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ABSTRACT

In this paper, the results of the Lie group method carried out by Ovsiannikov are utilized to study the one-dimensional hydrodynamic equations governing compressible, inviscid fluid flow in the absence of heat conduction. One-parameter subgroups of the admissible *R*-parameter Lie group of point transformations of the system are applied to reduce the first-order, non-linear system of partial differential equations (PDE)s to a first-order system of ordinary differential equations (ODE)s. Closed-form solutions to the reduced ODE systems are subsequently determined using a linear velocity profile ansatz. These solutions are valid in one-dimensional (1D) planar, cylindrical and spherical geometries and are connected to solutions of the governing system of PDEs through an inverse map. The linear velocity type solutions were first considered by Sedov in 1953 and constitute a subclass of all possible similarity solutions of the compressible hydrodynamics equations. They further serve as illustrative examples of using solutions obtained for the reduced ODEs to find solutions of the associated system of PDEs. Consequently, the reduced systems of ODEs are provided in their entirety inviting additional solutions to be determined via alternative explicit or numerical solution techniques.

I. INTRODUCTION

The Lie group methods discussed in this paper, otherwise known as symmetry methods, exploit intrinsic features of mathematical objects (e.g. the differential equations defining fundamental laws of physics) to obtain special types of solutions. These intrinsic features are connected to the notions of symmetry and invariance and the resulting solutions will subsequently be referred to as group-invariant solutions. Group-invariant solutions are a subject of interest for a number of reasons. Principally, they can be viewed as intermediate-asymptotic solutions to a much larger class of flows originating from a variety of initial and/or boundary conditions, i.e., they describe solutions to these flows at times when the dependence on the initial conditions has disappeared but the solution has not yet proceeded to an equilibrium state.[1] As a result, they can provide insight into the expected local evolutionary behaviour of a wide variety of problems. Additionally, group-invariant solutions can serve as verification tests to benchmark numerical methods, especially if the solutions are obtained explicitly in closed form. Also, due to their inherent invariant properties, the solutions can be used to check the symmetry preserving capabilities of said numerical approaches or even to help design new numerical schemes with symmetry preservation in mind.[2–4]

In the analysis section of this paper we outline intrinsic structures, known as symmetries, present in the system of non-linear, first-order partial differential equations governing one-dimensional, compressible, inviscid hydrodynamics. These symmetries are identifiable through invariance of the differential equations under a one- or multi-parameter Lie group of point transformations. A criterion defining this invariance will be provided later. Ultimately, the existence of such structures enables the realisation of a new local canonical co-ordinate system in which the system of partial differential equations (PDE)s may be reduced to a simpler form along certain directions. Such a simplified system is likely to be more easily solvable and the simplification may potentially lead to the attainment of a solution. Typically, one-parameter group invariance facilitates the reduction of a k^{th} order ordinary differential equation (ODE) by order one and reduction of a k^{th} order PDE with n independent variables to a k^{th} order PDE with $n-1$ independent variables. For the first-order system of PDEs considered in this paper, which possess two independent variables, various reductions to systems of ODEs are achieved.

Our analysis complements a subsection of prior work performed by Coggeshall and Axford.[5] In said paper, symmetry analysis is conducted on the governing equations expressed in terms of the dependent variables of velocity, density and temperature. Admissible symmetries of the equations are determined and used to obtain corresponding reduced systems of ODEs. Additional analysis is also performed in Ref. 5 to consider the effects of radiation transfer and heat conduction models. For simplicity, we neglect these additional mechanisms and proceed to perform the symmetry analysis instead using velocity, density and pressure as the dependent variables thus obtaining alternatively formulated systems of ODEs. For illustrative purposes we show how these systems of ODEs can be solved using an ansatz for the velocity and map the results to solutions of the original system. This is achieved by assuming that the velocity profile is linearly proportional to the distance from the center of symmetry. Such a class of solutions was first considered in work by Sedov[6] and later by Ramsey[7,8] which connected a number of linear velocity profile solutions appearing in fusion modelling literature.[9–14] By working in terms of velocity, density and pressure, it is easier to connect the example solutions derived in the current paper to members of the linear velocity class appearing elsewhere in the literature thereby placing these solutions within the context of symmetry analysis. Each solution we provide is explicitly connected to a particular symmetry for which the corresponding group generator is also given. Furthermore, the choice of dependent variables used in this paper also reflects the typical set of variables used in many multi-physics hydrocode solvers. This easily enables the direct use of any of the presented solutions for verification purposes. Further details on the linear velocity class will be given later.

Aside from the references provided above, many shock-less, linear velocity profile solutions also appear in other work by Coggeshall.[15] The equivalence between particular solutions derived in this work and the Coggeshall solutions is demonstrated by expressing the solutions in terms of the velocity, density and temperature. We also obtain examples of the self-similar solutions discussed by Zel'dovich and Raizer[16] in their chapter on self-similar processes in gas dynamics as well as those of exponential self-similar type due to Stanyukovich.[17]

In summary, Section II of this paper introduces the governing system of equations studied and a derivation of the governing equations is provided in Appendix A. In Section III, the invariance criterion defining an admissible Lie group of point transformations of a differential equation is discussed. We then highlight the results of application of the symmetry method to the hydrodynamics equations performed by Ovsiannikov. Next, in Section IV, various one-parameter symmetry groups of transformations are used to obtain reduced systems of equations. Details of the reductions performed are provided in Appendixes B–D. The reduced systems are then solved using the spatially-linear velocity profile anzatz introduced in Section V, and solutions are presented in Section VI. Further information on the steps involved in the calculation of the solutions are provided in Appendix E. Finally, Section VII connects pre-existing solutions in the literature with the presented work.

II. 1D, INVISCID, COMPRESSIBLE EQUATIONS OF MOTION

This paper concerns itself with solutions to the one-dimensional (1D) equations of motion, describing conservation of mass, momentum and entropy, for compressible, inviscid fluid flow in the absence of heat conduction. The respective equations are

$$\rho_t + u\rho_r + \rho\left(u_r + \frac{ku}{r}\right) = 0, \tag{1}$$

$$p_r + \rho(u_t + uu_r) = 0, \tag{2}$$

$$p_t + up_r + A_s(p,\rho)\left(u_r + \frac{ku}{r}\right) = 0, \tag{3}$$

where u, p and ρ account for the velocity, pressure and density profiles of the fluid and are functions of the spatial coordinate r and time t, respectively. The thermodynamic variable $A_s(p, \rho)$ represents the isentropic bulk modulus and the constant k has admissible values of 0, 1 or 2 for fluid motion with planar, cylindrical or spherical symmetry, respectively. A derivation of these equations is provided in Appendix A.

Due to the unspecified form of the isentropic bulk modulus as a function of pressure and density, $A_s(p, \rho)$, this system of equations permits the use of a broad range of equations of state (EOS). An isentropic bulk modulus of this form corresponds to a specific internal energy specified as a function of pressure and density, $e := e(p, \rho)$ and can be obtained directly using

$$A_s = \frac{1}{\left.\frac{\partial e}{\partial p}\right|_\rho}\left(\frac{p}{\rho} - \rho\left.\frac{\partial e}{\partial \rho}\right|_p\right). \tag{4}$$

An EOS of this kind is typically described as being thermodynamically incomplete since, for example, it is not possible to obtain information about the temperature behaviour without supplementary information

$$\left.\frac{\partial e}{\partial s}\right|_\rho = T. \tag{5}$$

Such an EOS is however sufficient to provide a closure model for the sytem of equations and facilitate the attainment of a solution. More information about thermodynamically complete equations of state or fundamental thermodynamic relations can be found in Refs. 18 (Chap. 5) and 19.

III. LIE GROUP INVARIANCE

We are interested in certain types of solutions referred to as group-invariant solutions. The term "group invariant solution" stems from the invariance of the solution surface under the action of an element, g, of the full group, G, admitted by the governing equations. Solutions of this type can be obtained systematically by leveraging known symmetries of differential equations. The method applied in the following sections is originally due to the work of Lie[20] and comprehensive presentations of the approach can be found in Bluman and Anco,[21] Olver,[22] and Ovsiannikov.[23] For a basic introduction to the topic see Albright et al.[24] Some of the most common examples of group-invariant solutions appearing in the literature are those that are invariant under scaling or stretching transformations, i.e., scaling group-invariant solutions. Typically, these solutions are often categorized as similarity solutions of the first or second kind depending on whether the exponents of the similarity variables used to construct the solution can or cannot be obtained through dimensional considerations alone.[25–27] Using Lie group methods, both scaling solutions and solutions invariant under alternative kinds of transformations can be sought. Together, these transformations constitute broader groups of transformations known as continuous Lie groups of point transformations.

In short, a Lie group of point transformations is a set of elements, g, combined with a group operation which satisfies the algebraic axioms associated with an abstract group[28] and also possesses the structure of a topological space associated with a smooth differentiable manifold, M. The Lie group operation itself is a diffeomorphism (a one-to-one smooth, differentiable, invertible map whose inverse is also differentiable). For every element of a Lie group of transformations acting on a manifold, there is an associated mapping from the manifold to itself. The properties of the equations we are interested in should be independent of the coordinate system used to formulate the equations. Since smooth manifolds provide a means for studying differential equations in a context free of local coordinates, Lie groups provide a powerful tool for identifying the structures that remain invariant under invertible changes of coordinates.[29]

In this section, we outline the admissible R-*parameter* group of transformations under which the equations of motion, Equations (1)–(3) are defined to be invariant [Ref. 21, pp. 73-74]. To achieve this, it is first necessary to define what constitutes *invariance* of a system of differential equations under a group of transformations.

A. Invariance criterion

The first step to establishing a definition for invariance requires outlining a concise notation in which to present the ideas. First, the system of governing equations is denoted using

$$F_l(\boldsymbol{x},\boldsymbol{y},\boldsymbol{q}) = 0, \qquad l = 1,\dots,L, \tag{6}$$

where $\boldsymbol{x} = (x_1, \dots, x_J)$ and $\boldsymbol{y} = (y^1, \dots, y^I)$ denote the vectors of independent and dependent variables x_j and y_i, respectively and $\boldsymbol{q}$ is the tensor of first order partial derivatives

$$q^i_j = \frac{\partial y^i}{\partial x_j}, \qquad i = 1,\dots,I \qquad j = 1,\dots,J.$$

For Equations (1)–(3), excluding initial and boundary conditions for the current discussion, $L = 3$, $\boldsymbol{x} = (t, r)$, $\boldsymbol{y} = (u, p, \rho)$ and therefore

$$F_1 = q^3_1 + y^1 q^3_2 + y^3\left(q^1_2 + \frac{ky^1}{x_2}\right) = 0, \tag{7}$$

$$F_2 = q^2_2 + y^3\left(q^1_1 + y^1 q^1_2\right) = 0, \tag{8}$$

$$F_3 = q^2_1 + y^1 q^2_2 + A_s(y^2, y^3)\left(q^1_2 + \frac{ky^1}{x_2}\right) = 0. \tag{9}$$

Next, consider an R-parameter group of point transformations of the independent and dependent variables denoted by

$$\tilde{\boldsymbol{x}} = \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y};\epsilon), \qquad \tilde{\boldsymbol{y}} = \boldsymbol{\beta}(\boldsymbol{x},\boldsymbol{y};\epsilon), \tag{10}$$

where the tilde denotes a transformation belonging to the group which takes old variables to new variables. These transformations are smoothly parameterized by a vector $\epsilon = (\epsilon_1, \dots, \epsilon_R)$ of R independent parameters. In order to determine the particular admissible R-parameter transformation group of the system of differential equations, we re-parameterize ϵ in terms of a single common parameter, s.[21] Consequently, it is only necessary to consider invariance of the system under one-parameter transformations. These transformations are denoted

$$\tilde{\boldsymbol{x}} = \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y};\epsilon(s)), \qquad \tilde{\boldsymbol{y}} = \boldsymbol{\beta}(\boldsymbol{x},\boldsymbol{y};\epsilon(s)). \tag{11}$$

Following Lie's first fundamental theorem, [Ref. 21, pp. 39-40], the identity transformation is defined for $s = 0$ so that

$$\tilde{\boldsymbol{x}}\big|_{s=0} = \boldsymbol{x}, \qquad \tilde{\boldsymbol{y}}\big|_{s=0} = \boldsymbol{y}. \tag{12}$$

Next, the transformations with respect to s are expanded to give a Taylor series around the identity generating $I + J$ equations

$$\tilde{x}_j = x_j + s\left.\frac{\partial \alpha_j(\boldsymbol{x},\boldsymbol{y};\epsilon(s))}{\partial s}\right|_{s=0} + \mathcal{O}(s^2), \quad j = 1,\dots,J, \tag{13}$$

$$\tilde{y}^i = y^i + s\left.\frac{\partial \beta^i(\boldsymbol{x},\boldsymbol{y};\epsilon(s))}{\partial s}\right|_{s=0} + \mathcal{O}(s^2), \quad i = 1,\dots,I. \tag{14}$$

Via application of the chain rule

$$\tilde{x}_j = x_j + s\sum_{r=1}^{R}\left.\frac{\partial \alpha_j(\boldsymbol{x},\boldsymbol{y};\epsilon)}{\partial \epsilon_r}\frac{\partial \epsilon_r}{\partial s}\right|_{s=0} + \mathcal{O}(s^2), \tag{15}$$

$$\tilde{y}^i = y^i + s\sum_{r=1}^{R}\left.\frac{\partial \beta^i(\boldsymbol{x},\boldsymbol{y};\epsilon)}{\partial \epsilon_r}\frac{\partial \epsilon_r}{\partial s}\right|_{s=0} + \mathcal{O}(s^2). \tag{16}$$

For the sake of conciseness, the partial derivatives of the transformations with respect to the parameters ϵ_r will subsequently be denoted using

$$\eta_{j,r}(\boldsymbol{x},\boldsymbol{y}) := \left.\frac{\partial \alpha_j(\boldsymbol{x},\boldsymbol{y};\epsilon)}{\partial \epsilon_r}\right|_{s=0}, \tag{17}$$

$$\phi^{i,r}(\boldsymbol{x},\boldsymbol{y}) := \left.\frac{\partial \beta^i(\boldsymbol{x},\boldsymbol{y};\epsilon)}{\partial \epsilon_r}\right|_{s=0}. \tag{18}$$

The partial derivatives of ϵ_r with respect to the parameter s will be denoted using

$$a_r := \left.\frac{\partial \epsilon_r}{\partial s}\right|_{s=0}, \tag{19}$$

where the a_r are constants. The transformations of the independent and dependent variables, defined by Equations (15) and (16), can now be written succinctly as

$$\tilde{x}_j = x_j + s\sum_{r=1}^{R} a_r \eta_{j,r}(\boldsymbol{x},\boldsymbol{y}) + \mathcal{O}(s^2), \tag{20}$$

$$\tilde{y}^i = y^i + s\sum_{r=1}^{R} a_r \phi^{i,r}(\boldsymbol{x},\boldsymbol{y}) + \mathcal{O}(s^2). \tag{21}$$

The terms $\eta_{j,r}$ and $\phi^{i,r}$ are called the one-parameter coordinate functions and they are integral to defining a criterion for invariance. In addition to the transformations defined by Equations (20) and (21), transformations of the derivatives appearing in the differential equations must also be considered. These transformations are defined in such a way so as to preserve the relationship between the independent and dependent variables and the partial derivative. They are denoted using

$$\tilde{q}^i_j = \frac{\partial \tilde{y}^i}{\partial \tilde{x}_j} = \gamma^i_j(\boldsymbol{x},\boldsymbol{y},\boldsymbol{q};\epsilon(s)), \quad i = 1,\ldots,I, \quad j = 1,\ldots,J. \tag{22}$$

The corresponding Taylor series is

$$\tilde{q}^i_j = q^i_j + s\left.\frac{\partial \gamma^i_j(\boldsymbol{x},\boldsymbol{y},\boldsymbol{q};\epsilon(s))}{\partial s}\right|_{s=0} + \mathcal{O}(s^2), \tag{23}$$

$$= q^i_j + s\sum_{r=1}^{R}\left.\frac{\partial \gamma^i_j(\boldsymbol{x},\boldsymbol{y},\boldsymbol{q};\epsilon)}{\partial \epsilon_r}\frac{\partial \epsilon_r}{\partial s}\right|_{s=0} + \mathcal{O}(s^2), \tag{24}$$

$$= q^i_j + \sum_{r=1}^{R} a_r \zeta^i_{j,r}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{q}) + \mathcal{O}(s^2), \tag{25}$$

where

$$\zeta^i_{j,r}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{q}) := \left.\frac{\partial \gamma^i_j(\boldsymbol{x},\boldsymbol{y},\boldsymbol{q};\epsilon)}{\partial \epsilon_r}\right|_{s=0}.$$

It is possible to calculate the $\zeta^i_{j,r}$ directly from $\eta_{j,r}$ and $\phi^{i,r}$ using the *prolongation* formula.[21,22,24] A derivation will not be given here, instead we simply state the formula

$$\zeta^i_{j,r} = \frac{\partial \phi^{i,r}}{\partial x_j} + \sum_{k=1}^{I}\frac{\partial \phi^{i,r}}{\partial u^k}q^k_j - \sum_{k=1}^{J}\frac{\partial \eta_{k,r}}{\partial x_j}q^i_k - \sum_{k=1}^{J}\sum_{l=1}^{I}\frac{\partial \eta_{k,r}}{\partial u^l}q^i_k q^l_j, \quad i = 1,\ldots,I, \quad j = 1,\ldots,J, \tag{26}$$

where k and l are dummy indices in this context.

Now that a fairly compact notation for the transformations of the variables has been outlined, we turn our attention to determining how these transformations alter the equations of motion $F_l(\boldsymbol{x},\boldsymbol{y},\boldsymbol{q}) \to F_l(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}},\tilde{\boldsymbol{q}})$. Once again, expanding around the identity transformation, the Taylor series is

$$F_l(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}},\tilde{\boldsymbol{q}}) - F_l(\boldsymbol{x},\boldsymbol{y},\boldsymbol{q}) = s\left.\frac{\partial F_l(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}},\tilde{\boldsymbol{q}})}{\partial s}\right|_{s=0} + \mathcal{O}(s^2), \tag{27}$$

$$= s\sum_{r=1}^{R}\left.\frac{\partial F_l(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}},\tilde{\boldsymbol{q}})}{\partial \epsilon_r}\frac{\partial \epsilon_r}{\partial s}\right|_{s=0} + \mathcal{O}(s^2). \tag{28}$$

By application of the chain rule,

$$\left.\frac{\partial F_l(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}},\tilde{\boldsymbol{q}})}{\partial \epsilon_r}\right|_{s=0} = \sum_{j=1}^{J}\left.\frac{\partial F_l}{\partial \tilde{x}_j}\frac{\partial \tilde{x}_j}{\partial \epsilon_r}\right|_{s=0} + \sum_{i=1}^{I}\left.\frac{\partial F_l}{\partial \tilde{y}^i}\frac{\partial \tilde{y}^i}{\partial \epsilon_r}\right|_{s=0} + \sum_{j=1}^{J}\sum_{i=1}^{I}\left.\frac{\partial F_l}{\partial \tilde{q}^i_j}\frac{\partial \tilde{q}^i_j}{\partial \epsilon_r}\right|_{s=0}, \tag{29}$$

$$= \sum_{j=1}^{J}\eta_{j,r}\frac{\partial F_l}{\partial x_j} + \sum_{i=1}^{I}\phi^{i,r}\frac{\partial F_l}{\partial y^i} + \sum_{i=1}^{I}\sum_{i=1}^{I}\zeta^i_{j,r}\frac{\partial F_l}{\partial q^i_j}. \tag{30}$$

From the Taylor series, the following differential operator can be defined

$$\mathrm{pr}^{(1)}V_r = V_r + V^{(1)}_r, \tag{31}$$

where

$$V_r = \sum_{j=1}^{J}\eta_{j,r}\frac{\partial}{\partial x_j} + \sum_{i=1}^{I}\phi^{i,r}\frac{\partial}{\partial y^i}, \tag{32}$$

and

$$V^{(1)}_r = \sum_{i=1}^{I}\sum_{i=1}^{I}\zeta^i_{j,r}\frac{\partial}{\partial q^i_j}. \tag{33}$$

V_r is known as the one-parameter infinitesimal group generator [Ref. 22, pp. 27-28] and $V^{(1)}_r$ denotes its first-order prolongation [Ref. 22, pp. 101-103]. Together they form the first prolongation of V_r, denoted by $\mathrm{pr}^{(1)}V_r$ which acts on the larger jet space comprised of both the independent and dependent variables, and all first order derivatives.

At this stage it is possible to present a clear definition for local invariance of a first-order system of differential equations under a given group of transformations. For a differential equation to be invariant

$$F_l(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}},\tilde{\boldsymbol{q}}) = F_l(\boldsymbol{x},\boldsymbol{y},\boldsymbol{q}). \tag{34}$$

Therefore, using the definition of $\mathrm{pr}^{(1)}V_r$ and Equation (28),

$$F_l(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}},\tilde{\boldsymbol{q}}) - F_l(\boldsymbol{x},\boldsymbol{y},\boldsymbol{q}) = s\sum_{r=1}^{R}a_r\mathrm{pr}^{(1)}V_rF_l\big|_{s=0} + \mathcal{O}(s^2), \tag{35}$$

Invariance requires the right hand side (r.h.s) of Equation (35) to be equal to zero. Recognising that the higher order terms in the Taylor expansion will include terms such as

$$\sum_{r=1}^{R}\sum_{r=1}^{R}a_r^2\left(\mathrm{pr}^{(1)}V_r\right)^2F_l = \sum_{r=1}^{R}a_r\mathrm{pr}^{(1)}V_r\left(\sum_{r=1}^{R}a_r\mathrm{pr}^{(1)}V_rF_l\right), \tag{36}$$

it is sufficient to conclude the r.h.s of Equation (35) is zero provided

$$\left.\sum_{r=1}^{R}a_r\mathrm{pr}^{(1)}V_rF_l\right|_{s=0} = 0. \tag{37}$$

From here, the *multi*-parameter, first-order prolongation of the group generator is defined by the linear combination

$$\mathrm{pr}^{(1)}V = \sum_{r=1}^{R}a_r\mathrm{pr}^{(1)}V_r, \tag{38}$$

where the a_r are real valued constants. The constants are named group parameters. Similarly, from Equation (31), $\mathrm{pr}^{(1)}V$ also consists of a linear combination of two components

$$\mathrm{pr}^{(1)}V = V + V^{(1)}, \tag{39}$$

where V and $V^{(1)}$ are simply

$$V = \sum_{j=1}^{J} \eta_j \frac{\partial}{\partial x_j} + \sum_{i=1}^{I} \phi^i \frac{\partial}{\partial y^i}, \tag{40a}$$

$$V^{(1)} = \sum_{i=1}^{I}\sum_{i=1}^{I} \zeta^i_j \frac{\partial}{\partial q^i_j}, \tag{40b}$$

with

$$\eta_j = \sum_{r=1}^{R} a_r \eta_{j,r}, \tag{41a}$$

$$\phi^i = \sum_{r=1}^{R} a_r \phi^{i,r}, \tag{41b}$$

$$\zeta^i_j = \sum_{r=1}^{R} a_r \zeta^i_{j,r}. \tag{41c}$$

In summary, using Equations (37) and (38), the criterion for invariance of a system of differential equations becomes

$$\mathrm{pr}^{(1)}VF_l\big|_{F_l=0} = 0, \qquad l = 1,\ldots,3. \tag{42}$$

From a practical point of view, this result can be applied in a systematic approach to determine the R-parameter Lie group of point transformations under which the system of equations F_l are invariant. The approach requires first constructing the system of determining equations specified in Equation (42) and solving them for the coordinate functions η_j and ϕ^i defining the group generator V. All of the information about the global action of the group transformations or flow is contained within the group generator and hence, it can be used to define the group. Consequently, in the following discussion, we refer to the group generator V in place of the group.

B. The Lie group of the governing equations

Analysis performed by Ovsiannikov [Ref. 23, p. 130-137] demonstrates that the system of Equations (1)–(3) is invariant under the group generator

$$V = \eta_1 \frac{\partial}{\partial t} + \eta_2 \frac{\partial}{\partial r} + \phi^1 \frac{\partial}{\partial u} + \phi^2 \frac{\partial}{\partial p} + \phi^3 \frac{\partial}{\partial \rho}, \tag{43}$$

where η_j and ϕ^i are the independent and dependent coordinate functions given by

$$\eta_1 = a_1 + (a_2 + a_5)t + a_8 t^2, \tag{44}$$

$$\eta_2 = a_3 + a_2 r + a_4 t + a_8 tr, \tag{45}$$

$$\phi^1 = a_4 - a_5 u + a_8(r - ut), \tag{46}$$

$$\phi^2 = a_7 + a_6 p - a_8(k+3)pt, \tag{47}$$

$$\phi^3 = (2a_5 + a_6)\rho - a_8(k+1)\rho t. \tag{48}$$

An alternative derivation of these coordinate functions, implementing the widely available, open source computational software package Symgrp[30] is also presented in work by Albright and McHardy.[31]

Recalling the definition for the group parameters a_r

$$a_r = \frac{\partial \epsilon_r}{\partial s}\bigg|_{s=0}, \tag{49}$$

it can be shown that whenever s corresponds to a particular ϵ_r, the multi-parameter group generator V reduces to the V_r associated with the particular ϵ_r. For example, if $s = \epsilon_1$

$$a_1 = \frac{\partial \epsilon_1}{\partial \epsilon_1}\bigg|_{\epsilon_1=0} = 1, \tag{50}$$

and the remaining group parameters a_r are zero

$$a_r = \frac{\partial \epsilon_r}{\partial \epsilon_1}\bigg|_{\epsilon_1=0} = 0, \qquad r = 2,\ldots,R. \tag{51}$$

Thus V, given by Equation (40), becomes

$$V = \eta_{j,1}\frac{\partial}{\partial x_j} + \phi^{i,1}\frac{\partial}{\partial y^i} = V_1. \tag{52}$$

Evaluating Equation (52) using Equations (43)–(48) yields

$$V_1 = \frac{\partial}{\partial t}. \tag{53}$$

For each independent parameter ϵ_r, there is a corresponding differential operator V_r. These differential operators form the basis vectors of the R-dimensional vector space known as the Lie Algebra.

By Equation (43), the 8-dimensional Lie basis of algebra operators is

$$V_1 = \frac{\partial}{\partial t}, \tag{54}$$

$$V_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}, \tag{55}$$

$$V_3 = \frac{\partial}{\partial r}, \tag{56}$$

$$V_4 = t\frac{\partial}{\partial r} + \frac{\partial}{\partial u}, \tag{57}$$

$$V_5 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} + 2\rho\frac{\partial}{\partial \rho}, \tag{58}$$

$$V_6 = p\frac{\partial}{\partial p} + \rho\frac{\partial}{\partial \rho}, \tag{59}$$

$$V_7 = \frac{\partial}{\partial p}, \tag{60}$$

$$V_8 = t^2\frac{\partial}{\partial t} + tr\frac{\partial}{\partial r} + (r - ut)\frac{\partial}{\partial u} - (k+1)t\rho\partial\rho - (k+3)tp\frac{\partial}{\partial p}. \tag{61}$$

This basis is subject to the restrictions that the operators V_3 and V_4 are only admissible in planar geometry which agrees with the work of Axford[32] and Boyd[33] and is physically sensible. The existence of other symmetries in the basis are also conditional on the equation of state used to close the system of governing equations. Formulated in terms of the isentropic bulk modulus, the constraints are encapsulated in the equation

$$A_s \frac{\partial \phi^2}{\partial p} - \frac{\partial A_s}{\partial p}\phi^2 - \frac{\partial A_s}{\partial \rho}\phi^3 = 0. \tag{62}$$

For all the example solutions presented in the subsequent discussion, the polytropic gas equation of state is used which corresponds to bulk moduli of the form

$$A_s = \gamma p, \tag{63}$$

where γ is a constant representing the adiabatic parameter. Under this assumption, (62) is only satisfied provided $a_7 = 0$ for any γ and $a_8 = 0$ if $\gamma \neq (k+3)/(k+1)$. For example, in spherical geometry with $k = 2$, this constraint results in a 5-dimensional Lie algebra when $\gamma = 5/3$ and a 4-dimensional algebra otherwise. This concurs with the basis provided in Ref. 15.

IV. REDUCED SYSTEMS

In this section, the admissible group transformations of the Equations (1)–(3), discussed in Section III B and encapsulated by the group generator V, are leveraged to systematically reduce the complexity of the governing system of equations. A complexity reduction is achieved by determining a new set of variables in terms of which the system can be re-expressed. The new variables correspond to the constants of integration of solutions to a system of characteristic equations. They amount to some combination of the original variables and we refer to them as invariants. As discussed in Olver [Ref. 22, p. 87], the number of functionally independent invariants obtainable is one fewer than the number of variables appearing in the original system. Thus, expressing the system in terms of the invariants typically results in a reduction of complexity, e.g. a reduction from first-order partial to first-order ordinary differential equations if the number of independent variables is reduced. Reduction of the system is desirable because the new systems are simpler to solve and any solutions of the reduced systems can be mapped back into the original coordinate space yielding solutions to the original system. Details of the reduction steps are provided in Appendixes B–D.

A. The characteristic system

The characteristic system used to obtain a new set of variables can be constructed directly from an admissible group generator of the original system. From the group analysis performed by Ovsiannikov and resulting group generator given by Equation (43), it is possible to write down the general characteristic system as

$$\begin{aligned}\frac{\mathrm{d}t}{a_1 + (a_2+a_5)t + a_8 t^2} &= \frac{\mathrm{d}r}{a_3 + a_2 r + a_4 t + a_8 t r} \\ &= \frac{\mathrm{d}u}{a_4 - a_5 u + a_8(r - ut)} \\ &= \frac{\mathrm{d}p}{a_7 + a_6 p - a_8(k+3)pt} \\ &= \frac{\mathrm{d}\rho}{(2a_5 + a_6)\rho - a_8(k+1)\rho t}.\end{aligned} \tag{64}$$

As written, this system cannot be solved without imposing constraints between the group parameters, a_r. In the discussion which follows, we systematically apply various constraints and solve the characteristic system to obtain the invariants in each case. The governing equations are then re-expressed in terms of the new variables. Each constraint imposed amounts to restrictions on the possible linear combinations of the differential basis operators in the Lie algebra, Equations (54)–(61). For all cases considered, we assume the material behaves according to the polytropic gas equation of state which requires $a_7 = 0$ in order to satisfy Equation (62). Additionally, we narrow the focus down to symmetries admissible in cylindrical and spherical geometries and so $a_3 = a_4 = 0$. The simplified characteristic system is

$$\begin{aligned}\frac{\mathrm{d}t}{a_1 + (a_2+a_5)t + a_8 t^2} &= \frac{\mathrm{d}r}{a_2 r + a_8 t r} = \frac{\mathrm{d}u}{-a_5 u + a_8(r - ut)} \\ &= \frac{\mathrm{d}p}{a_6 p - a_8(k+3)pt} \\ &= \frac{\mathrm{d}\rho}{(2a_5 + a_6)\rho - a_8(k+1)\rho t}.\end{aligned} \tag{65}$$

The flow chart in Figure 1 illustrates how the constraints considered lead to the different cases.

B. Reduction A. $a_8 = 0$ and if $a_1 = 0$, $a_2 + a_5 \neq 0$

With this choice of constraints the characteristic system is

$$\frac{\mathrm{d}t}{a_1 + (a_2+a_5)t} = \frac{\mathrm{d}r}{a_2 r} = \frac{\mathrm{d}u}{-a_5 u} = \frac{\mathrm{d}p}{a_6 p} = \frac{\mathrm{d}\rho}{(2a_5 + a_6)\rho}. \tag{66}$$

Solving the equation consisting of the first and second members of this system yields the following integration constant

$$\xi = r(a_1 + (a_2+a_5)t)^{-\frac{a_2}{a_2+a_5}}. \tag{67}$$

This integration constant represents the first new invariant. Similarly, solving the equations, obtained by pairing the first and third, first and fourth, and first and fifth members of the characteristic system, yields the additional solutions

$$u(t,r) = \hat{u}(\xi)[a_1 + (a_2+a_5)t]^{-\frac{a_5}{a_2+a_5}}, \tag{68}$$

$$p(t,r) = \hat{p}(\xi)[a_1 + (a_2+a_5)t]^{\frac{a_6}{a_2+a_5}}, \tag{69}$$

$$\rho(t,r) = \hat{\rho}(\xi)[a_1 + (a_2+a_5)t]^{\frac{2a_5+a_6}{a_2+a_5}}, \tag{70}$$

where $\hat{u}(\xi)$, $\hat{p}(\xi)$ and $\hat{\rho}(\xi)$ are the integration constants. As written, they are assumed to be unknown functions of the invariant ξ, and henceforth, the variables ξ, and $\hat{u}(\xi), \hat{p}(\xi)$ and $\hat{\rho}(\xi)$, are referred to as the independent and dependent invariants of the characteristic system, respectively.

The governing equations can now be expressed entirely in terms of these new invariants. The following system of ODEs are obtained

$$\hat{\rho}'(\hat{u} - a_2\xi) + \hat{\rho}\left(2a_5 + a_6 + \hat{u}' + \frac{k\hat{u}}{\xi}\right) = 0, \tag{71}$$

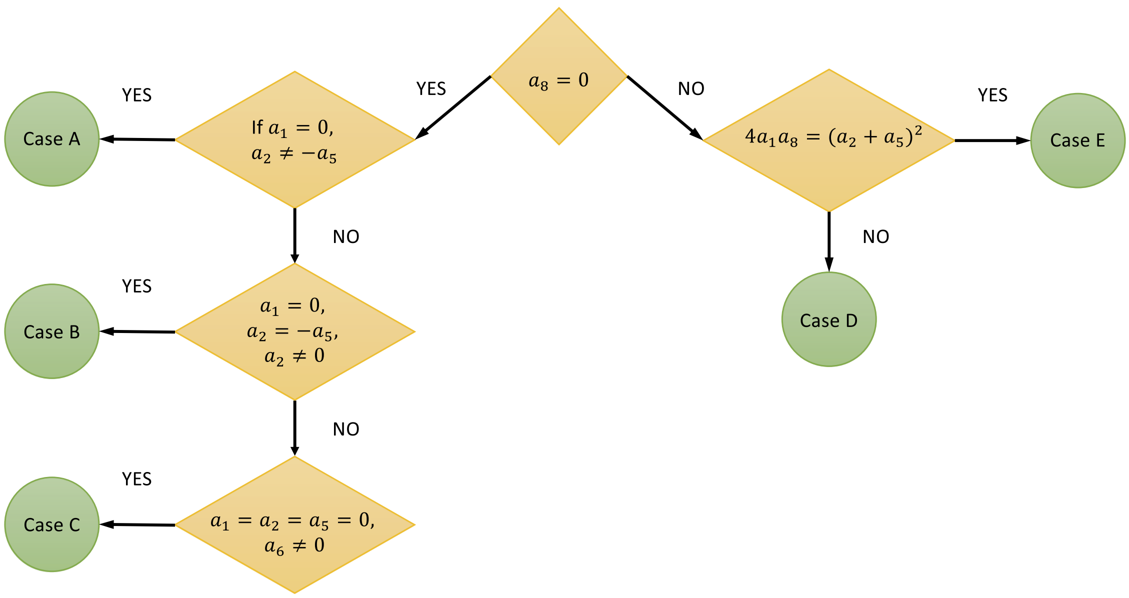

FIG. 1. Flow chart illustrating constraints imposed between the group parameters.

$$\hat{p}' + \hat{\rho}\big(\hat{u}'(\hat{u} - a_2\xi) - a_5\hat{u}\big) = 0, \tag{72}$$

$$\hat{\rho}'(\hat{u} - a_2\xi) + \hat{\rho}\left(a_6 + \gamma\left(\hat{u}' + \frac{k\hat{u}}{\xi}\right)\right) = 0, \tag{73}$$

where primes denote differentiation with respect to ξ. Details of this reduction are presented in Appendix B.

C. Reduction B. $a_8 = 0$, $a_1 = 0$, $a_2 = -a_5$, $a_2 \neq 0$

Without loss of generality, by restricting $a_2 \neq 0$ we can set $a_2 = 1$ by re-scaling the group generator as necessary. The characteristic system is

$$\frac{dt}{0} = \frac{dr}{r} = \frac{du}{u} = \frac{dp}{a_6 p} = \frac{d\rho}{(a_6 - 2)\rho}. \tag{74}$$

The first two members of (74) lead to

$$\frac{dt}{dr} = 0, \tag{75}$$

and thus $t = \text{constant} = \xi$. The second and third, second and fourth, and second and fifth members yield

$$\hat{u}(\xi) = \frac{u(t,r)}{r}, \tag{76a}$$

$$\hat{p}(\xi) = \frac{p(t,r)}{r^{a_6}}, \tag{76b}$$

$$\hat{\rho}(\xi) = \frac{\rho(t,r)}{r^{a_6-2}}. \tag{76c}$$

In terms of these new invariant variables the reduced governing equations are

$$\hat{\rho}' + \hat{\rho}\hat{u}(a_6 + k - 1) = 0, \tag{77}$$

$$a_6\hat{p} + \hat{\rho}\hat{u}' + \hat{\rho}\hat{u}^2 = 0, \tag{78}$$

$$\hat{p}' + \hat{p}\hat{u}[a_6 + \gamma(k+1)] = 0, \tag{79}$$

D. Reduction C. $a_8 = 0$, $a_1 = a_2 = a_5 = 0$, $a_6 \neq 0$

Again without loss of generality we can choose $a_6 = 1$ by rescaling the group generator. The characteristic system is therefore

$$\frac{dt}{0} = \frac{dr}{0} = \frac{du}{0} = \frac{dp}{p} = \frac{d\rho}{\rho}. \tag{80}$$

As discussed in Section VI of Coggeshall,[34] not all group transformations result in a reduction of the system to ODEs. In this case, solving for the invariants of the characteristic system does not specify a means by which the number of independent variables appearing in the system of equations can be reduced. For clarity and completeness, the invariants associated with this case are determined. Solving the equations consisting of the first and fourth, second and fourth, third and fourth, and fourth and fifth members of the characteristic system yields the following invariants along the characteristic direction

$$\frac{dt}{dp} = 0 \Rightarrow t = \text{invariant}, \tag{81}$$

$$\frac{dr}{dp} = 0 \Rightarrow r = \text{invariant}, \tag{82}$$

$$\frac{du}{dp} = 0 \Rightarrow u = \text{invariant}, \tag{83}$$

$$\frac{dp}{d\rho} = \frac{p}{\rho} \Rightarrow c = \frac{p}{\rho} = \text{invariant}. \tag{84}$$

In comparison to the previous examples which generated one independent and three dependent invariants, there are two independent and two dependent invariants t and r, and $u := u(t, r)$ and $c := c(t, r)$, respectively. This is an example of a group action which violates the transversality condition defined in Ref. 22 (pp. 228-230). This particular group transformation will not be considered further.

E. Reduction D. $a_8 \neq 0$, $4a_1a_8 \neq (a_2 + a_5)^2$

The characteristic system is

$$\frac{dt}{a_1 + (a_2 + a_5)t + a_8t^2} = \frac{dr}{a_2r + a_8tr} = \frac{du}{-a_5u + a_8(r - ut)} = \frac{dp}{a_6p - a_8(k+3)pt} = \frac{d\rho}{(2a_5 + a_6)\rho - a_8(k+1)\rho t}. \tag{85}$$

From the equation consisting of the first and second members of the characteristic system, the independent invariant is

$$\xi = \frac{r}{(a_1 + (a_2 + a_5)t + a_8t^2)^{1/2}} \exp\left(\frac{(a_5 - a_2)\arctan(\beta)}{\sqrt{4a_1a_8 - (a_2 + a_5)^2}}\right), \tag{86}$$

where

$$\beta = \frac{a_2 + a_5 + 2a_8t}{\sqrt{4a_1a_8 - (a_2 + a_5)^2}}. \tag{87}$$

In order to solve the equation consisting of the first and third members

$$\frac{du}{dt} = \frac{a_8r - u(a_5 + a_8t)}{a_1 + (a_2 + a_5)t + t^2}, \tag{88}$$

r must be eliminated using the independent invariant. From Equation (86)

$$r = \xi(a_1 + (a_2 + a_5)t + a_8t^2)^{\frac{1}{2}} \exp\left(\frac{(a_2 - a_5)\arctan(\beta)}{\sqrt{4a_1a_8 - (a_2 + a_5)^2}}\right). \tag{89}$$

Substituting into Equation (88)

$$\frac{du}{dt} = \frac{\xi}{(a_1 + (a_2 + a_5)t + a_8t^2)^{\frac{1}{2}}} \exp\left(\frac{(a_2 - a_5)\arctan(\beta)}{\sqrt{4a_1a_8 - (a_2 + a_5)^2}}\right) - \frac{u(a_5 + a_8t)}{a_1 + (a_2 + a_5)t + t^2}, \tag{90}$$

which has solution

$$u(t, r) = \frac{\hat{u}(\xi) + a_8t\xi}{(a_1 + (a_2 + a_5)t + a_8t^2)^{\frac{1}{2}}} \exp\left(\frac{(a_2 - a_5)\arctan(\beta)}{\sqrt{4a_1a_8 - (a_2 + a_5)^2}}\right). \tag{91}$$

Re-arranging for $\hat{u}$ and similarly solving the equations of the first and fourth and first and fifth members of the characteristic system results in the following dependent invariants

$$\hat{u}(\xi) = u(t, r)(a_1 + (a_2 + a_5)t + a_8t^2)^{\frac{1}{2}} \times \exp\left(\frac{(a_5 - a_2)\arctan(\beta)}{\sqrt{4a_1a_8 - (a_2 + a_5)^2}}\right) - a_8t\xi, \tag{92a}$$

$$\hat{p}(\xi) = p(t, r)(a_1 + (a_2 + a_5)t + a_8t^2)^{\frac{k+3}{2}} \times \exp\left(\frac{-(2a_6 + (a_2 + a_5)(k+3))\arctan(\beta)}{\sqrt{4a_1a_8 - (a_2 + a_5)^2}}\right), \tag{92b}$$

$$\hat{\rho}(\xi) = \rho(t, r)(a_1 + (a_2 + a_5)t + a_8t^2)^{\frac{k+1}{2}} \times \exp\left(\frac{-(2a_6 + 4a_5 + (a_2 + a_5)(k+1))\arctan(\beta)}{\sqrt{4a_1a_8 - (a_2 + a_5)^2}}\right). \tag{92c}$$

Expressing the governing system in terms of these invariants gives

$$\hat{\rho}'(\hat{u} - a_2\xi) + \hat{\rho}\left(2a_5 + a_6 + \hat{u}' + \frac{k\hat{u}}{\xi}\right) = 0, \tag{93}$$

$$\hat{p}' + \hat{\rho}\left(\hat{u}'(\hat{u} - a_2\xi) - a_5\hat{u} + a_1a_8\xi\right) = 0, \tag{94}$$

$$\hat{p}'(\hat{u} - a_2\xi) + \hat{p}\left(a_6 + \frac{k+3}{k+1}\left(\hat{u}' + \frac{k\hat{u}}{\xi}\right)\right) = 0. \tag{95}$$

For the details of this reduction see Appendix C.

F. Reduction E. $a_8 \neq 0$, $4a_1a_8 = (a_2 + a_5)^2$

The characteristic system with $4a_1a_8 = (a_2 + a_5)^2$ is

$$\frac{dt}{\left(\sqrt{a_8}t + \frac{a_2 + a_5}{2\sqrt{a_8}}\right)^2} = \frac{dr}{a_2r + a_8tr} = \frac{du}{-a_5u + a_8(r - ut)} = \frac{dp}{a_6p - a_8(k+3)pt} = \frac{d\rho}{(2a_5 + a_6)\rho - a_8(k+1)\rho t}. \tag{96}$$

The independent invariant is

$$\xi = \frac{r}{a_2 + a_5 + 2a_8t} \exp\left(\frac{a_2 - a_5}{a_2 + a_5 + 2a_8t}\right). \tag{97}$$

The additional dependent invariants are

$$\hat{u}(\xi) = u(t, r)(a_2 + a_5 + 2a_8t) \exp\left(\frac{a_2 - a_5}{a_2 + a_5 + 2a_8t}\right) - 4a_8^2t\xi, \tag{98a}$$

$$\hat{p}(\xi) = p(t, r)(a_2 + a_5 + 2a_8t)^{k+3} \exp\left(\frac{2a_6 + (a_2 + a_5)(k+3)}{a_2 + a_5 + 2a_8t}\right), \tag{98b}$$

$$\hat{\rho}(\xi) = \rho(t, r)(a_2 + a_5 + 2a_8t)^{k+1} \exp\left(\frac{2a_6 + 4a_5 + (a_2 + a_5)(k+1)}{a_2 + a_5 + 2a_8t}\right). \tag{98c}$$

The reduced system of equations is

$$\hat{p}'(\hat{u} - 4a_8a_2\xi) + \hat{\rho}(4a_8(2a_5 + a_6) + \hat{u}' + \frac{k\hat{u}}{\xi}) = 0, \tag{99}$$

$$\hat{p}' + \hat{\rho}(\hat{u}'(\hat{u} - 4a_8a_2\xi) - 4a_8a_5\hat{u} + 4a_8^2(a_2 + a_5)^2\xi) = 0, \tag{100}$$

$$\hat{p}'(\hat{u} - 4a_8a_2\xi) + \hat{p}\left(4a_8a_6 + \frac{k+3}{k+1}\left(\hat{u}' + \frac{k\hat{u}}{\xi}\right)\right) = 0. \tag{101}$$

V. THE LINEAR VELOCITY CLASS OF GROUP-INVARIANT SOLUTIONS

Having performed symmetry reduction on the system of governing equations, further progression is now made under the assumption that the velocity profile of the fluid is restricted to

$$u(t,r) = r\frac{\dot{R}(t)}{R(t)}, \tag{102}$$

where r is the spatial coordinate denoting the distance from the flow's center of symmetry, $R(t)$ is a function of time with dimensions of length which acts as a scaling term in the problem and the overdot represents differentiation with respect to time. Solutions pertaining to velocity profiles of this type are denoted linear velocity solutions due to the proportionality of the velocity with respect to the spatial coordinate r. Initial work appearing In Sedov[6] and Ref. 35 (Chap. 15) determined that velocity profiles of this kind correspond to *self-similar* flows i.e., flows for which the spatial distributions of the dependent variables, velocity, density and pressure, remain static when viewed using a length scale evolved appropriately with time. This self-similar behaviour is analogous to that identified in the analysis by Taylor during his estimation of the yield of a very intense point explosion [Refs. 25, pp. 1-11].[36,37]

In more recent work,[8] Ramsey applied the linear velocity ansatz to Equations (1)–(3) and determined restrictions on the function $R(t)$ as well as constraints on the admissible pressure and density profiles. Additional examples of linear velocity solutions also appear in Coggeshall.[15] In the latter sections of this paper, we compare the solutions discussed by Ramsey and Coggeshall with the results we obtain. Since in the current work we apply the linear velocity ansatz following the symmetry reductions, all the results presented are housed within the context of the Lie group theory. In the majority of cases, it is observed that the symmetry reduction performed dictates the resulting form of the function $R(t)$.

The class of solutions corresponding to the ansatz in Equation (102) also overlaps with self-similar solutions considered by Zel'dovich and Raizer in their chapter on self-similar solutions in gas dynamics [Refs. 16, pp. 788-790]. This chapter discusses velocities of the form

$$u(t,r) = \dot{R}(t)f(\xi),$$

where f is an arbitrary function of its argument $\xi = r/R(t)$. Using dimensional arguments, power-law type solutions were presented in this chapter where

$$R(t) = At^{\alpha},$$

and A and α are constants. Since the scope of the solutions in the current paper is limited to the linear velocity class, the results only coincide with the discussion in Zel-dovich and Raizer when $f(\xi) \propto \xi$. However, in comparison, by using Lie group methods and not simply dimensional analysis, a richer class of $R(t)$ functions leading to linear velocity solutions is obtained. For example, included is a particular member of the exponential type of similarity solutions discussed in Stanyukovich [Refs. 17, pp. 79-90] which take the form

$$R(t) = Ae^{\alpha t},$$

where A and α are again constants.

To conclude this section, it should be emphasized that the linear velocity solutions considered only constitute a subset of the possible group-invariant solutions. The reduced systems provided in Section IV may be used by the reader as a starting point in the search for alternative solutions. These additional solutions may be obtainable either explicitly or numerically by first making an alternative ansatz about the nature of the flow.

VI. GROUP INVARIANT LINEAR VELOCITY SOLUTIONS

In this section, a series of group-invariant solutions to the inviscid Euler equations are presented which correspond to flows of the linear velocity type discussed in Section V. The solutions are organsized according to the various symmetry reductions performed in Section IV in addition to any other required constraints on the group parameters resulting from the solution analysis. The solution method applied consisted of imposing a restriction on $\hat{u}$ consistent with the linear velocity ansatz of Equation (102) which enabled the identification of restrictions on the function $R(t)$. Following this the reduced systems were solved for the remaining unknowns $\hat{p}$ and $\hat{\rho}$ which were used to determine the desired pressure and density fields, respectively. For each of the solutions presented the associated $R(t)$ function is given explicitly whenever possible. Further details of the working can be found in Appendix E.

A. Solutions of system A

The following solutions were constructed following the symmetry analysis performed in Section IV B. The group generator for this case is

$$V = (a_1 + (a_2 + a_5)t)\frac{\partial}{\partial t} + a_2 r\frac{\partial}{\partial r} - a_5 u\frac{\partial}{\partial u} + a_6 p\frac{\partial}{\partial p} + (2a_5 + a_6)\rho\frac{\partial}{\partial \rho}, \tag{103}$$

which is obtained from Equation (43) by setting $a_3 = a_4 = a_8 = 0$. If $a_1 = 0$, the solutions in this section also require $a_2 + a_5 \neq 0$.
A.1.

$$R(t) = (a_1 + (a_2 + a_5)t)^{\frac{B}{a_2 + a_5}}, \tag{104}$$

$$u(t,r) = \frac{\dot{R}}{R}r = \frac{Br}{a_1 + (a_2 + a_5)t}, \tag{105}$$

$$p(t,r) = c_2 r^{\alpha_1}(a_1 + (a_2 + a_5)t)^{\alpha_2}, \tag{106}$$

$$\rho(t,r) = c_1 r^{\alpha_3}(a_1 + (a_2 + a_5)t)^{\alpha_4}, \tag{107}$$

where c_1 is a free parameter,

$$\alpha_1 = \frac{a_6 + B\gamma(k+1)}{a_2 - B}, \tag{108}$$

$$\alpha_2 = \frac{B(a_6 + a_2\gamma(k+1))}{(a_2+a_5)(B-a_2)}, \tag{109}$$

$$\alpha_3 = \frac{2a_5 + a_6 + B(k+1)}{a_2 - B}, \tag{110}$$

$$\alpha_4 = \frac{B(2a_5 + a_6 + a_2(k+1))}{(a_2+a_5)(B-a_2)}, \tag{111}$$

and

$$c_2 = \frac{c_1 B(a_2 + a_5 - B)(a_2 - B)}{2(a_2+a_5) + a_6 + B(k-1)}. \tag{112}$$

This solution is subject to the additional constraints,

$$B \neq a_2, \tag{113}$$

$$2(a_2+a_5) + a_6 + B(k-1) \neq 0, \tag{114}$$

$$2(a_2+a_5) + B(k-1-\gamma(k+1)) = 0. \tag{115}$$

A.2.

$$R(t) = (a_1 + (a_2+a_5)t), \tag{116}$$

$$u(t,r) = \frac{\dot{R}}{R}r = \frac{(a_2+a_5)r}{a_1 + (a_2+a_5)t}, \tag{117}$$

$$p(t,r) = \frac{c_2}{(a_1 + (a_2+a_5)t)^{k+1}}, \tag{118}$$

$$\rho(t,r) = \frac{c_1}{r^2(a_1 + (a_2+a_5)t)^{k-1}}, \tag{119}$$

where c_1 and c_2 are free parameters. Additionally, this solution requires

$$\gamma = 1, \tag{120}$$

$$a_6 = -(a_2+a_5)(k+1), \tag{121}$$

$$a_5 \neq 0. \tag{122}$$

Note the unphysical value of the adiabatic parameter required for the existence of this solution.

A.3.

$$R(t) = \left(a_1 + \frac{a_2 t}{2}(2-\eta)\right)^{\frac{2}{2-\eta}}, \tag{123}$$

$$u(t,r) = \frac{\dot{R}}{R}r = \frac{2a_2 r}{2a_1 + a_2 t(2-\eta)}, \tag{124}$$

$$p(t,r) = \frac{a_2^2(\gamma-1)(k+1)}{2R(t)^{\gamma(k+1)}} \int \hat{\rho}(\xi)\,\xi\,\mathrm{d}\xi, \tag{125}$$

$$\rho(t,r) = \frac{\hat{\rho}(\xi)}{R(t)^{k+1}}, \tag{126}$$

$$\xi = \frac{r}{R(t)}. \tag{127}$$

where $\hat{\rho}(\xi)$ is an arbitrary function of its argument and

$$\eta = (1-\gamma)(k+1) \tag{128}$$

For this solution we require

$$a_5 = \frac{a_2(\gamma-1)(k+1)}{2}, \tag{129}$$

$$a_6 = -a_2\gamma(k+1). \tag{130}$$

B. Solutions of system B

The following solutions were constructed following the symmetry analysis performed in Section IV C. The group generator for this case is

$$V = r\frac{\partial}{\partial r} + u\frac{\partial}{\partial u} + a_6 p\frac{\partial}{\partial p} + (a_6 - 2)\rho\frac{\partial}{\partial \rho}, \tag{131}$$

which can be obtained from Equation (43) by setting $a_1 = a_3 = a_4 = a_8 = 0$, $a_2 = -a_5$ and $a_2 = 1$.

B.1.

$$u(t,r) = \frac{\dot{R}(t)}{R(t)}r, \tag{132}$$

$$p(t,r) = \frac{c_2 r^{a_6}}{R(t)^{a_6+\gamma(k+1)}}, \tag{133}$$

$$\rho(t,r) = \frac{c_1}{r^{2-a_6}R(t)^{a_6+k-1}}, \tag{134}$$

where $R(t)$ must satisfy the constraint

$$\ddot{R}(t)R(t)^{1-\eta} + \frac{a_6 c_2}{c_1} = 0, \tag{135}$$

$\eta = (1-\gamma)(k+1)$ and c_1 and c_2 are free parameters.

B.2.

$$R(t) = \dot{R}_0 t + R_0, \tag{136}$$

$$u(t,r) = \frac{r}{t + R_0/\dot{R}_0}, \tag{137}$$

$$p(t,r) = \frac{c_2}{(\dot{R}_0 t + R_0)^{\gamma(k+1)}}, \tag{138}$$

$$\rho(t,r) = \frac{c_1}{r^2(\dot{R}_0 t + R_0)^{k-1}}, \tag{139}$$

where $\dot{R}_0$ and R_0 are the velocity and position of the boundary $R(t)$ at time $t = 0$. This solution is simply a particular subcase of solution B.1 where we have selected $a_6 = 0$ facilitating a trivial solution for $R(t)$.

C. Solutions of system D

The following solutions were constructed following the symmetry analysis performed in Section IV E. The group generator for this case is

$$V = (a_1 + (a_2+a_5)t + a_8 t^2)\frac{\partial}{\partial t} + (a_2 r + a_8 t r)\frac{\partial}{\partial r} + (-a_5 u + a_8(r - ut))\frac{\partial}{\partial u} + (a_6 - a_8(k+3)t)p\frac{\partial}{\partial p} + (2a_5 + a_6 - a_8(k+1)t)\rho\frac{\partial}{\partial \rho}, \tag{140}$$

which can be obtained from Equation (43) by setting $a_3 = a_4 = 0$. The solutions in this section are also further constrained by the requirements $\gamma = (k+3)/(k+1)$, $4a_1a_8 \neq (a_2+a_5)^2$ and $a_8 \neq 0$.
D.1.

$$R(t) = (a_1 + (a_2+a_5)t + a_8t^2)^{\frac{1}{2}}, \tag{141}$$

$$u(t,r) = \frac{(a_2+a_5+2a_8t)r}{2(a_1+(a_2+a_5)t+a_8t^2)}, \tag{142}$$

$$p(t,r) = \frac{c_2 r^{\alpha_1}}{R(t)^{\alpha_2}}, \tag{143}$$

$$\rho(t,r) = \frac{c_1 r^{\alpha_3}}{R(t)^{\alpha_4}}, \tag{144}$$

where

$$\alpha_1 = \frac{2a_6 + (a_2+a_5)(k+3)}{a_2 - a_5}, \tag{145}$$

$$\alpha_2 = \frac{2a_6 + 2a_2(k+3)}{a_2-a_5}, \tag{146}$$

$$\alpha_3 = \frac{4a_5 + 2a_6 + (a_2+a_5)(k+1)}{a_2-a_5}, \tag{147}$$

$$\alpha_4 = \frac{2(2a_5 + a_6 + a_2(k+1))}{a_2-a_5}, \tag{148}$$

and

$$c_2 = \frac{c_1(a_2-a_5)((a_2+a_5)^2 - 4a_1a_8)}{4(2a_6 + (k+5)(a_2+a_5))}. \tag{149}$$

This particular solution is additionally subject to $a_2 \neq a_5$ and $2a_6 + (a_2+a_5)(k+3) \neq 0$.
D.2.

$$R(t) = (a_1 + (a_2+a_5)t + a_8t^2)^{\frac{1}{2}}, \tag{150}$$

$$u(t,r) = \frac{(a_2+a_5+2a_8t)r}{2(a_1+(a_2+a_5)t+a_8t^2)}, \tag{151}$$

$$p(t,r) = \frac{c_2}{R(t)^{k+3}}, \tag{152}$$

$$\rho(t,r) = 0, \tag{153}$$

where c_2 is a free parameter. This solution additionally requires $a_2 \neq a_5$ and $2a_6 + (a_2+a_5)(k+3) = 0$. We include this solution for completeness despite the non-physical value of the density and corresponding internal energy.
D.3.

$$R(t) = (a_1 + 2a_2t + a_8t^2)^{\frac{1}{2}}, \tag{154}$$

$$u(t,t) = \frac{(a_2+a_8t)r}{a_1+2a_2t+a_8t^2}, \tag{155}$$

$$p(t,r) = \frac{(a_2^2 - a_1a_8)}{(a_1+2a_2t+a_8t^2)^{\frac{k+3}{2}}} \int \hat{\rho}(\xi)\xi d\xi, \tag{156}$$

$$\rho(t,r) = \frac{\hat{\rho}(\xi)}{(a_1+2a_2t+a_8t^2)^{\frac{k+1}{2}}}, \tag{157}$$

$$\xi = \frac{r}{R(t)}, \tag{158}$$

where $\rho(\xi)$ is an arbitrary function of its argument. This solution is additionally subject to $a_2 = a_5$ and $a_6 = -a_2(k+3)$.

D. Solutions of system E

The following solutions were constructed following the symmetry analysis performed in Section IV F. The group generator for this case is

$$\begin{aligned} V = &\left(\sqrt{a_8}t + \frac{a_2+a_5}{2\sqrt{a_8}}\right)^2 \frac{\partial}{\partial t} + (a_2+a_8t)r\frac{\partial}{\partial r} \\ &+ (-a_5u + a_8(r-ut))\frac{\partial}{\partial u} + (a_6 - a_8(k+3)t)p\frac{\partial}{\partial p} \\ &+ (2a_5 + a_6 - a_8(k+1)t)\rho\frac{\partial}{\partial \rho}, \end{aligned} \tag{159}$$

which can be obtained from Equation (43) by setting $a_3 = a_4 = 0$ and $4a_1a_8 = (a_2+a_5)^2$. The solutions in this section also require $\gamma = (k+3)/(k+1)$ and $a_8 \neq 0$.
E.1.

$$R(t) = a_2 + a_5 + 2a_8t, \tag{160}$$

$$u(t,r) = \frac{2a_8r}{a_2+a_5+a_8t}, \tag{161}$$

$$p(t,r) = 0, \tag{162}$$

$$\rho(t,r) = c_1\frac{r^{\alpha_1}}{R(t)^{\alpha_2}}, \tag{163}$$

where

$$\alpha_1 = \frac{2(2a_5+a_6) + (a_2+a_5)(k+1)}{a_2-a_5}, \tag{164}$$

$$\alpha_2 = \frac{2(2a_5+a_6+a_2(k+1))}{a_2-a_5}, \tag{165}$$

and c_1 is a free parameter. This solution is further subject to $a_2 \neq a_5$ and $2a_6 + (a_2+a_5)(k+3) \neq 0$.
E.2.

$$R(t) = a_2 + a_5 + 2a_8t, \tag{166}$$

$$u(t,r) = \frac{2a_8r}{a_2+a_5+a_8t}, \tag{167}$$

$$p(t,r) = \frac{c_2}{R(t)^{k+3}}, \tag{168}$$

$$\rho(t,r) = \frac{c_1}{r^2R(t)^{k-1}}, \tag{169}$$

where c_1 and c_2 are free parameters. This solution is further subject to $a_2 \neq a_5$ and $2a_6 + (a_2+a_5)(k+3) = 0$.

E.3.

$$R(t) = 2(a_2 + a_8 t), \tag{170}$$

$$u(t,r) = \frac{a_8 r}{a_2 + a_8 t}, \tag{171}$$

$$p(t,r) = \frac{c_1}{R(t)^{k+3}}, \tag{172}$$

$$\rho(t,r) = \frac{\hat{\rho}(\xi)}{R(t)^{k+1}}, \tag{173}$$

$$\xi = \frac{r}{R(t)}, \tag{174}$$

where c_1 is a free parameter and $\hat{\rho}(\xi)$ is an arbitrary function of its argument. This solution is subject to the additional requirements $a_2 = a_5$ and $a_6 = -a_2(k+3)$.

VII. SOLUTIONS IN THE LITERATURE

As briefly outlined in Section V there are many linear velocity solutions appearing throughout the existing literature. In the following discussion we first connect with the discussion of linear velocity solutions appearing in Ramsey.[8] Following this we focus on solutions presented in Coggeshall[15] and, in particular, derive Coggeshall solutions #2, #3, #6, #7 and a subcase of #4 of this paper starting from either solutions A.3 or D.3 in Section VI.

A. Ramsey solutions

As previously discussed, Ramsey[8] used the linear velocity ansatz of Equation (102) to determine constraints on the admissible $R(t)$ function and the corresponding density and pressure profiles. In said paper, the velocity, pressure and density were given by

$$u(t,r) = \frac{\dot{R}(t)}{R(t)} r, \tag{175}$$

$$p(t,r) = -\frac{\ddot{R}(t)}{R(t)^k} \int \xi \mathcal{R}(\xi) \mathrm{d}\xi, \tag{176}$$

$$\rho(t,r) = \frac{\mathcal{R}(\xi)}{R(t)^{k+1}}, \tag{177}$$

where $R(t)$ must satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\ddot{R} R^{1-\eta}\right] = 0, \tag{178}$$

with $\eta = (1-\gamma)(k+1)$, $\xi = r/R(t)$ and $\mathcal{R}(\xi)$ is an arbitrary function of its argument. Identifying symmetry properties of some of these solutions is achievable through comparison to the group-invariant solutions discussed in Section VI. In the majority of cases, the associated symmetry dictates the admissible function of $R(t)$. For example, in the case $\eta = -2$, using Equation (178) Ramsey determined

$$R(t) = R_0 \left(1 + \frac{\ddot{R}_0}{R_0} t^2\right)^{\frac{1}{2}}, \tag{179}$$

where R_0 and $\ddot{R}_0$ are constants. This function of $R(t)$ is analogous to that of Solution D.3 subject to the group parameter $a_2 = 0$. The various density profiles discussed in Ramsey for this solution correspond to different choices of $\hat{\rho}(\xi)$ in Solution D.3.

As another example, a large number of the Ramsey solutions possess the symmetry associated with Solution B.1. In particular, solutions corresponding to $\mathcal{R}(\xi) = A\xi^{b-2}$ where A and b are constants with the exception $b \neq 0$. Substituting this choice into Equations (175)–(177), the solution in the Ramsey formalism is given by

$$u(t,r) = \frac{\dot{R}(t)}{R(t)} r, \tag{180}$$

$$p(t,r) = -\frac{A\ddot{R} r^b}{b R^{b+k}}, \tag{181}$$

$$\rho(t,r) = \frac{A r^{b-2}}{R^{b+k-1}}, \tag{182}$$

along with Equation (178). The equivalence of this solution with Solution B.1 can be demonstrated by first considering Equation (135). Taking the time derivative of Equation (135) results in the constraint on $R(t)$ outlined by Ramsey. Next, assuming $a_6 \neq 0$, Equation (135) can be re-arranged for c_2 giving

$$c_2 = -\frac{c_1}{a_6} \ddot{R} R^{1-\eta}. \tag{183}$$

Substituting this result into Equations (132)–(134) of Solution B.1 gives

$$u(t,r) = \frac{\dot{R}(t)}{R(t)} r, \tag{184}$$

$$p(t,r) = -\frac{c_1 \ddot{R} r^{a_6}}{a_6 R^{a_6+k}}, \tag{185}$$

$$\rho(t,r) = \frac{c_1 r^{a_6-2}}{R^{a_6+k-1}}. \tag{186}$$

Equations (184)–(186) are equivalent to Equations (180)–(182) if we equate the free parameters $A = c_1$ and $b = a_6$. Solution B.2 applies in the case $a_6 = 0$.

In some cases, the solutions presented by Ramsey may be invariant under more than one symmetry transformation. For example, if the $R(t)$ function was given by Equation (179) and the $\mathcal{R}(\xi)$ function was of the form $\mathcal{R}(\xi) = A\xi^{b-2}$, the corresponding solution would possess the symmetries associated with both Solutions B.1 and D.3. Such solutions could therefore be derived using either one of the group generators coupled to B.1 or D.3.

B. Coggeshall #2

Next, solution #2 presented in Coggeshall[15] is derived. Starting with Solution A.3, set $a_1 = 0$, $a_2 = 1$ and choose $\hat{\rho} = \xi^b$ where b is a constant. This gives

$$R(t) = \left(\frac{2 + (\gamma-1)(k+1)}{2} t\right)^{\frac{2}{2+(\gamma-1)(k+1)}}, \tag{187}$$

$$u(t,r) = \frac{\dot{R}}{R}r = \left(\frac{2}{2+(\gamma-1)(k+1)}\right)\frac{r}{t}, \tag{188}$$

$$\rho(t,r) = \frac{\xi^b}{R(t)^{k+1}}, \tag{189}$$

$$p(t,r) = \frac{(\gamma-1)(k+1)}{2R(t)^{\gamma(k+1)}}\int \xi^{b+1}\mathrm{d}\xi, \tag{190}$$

$$\xi = \frac{r}{R(t)}, \tag{191}$$

Evaluating the pressure integral gives

$$p(t,r) = \frac{(\gamma-1)(k+1)\xi^{b+2}}{2(b+2)R(t)^{\gamma(k+1)}} + C, \tag{192}$$

where C is the constant of integration. Setting $C = 0$ and substituting for ξ in the pressure and density terms generates

$$p(t,r) = \frac{(\gamma-1)(k+1)r^{b+2}}{2(b+2)R(t)^{\gamma(k+1)+b+2}}, \tag{193}$$

$$\rho(t,r) = \frac{r^b}{R(t)^{b+k+1}}. \tag{194}$$

Next, substituting for $R(t)$ using Equation (187)

$$p(t,r) = \frac{2\rho_0(\gamma-1)(k+1)r^{b+2}}{2(b+2)[2+(\gamma-1)(k+1)]^2 t^{\frac{(b+k+1)}{2+(\gamma-1)(k+1)}} t^2}, \tag{195}$$

$$\rho(t,r) = \frac{\rho_0 r^b}{t^{\frac{2(b+k+1)}{2+(\gamma-1)(k+1)}}}, \tag{196}$$

where

$$\rho_0 = \left(\frac{2}{2+(\gamma-1)(k+1)}\right)^{\frac{2(b+k+1)}{2+(\gamma-1)(k+1)}}. \tag{197}$$

Finally, using the ideal gas equation of state, the corresponding temperature profile can be computed

$$T = \frac{p}{\Gamma\rho} = \frac{2(\gamma-1)(k+1)}{\Gamma(b+2)[2+(\gamma-1)(k+1)]^2}\frac{r^2}{t^2}, \tag{198}$$

where Γ is the gas constant.

C. Coggeshall #3

Beginning once again with Solution A.3, solution #3 of Coggeshall[15] is derived. We define

$$n = \frac{2}{2+(\gamma-1)(k+1)}, \tag{199}$$

and choose $a_1 = 1$. These choices lead to

$$R(t) = \left(1 + \frac{a_2}{n}t\right)^n. \tag{200}$$

Next the limit $n \to \infty$ is taken which is equivalent to

$$\gamma \to \frac{k-1}{k+1}. \tag{201}$$

This limiting value of gamma results in

$$R(t) = e^{a_2 t}, \tag{202}$$

and consequently this particular solution corresponds to a similarity solution of exponential type discussed by Stanyukovich.[17] To obtain the result given by Coggeshall it is necessary to further set

$$a_2 = -\frac{b}{\nu}, \quad \text{and} \quad \hat{\rho} = \rho_0\xi^{\nu-k-1}, \tag{203}$$

where ρ_0, b and ν are constants. These choices yield

$$\xi = \frac{r}{R(t)} = re^{\frac{bt}{\nu}}, \tag{204}$$

$$u(t,r) = \frac{\dot{R}}{R}r = -\frac{br}{\nu}, \tag{205}$$

$$\rho(t,r) = \frac{\rho_0\xi^{\nu-k-1}}{R(t)^{k+1}} = \frac{\rho_0 r^{\nu-k-1}}{R(t)^{\nu}} = \rho_0 r^{\nu-k-1}e^{bt}, \tag{206}$$

$$p(t,r) = \frac{\rho_0 b^2}{\nu^2 R(t)^{k-1}}\int \xi^{\nu-k}\mathrm{d}\xi = \frac{\rho_0 b^2 r^{\nu-k+1}e^{bt}}{\nu^2(k-\nu-1)}, \tag{207}$$

where the integration constant from the pressure integral is again chosen to be zero. Using Equations (206), (207) and the ideal gas EOS the temperature can be computed

$$T = \frac{p}{\Gamma\rho} = \frac{b^2r^2}{\Gamma\nu^2(k-\nu-1)}. \tag{208}$$

D. Coggeshall #4

Only a particular case of solution #4 Coggeshall[15] corresponds to a linear velocity solution, the case where $\gamma = (k-1)/(k+1)$ for which Coggeshall solution #4 becomes

$$u(t,r) = u_0 r, \tag{209}$$

$$\rho(t,r) = \frac{\rho_0}{r^{k+1}}, \tag{210}$$

$$T(t,r) = \frac{u_0^2 r^2}{\Gamma(k-1)}. \tag{211}$$

This solution can again be derived by starting from Solution A.3. As before, $a_1 = 1$ is chosen and

$$n = \frac{2}{2+(\gamma-1)(k+1)}. \tag{212}$$

The required choice of γ implies $n \to \infty$ which again results in

$$R(t) = e^{a_2 t}. \tag{213}$$

To obtain the velocity profile in Coggeshall's notation requires $a_2 = u_0$ and to generate the appropriate density requires $\hat{\rho} = \rho_0\xi^{-k-1}$ where u_0 and ρ_0 are again constants. Consequently

$$\xi = \frac{r}{R(t)} = re^{-a_2 t}, \tag{214}$$

$$u(t,r) = \frac{\dot{R}}{R}r = a_2 r = u_0 r, \tag{215}$$

$$\rho(t,r) = \frac{\rho_0}{R(t)^{k+1}\xi^{k+1}} = \frac{\rho_0}{r^{k+1}}, \tag{216}$$

$$p(t,r) = -\frac{a_2^2\rho_0}{R(t)^{k-1}}\int \xi^{-k}\mathrm{d}\xi = \frac{u_0^2\rho_0 r^{1-k}}{k-1}, \tag{217}$$

where upon evaluation of the integral it is assumed $k \neq 1$ and the integration constant is zero. The corresponding temperature is therefore

$$T = \frac{p}{\Gamma\rho} = \frac{u_0^2 r^2}{\Gamma(k-1)}, \tag{218}$$

which is again consistent with the Coggeshall result.

Furthermore, separately considering the case with $k = 1$, yields the solution

$$u(t,r) = u_0 r, \tag{219}$$

$$\rho(t,r) = \frac{\rho_0}{r^2}, \tag{220}$$

$$\begin{aligned} p(t,r) &= -u_0^2\rho_0\ln(\xi) + C = u_0^2\rho_0[\ln(R(t)) - \ln(r)] + C \\ &= u_0^3\rho_0 t - u_0^2\rho_0\ln(r) + C, \end{aligned} \tag{221}$$

where C is some constant of integration. The free parameters here are ρ_0, u_0 and C. Note however that this solution corresponds to an adiabatic parameter of $\gamma = 0$ and is therefore non-physical.

E. Coggeshall #6 and #7

Coggeshall solutions #6 and #7[15] are referred to by Coggeshall as extensions of Kidder's 1974 solution and 1976 hollow shell solution, respectively.[11,12] Both solutions reside under case D.3 and can be obtained by first setting $a_2 = 0$ (which further implies $a_5 = a_6 = 0$), setting $a_8 = -1$ and defining $a_1 = \tau^2$ where τ is a constant. The simplified D.3 solution is

$$R(t) = \sqrt{\tau^2 - t^2}, \tag{222}$$

$$u(t,r) = \frac{\dot{R}}{R}r = -\frac{rt}{\tau^2 - t^2}, \tag{223}$$

$$\rho(t,r) = \frac{\hat{\rho}(\xi)}{(\tau^2 - t^2)^{\frac{k+1}{2}}}, \tag{224}$$

$$p(t,r) = \frac{\tau^2}{(\tau^2 - t^2)^{\frac{k+3}{2}}}\int \hat{\rho}(\xi)\xi\mathrm{d}\xi, \tag{225}$$

$$\xi = \frac{r}{\sqrt{\tau^2 - t^2}}. \tag{226}$$

Solution #6 results by choosing $\hat{\rho} = \rho_0\xi^b$ where ρ_0 and b are constants. This choice yields

$$u(t,r) = -\frac{rt}{\tau^2 - t^2}, \tag{227}$$

$$\rho(t,r) = \frac{\rho_0 r^b}{(\tau^2 - t^2)^{\frac{b+k+1}{2}}}, \tag{228}$$

$$p(t,r) = \frac{\rho_0\tau^2 r^{b+2}}{(b+2)(\tau^2 - t^2)^{\frac{b+k+5}{2}}} + C, \tag{229}$$

where C is the integration constant. Setting $C = 0$, the resulting temperature is

$$T = \frac{p}{\Gamma\rho} = \frac{\tau^2 r^2}{\Gamma(b+2)(\tau^2 - t^2)^2}, \tag{230}$$

which is the temperature function presented by Coggeshall.

Solution #7 is generated from a more complicated function for $\hat{\rho}$

$$\hat{\rho}(\xi) = \left(\frac{R_0}{\xi}\right)^{\frac{b}{\gamma}}\left(\frac{\tau^{\gamma(k+1)-1-b}}{\left(R_0^{2-\frac{b}{\gamma}} - R_i^{2-\frac{b}{\gamma}}\right)}\right)^{\frac{1}{\gamma-1}}\left[\xi^{2-\frac{b}{\gamma}} - \left(\frac{R_i}{\tau}\right)^{2-\frac{b}{\gamma}}\right]^{\frac{1}{\gamma-1}}, \tag{231}$$

where b, τ, R_i, R_o are constants. Physically, R_i and R_0 correspond to the inner and outer radii of a hollow shell. Using Equations (224), (226) and (231), the density profile is given by

$$\begin{aligned} \rho(t,r) = \frac{R_0^{\frac{b}{\gamma}}}{r^{k+1}}\left(\frac{r}{\sqrt{\tau^2 - t^2}}\right)^{k+1-\frac{b}{\gamma}}\left(\frac{\tau^{\gamma(k+1)-1-b}}{\left(R_0^{2-\frac{b}{\gamma}} - R_i^{2-\frac{b}{\gamma}}\right)}\right)^{\frac{1}{\gamma-1}} \\ \times\left[\left(\frac{r}{\sqrt{\tau^2 - t^2}}\right)^{2-\frac{b}{\gamma}} - \left(\frac{R_i}{\tau}\right)^{2-\frac{b}{\gamma}}\right]^{\frac{1}{\gamma-1}}, \end{aligned} \tag{232}$$

and similarly from Equation (225) the pressure profile is

$$\begin{aligned} p(t,r) = \frac{\tau^2 R_0^{\frac{b}{\gamma}}}{(\tau^2 - t^2)^{\frac{k+3}{2}}}\left(\frac{\tau^{\gamma(k+1)-1-b}}{\left(R_0^{2-\frac{b}{\gamma}} - R_i^{2-\frac{b}{\gamma}}\right)}\right)^{\frac{1}{\gamma-1}} \\ \times\int \xi^{1-\frac{b}{\gamma}}\left(\xi^{2-\frac{b}{\gamma}} - \left(\frac{R_i}{\tau}\right)^{2-\frac{b}{\gamma}}\right)^{\frac{1}{\gamma-1}}\mathrm{d}\xi. \end{aligned} \tag{233}$$

Evaluating the integral, setting the integration constant equal to zero and substituting for ξ the pressure becomes

$$\begin{aligned} p(t,r) = \frac{R_0^{\frac{b}{\gamma}}}{r^{k+1}}\left(\frac{r}{\sqrt{\tau^2 - t^2}}\right)^{k+3}\left(\frac{\tau^{\gamma(k+1)-1-b}}{\left(R_0^{2-\frac{b}{\gamma}} - R_i^{2-\frac{b}{\gamma}}\right)}\right)^{\frac{1}{\gamma-1}} \\ \times\left[\left(\frac{r}{\sqrt{\tau^2 - t^2}}\right)^{2-\frac{b}{\gamma}} - \left(\frac{R_i}{\tau}\right)^{2-\frac{b}{\gamma}}\right]^{\frac{1}{\gamma-1}} \\ \times\frac{\tau^2(\gamma-1)}{r^2(2\gamma-b)}\left[\left(\frac{r}{\sqrt{\tau^2 - t^2}}\right)^{2-\frac{b}{\gamma}} - \left(\frac{R_i}{\tau}\right)^{2-\frac{b}{\gamma}}\right] \end{aligned} \tag{234}$$

Finally, using Equation (232) and (234) to compute the temperature function produces the result outlined by Coggeshall

$$\begin{aligned} T &= \frac{p}{\Gamma\rho}, \\ &= \frac{\tau^2(\gamma-1)}{\Gamma r^2(2\gamma-b)}\left(\frac{r}{\sqrt{\tau^2 - t^2}}\right)^{2+\frac{b}{\gamma}}\left[\left(\frac{r}{\sqrt{\tau^2 - t^2}}\right)^{2-\frac{b}{\gamma}} - \left(\frac{R_i}{\tau}\right)^{2-\frac{b}{\gamma}}\right]. \end{aligned} \tag{235}$$

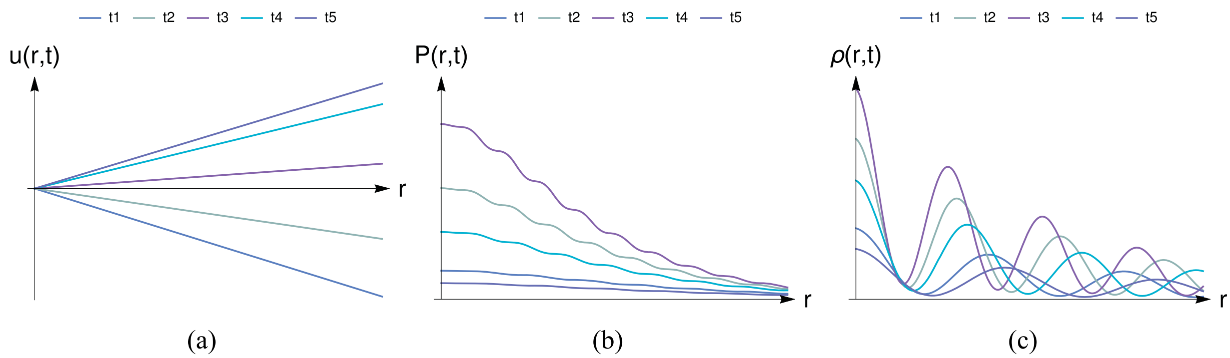

FIG. 2. D.3 example time snapshots at times t_1 through t_5 of dependent variable spatial distributions. (a) Velocity. (b) Pressure. (c) Density.

VIII. DISCUSSION

In this paper, the symmetries of the Equations (1)–(3) admissible in planar, cylindrical and spherical geometries coupled with the ideal gas equation of state were considered. Using these symmetries, the governing equations were systematically reduced to ODEs in Section IV. Following this, the linear velocity ansatz of Equation (102) was applied leading to the attainment of solutions to the systems of ODEs. These solutions were then mapped into group-invariant solutions of the original system and the results were presented in Section VI. In Section VII, several of the acquired group-invariant solutions were connected to solutions discussed by both Coggeshall and Ramsey which had previously been related to solutions in fusion literature by said authors.

Consequently, this paper serves as an example of the utility of Lie group methods to constructing solutions satisfying complex systems of non-linear PDEs. Some fairly complex flows are encapsulated in the results provided in Section VI. For example, consider Solution D.3 along with the particular choice

$$\hat{\rho}(\xi) = Ae^{-B\xi}(\cos^2 C\xi + D), \tag{236}$$

where A, B, C and D are constants. For this case, the product $\xi\hat{\rho}(\xi)$ is integrable enabling a solution to be determined for the pressure profile. Setting values for the group parameters, in addition to the constants $A - D$, further allows the spatial profiles of the dependent variables to be plotted at snapshots in time. As an example, the time-series profiles corresponding to the following values are displayed in Figure 2

$$a_1 = 9, \quad a_2 = -2.92, \quad a_8 = 1, \tag{237}$$

$$A = 4, \quad B = 0.3, \quad C = 2, \quad D = 0.1. \tag{238}$$

In addition to the time snapshots, the solution surface plots in $t - r$ space are also given in Figure 3.

Up until this point, solutions to the system of governing equations have been presented without consideration of the domain or boundary conditions that might be associated with a problem. We address this in the following discussion. As reviewed in Bluman and Anco [Refs. 21, pp. 351-86], invariance of a boundary-value problem requires that both the boundary conditions of the problem and the domain on which they are specified must be invariant under the group of transformations. As a consequence, the number of symmetries available to analyse a system of equations is often reduced when the system is coupled to boundary conditions. Sometimes no symmetries are available for certain boundary-value problems even though the governing equations themselves possess several. One method to ensure the boundary conditions are compatible is to construct the boundary conditions directly from the group-invariant solutions already obtained in Section VI. For example, from the D.3 solution corresponding to Equation (236), the velocity, pressure and density at the inner and outer boundary locations, $r = 0$ and $r = \alpha R(t)$, respectively, where α is some constant, are trivial to define as a function of time. For the velocity boundary condition at $r = 0$, one simply needs to substitute $r = 0$ into Equation (155) which results

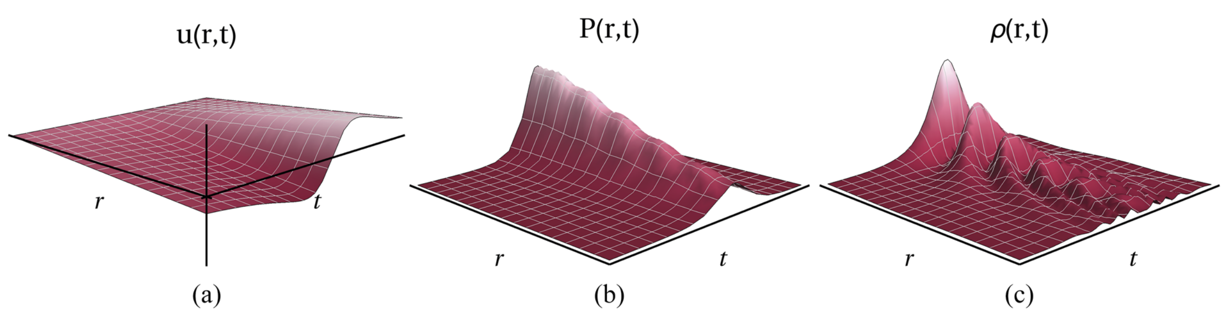

FIG. 3. D.3 example solution surfaces in $t - r$ space. (a) Velocity. (b) Pressure. (c) Density.

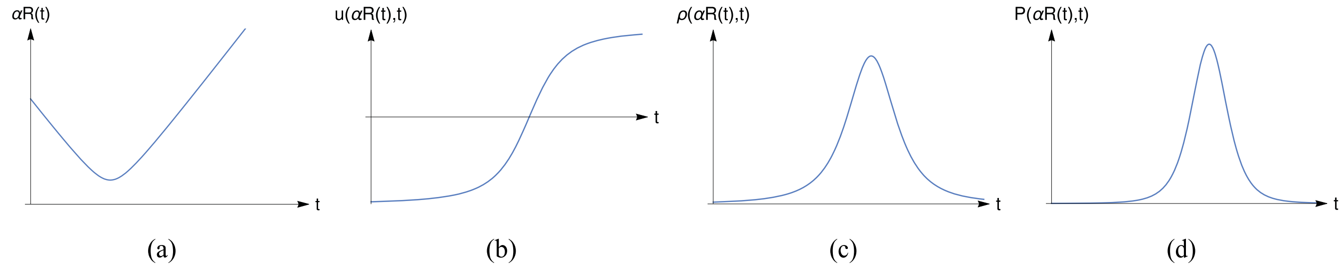

FIG. 4. The boundary conditions at $\alpha R(t)$ for solution D.3 supplemented by Equation (236) and the parameters of Equations (237) and (238). (a) Boundary vs time. (b) Velocity vs time. (c) Density vs time. (d) Pressure vs time.

in

$$u(r,t) = 0. \tag{239}$$

Similarly, the outer velocity boundary condition at $r = \alpha R(t)$ is given by substituting $r = \alpha R(t)$ into Equation (155) to obtain

$$u(\alpha R(t),t) = \frac{\alpha(a_2 + a_8 t)}{(a_1 + 2a_2 t + a_8 t^2)^{\frac{1}{2}}}. \tag{240}$$

In Figure 4, we plot the outer boundary location as a function of time for the D.3 example as well as the velocity, density and pressure at the boundary. By applying such a method, an appropriate boundary-value problem can be constructed for any of the solutions presented in Section VI.

Aside from the series of linear velocity solutions presented, many alternative group invariant solutions residing outside the scope of the linear velocity class can be constructed. Using the systems of ODEs presented as a basis, one approach to generate solutions may involve making a different ansatz for any of the dependent variable profiles. In some cases, the ansatz may allow additional explicit solutions to be obtained. In others, the resulting equations may have to be solved using a numerical solution technique. Regardless of the method used to obtain a solution to the ODEs, the solution can be mapped under the inverse group transformation to provide either an explicit or numeric solution to the original system of PDEs.

In addition to generating more group invariant solutions, extensions of this work could expand on the number of reduced systems of ODEs obtained by performing reductions using the symmetries solely admissible in planar geometry. These reduced systems can be obtained by using transformations that include the symmetries corresponding to the group parameters a_3 and a_4 which were previously disregarded in Section IV. Investigation could also be made into the reduced systems of equations obtained for equation of state models different from the ideal gas. One could also manipulate the governing system to account for heat conduction, radiation transport or electro-magnetic field effects and investigation could be made into how the admissible symmetries are changed. Alternatively, further investigation of the connections between the class of linear velocity solutions considered by Ramsey and those obtainable through a Lie group analysis approach could be made. Although many examples of the Ramsey solutions have been shown to be equivalent to solutions derived in Section VI, at present a number have not. In Section IV it was stated that using the systematic reduction technique applied in the other cases reduced systems of ODEs could not be determined for the group generator corresponding to group parameter a_6, i.e., $V = p\frac{\partial}{\partial p} + \rho\frac{\partial}{\partial \rho}$. The question remains as to which linear velocity solutions discussed by Ramsey are encapsulated by this symmetry and whether the symmetry can be applied to obtain them.

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APPENDIX A: SYSTEM DERIVATION

The system of Equations (1)–(3) stem from the more general flow equations combined with the simplifying assumptions made in the problem formulation. These general equations can be found in compressible fluid dynamics texts such as Thompson.[38] In three dimensions, the general fundamental equations of motion accounting for balance of mass, momentum and energy respectively within an arbitrary control volume are

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \boldsymbol{u} = 0, \tag{A1}$$

$$\nabla p + \rho(\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) = 0, \tag{A2}$$

$$\rho\frac{De}{Dt} - \sigma_{ik} D_{ik} + \nabla \cdot \boldsymbol{q} = 0, \tag{A3}$$

where $\boldsymbol{u}$, p and ρ are again the velocity, pressure and density, respectively, e denotes the specific internal energy, σ_{ik} and D_{ik} are components of the stress and deformation tensors, respectively, $\boldsymbol{q}$ is the heat flux vector, the Einstein summation convention over repeated indices is inferred and the subscripts denote partial differentiation with respect to the indicated variable. Additionally, $D(*)/Dt$ represents the material or Lagrangian derivative. This derivative refers to the time rate of change of a material property of a parcel of the fluid as it moves about in space. It can be decomposed into two parts

$$\frac{D(*)}{Dt} = \frac{\partial(*)}{\partial t} + \boldsymbol{u}\cdot\nabla(*). \tag{A4}$$

The first term on the r.h.s of the equality corresponds to the change in the material property at a fixed spatial position whereas the latter term accounts for the difference between the material property at two points in space at some coincident time.[39] Finally, the product $\sigma_{ik}D_{ik}$ accounts for the rate of work on a fluid parcel via surface forces. These surface forces include pressure and viscous forces that result in volume changes and deformations, respectively. By interpreting $\sigma_{ik}D_{ik}$ as a work term, Equation (A3) is analogous to the first law of thermodynamics

$$\mathrm{d}E = \delta W + \delta Q. \tag{A5}$$

By, separating the stress tensor into two components

$$\sigma_{ik} = -p\delta_{ik} + \Sigma_{ik}, \tag{A6}$$

where δ_{ik} is the Kronecker delta function, Equation (A3) becomes

$$\rho\frac{De}{Dt} + p\delta_{ik}D_{ik} - \Sigma_{ik}D_{ik} + \nabla\cdot\boldsymbol{q} = 0. \tag{A7}$$

Using the sifting property of the Kronecker delta function and the fact that the trace of the deformation tensor is equal to the divergence of the velocity field

$$\delta_{ik}D_{ik} = D_{mm} = \nabla\cdot\boldsymbol{u}. \tag{A8}$$

From the continuity equation, Equation (A1)

$$\nabla\cdot\boldsymbol{u} = -\frac{1}{\rho}\frac{D\rho}{Dt} = \rho\frac{Dv}{Dt}, \tag{A9}$$

where v is the specific volume. Thus, Equation (A7) becomes

$$\rho\left(\frac{De}{Dt} + p\frac{Dv}{Dt}\right) = \Sigma_{ik}D_{ik} - \nabla\cdot\boldsymbol{q}, \tag{A10}$$

$$= \Upsilon - \nabla\cdot\boldsymbol{q}, \tag{A11}$$

where Υ is simply denoted the dissipation function and is a term resulting from viscous forces. Finally, using the Gibbs equation

$$T\mathrm{d}s = \mathrm{d}e + P\mathrm{d}v, \tag{A12}$$

Equation (A11) can be expressed as

$$\rho T\frac{Ds}{Dt} = \Upsilon - \nabla\cdot\boldsymbol{q}. \tag{A13}$$

For a thorough presentation of the above discussion, the reader is encouraged to consult [Refs. 38, p. 56-59].

For the purposes of this paper, it will be assumed that both heat conduction and viscosity are negligible, i.e. the fluid is *inviscid*, and any external body forces such as gravitation are ignored. Therefore, all terms on the r.h.s of (A13) are zero and we are simply left with

$$\frac{Ds}{Dt} = 0, \tag{A14}$$

assuming $T \neq 0$. As a consequence, isentropic or reversible adiabatic fluid flow is the focus. This is not to be confused with homentropic flow for which $Ds/Dt = \nabla s = 0$.

Equations (A1), (A2) and (A14) reduce to the following system of differential equations when only one spatial dimension is considered

$$\rho_t + u\rho_r + \rho\left(u_r + \frac{ku}{r}\right) = 0, \tag{A15}$$

$$p_r + \rho(u_t + uu_r) = 0, \tag{A16}$$

$$\frac{Ds}{Dt} = 0, \tag{A17}$$

where the geometry based constant k is a consequence of evaluating the divergence of the velocity in planar, cylindrical or spherical polar coordinate systems. This system presents three equations in four unknown variables. To close the system, an incomplete EOS, $e := e(\rho, p)$, is incorporated.

Starting from Gibbs equation in the form

$$\mathrm{d}e = T\mathrm{d}s + \frac{p}{\rho^2}\mathrm{d}\rho, \tag{A18}$$

by assuming the temperature is greater than zero

$$T = \left.\frac{\partial e}{\partial s}\right|_\rho > 0,$$

it is inferred that a fundamental potential $e(s, \rho)$ is invertible and the specific entropy can be expressed in terms of the specific internal energy and density $s := s(e, \rho)$. Consequently, an EOS of the form $e = e(\rho, p)$ implies $s = s(\rho, p)$. Using this result and following the work of Ovsiannikov,[23] Holm,[40] Axford[32] and Boyd et al.[33] the isentropic flow equation, Equation (A14), can be expanded via the chain rule

$$\frac{Ds(\rho,p)}{Dt} = \left.\frac{\partial s}{\partial \rho}\right|_p\frac{D\rho}{Dt} + \left.\frac{\partial s}{\partial p}\right|_\rho\frac{Dp}{Dt} = 0. \tag{A19}$$

Substituting for $D\rho/Dt$ from Equation (1), Equation (A19) becomes

$$\frac{Dp}{Dt} - \rho\frac{\left.\frac{\partial s}{\partial \rho}\right|_p}{\left.\frac{\partial s}{\partial p}\right|_\rho}\left(\frac{\partial u}{\partial r} + \frac{ku}{r}\right) = 0, \tag{A20}$$

or equivalently

$$\frac{Dp}{Dt} + A_s(\rho,p)\left(\frac{\partial u}{\partial r} + \frac{ku}{r}\right) = 0, \tag{A21}$$

where A_s is the isentropic bulk modulus defined by

$$A_s(\rho,p) = -\rho\frac{\left.\frac{\partial s}{\partial \rho}\right|_p}{\left.\frac{\partial s}{\partial p}\right|_\rho}. \tag{A22}$$

Equation (A21) is equivalent to Equation (3).

APPENDIX B: CASE A REDUCTION TO ODEs

First the following function of time is defined

$$W(t) = (a_1 + (a_2 + a_5)t)^{\frac{1}{a_2 + a_5}}. \tag{B1}$$

The independent invariant ξ, given by Equation (67), in addition to the velocity, pressure and density profiles in Equations (68)–(70) can be re-expressed using this new function

$$\xi = rW^{-a_2}, \tag{B2a}$$

$$u(t,r) = \hat{u}(\xi)W^{-a_5}, \tag{B2b}$$

$$p(t,r) = \hat{p}(\xi)W^{\frac{a_6}{B}}, \tag{B2c}$$

$$\rho(t,r) = \hat{\rho}(\xi)W^{2a_5+a_6}. \tag{B2d}$$

Through application of the chain rule and simplification the partial derivatives are determined

$$\xi_t = -a_2\frac{\dot{W}}{W}\xi, \tag{B3a}$$

$$\xi_r = \frac{1}{W^{a_2}}, \tag{B3b}$$

$$u_t = -\frac{\dot{W}}{W}W^{a_5}\big(a_5\hat{u} + a_2\xi\hat{u}'\big), \tag{B3c}$$

$$u_r = \frac{\hat{u}'}{W^{a_2+a_5}}, \tag{B3d}$$

$$p_t = \frac{\dot{W}}{W}W^{a_6}\big(a_6\hat{p} - a_2\xi\hat{p}'\big), \tag{B3e}$$

$$p_r = W^{a_6-a_2}\hat{p}', \tag{B3f}$$

$$\rho_t = \frac{\dot{W}}{W}W^{2a_5+a_6}\big((2a_5+a_6)\hat{\rho} - a_2\xi\hat{\rho}'\big), \tag{B3g}$$

$$\rho_r = W^{2a_5+a_6-a_2}\hat{\rho}'. \tag{B3h}$$

Upon substitution of Equations (B2) and (B3) into the governing equations (1)–(3) and also recognising

$$\frac{\dot{W}}{W} = \frac{1}{W^{a_2+a_5}}, \tag{B4}$$

it is possible to obtain the reduced system given by Equations (71)–(73).

APPENDIX C: CASE D REDUCTION TO ODEs

For this case, the reduction is simplified by defining

$$W(t) = (a_1 + (a_2 + a_5)t + a_8t^2)^{\frac{1}{2}}, \tag{C1}$$

$$X(t) = \frac{\arctan\left(\frac{a_2 + a_5 + 2a_8t}{\sqrt{4a_1a_8 - (a_2+a_5)^2}}\right)}{\sqrt{4a_1a_8 - (a_2+a_5)^2}}, \tag{C2}$$

which have the following first order time derivatives

$$\dot{W} = \frac{a_2 + a_5 + 2a_8t}{2W}, \qquad \dot{X} = \frac{1}{2W^2}. \tag{C3}$$

The Case D independent invariant, Equation (86), and velocity, pressure and density fields defined by Equations (92a)–(92c) are re-expressed using these functions

$$\xi = \frac{r}{W}\exp[X(a_5 - a_2)], \tag{C4a}$$

$$u(t,r) = \frac{\hat{u} + a_8t\xi}{W}\exp[X(a_2 - a_5)], \tag{C4b}$$

$$p(t,r) = \frac{\hat{p}(\xi)}{W^{k+3}}\exp[X(2a_6 + (a_2+a_5)(k+3))], \tag{C4c}$$

$$\rho(t,r) = \frac{\hat{\rho}(\xi)}{W^{k+1}}\exp[X(4a_5 + 2a_6 + (a_2+a_5)(k+1))]. \tag{C4d}$$

Next, from Equations (C1) the following partial derivatives are determined via application of the chain rule and simplification

$$\xi_t = -\frac{\xi(a_2 + a_8t)}{W^2}, \tag{C5a}$$

$$\xi_r = \frac{\exp(X(a_5 - a_2))}{W}, \tag{C5b}$$

$$u_t = \frac{\exp[X(a_2 - a_5)]}{W^3}\big(-\hat{u}(a_5 + a_8t) - \hat{u}'\xi(a_2 + a_8t) - a_8t\xi(a_2 + a_5 + 2a_8t) + a_8W^2\xi\big), \tag{C5c}$$

$$u_r = \frac{\hat{u}' + a_8t}{W^2}, \tag{C5d}$$

$$p_t = \frac{\exp[X(2a_6 + (a_2+a_5)(k+3))]}{W^{k+5}} \times \big(\hat{p}(a_6 - a_8(k+3)t) - \hat{p}'\xi(a_2 + a_8t)\big), \tag{C5e}$$

$$p_r = \frac{\exp[X(2a_6 + (a_2+a_5)(k+3) + a_5 - a_2)]}{W^{k+4}}\hat{p}', \tag{C5f}$$

$$\rho_t = \frac{\exp[X(4a_5 + 2a_6 + (a_2+a_5)(k+1))]}{W^{k+3}} \times \big(\hat{\rho}(2a_5 + a_6 - a_8(k+1)t) - \hat{\rho}'\xi(a_2 + a_8t)\big), \tag{C5g}$$

$$\rho_r = \frac{\exp[X(4a_5 + 2a_6 + (a_2+a_5)(k+1) + a_5 - a_2)]}{W^{k+2}}\hat{\rho}'. \tag{C5h}$$

Proceeding to substitute these partial derivatives along with Equations (C4b)–(C4d) into the governing equations yields the reduced system given by Equations (93)–(95).

APPENDIX D: CASE E REDUCTION TO ODEs

For this case, the function of time is

$$W(t) = a_2 + a_5 + 2a_8 t, \tag{D1}$$

which has the first order time derivative

$$\dot{W} = 2a_8. \tag{D2}$$

Once again, the independent invariant as well as the velocity, pressure and density fields are re-expressed using this function

$$\xi = \frac{r}{W}\exp\left[\frac{a_2 - a_5}{W}\right] \tag{D3a}$$

$$u(t,r) = \frac{\hat{u}(\xi) + 4a_8^2 t\xi}{W}\exp\left[\frac{a_5 - a_2}{W}\right], \tag{D3b}$$

$$p(t,r) = \frac{\hat{p}(\xi)}{W^{k+3}}\exp\left[\frac{-2a_6 - (a_2 + a_5)(k+3)}{W}\right], \tag{D3c}$$

$$\rho(t,r) = \frac{\hat{\rho}(\xi)}{W^{k+1}}\exp\left[\frac{-2(a_6 + 2a_5) - (a_2 + a_5)(k+1)}{W}\right]. \tag{D3d}$$

Using Equations (D1)–(D3d) the following partial derivatives are computed via application of the chain rule and simplification

$$\xi_t = -\frac{\xi}{W^2}\left(4a_8 a_2 + 4a_8^2 t\right), \tag{D4a}$$

$$\xi_r = \frac{1}{W}\exp\left[\frac{a_2 - a_5}{W}\right], \tag{D4b}$$

$$u_t = \frac{1}{W^3}\exp\left[\frac{a_5 - a_2}{W}\right]\left(4a_8^2\xi(a_2 + a_5)^2 - \hat{u}'\xi(4a_2 a_8 + 4a_8^2 t) - \hat{u}(4a_5 a_8 + 4a_8^2 t) - 16a_8^4\xi t^2\right) \tag{D4c}$$

$$u_r = \frac{1}{W^2}\left(\hat{u}' + 4a_8^2 t\right), \tag{D4d}$$

$$p_t = \frac{1}{W^{k+5}}\exp\left[\frac{-2a_6 - (a_2 + a_5)(k+3)}{W}\right]\left(\hat{p}(4a_8 a_6 - 4a_8^2 t(k+3)) - \hat{p}'\xi(4a_8 a_2 + 4a_8^2 t)\right), \tag{D4e}$$

$$p_r = \frac{1}{W^{k+4}}\exp\left[\frac{-2a_6 - (a_2 + a_5)(k+3)}{W}\right]\exp\left[\frac{a_2 - a_5}{W}\right]\hat{p}', \tag{D4f}$$

$$\rho_t = \frac{1}{W^{k+3}}\exp\left[\frac{-2(a_6 + 2a_5) - (a_2 + a_5)(k+1)}{W}\right] \times \left(\hat{\rho}(4a_8(a_6 + 2a_5) - 4a_8^2 t(k+1)) - \hat{\rho}'\xi(4a_8 a_2 + 4a_8^2 t)\right), \tag{D4g}$$

$$\rho_r = \frac{1}{W^{k+2}}\exp\left[\frac{-2(a_6 + 2a_5) - (a_2 + a_5)(k+1)}{W}\right]\exp\left[\frac{a_2 - a_5}{W}\right]\hat{\rho}'. \tag{D4h}$$

Proceeding to substitute these partial derivatives into the governing equations in addition to Equations (D3b)–(D3d) yields the reduced system given by Equations (99)–(101).

APPENDIX E: DETERMINING LINEAR VELOCITY SOLUTIONS

In this section the reduced systems of equations are solved under the linear velocity ansatz. The results are used to construct the solutions presented in Section VI.

1. A solutions

By inspection of Equations (67) and (68), consideration of solutions of type (102), restricts the choice of the velocity to $\hat{u}(\xi) = B\xi$, where B is a constant. It is assumed $B \neq 0$, and therefore trivial, static flows are excluded from the analysis. Substituting $B\xi$ for $\hat{u}$ and then Equation (67) for ξ gives

$$u(t,r) = B\xi(a_1 + (a_2 + a_5)t)^{-\frac{a_5}{a_2 + a_5}} \tag{E1}$$

$$= \frac{Br}{a_1 + (a_2 + a_5)t}. \tag{E2}$$

Comparing this result to Equation (102)

$$u(t,r) = \frac{\dot{R}}{R}r, \tag{E3}$$

yields the differential equation

$$\frac{\dot{R}}{R} = \frac{B}{a_1 + (a_2 + a_5)t}, \tag{E4}$$

which has solution

$$R(t) = A(a_1 + (a_2 + a_5)t)^{\frac{B}{a_2 + a_5}}, \tag{E5}$$

where A is the constant of integration. Without loss of generality set $A = 1$ since it can be absorbed into the group parameters and its value does not influence the result of $\dot{R}/R$.

Equations (67)–(70) can be written succinctly in terms of this new $R(t)$ function

$$\xi = \frac{r}{R^{\frac{a_2}{B}}}, \tag{E6a}$$

$$u(t,r) = \frac{\dot{R}}{R}r, \tag{E6b}$$

$$p(t,r) = \hat{p}R^{\frac{a_6}{B}}, \tag{E6c}$$

$$\rho(t,r) = \hat{\rho}R^{\frac{2a_5 + a_6}{B}}. \tag{E6d}$$

The task of determining the unknown functions $\hat{p}$ and $\hat{\rho}$ remains. We proceed by substituting $\hat{u} = B\xi$ and $\hat{u}' = B$ into Equations (71)–(73) to get

$$\hat{\rho}'\xi(B - a_2) + \hat{\rho}(2a_5 + a_6 + B(k+1)) = 0 \tag{E7}$$

$$\hat{p}' + \hat{\rho}\xi B(B - a_2 - a_5) = 0, \tag{E8}$$

$$\hat{p}'\xi(B - a_2) + \hat{p}(a_6 + B\gamma(k+1)) = 0. \tag{E9}$$

The subsequent analysis now bifurcates and requires additional assumptions about the value of B. Three cases are considered in the following sections.

a. Solution A.1: $B \neq a_2$, exponent $\neq -1$

For this first case, it is assumed $B \neq a_2$ and Equations (E7) and (E9) are re-arranged to obtain

$$\frac{d\hat{\rho}}{\hat{\rho}} = \frac{2a_5 + a_6 + B(k+1)}{\xi(a_2 - B)} d\xi, \tag{E10}$$

$$\frac{d\hat{p}}{\hat{p}} = \frac{a_6 + B\gamma(k+1)}{\xi(a_2 - B)} d\xi, \tag{E11}$$

which have solutions

$$\hat{\rho} = c_1 \xi^{\frac{2a_5 + a_6 + B(k+1)}{a_2 - B}}, \tag{E12}$$

$$\hat{p} = c_2 \xi^{\frac{a_6 + B\gamma(k+1)}{a_2 - B}}, \tag{E13}$$

where c_1 and c_2 are integration constants.

Equation (E8) is written as

$$\frac{d\hat{p}}{d\xi} = B(a_2 + a_5 - B)\hat{\rho}\xi, \tag{E14}$$

which upon integration with respect to ξ gives

$$\hat{p} = B(a_2 + a_5 - B) \int \hat{\rho}\xi. \tag{E15}$$

Substituting for $\hat{\rho}$ using the Equation (E12)

$$\hat{p} = c_1 B(a_2 + a_5 - B) \int \xi^{\frac{2a_5 + a_6 + a_2 + Bk}{a_2 - B}} d\xi. \tag{E16}$$

At this point there is an additional bifurcation of the solution paths. For the A.1 case, it is further assumed that the exponent in the integral is not equal to -1 which is equivalent to the constraint

$$2(a_2 + a_5) + a_6 + B(k-1) \neq 0. \tag{E17}$$

This constraint is conditional upon the choice of geometry parameter k since we have already assumed $B \neq 0$. If $k = 1$

$$a_6 \neq -2(a_2 + a_5), \tag{E18}$$

otherwise

$$B \neq \frac{2(a_2 + a_5) + a_6}{1 - k}. \tag{E19}$$

Integrating under this assumption, the solution is

$$\hat{p} = \frac{c_1 B(a_2 + a_5 - B)(a_2 - B)}{2(a_2 + a_5) + a_6 + B(k-1)} \xi^{\frac{2(a_2 + a_5) + a_6 + B(k-1)}{a_2 - B}} + c_3, \tag{E20}$$

where c_3 is another constant of integration.

This solution for $\hat{p}$ must be self-consistent with Equation (E13) and therefore it is also required that

$$c_3 = 0, \tag{E21a}$$

$$c_2 = \frac{c_1 B(a_2 + a_5 - B)(a_2 - B)}{2(a_2 + a_5) + a_6 + B(k-1)}, \tag{E21b}$$

$$a_6 + \gamma B(k+1) = 2(a_2 + a_5) + a_6 + B(k-1), \tag{E21c}$$

From the last of these constraints it is inferred

$$2(a_2 + a_5) = B(\gamma(k+1) - k + 1), \tag{E22}$$

which is conditional on the choice of γ. If

$$\gamma = \frac{k-1}{k+1}, \tag{E23}$$

we require

$$a_2 + a_5 = 0, \tag{E24}$$

otherwise

$$B = \frac{2(a_2 + a_5)}{\gamma(k+1) - k + 1}. \tag{E25}$$

Without setting values for k and γ, the velocity, density and pressure fields can be defined by evaluating Equation (E6) using Equations (E12), (E13) and (E5) with $A = 1$. This solution requires the constraints of Equations (E17) and (E22) to be satisfied and is summarized in Section VI A.

b. Solution A.2: $B \neq a_2$, exponent $= -1$

Next, the integration step in Equation (E16) is conducted again assuming this time that the exponent of ξ within the integral has a value of -1. This condition is equivalent to

$$a_6 + B(k-1) + 2(a_2 + a_5) = 0. \tag{E26}$$

Equation (E16) becomes

$$\hat{p} = c_1 B(a_2 + a_5 - B) \int \frac{d\xi}{\xi}, \tag{E27}$$

which upon integrating yields

$$\hat{p} = \ln(\xi^{c_1 B(a_2 + a_5 - B)}) + c_3, \tag{E28}$$

where c_3 is the integration constant. As before, this solution for $\hat{p}$ must be self-consistent with Equation (E13) which is only true provided

$$c_2 = c_3, \tag{E29a}$$

$$c_1 B(a_2 + a_5 - B) = 0, \tag{E29b}$$

$$a_6 + B\gamma(k+1) = 0. \tag{E29c}$$

Assuming the velocity and density fields are non-trivial, which requires $B \neq 0$ and $c_1 \neq 0$, we can infer

$$B = a_2 + a_5 \neq 0, \tag{E30}$$

which upon substitution into Equations (E26) and (E29) gives two constraints on a_6

$$a_6 = -(a_2 + a_5)(k+1), \tag{E31}$$

$$a_6 = -\gamma(a_2 + a_5)(k+1). \tag{E32}$$

Assuming $k \neq -1$, these are only simultaneously satisfied for $\gamma = 1$. Furthermore, the constraint $B \neq a_2$ reduces to $a_5 \neq 0$ under Equation (E30).

Consequently, Equations (E12) and (E13) for this case become

$$\hat{\rho} = \frac{c_1}{\xi^2}, \qquad \hat{p} = c_2. \tag{E33}$$

Evaluating Equation (E6) using Equation (E33) yields the A.2 result summarized in Section VI A.

c. Solution A.3: $B = a_2$

The final solution generated from case A is obtained by setting $B = a_2$ where it is further assumed $a_2 \neq 0$ since the focus is on solutions that generate a non-trivial velocity. In this case the reduced Equations (E7)–(E9) become

$$2a_5 + a_6 + a_2(k+1) = 0, \tag{E34}$$

$$\hat{p}' - a_2 a_5 \hat{\rho}\xi = 0, \tag{E35}$$

$$a_6 + a_2\gamma(k+1) = 0. \tag{E36}$$

Integrating Equation (E35) with respect to ξ gives

$$\hat{p} = a_2 a_5 \int \hat{\rho}\,\xi\,\mathrm{d}\xi. \tag{E37}$$

Solving both Equations (E34) and (E36) for a_6 and equating gives

$$a_6 = -2a_5 - a_2(k+1) = -a_2\gamma(k+1), \tag{E38}$$

from which it is inferred

$$a_5 = \frac{a_2(\gamma-1)(k+1)}{2}. \tag{E39}$$

Imposing the constraints of Equation (E39) and $B = a_2$ on the function $R(t)$ specified in Equation (E5) yields

$$R(t) = \left(a_1 + \frac{a_2 t}{2}[2 + (\gamma-1)(k+1)]\right)^{\frac{2}{2+(\gamma-1)(k+1)}}, \tag{E40}$$

and the independent invariant is simply

$$\xi = \frac{r}{R(t)}. \tag{E41}$$

Finally, evaluating the expressions for the velocity, pressure and density fields in Equation (E6) using Equations (E38)–(E41) determines the final solution presented in Section VI A.

2. B solutions

In this section, solutions are obtained to the reduced system of Equations (77)–(79) derived in Section IV C

$$\hat{\rho}' + \hat{\rho}\hat{u}(a_6 + k - 1) = 0, \tag{E42}$$

$$a_6\hat{p} + \hat{\rho}\hat{u}' + \hat{\rho}\hat{u}^2 = 0, \tag{E43}$$

$$\hat{p}' + \hat{p}\hat{u}[a_6 + \gamma(k+1)] = 0. \tag{E44}$$

a. Solution B.1

Equations (E42) and (E44) can be re-expressed as

$$\frac{\mathrm{d}\hat{\rho}}{\hat{\rho}} = -\hat{u}(a_6 + k - 1)\mathrm{d}\xi, \tag{E45}$$

$$\frac{\mathrm{d}\hat{p}}{\hat{p}} = -\hat{u}[a_6 + \gamma(k+1)]\mathrm{d}\xi, \tag{E46}$$

and have solutions

$$\hat{\rho} = \exp\left(-(a_6 + k - 1)\int \hat{u}\mathrm{d}\xi\right), \tag{E47a}$$

$$\hat{p} = \exp\left(-[a_6 + \gamma(k+1)]\int \hat{u}\mathrm{d}\xi\right). \tag{E47b}$$

These solutions must also satisfy Equation (E43) which can be rearranged to

$$-a_6\frac{\hat{p}}{\hat{\rho}} = \hat{u}' + \hat{u}^2. \tag{E48}$$

At this point the ansatz for linear velocity flow is made which requires comparing Equation (76) to Equation (102)

$$u(t,r) = \frac{\dot{R}(t)}{R(t)}r = \hat{u}(\xi)r. \tag{E49}$$

Since $\xi = t$ for this particular case

$$\frac{\dot{R}}{R} = \hat{u}. \tag{E50}$$

Re-expressing Equations (E47) and (E48) using this result

$$\hat{\rho} = \exp(-(a_6 + k - 1)\int \frac{\dot{R}}{R}\mathrm{d}t) = \frac{c_1}{R^{a_6+k-1}}, \tag{E51}$$

$$\hat{p} = \exp(-[a_6 + \gamma(k+1)]\int \frac{\dot{R}}{R}\mathrm{d}t) = \frac{c_2}{R^{a_6+\gamma(k+1)}}, \tag{E52}$$

$$-a_6\frac{\hat{p}}{\hat{\rho}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{R}}{R}\right) + \frac{\dot{R}^2}{R^2} = \frac{\ddot{R}}{R}. \tag{E53}$$

where c_1 and c_2 are integration constants. Substituting Equations (E51) and (E52) into (E53) gives

$$\ddot{R}R^{1-\eta} = -a_6\frac{c_2}{c_1}, \tag{E54}$$

where $\eta = (1-\gamma)(k+1)$.

The solution to the original system of equations is then obtained using Equation (76)

$$u(t,r) = \hat{u}(\xi)r, \tag{E55a}$$

$$p(t,r) = \hat{p}(\xi)r^{a_6}, \tag{E55b}$$

$$\rho(t,r) = \hat{\rho}(\xi)r^{a_6-2}, \tag{E55c}$$

and the results given by Equations (E50), (E51) and (E52). The result is presented as solution B.1 in Section VI B.

b. Solution B.2: $a_6 = 0$

From Equation (E54), if $a_6 = 0$ and assuming a non trivial function for $R(t)$, by extension

$$\ddot{R} = 0, \tag{E56}$$

and therefore

$$R(t) = \dot{R}_0 t + R_0, \tag{E57}$$

where $\dot{R}_0$ and R_0 are the velocity and position of the boundary $R(t)$ at time $t = 0$. This step yields the solution B.2 presented in Section VI B upon substitution for a_6 and $R(t)$.

3. D solutions

In Section IV E, the velocity profile was formulated according to Equation (91). Therefore, once again setting $\hat{u} = B\xi$ restricts the analysis to flows of type Equation (102). The resulting velocity profile is

$$u(t,r) = \frac{(B + a_8 t)\xi}{(a_1 + (a_2 + a_5)t + a_8 t^2)^{\frac{1}{2}}} \exp\Big(\frac{(a_2 - a_5)\arctan(\alpha)}{\sqrt{4a_1 a_8 - (a_2 + a_5)^2}}\Big), \tag{E58}$$

where

$$\alpha = \frac{a_2 + a_5 + 2a_8 t}{\sqrt{4a_1 a_8 - (a_2 + a_5)^2}}. \tag{E59}$$

Substitution of Equation (86) for ξ gives

$$u(t,r) = \frac{(B + a_8 t)r}{a_1 + (a_2 + a_5)t + a_8 t^2}. \tag{E60}$$

As before, this velocity profile is compared to the ansatz of Equation (102) to determine the unknown $R(t)$. The following differential equation is obtained

$$\frac{\dot{R}}{R} = \frac{(B + a_8 t)}{a_1 + (a_2 + a_5)t + a_8 t^2}, \tag{E61}$$

which has solution

$$R(t) = A\big(a_1 + (a_2 + a_5)t + a_8 t^2\big)^{\frac{1}{2}} \exp\left(\frac{[2B - (a_2 + a_5)]\arctan(\alpha)}{\sqrt{4a_1 a_8 - (a_2 + a_5)^2}}\right), \tag{E62}$$

where A is the constant of integration. Again, without loss of generality, A=1. The task of solving the reduced system for $\hat{p}$ and $\hat{\rho}$ under the restriction $\hat{u} = B\xi$ remains and is now addressed.

Substituting $\hat{u} = B\xi$ into Equations (93)–(95) gives

$$\hat{\rho}'\xi(B - a_2) + \hat{\rho}(2a_5 + a_6 + B(k+1)) = 0, \tag{E63}$$

$$\hat{p}' + \hat{\rho}\xi(B(B - a_2 - a_5) + a_1 a_8) = 0, \tag{E64}$$

$$\hat{p}'\xi(B - a_2) + \hat{p}(a_6 + B(k+3)) = 0. \tag{E65}$$

This system is almost identical to that solved in Section E1 with the exception of the additional latter term in Equation (E64) and the restricted value of $\gamma = (k+3)/(k+1)$. Consequently, as before, the analysis is separated according to the value of the parameter B.

a. Solution D.1: $B \neq a_2$, exponent $\neq -1$

Assuming $B \neq a_2$, Equations (E63) and (E65) are re-arranged giving

$$\frac{\mathrm{d}\hat{\rho}}{\hat{\rho}} = \frac{2a_5 + a_6 + B(k+1)}{\xi(a_2 - B)}\mathrm{d}\xi, \tag{E66}$$

$$\frac{\mathrm{d}\hat{p}}{\hat{p}} = \frac{a_6 + B(k+3)}{\xi(a_2 - B)}\mathrm{d}\xi, \tag{E67}$$

with solutions

$$\hat{\rho} = c_1 \xi^{\frac{2a_5 + a_6 + B(k+1)}{a_2 - B}}, \tag{E68}$$

$$\hat{p} = c_2 \xi^{\frac{a_6 + B(k+3)}{a_2 - B}}, \tag{E69}$$

where c_1 and c_2 are integration constants. Equation (E64) can also be re-arranged and integrated to

$$\hat{p} = (B(a_2 + a_5 - B) - a_1 a_8)\int \hat{\rho}\xi \mathrm{d}\xi, \tag{E70}$$

which after substituting for $\hat{\rho}$ using Equation (E68) becomes

$$\hat{p} = c_1(B(a_2 + a_5 - B) - a_1 a_8)\int \xi^{\frac{2a_5 + a_6 + a_2 + Bk}{a_2 - B}} \mathrm{d}\xi. \tag{E71}$$

Once again, to proceed assumptions must be made about the value of the exponent of ξ. First it is assumed the exponent is not equal to -1 which is equivalent to the restriction obtained previously in Equation (E17)

$$2(a_2 + a_5) + a_6 + B(k-1) \neq 0. \tag{E72}$$

Integrating under this assumption yields

$$\hat{p} = \frac{c_1(B(a_2 + a_5 - B) - a_1 a_8)(a_2 - B)}{2(a_2 + a_5) + a_6 + B(k-1)} \xi^{\frac{2(a_2 + a_5) + a_6 + B(k-1)}{a_2 - B}} + c_3, \tag{E73}$$

where c_3 is the constant of integration. Requiring self-consistency between Equations (E69) and (E73) enforces

$$c_3 = 0, \tag{E74a}$$

$$c_2 = \frac{c_1(B(a_2 + a_5 - B) - a_1 a_8)(a_2 - B)}{2(a_2 + a_5) + a_6 + B(k-1)}, \tag{E74b}$$

$$a_6 + B(k+3) = 2(a_2 + a_5) + a_6 + B(k-1). \tag{E74c}$$

From the last of the constraints in Equation (E74) we infer

$$B = \frac{a_2 + a_5}{2}, \tag{E75}$$

and by extension

$$c_2 = \frac{c_1(a_2 - a_5)((a_2 + a_5)^2 - 4a_1 a_8)}{4(2a_6 + (k+5)(a_2 + a_5))}. \tag{E76}$$

Furthermore, by Equation (E75), the constraint $B \neq a_2$ becomes

$$a_2 \neq a_5, \tag{E77}$$

Equation (E72) becomes

$$2a_6 + (k+3)(a_2 + a_5) \neq 0, \tag{E78}$$

and there is a large simplification in the expression for $R(t)$ given by Equation (E62)

$$R(t) = (a_1 + (a_2 + a_5)t + a_8 t^2)^{\frac{1}{2}}. \tag{E79}$$

For this particular subcase, which is now associated with Equation (E78), the solutions for $\hat{p}$ and $\hat{\rho}$ are expressed by

$$\hat{p} = c_2 \xi^{\frac{2a_6 + (a_2 + a_5)(k+3)}{a_2 - a_5}}, \tag{E80}$$

$$\hat{\rho} = c_1 \xi^{\frac{4a_5 + 2a_6 + (a_2 + a_5)(k+1)}{a_2 - a_5}}. \tag{E81}$$

All that remains is to back substitute the results to obtain the velocity, pressure and density fields, $u(t,r)$, $p(t,r)$ and $\rho(t,r)$ respectively.

The velocity profile is easily obtained by combining Equations (E60) and (E75) to get

$$u(t,r) = \frac{(a_2 + a_5 + 2a_8 t)r}{2(a_1 + (a_2 + a_5)t + a_8 t^2)}. \tag{E82}$$

Next, for the sake of tractability, Equations (93) and (94) are rearranged for $p(t,r)$ and $\rho(t,r)$ and expressed in terms of the functions $R(t)$ and $X(t)$

$$p(t,r) = \frac{\hat{p}}{R(t)^{k+3}} \exp[X(t)(2a_6 + (a_2 + a_5)(k+3))], \tag{E83}$$

$$\rho(t,r) = \frac{\hat{\rho}}{R(t)^{k+1}} \exp[X(t)(4a_5 + 2a_6 + (a_2 + a_5)(k+1))], \tag{E84}$$

where $R(t)$ is given by Equation (E79) and

$$X(t) = \frac{\arctan\left(\frac{a_2 + a_5 + 2a_8 t}{\sqrt{4a_1 a_8 - (a_2 + a_5)^2}}\right)}{\sqrt{4a_1 a_8 - (a_2 + a_5)^2}}. \tag{E85}$$

ξ is also expressed in the same format using Equation (86)

$$\xi = \frac{r}{R(t)} \exp[X(t)(a_5 - a_2)]. \tag{E86}$$

Substituting Equations (E80) and (E81) for $\hat{p}$ and $\hat{\rho}$ into Equations (E83) and (E84) gives

$$p(t,r) = \frac{c_2}{R(t)^{k+3}} \xi^{\frac{2a_6 + (a_2 + a_5)(k+3)}{a_2 - a_5}} \exp[X(t)(2a_6 + (a_2 + a_5)(k+3))], \tag{E87}$$

$$\rho(t,r) = \frac{c_1}{R(t)^{k+1}} \xi^{\frac{4a_5 + 2a_6 + (a_2 + a_5)(k+1)}{a_2 - a_5}} \times \exp[X(t)(4a_5 + 2a_6 + (a_2 + a_5)(k+1))]. \tag{E88}$$

Finally, upon substitution of Equation (E86) for ξ, the argument of the exponential term is zero and the final solution simplifies to Solution D.1 presented in Section VI C.

2. Solution D.2: $B \neq a_2$, exponent = −1

Returning to Equation (E71), a solution is obtained assuming that the exponent of ξ within the integral is equal to -1. This is equivalent to the constraint

$$2(a_2 + a_5) + a_6 + B(k-1) = 0. \tag{E89}$$

Under this assumption Equation (E71) becomes

$$\hat{p} = c_1(B(a_2 + a_5 - B) - a_1 a_8) \int \frac{1}{\xi} \mathrm{d}\xi, \tag{E90}$$

which has solution

$$\hat{p} = \ln \xi^{c_1(B(a_2 + a_5 - B) - a_1 a_8)} + c_3, \tag{E91}$$

where c_3 is some constant of integration.

Once again, requiring self-consistency between Equations (E69) and (E91) yields

$$\ln \xi^{c_1(B(a_2 + a_5 - B) - a_1 a_8)} + c_3 = c_2 \xi^{\frac{a_6 + B(k+3)}{a_2 - B}}, \tag{E92}$$

which is only satisfied provided

$$c_2 = c_3, \tag{E93a}$$

$$c_1(B(a_2 + a_5 - B) - a_1 a_8) = 0, \tag{E93b}$$

$$a_6 + B(k+3) = 0. \tag{E93c}$$

Rearranging Equation (E89) as well as the final constraint in Equation (E93) for a_6 and equating the results gives

$$a_6 = -B(k+3) = -2(a_2 + a_5) + B(1-k), \tag{E94}$$

from which it is inferred

$$B = \frac{a_2 + a_5}{2}. \tag{E95}$$

Substituting this result into the second constraint in Equation (E93) yields

$$c_1\left(\frac{(a_2 + a_5)^2}{4} - a_1 a_8\right) = 0, \tag{E96}$$

which requires either $c_1 = 0$ or $4a_1 a_8 = (a_2 + a_5)^2$. Since the latter of these solutions is already excluded, c_1 must equal zero. Substituting the former results into Equations (E68) and (E69) determines a trivial $\hat{\rho}$ and constant $\hat{p}$

$$\hat{\rho} = 0, \qquad \hat{p} = c_2. \tag{E97}$$

As was true for the previous D.1 solution, Equation (E95) infers that the constraint $B \neq a_2$ becomes

$$a_2 \neq a_5, \tag{E98}$$

and the expression for $R(t)$ given by Equation (E62) simplifies to

$$R(t) = (a_1 + (a_2 + a_5)t + a_8 t^2). \tag{E99}$$

The solution for the velocity field is therefore equivalent to that given in Equation (E82).

If the pressure and temperature fields are again prescribed by Equations (E83) and (E84), respectively, then upon substitution for $\hat{p}$ and $\hat{\rho}$ using Equation (E97), in addition to $a_6 = (a_2 + a_5)(k+3)/2$

$$p(t,r) = \frac{c_2}{R(t)^{k+3}}, \tag{E100}$$

$$\rho(t,r) = 0. \tag{E101}$$

This solution is presented as D.2 in Section VI C.

c. Solution D.3: $B = a_2$

The final solution obtainable from the reduced system consisting of Equations (93)–(95) is constructed by setting $B = a_2$. Under this assumption, the system simplifies to

$$2a_5 + a_6 + a_2(k+1) = 0, \tag{E102}$$

$$\hat{p}' + \hat{\rho}\xi(a_1 a_8 - a_2 a_5) = 0, \tag{E103}$$

$$a_6 + a_2(k+3) = 0. \tag{E104}$$

Equations (E102) and (E104) are solved for a_6 and giving

$$a_6 = -2a_2 - a_2(k+1) = -a_2(k+3), \tag{E105}$$

from which it is inferred

$$a_2 = a_5. \tag{E106}$$

Equation (E103) can then be integrated with respect to ξ to obtain

$$\hat{p} = (a_2^2 - a_1 a_8) \int \hat{\rho}\xi \mathrm{d}\xi. \tag{E107}$$

By Equation (E106) and with $B = a_2$, the velocity profile is easily obtained from Equation (E60)

$$u(t,r) = \frac{(a_2 + a_8 t)r}{a_1 + 2a_2 t + a_8 t^2}, \tag{E108}$$

$R(t)$ given in Equation (E62) with $A = 1$ becomes

$$R(t) = (a_1 + 2a_2 t + a_8 t^2)^{\frac{1}{2}}, \tag{E109}$$

and the independent invariant is

$$\xi = \frac{r}{R(t)}. \tag{E110}$$

Additionally, with the constraint on a_6 determined by Equation (E105), Equations (E69) and (E68) simplify to

$$p(t,r) = \frac{\hat{p}}{R(t)^{k+3}}, \tag{E111}$$

$$\rho(t,r) = \frac{\hat{\rho}}{R(t)^{k+1}}, \tag{E112}$$

since the argument of the exponential of both is zero. Finally substituting Equation (E107) for $\hat{p}$ and Equation (E109) for $R(t)$ yields the solution D.3 given in Section VI C.

4. E solutions

The solutions in this section are obtained from the reduced system of Equations (99)–(101). Rearranging Equation (98a) for $u(t,r)$

$$u(t,r) = \exp\Big(\frac{a_5 - a_2}{a_2 + a_5 + 2a_8 t}\Big)\frac{(4a_8^2 t\xi + \hat{u}(\xi))}{a_2 + a_5 + 2a_8 t}, \tag{E113}$$

it is apparent that the linear velocity ansatz will again restrict the focus to $\hat{u} = B\xi$ where B is a constant. Substituting for $\hat{u}$ and then for ξ using Equation (97)

$$u(t,r) = \exp\Big(\frac{a_5 - a_2}{a_2 + a_5 + 2a_8 t}\Big)\frac{(4a_8^2 t + B)\xi}{a_2 + a_5 + 2a_8 t}, \tag{E114}$$

$$= \exp\Big(\frac{a_5 - a_2}{a_2 + a_5 + 2a_8 t}\Big)\frac{(4a_8^2 t + B)}{a_2 + a_5 + 2a_8 t}\frac{r}{a_2 + a_5 + 2a_8 t} \times \exp\Big(\frac{a_2 - a_5}{a_2 + a_5 + 2a_8 t}\Big), \tag{E115}$$

$$= \frac{(4a_8^2 t + B)r}{(a_2 + a_5 + 2a_8 t)^2}. \tag{E116}$$

By comparison with the anzatz

$$u(t,r) = \frac{\dot{R}}{R}r = \frac{(4a_8^2 t + B)r}{(a_2 + a_5 + 2a_8 t)^2}, \tag{E117}$$

we obtain

$$\frac{\dot{R}}{R} = \frac{4a_8^2 t + B}{(a_2 + a_5 + 2a_8 t)^2}, \tag{E118}$$

and therefore

$$R(t) = A(a_2 + a_5 + 2a_8 t)\exp\Big(\frac{a_2 + a_5 - B/2a_8}{a_2 + a_5 + 2a_8 t}\Big). \tag{E119}$$

Without loss of generality, again simplification is made by setting $A = 1$.

Proceeding to substitute the linear velocity assumption into Equations (99)–(101) generates the following reduced system

$$\hat{p}'\xi(B - 4a_8 a_2) + \hat{p}(4a_8(2a_5 + a_6) + B(k+1)) = 0, \tag{E120}$$

$$\hat{p}' + \hat{\rho}\xi(B(B - 4a_8 a_2 - 4a_8 a_5) + 4a_8^2(a_2 + a_5)^2) = 0, \tag{E121}$$

$$\hat{\rho}'\xi(B - 4a_8 a_2) + \hat{\rho}(4a_8 a_6 + B(k+3)) = 0. \tag{E122}$$

The structure of this system is again very similar to that encountered in Section E3 and thus we proceed in an analogous fashion.

a. Solution E.1: $B \neq 4a_8 a_2$, exponent $\neq -1$

Assuming $B \neq 4a_8 a_2$, Equations (E120) and (E122) are rearranged to give

$$\frac{\mathrm{d}\hat{p}}{\hat{p}} = \frac{4a_8(2a_5 + a_6) + B(k+1)}{\xi(4a_8 a_2 - B)}\mathrm{d}\xi, \tag{E123}$$

$$\frac{\mathrm{d}\hat{\rho}}{\hat{\rho}} = \frac{4a_8 a_6 + B(k+3)}{\xi(4a_8 a_2 - B)}\mathrm{d}\xi, \tag{E124}$$

which have solutions

$$\hat{p} = c_1 \xi^{\frac{4a_8(2a_5 + a_6) + B(k+1)}{4a_8 a_2 - B}}, \tag{E125}$$

$$\hat{\rho} = c_2 \xi^{\frac{4a_8 a_6 + B(k+3)}{4a_8 a_2 - B}}, \tag{E126}$$

where c_1 and c_2 are constants of integration.

Integration of Equation (E121) with respect to ξ yields

$$(B(4a_8(a_2 + a_5) - B) - 4a_8^2(a_2 + a_5)) \int \hat{\rho}\xi \mathrm{d}\xi, \tag{E127}$$

which upon substitution for $\hat{\rho}$ using Equation (E125) gives

$$\hat{p} = c_1(B(4a_8(a_2 + a_5) - B) - 4a_8^2(a_2 + a_5)^2) \int \xi^{\frac{4a_8(2a_5 + a_6 + a_2) + Bk}{4a_8 a_2 - B}}. \tag{E128}$$

As before, assuming the exponent on ξ is not equal to -1, which is equivalent to

$$4a_8(2(a_2 + a_5) + a_6) + B(k-1) \neq 0, \tag{E129}$$

Equation (E128) becomes

$$\hat{p} = \frac{c_1(B(4a_8(a_2 + a_5) - B) - 4a_8^2(a_2 + a_5)^2)}{4a_8(2(a_2 + a_5) + a_6) + B(k-1)} \times (4a_8 a_2 - B)\xi^{\frac{4a_8(2(a_2 + a_5) + a_6) + B(k-1)}{4a_8 a_2 - B}} + c_3, \tag{E130}$$

where c_3 is the integration constant.

Requiring self-consistency in terms of powers of ξ between Equations (E126) and (E130) generates a constraint between the exponents

$$4a_8a_6 + B(k+3) = 4a_8(2(a_2+a_5)+a_6) + B(k-1), \quad \text{(E131)}$$

from which it is inferred

$$B = 2a_8(a_2+a_5). \quad \text{(E132)}$$

Back substitution of this result into Equations (E126) and (E130) yields

$$\hat{p} = c_2\xi^{\frac{2a_6+(a_2+a_5)(k+3)}{a_2-a_5}} \text{ and } \hat{p} = c_3. \quad \text{(E133)}$$

The constraint given by Equation (E129) also becomes

$$2a_6 + (a_2+a_5)(k+3) \neq 0. \quad \text{(E134)}$$

From this it is concluded that the exponent of ξ in Equation (E133) is non-zero and thus self-consistency between the two expressions for $\hat{p}$ is only achieved provided

$$c_2 = c_3 = 0, \quad \text{(E135)}$$

and by extension

$$\hat{p} = 0. \quad \text{(E136)}$$

Furthermore, from Equation (E132), the constraint $B \neq 4a_8a_2$ becomes

$$a_2 \neq a_5, \quad \text{(E137)}$$

and $R(t)$ given by Equation (E119) simplifies to

$$R(t) = a_2 + a_5 + 2a_8t. \quad \text{(E138)}$$

Combining these results with Equation (E117)

$$u(t,r) = \frac{2a_8r}{a_2+a_5+a_8t}, \quad \text{(E139)}$$

and upon substitution of the results into Equations (E125) and (E126)

$$\hat{p} = 0, \qquad \hat{\rho} = c_1\xi^{\frac{4a_5+2a_6+(a_2+a_5)(k+1)}{a_2-a_5}}. \quad \text{(E140)}$$

Finally, re-arranging Equations (98b) and (98c) for the pressure and density profiles gives

$$p(t,r) = \frac{\hat{p}(\xi)}{R(t)^{k+3}}\exp\left[\frac{-2a_6-(a_2+a_5)(k+3)}{R(t)}\right], \quad \text{(E141)}$$

$$\rho(r,t) = \frac{\hat{\rho}(\xi)}{R(t)^{k+1}}\exp\left[\frac{-2(a_6+2a_5)-(a_2+a_5)(k+1)}{R(t)}\right], \quad \text{(E142)}$$

where $R(t)$ is given by Equation (E138). Similarly, the independent invariant ξ given in Equation (97) becomes

$$\xi = \frac{r}{R(t)}\exp\left[\frac{a_2-a_5}{R(t)}\right]. \quad \text{(E143)}$$

Upon substitution of Equation (E140) for $\hat{p}$ and $\hat{\rho}$ the exponential argument in the density expression equates to zero and the solutions are

$$p(t,r) = 0, \quad \text{(E144)}$$

$$\rho(t,r) = c_1\frac{r^{\alpha_1}}{R(t)^{\alpha_2}}, \quad \text{(E145)}$$

where

$$\alpha_1 = \frac{2(2a_5+a_6)+(a_2+a_5)(k+1)}{a_2-a_5}, \quad \text{(E146)}$$

$$\alpha_2 = \frac{2(2a_5+a_6+a_2(k+1))}{a_2-a_5}. \quad \text{(E147)}$$

This is presented as solution E.1 in Section VI D.

b. Solution E.2: $B \neq 4a_8a_2$, exponent = −1

Returning to Equation (E128) and proceeding under the assumption that the exponent of ξ within the integral is equal to -1, which is equivalent to

$$4a_8(2(a_2+a_5)+a_6) + B(k-1) = 0, \quad \text{(E148)}$$

the equation becomes

$$\hat{p} = c_1(B(4a_8(a_2+a_5)-B) - 4a_8^2(a_2+a_5)^2)\int\frac{1}{\xi}\mathrm{d}\xi, \quad \text{(E149)}$$

which has solution

$$\hat{p} = \ln\xi^{c_1(B(4a_8(a_2+a_5)-B)-4a_8^2(a_2+a_5)^2)} + c_3, \quad \text{(E150)}$$

where c_3 is the constant of integration. Requiring self-consistency between this result and Equation (E126) requires the exponents on the ξ terms appearing to be zero and therefore

$$c_1(B(4a_8(a_2+a_5)-B) - 4a_8^2(a_2+a_5)^2) = 0, \quad \text{(E151a)}$$

$$4a_8a_6 + B(k+3) = 0, \quad \text{(E151b)}$$

which further implies

$$c_2 = c_3. \quad \text{(E152)}$$

Equating the final constraint of Equation (E151) and Equation (E148) since they are both zero results in

$$4a_8a_6 + B(k+3) = 4a_8(2(a_2+a_5)+a_6) + B(k-1), \quad \text{(E153)}$$

from which it is determined

$$B = 2a_8(a_2+a_5). \quad \text{(E154)}$$

The previous expression for B results in a simplified $R(t)$ function via substitution into Equation (E119)

$$R(t) = a_2 + a_5 + 2a_8t, \quad \text{(E155)}$$

and the constraint $B \neq 4a_8a_2$ reduces once again to $a_2 \neq a_5$. Since Equation (E155) is equivalent to that obtained in Equation (E138), the velocity profile of Equation (E139) is obtained. The $\hat{p}$ and $\hat{\rho}$ solutions for this case stem from Equations (E125) and (E125) along with Equation (E154) and

$$a_6 = -\frac{(a_2+a_5)(k+3)}{2}. \quad \text{(E156)}$$

The results are

$$\hat{p} = c_2, \tag{E157}$$

$$\hat{\rho} = \frac{c_1}{\xi^2}, \tag{E158}$$

where c_1 and c_2 are arbitrary constants. Finally, substituting these results into Equations (E141) and (E142) gives

$$p(t,r) = \frac{c_2}{R(t)^{k+3}}, \tag{E159}$$

$$\rho(t,r) = \frac{c_1}{r^2 R(t)^{k-1}} \tag{E160}$$

since again the arguments of the exponentials in both terms equate to zero.

c. Solution E.3: $B = 4a_8a_2$

The final solution is determined by setting $B = 4a_8a_2$. By doing so the reduced system of Equations (E120)–(E122) undergoes further simplification to

$$2a_5 + a_6 + a_2(k+1) = 0, \tag{E161}$$

$$\hat{p}' + 4a_8^2\hat{\rho}\xi(a_2^2 + a_5^2 - 2a_2a_5) = 0, \tag{E162}$$

$$a_6 + a_2(k+3) = 0. \tag{E163}$$

From Equations (E161) and (E163) it is inferred

$$a_6 = -2a_5 - a_2(k+1) = -2a_2(k+3), \tag{E164}$$

and therefore

$$a_2 = a_5. \tag{E165}$$

Back substitution into Equation (E162) yields

$$\hat{p}' = 0, \tag{E166}$$

and therefore

$$\hat{p} = c_1, \tag{E167}$$

where c_1 is a constant.

By setting $B = 4a_8a_2$ and $a_2 = a_5$, the $R(t)$ function defined in Equation (E119) becomes

$$R(t) = 2(a_2 + a_8 t), \tag{E168}$$

and the independent invariant ξ defined by Equation (97) becomes

$$\xi = \frac{r}{R(t)}. \tag{E169}$$

Using Equation (E168) to determine the corresponding linear velocity profile

$$u(t,r) = \frac{a_8 r}{a_2 + a_8 t}. \tag{E170}$$

Furthermore, the pressure and density profiles can once again be obtained from Equations (98a) and (98c) and re-arranged to the forms previously presented in Equations (E141) and (E142). Upon substitution of $a_6 = -a_2(k+3)$ and $a_5 = a_2$ these expressions become

$$p(t,r) = \frac{\hat{p}}{R(t)^{k+3}}, \tag{E171}$$

$$\rho(t,r) = \frac{\hat{\rho}}{R(t)^{k+1}}, \tag{E172}$$

where $R(t)$ is given by Equation (E168). Finally, substituting $\hat{p} = c_1$ yields the solution E.3 presented in Section VI D.

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system of equations may be re-expressed. The number of new dimensionless variables is equal to the difference between the total number of dimensional variables in the system and the number of variables with independent dimensions. Since the quantity of new variables is fewer, a reduction of the system complexity is achieved and solutions are more readily accessible. See Ref. 25 for an excellent presentation of the fundamentals of dimensional analysis and application of scaling groups to construct self-similar solutions.
[28]The binary operation of a group must satisfy the closure property, associativity and the existence of both an identity element and an inverse element for every group element.
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