

# PT-symmetry entails pseudo-Hermiticity regardless of diagonalizability

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We prove that in finite dimensions, a Parity-Time (PT)-symmetric Hamiltonian is necessarily pseudo-Hermitian regardless of whether it is diagonalizable or not. This result is different from Mostafazadeh's, which requires the Hamiltonian to be diagonalizable. PT-symmetry breaking often occurs at exceptional points where the Hamiltonian is not diagonalizable. Our result implies that PT-symmetry breaking is equivalent to the onset of instabilities of pseudo-Hermitian systems, which was systematically studied by Krein et al. in 1950s. In particular, we show that the mechanism of PT-symmetry breaking is the resonance between two eigenmodes with opposite signs of actions.

In quantum physics, observables are assumed to be Hermitian operators. Bender and collaborators<sup>1-3</sup> proposed to relax this fundamental assumption and considered Parity-Time (PT)-symmetric operators. The concept and techniques of PT-symmetry have been applied to many branches of physics<sup>4-19</sup>. Although PT-symmetry was first studied in infinite-dimensional systems, many of the current applications are in finite dimensions.

When discussing PT-symmetry, a related property, pseudo-Hermiticity, is often considered. Pseudo-Hermitian operators were introduced by Dirac and Pauli as a class of non-Hermitian operators<sup>20-22</sup>. Investigating the relation between PT-symmetry and pseudo-Hermiticity may reveal important mathematical and physical structures of non-Hermitian operators. In this regard, Mostafazadeh proved that a diagonalizable PT-symmetric Hamiltonian is pseudo-Hermitian<sup>23-25</sup>.

In this paper, we prove that in finite dimensions, a PT-symmetric Hamiltonian is necessarily pseudo-Hermitian regardless of whether it is diagonalizable or not. We first prove that for a Hamiltonian  $H$ , a sufficient and necessary condition of pseudo-Hermiticity is that  $H$  is similar to its Hermitian conjugate  $\bar{H}$  (Theorem 2). Then because a PT-symmetric Hamiltonian is similar to its Hermitian conjugate, it is pseudo-Hermitian (Theorem 3). We emphasize that this result is different from Mostafazadeh's<sup>23-25</sup>. The difference is significant, because our result relaxes the diagonalizability requirement. As we know, most of the interesting PT-symmetry breaking happens at exceptional points where the Hamiltonian is not diagonalizable. Our result is applicable when studying these effects.

As such an application, we show that a theoretical description of PT-symmetry breaking, which is arguably the most important topic in PT-symmetry physics, can be built upon the mathematical work on the instabilities of pseudo-Hermitian systems developed by Krein, Gel'fand and Lidskii<sup>26-29</sup> in 1950s. For a pseudo-Hermitian Hamiltonian, its eigenvalues are symmetric with respect to the real axis. As the system parameters vary, a necessary and sufficient condition for the onset of instability is that two eigenmodes with opposite

signs of actions collide, which is the so-called Krein collision. These results can be directly applied to PT-symmetric Hamiltonians due to Theorem 3, implying that PT-symmetry breaking occurs when and only when different kinds of eigenmodes with opposite signs of actions collide. Note that when PT-symmetry breaking happens, the Hamiltonian can be either diagonalizable or non-diagonalizable. But PT-symmetry is often broken at the exceptional points where the Hamiltonian is not diagonalizable. As an example, we show that the governing equations of the classical Kelvin-Helmholtz instability, which was proven to be PT-symmetric<sup>19</sup>, is pseudo-Hermitian, and the Kelvin-Helmholtz instability is the result of PT-symmetry breaking triggered by the Krein collision.

We start from the definitions of PT-symmetry, pseudo-Hermiticity, and another related concept, i.e., G-Hamiltonian matrix. Consider the linear system specified by a Hamiltonian  $H$ ,

$$\dot{\mathbf{x}} = -iH\mathbf{x} = A\mathbf{x}, \quad (1)$$

where  $A$  is defined to be a shorthand notation of  $-iH$ .

The Hamiltonian  $H$  in Eq. (1) is called PT-symmetric<sup>1-3</sup> if it commutes with the parity-time operator  $PT$ , i.e.,

$$PTH - HPT = 0. \quad (2)$$

Here  $P$  is a linear operator satisfying  $P^2 = I$  and  $T$  is the complex conjugate operator. In the present study, we will focus on finite-dimensional systems, for which  $H$ ,  $A$  and  $P$  can be represented by matrices, and Eq. (2) is equivalent to

$$P\bar{H} - HP = 0, \quad (3)$$

where  $\bar{H}$  denotes the complex conjugates of  $H$ .

The Hamiltonian  $H$  in Eq. (1) is called pseudo-Hermitian<sup>20-22</sup> if there exists a non-singular Hermitian matrix  $G$  such that

$$H^\dagger G - GH = 0, \quad (4)$$

where  $H^\dagger$  is the conjugate transpose of the matrix  $H$ .

The matrix  $A = -iH$  in Eq. (1) is called G-Hamiltonian<sup>26-29</sup> if there exist a non-singular Hermitian matrix  $G$  and a Hermitian matrix  $S$  such that

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$$A = iG^{-1}S. \quad (5)$$

The concept of pseudo-Hermiticity was first introduced by Dirac and Pauli in 1940s<sup>20–22</sup>. G-Hamiltonian matrix was defined by Krein et al. in 1950s<sup>26–29</sup> in the study of linear dynamical systems satisfying the G-Hamiltonian condition (5). For finite-dimensional systems, these two concepts are equivalent.

**Theorem 1.** *For a finite-dimensional system  $\dot{\mathbf{x}} = -iH\mathbf{x} = A\mathbf{x}$ ,  $H$  is pseudo-Hermitian if and only if  $A$  is a G-Hamiltonian matrix.*

The proof of Theorem 1 is straightforward according to the definitions of pseudo-Hermitian and G-Hamiltonian matrices. But we give this fact the status of a theorem to highlight the exact equivalence between these two concepts independently defined by physicists and mathematicians. We will mostly use the terminology of pseudo-Hermiticity exclusively hereafter.

Now we establish a necessary and sufficient condition for pseudo-Hermiticity.

**Theorem 2.** *For a matrix  $H \in C^{m \times n}$ , it is pseudo-Hermitian if and only if it is similar to its complex conjugate  $\bar{H}$ .*

*Proof.* Necessity is easy to prove. If a Hamiltonian is pseudo-Hermitian, i.e., satisfying Eq. (4), then  $H = G^{-1}H^\dagger G$ . Thus matrix  $H$  is similar to  $H^\dagger$ . We know that  $H^\dagger$  can also be written as  $H^\dagger = \bar{H}^T$ . Since every matrix is similar to its own transpose,  $H^\dagger$  is similar to  $\bar{H}$ . In conclusion,  $H$  is similar to  $\bar{H}$ .

We prove the sufficiency by constructing the Hermitian matrix  $G$ . Matrix  $H$  can be written as

$$H = Q^{-1}JQ, \quad (6)$$

where  $J$  is its Jordan canonical form and  $Q$  is a reversible matrix. The Jordan canonical form consists of several Jordan blocks of the form

$$J(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}_{m \times m}. \quad (7)$$

When  $m = 1$ , the Jordan block  $J(\lambda)$  is reduced to  $\lambda$ . If  $H$  is similar to  $\bar{H}$ , then its eigenvalues are symmetric with respect to the real axis, and they are either real numbers or complex number pairs of the form  $\lambda = a + bi$  and  $\bar{\lambda} = a - bi$ , where  $a$  and  $b$  are real numbers. Accordingly, there are two kinds of matrix blocks

$$F_1 = J(a) = \begin{pmatrix} a & 1 & & \\ & \ddots & \ddots & \\ & & a & 1 \\ & & & a \end{pmatrix}_{m \times m} \quad (8)$$

$$F_2 = \begin{pmatrix} J(a+bi) & 0 \\ 0 & J(a-bi) \end{pmatrix}_{2l \times 2l}.$$

The Jordan matrix can now be expressed as  $J = \text{Diag}(M_1, M_2, \dots, M_k)$ , where  $M_j$  is in the form of  $F_1$  or  $F_2$ . In the following, we prove that both types of matrix blocks are pseudo-Hermitian. For both types of matrix blocks, we find that Hermitian matrix

$$G'_j = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & 1 & 0 \\ 0 & \ddots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} \quad (9)$$

satisfies the condition  $M_j^\dagger G'_j - G'_j M_j = 0$ , i.e.,  $M_j$  is pseudo-Hermitian. Next we construct a larger Hermitian matrix  $G'$  using  $G'_j$  as follows,

$$G' = \text{Diag}(G'_1, G'_2, \dots, G'_k), \quad (10)$$

and the Jordan canonical form of  $H$  satisfies  $J^\dagger G' - G' J = 0$ . Let

$$G = Q^\dagger G' Q, \quad (11)$$

and we obtain

$$\begin{aligned} H^\dagger G - GH &= (Q^{-1}JQ)^\dagger G - GQ^{-1}JQ \\ &= Q^\dagger J^\dagger Q^{-\dagger} Q^\dagger G' Q - Q^\dagger G' Q Q^{-1} JQ \\ &= Q^\dagger (J^\dagger G' - G' J) Q \\ &= 0, \end{aligned} \quad (12)$$

where  $G$  is a non-singular Hermitian matrix. This completes the proof that  $H$  is pseudo-Hermitian.  $\square$

The theorem is proved by constructing a non-singular Hermitian matrix  $G$  for the similarity transformation between  $H$  and  $\bar{H}$ . But  $G$  is not unique. For a given  $H$ , we can find more than one non-singular Hermitian matrices  $G$ . In practice, one does not need to follow the construction procedure given in Theorem 2 to find  $G$ . It is often found by direct calculation. According to Theorem 2, when  $H$  is a real square matrix, it is a pseudo-Hermitian matrix since it is similar to itself.

**Theorem 3.** *For finite-dimensional systems, a PT-symmetric Hamiltonian  $H$  is necessarily pseudo-Hermitian.*

*Proof.* By the definition of PT-symmetry, i.e., Eq. (3),  $H$  is similar to  $\bar{H}$ . Thus, according to Theorem 2, it is pseudo-Hermitian.  $\square$

Theorem 3 is the main theorem in this paper, and we would like to emphasize again that it holds regardless of whether  $H$  is diagonalizable or not. We note that Mostafazadeh's result<sup>23–25</sup>, which states that diagonalizable PT-symmetric Hamiltonians are pseudo-Hermitian, is different from Theorem 3.

As an application of Theorem 3, we investigate the mechanism of PT-symmetry breaking in the framework of pseudo-Hermiticity. Theorem 3 implies that PT-breaking is equivalent to the onset of instabilities of pseudo-Hermitian matrices, which was systematically studied by Krein, Gel'fand and

Lidskii<sup>26–29</sup> in 1950s. Specifically, the instability analysis of pseudo-Hermitian matrices gives a comprehensive description on how real eigenvalues of  $H$  evolve into conjugate pairs of complex eigenvalues as the system parameters vary. Here we briefly summarize the main results. (i) The eigenvalues of a pseudo-Hermitian Hamiltonian  $H$  are symmetric with respect to real axis. They are either real numbers or complex conjugate pairs. (ii) Let  $\psi$  be an eigenmode (or eigenvector) of  $H$ , the indefinite inner product associated with  $G$  can be defined as<sup>26–29</sup>

$$\langle \psi, \psi \rangle = \psi^\dagger G \psi.$$

It was found that the physical meaning of the indefinite inner product is action<sup>30</sup>, which is partially indicated by the fact that its dimension is [energy]×[time]. We will also refer to the indefinite inner product as action, especially in the context of physics. (ii) The eigenvalues of  $H$  can be classified according to the indefinite inner products of the corresponding eigenvectors. An  $r$ -fold real eigenvalue  $\lambda$  of  $H$  with its eigen-subspace  $V_\lambda$  is called the first kind if all eigenmodes of  $\lambda$  have positive actions, i.e.,  $\langle \mathbf{y}, \mathbf{y} \rangle > 0$  for any  $\mathbf{y} \neq 0$  in  $V_\lambda$ . It is called the second kind if all eigenmodes of  $\lambda$  have negative actions. If there exists a zero-action eigenmode, then  $\lambda$  is called an eigenvalue of mixed kind<sup>29</sup>. If an eigenvalue

is the first kind or the second kind, it's called definite. (iii) The number of each kind of eigenvalues is determined by the Hermitian matrix  $G$ . Let  $p$  be the number of positive eigenvalues and  $q$  be the number of negative eigenvalues of the matrix  $G$ , then any pseudo-Hermitian Hamiltonian has  $p$  eigenvalues of first kind and  $q$  eigenvalues of second kind (counting multiplicity). (iv) The finite-dimensional pseudo-Hermitian Hamiltonian is strongly stable if and only if all of its eigenvalues lie on the real axis and are definite. Here, a pseudo-Hermitian Hamiltonian is strongly stable means that eigenvalues of all pseudo-Hermitian Hamiltonians in an open neighborhood of the parameter space lie on the real axis. As a result, a pseudo-Hermitian Hamiltonian becomes unstable when and only when a positive-action mode resonates with a negative-action mode. This is a process known as the Krein collision.

Applying these results to PT-symmetric Hamiltonians, we see that PT-symmetry breaking can happen only when a repeated eigenvalue appears as a result of two eigenmodes resonate. However, if two eigenmodes with the same sign of action resonate, then there is no PT-symmetry breaking. PT-symmetry breaking is triggered only when a positive-action mode resonates with a negative-action mode.

Let's look at an example. The governing equations for the classical Kelvin-Helmholtz instability in fluid dynamics was shown to be a complex system with the following PT-symmetric Hamiltonian<sup>19</sup>

$$H = \left( \begin{array}{c|c} \frac{-k(-u_{10}\rho_{10} - 2u_{20}\rho_{20} + u_{10}\rho_{20})}{\rho_{10} + \rho_{20}} & \frac{-i|k|(u_{10} - u_{20})^2\rho_{20} + ig(\rho_{20} - \rho_{10})}{\rho_{10} + \rho_{20}} \\ \hline 0 & \frac{-i|k|}{ku_{10}} \end{array} \right). \quad (13)$$

According to Theorem 3, it is also a pseudo-Hermitian Hamiltonian satisfying Eq. (4). With straightforward calculation, we find the following Hermitian matrix

$$G = \left( \begin{array}{c|c} -|k| & 0 \\ \hline 0 & \frac{|k|(u_{10} - u_{20})^2\rho_{20} - g(\rho_{20} - \rho_{10})}{\rho_{10} + \rho_{20}} \end{array} \right) \quad (14)$$

such that  $H^\dagger G - GH = 0$ . The eigenvalues of  $H$  are

$$\begin{aligned} a_1 &= \frac{k(\rho_{10}u_{10} + \rho_{20}u_{20}) - \sqrt{\Delta}}{\rho_{10} + \rho_{20}}, \\ a_2 &= \frac{k(\rho_{10}u_{10} + \rho_{20}u_{20}) + \sqrt{\Delta}}{\rho_{10} + \rho_{20}}, \end{aligned} \quad (15)$$

and the corresponding eigenvectors are

$$\begin{aligned} \phi_1 &= \left( \frac{-ik\rho_{20}(u_{10} - u_{20}) + \sqrt{\Delta}}{|k|(\rho_{10} + \rho_{20})}, 1 \right), \\ \phi_2 &= \left( \frac{-ik\rho_{20}(u_{10} - u_{20}) - \sqrt{\Delta}}{|k|(\rho_{10} + \rho_{20})}, 1 \right), \end{aligned} \quad (16)$$

where  $\Delta = -|k|g(\rho_{10}^2 - \rho_{20}^2) - k^2\rho_{10}\rho_{20}(u_{10} - u_{20})^2$ . The signs of actions for the eigenvalues of  $H$  can be determined

by the Hermitian matrix  $G$ . When

$$\tau \equiv \frac{|k|(u_{10} - u_{20})^2\rho_{20} - g(\rho_{20} - \rho_{10})}{\rho_{10} + \rho_{20}} < 0,$$

both eigenvalues of  $G$  are negative and the PT-symmetric Hamiltonian  $H = iA$  is stable. When  $\tau > 0$ , one of the eigenvalues of  $G$  is positive and the other one is negative. Thus one eigenvalue of  $H$  have a positive action and the other one has a negative action, and the resonance between them will result in PT-symmetry breaking. Let's use a numerically calculated examples to observe the breaking of PT-symmetry. We plot the process in Fig. 1 by fixing  $u_{10} = 1$ ,  $\rho_{10} = 2$ ,  $\rho_{20} = 3$ ,  $k = 1$  and  $g = 3$ , and varying  $u_{20}$  from 2.3 to 2.7. When  $u_{20} = 2.3$ , the eigenvalues of  $H$  are all real numbers, one of which has a positive action (marked by  $M_+$ ) and the other one has a negative action (marked by  $M_-$ ) in Fig. 1(a). Fig. 1(b) shows that as  $u_{20}$  increases,  $M_+$  and  $M_-$  move towards each other. Increasing  $u_{20}$  to  $\sqrt{5}/2 + 1 = 2.58114$ , eigenmodes  $M_+$  and  $M_-$  collide on the real axis, as shown in Fig. 1(c). Because the resonance is between modes with different sign of actions, the eigenvalues of  $H$  split into a pair symmetric with respect to the real axis and the PT-symmetry is broken. Fig. 1(d) shows that the two eigenvalues of  $H$  move out of real axis when  $u_{20} = 2.7$ .

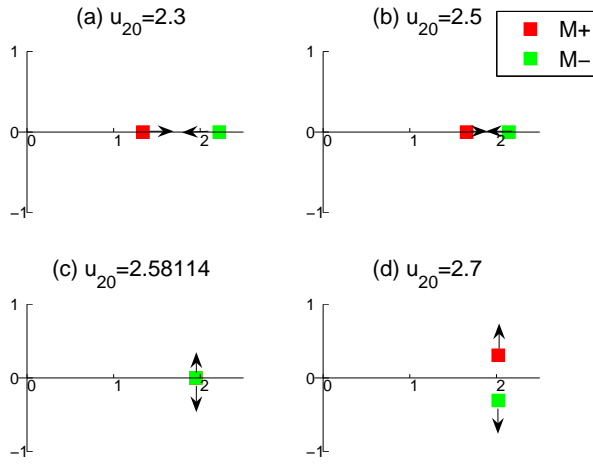


FIG. 1. PT-symmetry breaking occurs when a positive-action eigenmode (red) resonates with a negative-action eigenmode (green).

In summary, we have proved that for finite-dimensional systems, a PT-symmetric Hamiltonian is necessarily pseudo-Hermitian regardless of whether it is diagonalizable or not. This result is stronger than Mostafazadeh's<sup>23–25</sup>, which requires that the Hamiltonian is diagonalizable. As we know, PT-symmetry breaking often happens at exceptional points where the Hamiltonian is not diagonalizable. The fact that a PT-symmetric Hamiltonian is always pseudo-Hermitian implies that PT-symmetry breaking is equivalent to the onset of instabilities of pseudo-Hermitian matrices. Therefore, the systematic results by Krein et al. on how a pseudo-Hermitian system becomes unstable<sup>26–29</sup> can be directly applied to the process of PT-symmetry breaking. In particular, we showed that PT-symmetry breaking is triggered when and only when two eigenmodes with different signs of actions resonate. This process is illustrated using the example of the classical Kelvin-Helmholtz instability.

We finish our discussion with an observation. Theorem 3 asserts that a PT-symmetric matrix is necessarily pseudo-Hermitian. One wonders whether the reverse is true. If the  $P$  operator in the definition of PT-symmetry (2) is not required to be a parity transformation, i.e.,  $P^2 = I$ , then a pseudo-Hermitian matrix is also PT-symmetric according to Theorem 2. In this case, PT-symmetry and pseudo-Hermitian are equivalent, at least in finite dimensions. We note that essentially all the spectrum properties associated with PT-symmetry are still valid when the requirement of  $P^2 = I$  is removed.

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