat{\beta} + \epsilon) + \zeta_1^k(\hat{\beta} + \epsilon),$$

where

$$\nu_1^k(\hat{\beta} + \epsilon) = \nu_1^k(\hat{\beta}) + \sum_{i \in J^{+=}(\hat{\beta})} \eta_i^k a_i - \sum_{i \in J^{-=}(\hat{\beta})} \eta_i^k a_i, \tag{97}$$

$$\zeta_1^k(\hat{\beta} + \epsilon) = \zeta_1^k(\hat{\beta}) + \sum_{i \in J^{+=}(\hat{\beta})} d_i a_i - \sum_{i \in J^{-=}(\hat{\beta})} d_i a_i. \tag{98}$$

(97), (98) follows from equation (85). □

It is easy to see that Corollary 15 can be used to compute $\varphi_1^k(\hat{\beta} + \epsilon)$ by updating the value of $\varphi_1^k(\hat{\beta})$ by using $\nu_1^k(\hat{\beta})$ and $\zeta_1^k(\hat{\beta})$. Similarly, we compute $\varphi_2^k(\hat{\beta} + \epsilon)$ by updating the value of $\varphi_2^k(\hat{\beta})$ by using $\nu_2^k(\hat{\beta})$ and $\zeta_2^k(\hat{\beta})$.

3.2 The algorithm

In this subsection, we present the algorithm.

In order to make it easier to describe the algorithm as well as its proof of correctness, we will make use of the notation described as follows.

For two points $P_1, P_2 \in \mathbb{R}^2$, let us denote $\text{conv}(P_1, P_2)$ the set of all convex combinations of P_1 and P_2, i.e., $\text{conv}(P_1, P_2) = \{\lambda P_1 + (1 - \lambda)P_2 : \lambda \in [0, 1]\}$.

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If $P = (\alpha, \beta) \in \mathbb{R}^2$, then we will denote the components of P as P_α, P_β, i.e., $P = (P_\alpha, P_\beta)$. In the same way, if $\Gamma \subset \mathbb{R}^2$, Γ_β will denote the set composed of the second components of all elements that belong to Γ.

Algorithm 1 contains the pseudocode of a procedure to compute the projection of a point on the intersection of two hyperplanes and a box.

Fundamentally, Algorithm 1 consists of searching through the list of points that belong to S_1 which ones also belong to S_2. This is done by first obtaining the points that belong to S_1 by solving $\varphi_1^k(\alpha) = 0$ for each variable index k, using Algorithm 2. After the list of points in S_1 is retrieved, it is sorted on the β coordinate, so that we can find where in this sequence $\varphi_2(\cdot)$ is 0.

Line 5 of Algorithm 1 involves finding the tuples $(\alpha, \beta, \varphi_2^k(\beta))$ such that $\varphi_1^k(\beta) = 0$ and $\tilde{\varphi}_1^k(\beta) = 0$. The distinction between $\varphi_1^k(\beta)$ and $\tilde{\varphi}_1^k(\beta)$ is done by the parameter u. For $\varphi_1^k(\beta)$, $u = 0$ and for $\tilde{\varphi}_1^k(\beta)$, $u = d_k$. This step invokes Algorithm 2.

Algorithm 2 uses formulae (70) and (71) in order to compute the breakpoints $\varphi_1^k(\cdot)$ in Line 4. Then, it successively uses the assertions in Corollary 15 in order to compute both $\varphi_1^k(\beta)$ and $\varphi_2^k(\beta)$. For values of β for which $\varphi_1^k(\beta) = 0$, we have to save the tuple $(\alpha, \beta, \varphi_2^k(\beta))$ that will be used in Algorithm 1.

Algorithm 1 Proj2HB

1: Input: $a, d, \overline{x} \in \mathbb{R}^n, b_1, b_2 \in \mathbb{R}$.
2: Output: a point (α, β) s.t. $\varphi_1(\alpha, \beta) = 0$ and $\varphi_2(\alpha, \beta) = 0$.
3: $\Gamma \leftarrow \emptyset$;
4: **for** $k = 1, \ldots, n$ **do**
5: $\quad \Gamma \leftarrow \Gamma \cup$ Solve $\varphi_1(k, \varphi_1^k, \varphi_2^k, 0) \cup$ Solve $\varphi_1(k, \tilde{\varphi}_1^k, \tilde{\varphi}_2^k, d_k)$;
6: **end for**
7: Sort ascending Γ on the Γ_β component;
8: **if** $\exists(\hat{\alpha}, \hat{\beta}) \in \{(\alpha, \beta) \mid \varphi_1(\alpha, \beta) = 0, \varphi_2(\alpha, \beta) = 0, \ \alpha < (\Gamma_{(1)})_\alpha\}$ **then**
9: $\quad$ Stop: return $(\hat{\alpha}, \hat{\beta})$;
10: **end if**
11: **if** $(\Gamma_{(1)})_{\varphi_2} = 0\}$ **then**
12: $\quad$ Stop: return $(\Gamma_{(1)})_\alpha, (\Gamma_{(1)})_\beta$;
13: **end if**
14: **for** $\Gamma_{(j)} \in \Gamma - \{\Gamma_{(1)}\}$ **do**
15: $\quad$ **if** $(\Gamma_{(j)})_{\varphi_2} = 0\}$ **then**
16: $\quad\quad$ Stop: return $(\Gamma_{(j)})_\alpha, (\Gamma_{(j)})_\beta$;
17: $\quad$ **end if**
18: $\quad$ **if** $0 \in [\min((\Gamma_{(j-1)})_{\varphi_2}, (\Gamma_{(j)})_{\varphi_2}), \max((\Gamma_{(j-1)})_{\varphi_2}, (\Gamma_{(j)})_{\varphi_2})]$ **then**
19: $\quad\quad$ Stop: return $(\varphi_2)^{-1}([\min((\Gamma_{(j-1)})_{\varphi_2}, (\Gamma_{(j)})_{\varphi_2}), \max((\Gamma_{(j-1)})_{\varphi_2}, (\Gamma_{(j)})_{\varphi_2})) \cap \{0\}$;
20: $\quad$ **end if**
21: **end for**
22: **if** $\exists(\hat{\alpha}, \hat{\beta}) \in \{(\alpha, \beta) \mid \varphi_1(\alpha, \beta) = 0, \varphi_2(\alpha, \beta) = 0, \ \alpha > (\Gamma_{(|\Gamma|)})_\alpha\}$ **then**;
23: $\quad$ Stop: return $(\hat{\alpha}, \hat{\beta})$;
24: **end if**
25: return $\emptyset$;

Let Γ be a collection of points in $\mathbb{R}^3$, where the first two coordinates are points in the space of the Lagrangian multipliers and the last is the value of φ_2 applied on those multipliers. So, if $P \in \Gamma$, P_α, P_β are the multipliers and $P_{\varphi_2} = \varphi_2(P_\alpha, P_\beta)$.

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Algorithm 2 Solve φ_1

1: Input: k, $\varphi_1^k(\cdot)$, $\varphi_2^k(\cdot)$ and u.
2: Output: γ (set of tuples that contain the extreme points in the solution of $\varphi_1^k(\beta) = 0$ and the values of φ_2 on these points).
3: $\gamma \leftarrow \emptyset$;
4: $B \leftarrow \{\beta_j^k \mid \beta_j^k \text{ is a breakpoint of } \varphi_1^k(\beta)\}$;
5: sort B ascending $\beta_{(1)}^k \leq \cdots \leq \beta_{(2n-2)}^k$;
6: Compute $\varphi_1^k(\beta_{(1)}^k)$ and $\varphi_2^k(\beta_{(1)}^k)$;
7: **if** $\varphi_1^k(\beta_{(1)}^k) = 0$ **then**
8: $\quad \hat{\alpha} \leftarrow -\beta_{(1)}^k \frac{c_k}{a_k} + \frac{\overline{x}-u}{a_k}$;
9: $\quad \gamma \leftarrow \gamma \cup \{(\hat{\alpha}, \beta_{(1)}^k, \varphi_2^k(\beta_{(1)}^k))\}$;
10: **end if**
11: **for** j s.t. $\beta_{(j)}^k \in B - \{\beta_{(1)}^k\}$ **do**
12: $\quad$ Compute $\varphi_1^k(\beta_{(j)}^k)$ from $\varphi_1^k(\beta_{(j-1)}^k)$ using Corollary 15;
13: $\quad$ Compute $\varphi_2^k(\beta_{(j)}^k)$ from $\varphi_2^k(\beta_{(j-1)}^k)$ using Corollary 15;
14: $\quad$ **if** $\varphi_1^k(\beta_{(j)}^k) = 0$ **then**
15: $\quad\quad \hat{\alpha} \leftarrow -\beta_{(j)}^k \frac{c_k}{a_k} + \frac{\overline{x}-u}{a_k}$;
16: $\quad\quad \gamma \leftarrow \gamma \cup \{(\hat{\alpha}, \hat{\beta}, \varphi_2^k(\hat{\beta}))\}$;
17: $\quad$ **else**
18: $\quad\quad a = \min(\varphi_1^k(\beta_{(j-1)}^k), \varphi_1^k(\beta_{(j)}^k))$;
19: $\quad\quad b = \max(\varphi_1^k(\beta_{(j-1)}^k), \varphi_1^k(\beta_{(j)}^k))$;
20: $\quad\quad$ **if** $0 \in [a, b]$ **then**
21: $\quad\quad\quad \hat{\beta} \leftarrow (\varphi_1)^{-1}(\{0\}) \cap [\beta_{(j-1)}^k, \beta_{(j)}^k]$;
22: $\quad\quad\quad$ Compute $\varphi_2^k(\hat{\beta})$ from $\varphi_2^k(\beta_{(j-1)}^k)$ using Corollary 15;
23: $\quad\quad\quad \hat{\alpha} \leftarrow -\hat{\beta} \frac{c_k}{a_k} + \frac{\overline{x}-u}{a_k}$;
24: $\quad\quad\quad \gamma \leftarrow \gamma \cup \{(\hat{\alpha}, \hat{\beta}, \varphi_2^k(\hat{\beta}))\}$;
25: $\quad\quad$ **end if**
26: $\quad$ **end if**
27: **end for**
28: return γ;

Let $\tilde{\Gamma} = \{(P_\alpha, P_\beta) : P \in \Gamma\}$, i.e., $\tilde{\Gamma}$ contains the orthogonal projection of the points in Γ onto the space of the first two coordinates. Although this notation is not used in Algorithm 1, it will be useful in the next theorem.

In order to prove the correctness of Algorithm 1, given two points in sequence in the set $\tilde{\Gamma}$ ("in sequence" here means after the ordering on the β component (see Line 7 of Algorithm 1), since $\tilde{\Gamma}$ is obtained from Γ), we must prove that their convex combination solves $\varphi_1(\alpha, \beta) = 0$. The following theorem achieves that:

Theorem 16 *Let $P, Q \in \tilde{\Gamma}$ such that $relint(\text{conv}(P_\beta, Q_\beta)) \cap \tilde{\Gamma}_\beta = \emptyset$. Then, the following holds*

$$(\alpha, \beta) \in \text{conv}(P, Q) \Rightarrow \varphi_1(\alpha, \beta) = 0. \tag{99}$$

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Proof $P, Q \in \tilde{\Gamma}$. Suppose without loss of generality that $P_\beta < Q_\beta$.
It is easy to see that

$$\text{relint}(\text{conv}(P_\beta, Q_\beta)) \cap \tilde{\Gamma}_\beta = \emptyset \iff \nexists R \in \tilde{\Gamma} \text{ s.t. } P_\beta < R_\beta < Q_\beta. \quad (100)$$

We will prove by contradiction.
Suppose that the implication does not hold. Thus, $\exists \hat{\beta} \in \text{relint}(\text{conv}(P_\beta, Q_\beta))$ such that $\varphi_1(\alpha, \hat{\beta}) \neq 0$, for α such that $(\alpha, \hat{\beta}) \in conv(P, Q)$.

Let $S_1(P_\beta)$ be as in (14). Then, from Corollary 3, $S_1(P_\beta))$ is convex and $S_1(P_\beta) \subsetneq \mathbb{R}$. So, $S_1(P_\beta)$ contains an extreme point. It is easy to see that by construction, P_β is an extreme point of the set $S_1(P_\beta)$, since $P \in \Gamma$. Thus, at least one of the following is true (both may hold):

$$P_\beta = P_\beta^{\text{inf}}, \text{ where } P^{\text{inf}} := \inf_\alpha \{(\alpha, \beta) : (\alpha, \beta) \in S_1(P_\beta)\}, \quad (101)$$

$$P_\beta = P_\beta^{\text{sup}}, \text{where } P^{\text{sup}} := \sup_\alpha \{(\alpha, \beta) : (\alpha, \beta) \in S_1(P_\beta)\}. \quad (102)$$

The same holds for Q_β.
Suppose without loss of generality that $P_\beta = P_\beta^{\text{inf}}$. Then, there exist $\epsilon_P > 0$ such that the following function $\eta^P : \mathbb{R}^1 \to \mathbb{R}^2$ is well-defined $\forall \beta \in \text{conv}(P_\beta, P_\beta + \epsilon_P)$:

$$\eta^P(\beta) = \inf_\alpha \{(\alpha, \beta) : (\alpha, \beta) \in S_1(\beta)\}. \quad (103)$$

We will consider two cases: $Q_\beta = Q_\beta^{\text{inf}}$ and $Q_\beta = Q_\beta^{\text{sup}}$. Suppose $Q_\beta = Q_\beta^{\text{inf}}$. Thus, there exist $\epsilon_Q > 0$ such that the following function $\eta^Q : \mathbb{R}^1 \to \mathbb{R}^2$ is well-defined $\forall \beta \in \text{conv}(Q_\beta - \epsilon_Q, Q_\beta)$:

$$\eta^Q(\beta) = \inf_\alpha \{(\alpha, \beta) : (\alpha, \beta) \in S_1(\beta)\}. \quad (104)$$

It is clear that both $\eta^P(\cdot)$ and $\eta^Q(\cdot)$ are continuous. Because of continuity and the fact that $S_1(P_\beta) \neq \emptyset$ (Lemma 4), both $\eta^P(\cdot)$ and $\eta^Q(\cdot)$ are well defined in $\text{conv}(P_\beta, Q_\beta)$.

$\eta^P(\cdot)$ and $\eta^Q(\cdot)$ do not possess any breakpoint in $\text{conv}(P_\beta, Q_\beta)$; i.e., they are affine functions. Otherwise, any breakpoint of $\eta^P(\cdot)$ and $\eta^Q(\cdot)$ would also belong to Γ, and, therefore, there would exist $R \in \Gamma$ s.t. $P_\beta < R_\beta < Q_\beta$, contradicting our assumption.

Since both functions are well defined and affine in $\text{conv}(P_\beta, Q_\beta)$, they must be equal, since if for some $\hat{\beta} \in \text{conv}(P_\beta, Q_\beta), \eta^P(\hat{\beta}) \neq \eta^Q(\hat{\beta})$, either $\eta^P(\hat{\beta}_\alpha) > \eta^Q(\hat{\beta}_\alpha)$ or $\eta^P(\hat{\beta}_\alpha) < \eta^Q(\hat{\beta}_\alpha)$. The first case contradicts the definition of $\eta^P(\cdot)$, and the second case contradicts the definition of $\eta^Q(\cdot)$.

Now, suppose that $Q_\beta = Q_\beta^{\text{sup}}$. Thus, there exist $\epsilon_Q > 0$ such that the following function $\eta^Q : \mathbb{R}^1 \to \mathbb{R}^2$ is well-defined $\forall \beta \in \text{conv}(Q_\beta - \epsilon_Q, Q_\beta)$:

$$\eta^Q(\beta) = \sup_\alpha \{(\alpha, \beta) : (\alpha, \beta) \in S_1(\beta)\}. \quad (105)$$

□

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If $\eta^P(\cdot)$ and $\eta^Q(\cdot)$ are equal, then the result is proved. So, suppose that they are not equal.

Claim 17 *$\eta^P(\cdot)$ and $\eta^Q(\cdot)$ do not intersect.*

Proof Suppose $\eta^P(\cdot)$ and $\eta^Q(\cdot)$ intersect. Since $\eta^P(\cdot)$ and $\eta^Q(\cdot)$ are affine functions that are not equal, they can intersect at most at just one point.

Let T be such that $\eta^P(T_\beta) = \eta^Q(T_\beta)$, for $T_\beta \in \text{relint}(\text{conv}(P, Q))$.

From the definition of $\eta^P(\cdot)$ and $\eta^Q(\cdot)$, we must have $\eta^P(P_\beta)_\alpha < \eta^Q(P_\beta)_\alpha$ and $\eta^P(Q_\beta)_\alpha < \eta^Q(Q_\beta)_\alpha$.

Since $\eta^P(\cdot)$ is affine, $\eta^P(P_\beta)$, $\eta^P(T_\beta)$, $\eta^P(Q_\beta)$ are colinear.

The same must hold for the points $\eta^Q(P_\beta)$, $\eta^Q(T_\beta)$, $\eta^Q(Q_\beta)$, which is impossible, since in order for that to happen, we must have that either $\eta^P(P_\beta)_\alpha > \eta^Q(P_\beta)_\alpha$, or $\eta^P(Q_\beta)_\alpha > \eta^Q(Q_\beta)_\alpha$. □

Since $\eta^P(\cdot)$ and $\eta^Q(\cdot)$ do not intersect, then

$$\text{conv}(P, Q) \subset \bigcup_{\beta \in \text{conv}(P_\beta, Q_\beta)} \text{conv}(\min(\eta^P(\beta), \eta^Q(\beta)), \max(\eta^P(\beta), \eta^Q(\beta))). \tag{106}$$

Since $\eta^P(\beta) \in S_1(P_\beta)$ and $\eta^Q(\beta) \in S_1(P_\beta)$ and $S_1(P_\beta)$ is convex, we have:

$$\text{conv}(P, Q) \subset S_1(\hat{P}_\beta). \tag{107}$$

Thus, $\varphi_1(\alpha, \beta) = 0$, $\forall(\alpha, \beta) \in \text{conv}(P, Q)$, contradicting our assumption and, therefore, proving the result. □

Now we are ready to present the proof of correctness of Algorithm 1.

Theorem 18 *Algorithm 1 returns a pair of Lagrangian multipliers (α, β) if and only if there exists (α, β) such that*

$$\varphi_1(\alpha, \beta) = 0, \tag{108}$$
$$\varphi_2(\alpha, \beta) = 0. \tag{109}$$

Proof First, we will prove sufficiency; i.e., if a pair (α, β) is returned by Algorithm 1, then it must satisfy equalities (108), (109).

Suppose that $(\hat{\alpha}, \hat{\beta})$ is returned by Algorithm 1. It is clear that Γ contains the extreme points of the level sets of $\varphi_1^k(\cdot)$, $\tilde{\varphi}_1^k(\cdot)$ at zero, $\forall k \in J$. These extreme points are the breakpoints of the functions $\underline{\psi}_1(\cdot)$, $\overline{\psi}_1(\cdot)$, defined in (35), (36). After Line 7, Γ contains these breakpoints sorted. If $\hat{\beta} < (\Gamma_{(1)})_\beta$ or $\hat{\beta} > (\Gamma_{(|\Gamma|)})_\beta$, then, by construction, $(\hat{\alpha}, \hat{\beta})$ satisfies (108), (109). In the same way, if $\hat{\beta} = (\Gamma_{(j)})_\beta$ for some $j \in J$, the same conclusion holds. If $(\Gamma_{(j-1)})_\beta < \hat{\beta} < (\Gamma_{(j)})_\beta$ for $j \in J - \{1\}$, then, from Theorem 16, $(\hat{\alpha}, \hat{\beta})$ satisfies (108). Using the continuity of $\varphi_2(.,.)$, it is clear that (109) is also satisfied.

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We will now prove the necessity by contradiction. Suppose that there is a pair $(\hat{\alpha}, \hat{\beta})$ that solves (108), (109) but Algorithm 1 did not return any point. If $\hat{\beta} < (\Gamma_{(1)})_\beta$ or $\hat{\beta} > (\Gamma_{(|\Gamma|)})_\beta$, then the algorithm would have found the pair (see Lines 8 and 22). Therefore, there was no solution point in the convex combinations checked by the algorithm. Since $(\hat{\alpha}, \hat{\beta})$ is a solution, then it must be contained in some convex combination of points in Γ, as it contains all the breakpoints of the functions $\underline{\psi}_1(\cdot), \overline{\psi}_1(\cdot)$. Thus, $\exists j \in J - \{1\}$ such that $(\Gamma_{(j-1)})_\beta < \hat{\beta} < (\Gamma_{(j)})_\beta$. It is clear that $\nexists j \in J - \{1\}$ such that $(\hat{\alpha}, \hat{\beta}) \in \text{conv}(\Gamma_{(j-1)}, \Gamma_{(j)})$; otherwise, the algorithm would have returned this point, since the algorithm checks this convex combination. Thus, $(\hat{\alpha}, \hat{\beta}) \in \text{conv}(\Gamma_{(j)}, \Gamma_{(k)})$, with $j - k > 1$. Suppose without loss of generality that $j - k = 2$; i.e., there exists just one breakpoint between the two points that generate the solution. Thus, $(\hat{\alpha}, \hat{\beta}) \in \text{conv}(\Gamma_{(k)}, \Gamma_{(k+2)})$, for some $k \in J - \{|J|, |J| - 1\}$. Suppose without loss of generality that $(\Gamma_{(k+1)})_\alpha < \max((\Gamma_{(k)})_\alpha, (\Gamma_{(k+2)})_\alpha)$; i.e., $\Gamma_{(k+1)}$ is below the line segment connecting $\Gamma_{(k)}$ to $\Gamma_{(k+2)}$. Since $\varphi_2(\hat{\alpha}, \hat{\beta}) = 0$, then $S_2(\hat{\alpha}) \neq \emptyset$. Since $\varphi_2(.,.)$ is continuous and $S_2(\alpha) \neq \emptyset, \forall \alpha \in \mathbb{R}$ (see Corollary 3), $\exists \epsilon > 0$ such that $\varphi_2(\hat{\alpha} - \epsilon, \hat{\beta}) = 0$. Thus, there exists a point $P = (\hat{\alpha} - \epsilon, \hat{\beta})$ strictly contained in the simplex formed by the points $\Gamma_{(k)}, \Gamma_{(k+1)}, \Gamma_{(k+2)}$. Again, using the fact that $\varphi_2(.,.)$ is continuous and Corollary 3, we can take ϵ big enough such that $(\hat{\alpha} - \epsilon, \hat{\beta}) \in \text{conv}(\Gamma_{(k)}, \Gamma_{(k+1)})$ or $(\hat{\alpha} - \epsilon, \hat{\beta}) \in \text{conv}(\Gamma_{(k+1)}, \Gamma_{(k+2)})$. Since any point on both convex combinations also satisfies (108), we found a solution point contained on a convex combination of two subsequence breakpoints in Γ, leading to a contradiction and the result is proved. □

Lemma 19 *The complexity of Algorithm 1 is* $O(n^2 \log n)$.

Proof It is clear that Algorithm 2 has $O(n \log n)$, as its most costly step is the sorting of the breakpoint of the function $\varphi_1^k(\cdot)$ and there are at most $2n - 2$ (see proof of 13).

Since Line 5 is executed n times, loop (4–6) has complexity $O(n^2 \log n)$.

Γ contains the extreme points of the level sets of $\varphi_1(\cdot, \cdot)$ at 0. Since this level set is bounded by the functions $\underline{\psi}_1(\cdot)$, $\overline{\psi}_1(\cdot)$, using Lemma 13 we have that $|\Gamma| \leq 4n^2 - 4n$. Thus, sorting the elements of Γ (line 7) will take $O(n^2 \log n)$ operations.

Since every operation in the loop (14–21) takes constant time, (14–21) has quadratic complexity.

It is easy to see that lines 8 and 22 take linear time.

All of the remaining operations take constant time.

Therefore, the overall complexity of Algorithm 1 is $O(n^2 \log n + n^2 \log n + n^2 + 2n) = O(n^2 \log n)$. □

4 An example

In order to make the previous results as well as the algorithm proposed easier to understand, we will now present how the functions are used to compute the solution for a problem in $\mathbb{R}^3$.

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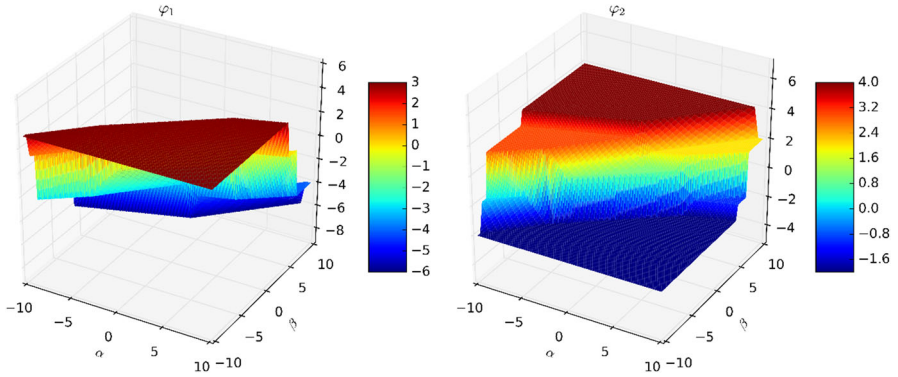

Fig. 1 $\varphi_1(.,.)$ and $\varphi_2(.,.)$

Consider the following problem:

$$\begin{aligned} \text{hyperplane 1:} \quad & a = (3, -4, 2), b_1 = 2, \\ \text{hyperplane 2:} \quad & c = (-1, 3, -2), b_2 = -1, \\ \text{upperbound on the variables:} \quad & d = (1, 1, 1), \\ \text{point to be projected:} \quad & \overline{x} = (5, 3, 2). \end{aligned}$$

Figure 1 shows functions $\varphi_1(.,.)$ and $\varphi_2(.,.)$. Recall that we want to find the point (α, β) such that $\varphi_1(\alpha, \beta) = 0$ and $\varphi_2(\alpha, \beta) = 0$.

The first step is to compute the functions defined in (68), (69).

$$\begin{aligned} \varphi_1^0 = & \min\left\{\max\left\{3 - \frac{5}{-4}3 - \left(3 - \frac{-1}{-4}3\right)\beta, 0\right\}, 1\right\}(-4) \\ & + \min\left\{\max\left\{3 - \frac{5}{-4}3 - \left(3 - \frac{-1}{-4}3\right)\beta, 0\right\}, 1\right\}(-4) \\ & + \min\left\{\max\left\{2 - \frac{5}{2}3 - \left(-2 - \frac{-1}{2}3\right)\beta, 0\right\}, 1\right\}2 \\ & + \min\left\{\max\left\{2 - \frac{5}{2}3 - \left(-2 - \frac{-1}{2}3\right)\beta, 0\right\}, 1\right\}2 - 2, \\ \tilde{\varphi}_1^0 = & \min\left\{\max\left\{3 - \frac{5}{-4}3 + \frac{1}{-4}3 - \left(3 - \frac{-1}{-4}3\right)\beta, 0\right\}, 1\right\}(-4) \\ & + \min\left\{\max\left\{3 - \frac{5}{-4}3 + \frac{1}{-4}3 - \left(3 - \frac{-1}{-4}3\right)\beta, 0\right\}, 1\right\}(-4) \\ & + \min\left\{\max\left\{2 - \frac{5}{2}3 + \frac{1}{2}3 - \left(-2 - \frac{-1}{2}3\right)\beta, 0\right\}, 1\right\}2 \\ & + \min\left\{\max\left\{2 - \frac{5}{2}3 + \frac{1}{2}3 - \left(-2 - \frac{-1}{2}3\right)\beta, 0\right\}, 1\right\}2 - 2. \end{aligned}$$

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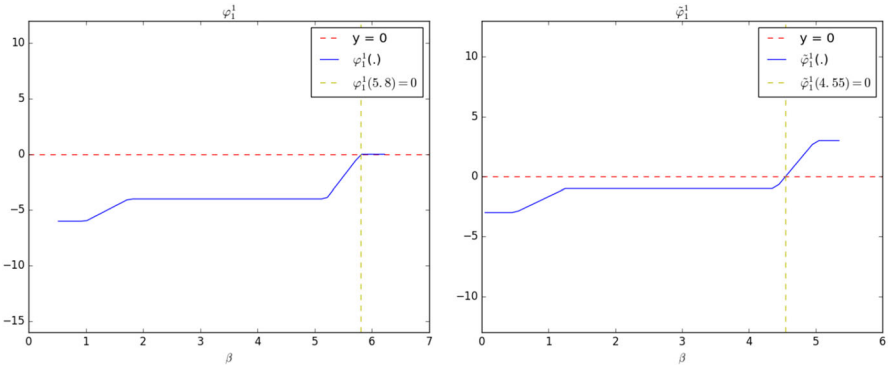

Fig. 2 $\varphi_1^1(.)$ and $\tilde{\varphi}_1^1(.)$

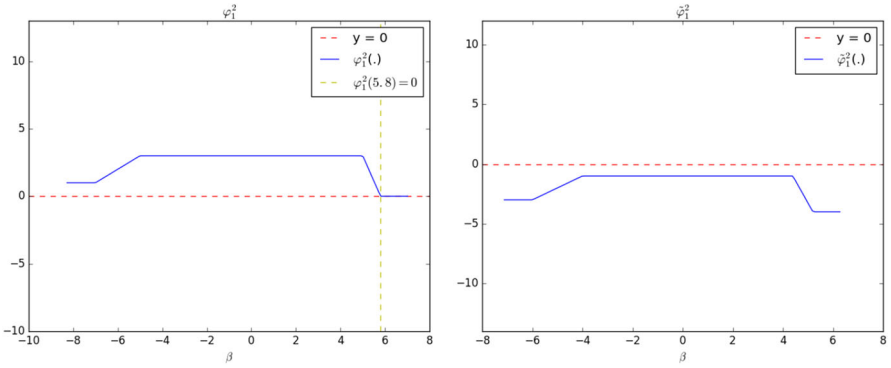

Fig. 3 $\varphi_1^2(.)$ and $\tilde{\varphi}_1^2(.)$

Similarly, we can compute φ_1^1, $\tilde{\varphi}_1^1$, φ_1^2, and $\tilde{\varphi}_1^2$.

The graphs of the functions $\varphi_1^k(.)$ and $\tilde{\varphi}_1^k(.)$ for $k = 1, \ldots, 3$ are shown in Figs. 2, 3, and 4. Since we need to find the values of β such that $\varphi_1^k(\beta) = 0$ and $\tilde{\varphi}_1^k(\beta) = 0$, for $k = 1, \ldots, 3$, as in Algorithm 2, these graphs also show the line that defines the level set that we are interested in $y = 0$ (in red) and for which value of β (if any) $\tilde{\varphi}_x^k(\beta) = 0$, for $k = 1, \ldots, 3$ (in yellow).

After executing Line 5 of Loop 4 of Algorithm 1, we have

$$\Gamma = \{(3.6, 5.8, -1.0), (2.85, 4.55, 0.25), (3.6, 5.8, -1.0), \\ (-5.75, -6.75, 0.75), (-3.75, -4.25, 0.25)\}.$$

It can be seen in Lines 9, 16 and 24 of Algorithm 2 that the tuples in Γ are of the form $(\alpha, \beta, \varphi_2^k(\beta))$. After Line 7, i.e., sorting on the β component, we have:

$$\Gamma = \{(-5.75, -6.75, 0.75), (-3.75, -4.25, 0.25), \\ (2.85, 4.55, 0.25), (3.6, 5.8, -1.0), (3.6, 5.8, -1.0)\}.$$

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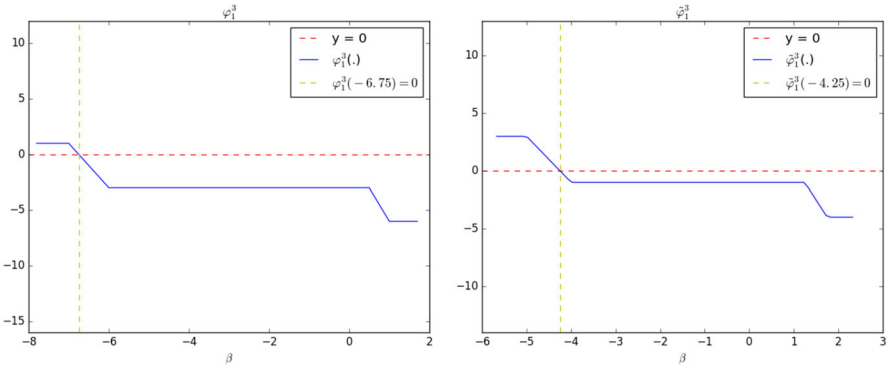

Fig. 4 $\varphi_1^3(.)$ and $\tilde{\varphi}_1^3(.)$

Since $0 \in [0.25, -1.0]$ (Line 18 of Algorithm 2), i.e., the solution is between the tuples (2.85, 4.55, 0.25) and (3.6, 5.8, −1.0). Thus, we can calculate β by using the coordinates (4.55, 0.25) and (5.8, −1.0). Let $y_0 = -1, y_1 = 0.25, y = 0, \beta_0 = 5.8, \beta_1 = 4.55$. Let $S_\beta = \frac{y_1 - y_o}{\beta_1 - \beta_o}$. Thus,

$$y - y_o = S_\beta(\beta - \beta_o) \Rightarrow \beta = \frac{-y_o + S_\beta \beta_o}{S_\beta} = \frac{-(-1) + (-1)5.8}{-1} = 4.8$$

Similarly, we can calculate α by using the coordinates (2.85, 0.25) and (3.6, −1.0). Let $y_0 = -1, y_1 = 0.25, y = 0, \alpha_0 = 3.6, \alpha_1 = 2.85$. Let $S_\alpha = \frac{y_1 - y_o}{\alpha_1 - \alpha_o}$. Thus

$$y - y_o = S_\alpha(\alpha - \alpha_o) \Rightarrow \beta = \frac{-y_o + S_\alpha \alpha_o}{S_\alpha} = \frac{-(-1) + \frac{5}{3}3.8}{-\frac{5}{3}} = 3.0.$$

Thus, the solution is:

$$\alpha = 3 \text{ and } \beta = 4.8.$$

The primal solution can be obtained from the values of α, β using relation (11) :

$$\begin{aligned} x_1 &= \min\{\max\{5.0 - 3(+3) - 4.8(-1), 0\}, 1\} = 0.8 \\ x_2 &= \min\{\max\{3.0 - 3(-4) - 4.8(+3), 0\}, 1\} = 0.6 \\ x_3 &= \min\{\max\{2.0 - 3(+2) - 4.8(-2), 0\}, 1\} = 1. \end{aligned}$$

5 Application

The method presented in this paper can be applied to the cart-centering problem [see Athans and Falb (1966)].

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The description of the problem involves a cart with mass m moving along an infinite one-dimensional frictionless track. The problem consists in centering the cart, in minimal time, by applying an external time-variant force (of maximal magnitude F) so as to accelerate the cart in either the positive or the negative direction of the track. Given the values for the position $x(t)$ and velocity $v(t)$ of the cart at time t, this problem can be further formulated as calculating a force $F(t) = F[x(t), v(t)]$ in order to place the cart in position $x(t_f) = 0$ with velocity $v(t_f) = 0$ in minimal t_f.

Though simple, this is a typical optimal control problem and has a well-known solution [the bang-bang rule Koza (1992)].

Let τ be the interval of discretization of time. This system is simulated using Euler's method, i.e., taking time steps of t seconds and applying Newton's Laws:

$$x(t+\tau) = x(t) + \tau v(t)$$
$$v(t+\tau) = v(t) + \tau \frac{F(t)}{m}$$

Let $f = (f_1, f_2, \ldots, f_n)^T \in \mathbb{R}^n$ be given. Mathematically, this problem can be formulated as a convex quadratic programming problem:

$$\begin{aligned} (Q) := \min \quad & \textstyle\sum_{i=1}^n f_i^2 \\ \text{s.t.} \quad & \\ & \textstyle\sum_{i=1}^n f_i = 0, \\ & \textstyle\sum_{i=1}^n (\frac{1}{2}(n-i)+1) f_i = 1, \\ & -1 \le f_i \le 1, \qquad i = 1 \cdots n. \end{aligned}$$

Setting $x_i = f_i + 1$:

$$\begin{aligned} (\hat{Q}) := \min \quad & \textstyle\sum_{i=1}^n (x_i - 1)^2 \\ \text{s.t.} \quad & \\ & \textstyle\sum_{i=1}^n x_i = n, \\ & \textstyle\sum_{i=1}^n (\frac{1}{2}(n-i)+1) x_i = 1 + n + \frac{1}{2}\sum_{i=1}^n (n-i), \\ & 0 \le x_i \le 2, \qquad i = 1 \cdots n. \end{aligned}$$

Problem $(\hat{Q})$ is now in a format which allows it to be solved by Algorithm 1.

6 Numerical results

We compared Algorithm 1 with CPLEX version 12.4 for instances randomly generated for different dimension sizes n, ranging from 10^3 to $3 \cdot 10^4$ with step $1 \cdot 10^3$. For each dimension, we generated 200 instances, from which we removed the 5 highest and 5 lowest values for each metric compared.

CPLEX was run using the following precisions: 10^{-4}, 10^{-8} and 10^{-12}, which are identified by the labels Cplex10^{-4}, Cplex10^{-8}, and Cplex10^{-12}, respectively. We

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Fig. 5 Execution time

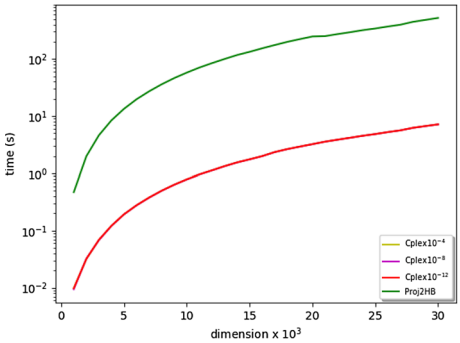
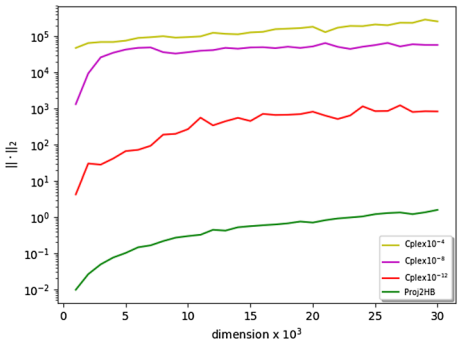


Fig. 6 Feasibility error



generated two graphs using log scale: execution time (Fig. 5) and feasibility error (Fig. 6).

CPLEX outperformed our approach for these instances significantly as regards execution time, being more than 2 orders of magnitude lower than our approach (see Fig. 5). There was no significant time difference among the CPLEX instances run with different precisions.

We also compared feasibility error. As we can see in Fig. 6, CPLEX instances behaved as expected being more accurate with the increase in precision. However, even for precision 10^{-12} (i.e., Cplex10^{-12}), our algorithm returned solutions more than 2 orders of magnitude more accurate than CPLEX's.

7 Concluding remarks and future work

In this document, we presented an efficient algorithm for the intersection of two hyperplanes and a box in $\mathbb{R}^n$. Different methods for solving this problem exist, but their complexity depends on the desired accuracy of the solution, making them weakly polynomial.

We demonstrated in this document a strongly polynomial algorithm whose complexity is lower to the existing methods in the literature. Our method relies on the KKT optimality conditions for this problem and consists of finding an optimal pair of Lagrangian multipliers.

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As seen in Sect. 6, we empirically showed that although our approach takes significantly more time than CPLEX, it reaches solutions orders of magnitude more accurate than CPLEX's solutions.

One possible future work related to this research is to try to extend this method to more hyperplanes, as long as the complexity remains inferior to those available in the literature.

Another interesting extension of this work would be to try to apply the same approach to more restrictive types of constraints, such as considering just positive hyperplanes or considering only nonnegative constraints, as opposed to upper and lower bounds. Considering more specific types of the problems may lead to an algorithm of lower complexity.

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