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Diamagnetic field states in cosmological plasmas

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Using a generally covariant Electro-Vortic (magnetofluid) formalism for relativistic plasmas, the dynamical evolution of a generalized vorticity (a combination of the magnetic and kinematic parts) is studied in a cosmological context. We derive macroscopic vorticity and magnetic field structures that can emerge in spatial equilibrium configurations of the relativistic plasma. These fields, however, evolve in time. These magnetic and velocity fields fields, self-consistently sustained in a plasma with arbitrary thermodynamics, constitute a diamagnetic state in the expanding Universe. In particular, we explore a special class of magnetic/velocity field structures supported by a plasma in which the generalized vorticity vanishes. We derive a highly interesting characteristic of such "superconductor–like" fields in a cosmological plasmas in the radiation–era in early Universe. In that case, the fields grow proportional to the scale factor, establishing a deep connection between the expanding universe and the primordial magnetic fields.

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I. INTRODUCTION

Exploring the interaction of gravitational fields and inhomogeneous plasma thermodynamics as a possible source of primordial magnetic fields has, recently, received considerable attention [1–9]. Much of this work has been carried out within the framework of what has been called a unified magneto–fluid, recently generalized to, Electro-Vortic (EV) formalism [10, 11]. The primary new construct of this formalism is the EV tensor $M^{\mu\nu} = F^{\mu\nu} + (m/q)S^{\mu\nu}$, a weighted sum of the Electromagnetic $F^{\mu\nu}$ and the Vortical $S^{\mu\nu}$ field tensors; the latter representing both the kinematic and thermal content of the relativistic hot fluid. In the EV formalism, the fluid dynamics reduces to a simple Helmholtz vortical form in terms of new composite variables, the most familiar being the so called generalized vorticity $\boldsymbol{\Omega}$ (such that $\Omega^i = \epsilon^{ijk}M_{jk}$) that has both magnetic and thermal-kinetic parts. The most important message is that in the considerably complicated dynamics of a hot relativistic fluid, $\boldsymbol{\Omega}$ plays the same role as the magnetic field $\boldsymbol{B}$ does in the much simpler magnetohydrodynamics (MHD).

The earlier special relativistic theory [10] was extended to explore general relativistic effects in Refs. [1–6]. It was shown that a combination of gravity modified Lorentz factor of the fluid element and the spatial variation of plasma thermodynamics, leads to an additional (to the special relativistic mechanism [12, 13]) source that creates a vorticity seed out of a state with no initial vorticity. This mechanism, true for arbitrary thermodynamics, can be viewed as a grand generalization of the MHD or Biermann battery mechanism which operates only for the special class of baroclinic thermodynamics.

As in the conventional scenario, this relativistic seed vorticity (and therefore a magnetic field) can then, be amplified, for example, by a dynamo mechanism. These calculations have shown that that a plasma around a Kerr black hole can produce a larger magnetic field seed than a corresponding Schwarzschild system [4].

In the cosmological context, plasma dynamics have been thoroughly studied [14]. Several authors have studied the problem of the generation and amplification of magnetic fields through a Biermann battery or similar mechanism showing that large scale magnetic fields can evolve in the early–Universe [15–20]. Also, models for magnetic field generation in Friedmann-Robertson-Walker (FRW) cosmology have been studied in precombination era [21], in the inflationary epoch [22–25], in $f(R)$ gravity [26], in nonlinear electrodynamics [27], or by gravitational waves [28].

In this work, we explore alternative mechanisms that might pertain more generally than the very specific baroclinic Biermann battery; in fact, the Biermann battery may be weak or absent in an ideal fluid plasma. In this case, we investigate magnetic field states that can emerge without invoking any Biermann-like mechanism, i.e, for a general thermodynamical properties, in which the large scale magnetic field be some sort of an equilibrium solution. It is now well established that a relativistic plasma can sustain self–consistent equilibrium configurations, states that represent self–consistent macroscopic structured magnetic and flow fields (see Refs. [11, 29, 30] and references therein). Amongst this set of self–consistent macroscopic fields, there does exist a class endowed with features characteristic of a superconductor [6] where the plasma displays perfect diamagnetism.

In previous generalizations to general relativity, [1–6], the gravitational field was specified as a static background (the space time metric is independent of time).

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Such calculations are generally adequate, for instance, to investigate plasmas in the accretion disks around compact objects. However, a purely static approach, when applied to the universe as a whole, misses out on a fundamental element of cosmology – the expansion of the universe through a time dependent metric.

The aim of this paper, then, is to rework the EV formalism, investigating the primordial magnetic field problem in an expanding universe described by the FRW metric; we will seek self–consistent field solutions (in the spirit of Ref. [11]) that evolve with the universe through an explicit dependence on the scale factor. We are focused on magnetic field states that can develop in plasmas where the Biermann mechanism is not operative. This new type of magnetic field structure emerges for any thermodynamical evolution of the plasma. The particular structure we deal with is accessible if the plasma system (expanding with the universe) behaves as a superconductor fluid. This occurs when the magnetic field is proportional to the fluid plasma vorticity, implying that the plasma dynamics enters in a diamagnetic phase. In particular, in the cosmological context, this solution is only possible when the total generalized vorticity of the plasma system vanishes fulfilling (see below). It is important to be emphasized that these solutions are not consequence of a dynamo mechanism, but they are simply the self–consistent set of fields that can be sustained in a plasma in a diamagnetic state for a given thermodynamics. These are equilibrium states in which the electromagnetic, kinematic and thermal forces have come to a balance in a curved spacetime, which is defined by the equilibrium state solution for the vanishing generalized vorticity. As a result, these magnetic field (having a diamagnetic plasma behavior) are very dependent on the cosmology in which it evolves. Hopefully this will provide further insights on the issue of the origin of primordial magnetic fields [15, 18].

Other formalims have been used to explore the generation of magnetic field in cosmology. For example, the gradient expansion formalism has been used to study kinematic vorticity and magnetic fields in an electron-ion-photon pre-decoupled plasma [8]. The two formalisms are different in that the former uses a unified formulation to construct a conserved (in the absence of the drive) generalized vorticity (combining the magnetic and the kinematic). The latter does not consider relativistic translations of the classical Biermann battery.

With all this in consideration, our results show that the magnetic field strength increases with the scale factor, at least partially compensating the automatic dilution caused by the expanding universe, and thus the magnetic field will dilute at slower rates compared with what is subject to the expansion of the Universe.

The paper is organized as follows. In Sec. II, we review some of the essentials of the the EV formalism. In Sec. III, we develop the 3+1 formalism used to describe the cosmological plasma, and derive the dynamical equation for the generalized vorticity. In Sec. IV, we study time varying spatial self–sustained solutions with their spatial structure resembling that of a perfect diamagnetic plasma state. The results are discussed in Sec. V.

II. UNIFIED PLASMA DYNAMICS IN CURVED SPACETIME

The dynamics of an ideal plasma (a charged fluid), immersed in an electromagnetic field $F_{\mu\nu}$, is contained in the conservation law

$$\nabla_\nu T_i^{\mu\nu} = q_i n_i F^{\mu\nu} U_{\nu i} \,, \tag{1}$$

where U_i^μ (such that $U_i^\mu U_{\mu i} = -1$) is the four velocity of each plasma fluid i, ∇_ν is the covariant derivative for the metric $g_{\mu\nu}$ describing a curved spacetimes ($c = 1$), and

$$T_i^{\mu\nu} = m_i n_i f_i U_i^\mu U_i^\nu + p_i g^{\mu\nu} \,, \tag{2}$$

is the energy-momentum tensor for an ideal plasma [10, 31]; the charge q_i and the mass m_i of each fluid element are scalar invariants. The definition of the energy-momentum (2) involves three thermodynamic scalars: the scalar number density n (the rest frame density), the scalar pressure p, and enthalpy density $h = mnf$, where f is a function of temperature T that in the special case of a relativistic Maxwell distribution becomes $f = K_3(m/T)/K_2(m/T)$ where K_j is the modified Bessel functions of order j (the Boltzmann constant is chosen $k_B = 1$). The system is completed with the continuity for each fluid

$$\nabla_\mu \left(n_i U_i^\mu\right) = 0 \,, \tag{3}$$

and Maxwell equations

$$\nabla_\nu F^{\mu\nu} = 4\pi \sum_i q_i n_i U_i^\mu \,. \tag{4}$$

The global dynamics given by Eqs. (1)-(4) is more conveniently studied in terms of a unified field tensor [1, 10, 31] for each plasma fluid,

$$M_i^{\mu\nu} = F^{\mu\nu} + \frac{m_i}{q_i} S_i^{\mu\nu} \,, \tag{5}$$

in which all kinematic and thermal (through f) aspects of the fluid are now represented by the antisymmetric tensor [1, 10, 31]

$$S^{\mu\nu} = \nabla^\mu \left(f U^\nu\right) - \nabla^\nu \left(f U^\mu\right) \,, \tag{6}$$

for the each corresponding fluid. The resulting equation of motion takes the form (∂^μ is the four derivative)

$$q_i \; U_{\nu i} M_i^{\mu\nu} = -T_i \partial^\mu \sigma_i \,. \tag{7}$$

where σ is the scalar entropy density of the fluid, and it is related to pressure through $nT\partial^\mu\sigma = mn\partial^\mu f - \partial^\mu p$. We can see that due to the antisymmetry of $M_{\mu\nu}$, the fluid description presented here is always isentropic for each plasma component

$$U_{\mu i}\partial^{\mu}\sigma_i = 0\,. \quad (8)$$

In general, the above covariant set of equations can be put in an explicit and intuitive form by performing a $3+1$ decomposition on them [1–4, 32]. This can be achieved when all physical quantities are measured by fiducial observers (FIDOs) [32]. In this case, the spacetime separates into absolute 3–dimensional space and the universal time t. This allows to describe every physical process in 3–dimensional space evolving in time t, measured by the FIDOs. For an irrotational spacetime [32], this is achieved by introducing a normalized timelike vector field $n_\mu = (\alpha, 0, 0, 0)$ and $n^\mu = (-1/\alpha, 0, 0, 0)$, obeying $n^\mu n_\mu = -1$, where α is known as the lapse function (for generalizations see Ref. [3]). Then, the spacetime metric can be written as

$$g_{\mu\nu} = \gamma_{\mu\nu} - n_\mu n_\nu\,, \quad (9)$$

where $\gamma_{\mu\nu}$ is the 3–dimensional spatial metric, with $n^\mu\gamma_{\mu\nu} = 0$.

Appropriate equations in the $3+1$ formalism can be constructed by projecting into timelike and spacelike hypersurfaces; by contracting every tensor with n^μ and $\gamma_{\mu\nu}$.

III. EXPLICIT PLASMA DYNAMICS IN COSMOLOGY

For an explicit formulation of plasma dynamics in the the cosmological background, we introduce the FRW metric. We model a two-fluid quasi-neutral cosmological plasma formed by electrons (e) and positrons (p). Both fluids are considered to have the same density. Since the focus of this calculation is to figure out how the plasma dynamics is affected by the expansion of the Universe, we will restrict ourselves to a spatially flat universe; the corresponding FRW metric is given by the element $ds^2 = -dt^2 + a^2\eta_{jk}dx^j dx^k$ $(j, k = 1, 2, 3)$ [33, 34], where $a = a(t)$ is the time-dependent scale factor of the Universe, and $\eta_{jk} = (1, 1, 1)$ is the 3-metric of the spacelike hypersurfaces of the flat spacetime. To perform the $3+1$ decomposition of the covariant fluid plasma equations (Eq. (7) with the FRW metric) we identify the lapse function $\alpha = 1$ and $\gamma_{\mu\nu} = a^2\eta_{\mu\nu}$, such that $n_\mu = (1, 0, 0, 0)$ and $n^\mu = (-1, 0, 0, 0)$.

In general, for each fluid, we may write the four-velocity measured for the FIDO as (we drop the specie indexes for simplicity)

$$U^\mu = -\Gamma n^\mu + \Gamma\eta^\mu{}_\nu v^\nu\,, \quad (10)$$

where $v^j = dx^j/dt$ corresponds to the spatial j-component of the fluid velocity $\boldsymbol{v}$, and Γ is the Lorentz factor, which is given by

$$n_\mu U^\mu = \Gamma = \left(1 - v^2\right)^{-1/2}\,, \quad (11)$$

by the normalization of the fluid four velocity, where $v^2 = \gamma_{jk}v^j v^k = \boldsymbol{v}\cdot\boldsymbol{v}$. Similarly, using the 3+1 decomposition (10) of the FRW metric, the continuity equation (3) becomes

$$\frac{1}{a^3}\frac{\partial}{\partial t}\left(a^3 n\Gamma\right) + \nabla\cdot(n\Gamma\boldsymbol{v}) = 0\,, \quad (12)$$

for each fluid. Here, ∇ is the flat spatial gradient operator.

Now, neglecting the plasma back-reaction on spacetime, one may write down the decomposition of the Maxwell and plasma field equations. The $3+1$ decomposed [35–45] electric and magnetic fields are obtained as

$$E^\mu = n_\nu F^{\nu\mu}\,, \qquad B^\mu = \frac{1}{2}n_\rho\epsilon^{\rho\mu\sigma\tau}F_{\sigma\tau}\,, \quad (13)$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is the totally antisymmetric tensor. Notice that the electric E^μ and the magnetic B^μ fields are spacelike tensors ($n_\mu E^\mu = 0$ and $n_\mu B^\mu = 0$), implying that $E^0 = 0 = B^0$ for a cosmological background. With the previous definitions we decompose the electromagnetic field tensor as

$$F^{\mu\nu} = E^\mu n^\nu - E^\nu n^\mu - \epsilon^{\mu\nu\rho\sigma}B_\rho n_\sigma\,. \quad (14)$$

This allow us to explicitly write the Maxwell equations (4) in terms of electric and magnetic fields for a cosmological background by projecting them into timelike and spacelike hypersurfaces. Substituting (14) into (4), and projecting it onto n_μ we find [32, 35, 36]

$$\nabla\cdot\boldsymbol{E} = 4\pi\sum_i q_i n_i\Gamma_i = 0\,, \quad (15)$$

as quasineutrality is assumed. As the two plasma fluids are oppositely charged with $q_p = e = -q_e$ (with positron charge e), we find that $\Gamma_e = \Gamma_p = \Gamma$ implying that both fluids have equal magnitude of velocity.

Also, projecting Eq. (4) onto spacelike hypersurfaces yields

$$\frac{1}{a}\nabla\times\boldsymbol{B} = 4\pi en\Gamma(\boldsymbol{v}_p - \boldsymbol{v}_e) + \frac{1}{a^3}\frac{\partial\left(a^3\boldsymbol{E}\right)}{\partial t}\,. \quad (16)$$

Similarly, one can define the dual electromagnetic tensor

$$F^{*\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\tau}F_{\rho\tau} = B^\mu n^\nu - B^\nu n^\mu - \epsilon^{\mu\nu\rho\tau}E_\rho n_\tau\,, \quad (17)$$

that satisfies $F^{*\mu\nu}{}_{;\nu} = 0$ by its antisymmetry. When projected onto n_μ, we find the timelike component [32, 35, 36]

$$\nabla\cdot\boldsymbol{B} = 0\,, \quad (18)$$

whereas the spacelike projection has the vectorial equivalent [32, 35, 36]

$$\frac{1}{a^3}\frac{\partial\left(a^3\boldsymbol{B}\right)}{\partial t} = -\frac{1}{a}\nabla\times\boldsymbol{E}\,. \quad (19)$$

Eqs. (15), (16), (18) and (19) correspond to the Maxwell equations for a cosmological plasma in an expanding universe. Furthermore, Eqs. (16) and (19) reflect the expansion of the Universe, through the extrinsic–curvature tensor $3\partial_t a/a$, proportional to the Hubble parameter. Also notice that Weyl invariance is explicit in Maxwell equations when the conformal time $\int dt/a$ is introduced, and Eqs. (15)–(19) are written in terms of comoving electric $a^3\boldsymbol{E}$ and magnetic $a^3\boldsymbol{B}$ fields.

Following with our description for plasma dynamics, a similar decomposition can be performed on the antisymmetric unified tensor (5) yielding the generalized electric ξ^μ and magnetic Ω^μ fields for each fluid

$$\xi^\mu = n_\nu M^{\nu\mu}\,, \qquad \Omega^\mu = \frac{1}{2} n_\rho \epsilon^{\rho\mu\sigma\tau} M_{\sigma\tau}\,, \tag{20}$$

that are both spacelike ($n_\mu \xi^\mu = 0$ and $n_\mu \Omega^\mu = 0$). Equivalently, $M^{\mu\nu}$ may be written as

$$M^{\mu\nu} = \xi^\mu n^\nu - \xi^\nu n^\mu - \epsilon^{\mu\nu\rho\sigma}\Omega_\rho n_\sigma\,. \tag{21}$$

The n_μ projection will give the (three-vector) generalized electric $\boldsymbol{\xi}$ and magnetic fields $\boldsymbol{\Omega}$ for each fluid (where we have droped the specie indexes)

$$\begin{aligned} \boldsymbol{\xi} &= \boldsymbol{E} + \frac{m}{qa^2}\left[\nabla\left(f\Gamma\right) + \frac{\partial}{\partial t}\left(f a^2 \Gamma \boldsymbol{v}\right)\right]\,, \\ \boldsymbol{\Omega} &= \boldsymbol{B} + \frac{m}{q\,a}\nabla\times\left(f\Gamma\boldsymbol{v}\right)\,, \end{aligned} \tag{22}$$

where now m is electron mass. Those vectors are the curved spacetime generalization of the corresponding vector fields defined in Refs. [12, 13]. We emphasize that the generalized magnetic field $\boldsymbol{\Omega}$ allows an interpretation as a generalized vorticity because it is, indeed, the curl of a potential ($\boldsymbol{\Omega} = \nabla\times\mathcal{A}$), where

$$\mathcal{A} = \boldsymbol{A} + \frac{mf\Gamma}{q\,a}\boldsymbol{v}\,, \tag{23}$$

for each fluid, where $\boldsymbol{A}$ is the vector potential of the electromagnetic field. From now, the names "generalized magnetic fields" and "generalized vorticity" are used interchangeably.

The generalized electric field (22) and the generalized magnetic field (22) are the key to writing the equation of motion (7) in an insightful form. With previous definitions, Eq. (7) becomes

$$\xi^\mu - a^2\gamma_{ij}\xi^i v^j n^\mu + n_\lambda \epsilon^{\lambda\mu\nu\rho} v_\nu \Omega_\rho = -\frac{T}{q\Gamma}\partial^\mu\sigma\,. \tag{24}$$

This is the covariant form of the equation of motion from where the $3+1$ equations can be obtained by appropriated projections on the timelike and spacelike hypersurfaces. The n^μ projection gives arise to the equation for energy conservation for each plasma component

$$a^2\boldsymbol{v}\cdot\boldsymbol{\xi} = \frac{T}{q\Gamma}\frac{\partial\sigma}{\partial t}\,, \tag{25}$$

while the spacelike $\gamma^\beta{}_\mu$ projection yields the vectorial momentum evolution equation

$$\boldsymbol{\xi} + a\,\boldsymbol{v}\times\boldsymbol{\Omega} = -\frac{T}{q\,a^2\,\Gamma}\nabla\sigma\,. \tag{26}$$

Eqs. (25) and (26) are equivalent to the usual 3+1 plasma equations [35, 36] invoked in plasma literature. Notice that there exist effects of the interaction of the fluid with the spacetime expansion hidden in the definition of the unified fields.

This unified magnetofluid approach leads us directly to the general vortical form of depicting the plasma dynamics. In this formalism the sources of general vorticity (where the magnetic field is just a component) are explicitly revealed. The vortical plasma dynamics can be completly described by using the antisymmetric properties of the unified tensor $M^{\mu\nu}$. Its dual tensor follows the conservation equation

$$\nabla_\nu M^{*\mu\nu} = 0\,, \tag{27}$$

where $M^{*\mu\nu} = (1/2)\epsilon^{\mu\nu\alpha\beta}M_{\alpha\beta}$ (in analogy with the electromagnetic tensor). A $3+1$ decomposition of this equation provides physical insights on vortical dynamics. The dual tensor,

$$M^{*\mu\nu} = \Omega^\mu n^\nu - \Omega^\nu n^\mu + \epsilon^{\mu\nu\alpha\beta}\xi_\alpha n_\beta\,, \tag{28}$$

on $3+1$ decomposition, leads to the equation for the timelike hypersurface

$$\nabla\cdot\boldsymbol{\Omega} = 0\,. \tag{29}$$

This equation represent the generalization of the divergence-free nature of the magnetic field. On the other hand, the spacelike projection of Eq. (27),

$$\frac{1}{a^3}\frac{\partial}{\partial t}\left(a^3\boldsymbol{\Omega}\right) = -\frac{1}{a}\nabla\times\boldsymbol{\xi}\,, \tag{30}$$

represents the constraint linking the generalized electric and magnetic fields (Generalized Faraday law) for each fluid.

Notice that the plasma dynamics is contained in Eqs. (26) and (30), whereas Eqs. (25) and (29) are constraints that establishes the conservation of entropy density (8) along each fluid motion

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\sigma = 0\,. \tag{31}$$

We can combine Eqs. (26) and (30) to obtain the complete cosmlogical plasma dynamics in terms of the generalized vorticity

$$\frac{1}{a^3}\frac{\partial}{\partial t}\left(a^3\boldsymbol{\Omega}\right) = \nabla\times\left(\boldsymbol{v}\times\boldsymbol{\Omega}\right) + \frac{1}{qa^3}\nabla\left(\frac{T}{\Gamma}\right)\times\nabla\sigma\,. \tag{32}$$

The last term is known as the Biermann battery, and it depends on the thermodynamical and kinetic properties of the plasma. This battery is usually invoked as the seed for cosmological magnetic fields [15, 16, 18]. However, if the velocity magnitude and plasma temperature have variations along the entropy gradients, then the Biermann battery vanishes, and the only posibility to have a magnetic field is a self–produced structure, as the one studied in this work.

The above system show how the unified plasma dynamics and Maxwell equations appear explicitly in the 3+1 decomposition in the FIDO frame for each plasma component. In order to keep the generality of the covariant equations, it is important to remark that other definitions for the fields can be constructed. For example, electric and magnetic fields can be defined in the comoving frame of the fluid as $E^\mu = U_\nu F^{\nu\mu}$ and $B^\mu = U_\rho \epsilon^{\rho\mu\sigma\tau} F_{\sigma\tau}/2$, such that $U_\mu E^\mu = 0 = U_\mu B^\mu$ [38–42]. Although these definitions are very useful in magnetohydrodinamics for example, they introduce terms in the Maxwell equations due to the general rotational nature of the fluid four–velocity. On the other hand, the $3+1$ decomposition keeps Maxwell equations in curved spacetime closer to its form in flat–spacetime. Besides, the 3+1 formalism in terms of FIDOs quantities has been shown to be a proper way to study connection and reconnection of high–energy plasmas in General Relativity [46].

IV. PLASMAS IN CLASSICAL PERFECT DIAMAGNETISM STATE – MAGNETIC FIELD STRUCTURES

In formulating plasma dynamics beyond MHD, we noticed that if the generalized vorticity were to replace the magnetic field, the MHD like vortical structure of the dynamics is fully retained. Since the velocity and magnetic fields are the measurables of interest, we will seek self–consistent solutions for $\boldsymbol{v}$ and $\boldsymbol{B}$ for a specified thermodynamics. A variety of such solutions for the special relativistic dynamics were worked out in Ref. [11]. In this section, we will explore the appropriate translations of some of these solutions in the context of cosmological plasmas in an expanding universe.

An interesting (and exact) class of solutions, accessible to cosmological plasmas, belong to the general category of states that display Classical Perfect Diamagnetism (CPD) [29]. In CPD states, the generalized vorticity is fully expelled from the plasma interior. It is worthwhile to remark here that it is the vanishing of the canonical vorticity that leads to the London equation (implying Meissner Ochsenfeld effect) describing a standard superconductor, when the plasma does not evolve thermodynamics. Let us now derive the equations that define the CPD state pertinent to the cosmological plasma. In its simplest manifestation in a homentropic plasma ($\partial^\mu \sigma = 0$), Eq. (7) allows the solution

$$M^{\mu\nu} = 0\,, \qquad F^{\mu\nu} = -\frac{m}{q} S^{\mu\nu}\,, \tag{33}$$

that has a vanishing generalized vorticty, $\boldsymbol{\Omega} = 0$. More explicitly, this condition relates the magnetic and velocity fields [see Eq. (22)] for each fluid,

$$\begin{aligned} \boldsymbol{B} &= -\frac{m}{e\,a}\nabla \times (f\Gamma \boldsymbol{v}_p)\,, \\ \boldsymbol{B} &= \frac{m}{e\,a}\nabla \times (f\Gamma \boldsymbol{v}_e)\,. \end{aligned} \tag{34}$$

Notice that in this CPD state, the Biermann battery vanishes. Also, note that Eq. (34), Maxwell equations (16)–(19), and Eq. (32) form a self–consistent system for the magnetic and velocity fields. Using (34) in Maxwell equations (16) and (19), we get

$$\nabla^2 \boldsymbol{B} = \frac{8\pi q^2 n}{mf} a^2 \boldsymbol{B} + \frac{1}{a^2}\frac{\partial}{\partial t}\left(a \frac{\partial}{\partial t}\left(a^3 \boldsymbol{B}\right)\right)\,, \tag{35}$$

that can be solved for the magnetic field as long as the thermodynamic functions and the scale parameter a are specified. There are no explicit external drives, relativistic or otherwise.

Unlike the standard equations associated with superconducting states, Eq. (35) is time dependent. An exactly solvable set emerges if we assume that the thermodynamic quantities (n and f) can be function of time (through the scale factor a). In such a case, the ansatz, $\boldsymbol{B}(\boldsymbol{x}, t) = \boldsymbol{b}(\boldsymbol{x})\mathcal{T}(t)/a(t)^3$ splits (35) into two ordinary differential equations

$$\nabla^2 \boldsymbol{b} = \lambda_p^2\, \boldsymbol{b}\,, \tag{36}$$

$$\frac{\partial^2 \mathcal{T}}{\partial \tau^2} + \omega_p^2 \left(\frac{n f_0}{n_0 f} a^2 - 1\right)\mathcal{T} = 0\,, \tag{37}$$

for the spatial $\boldsymbol{b}$ and the temporal $\mathcal{T}$ parts of the magnetic field, written in terms of the conformal time

$$\tau = \int \frac{dt}{a}\,. \tag{38}$$

Here, density n_0 and f_0 are the values of those functions at some known time t_0. Therefore, the complete temporal variation of density and temperature are given in n and f. The spatial part (36) is, precisely, the London equation predicting the spatial decay of $\boldsymbol{b}$ on a collisionless thermally corrected skin depth $\lambda_p = 1/\omega_p$, where

$$\omega_p = \sqrt{\frac{8\pi q^2 n_0}{m f_0}}\,, \tag{39}$$

is the thermally corrected plasma frequency for an electron-positron plasma.

To explicitly evaluate the temporal behavior of the magnetic field, let us study the solution in the radiation–dominated era when the scale factor increases as $a = (t/t_0)^{1/2}$ [33], where t_0 is the age in a spatially flat radiation–dominated Universe. In this model, a hot plasma has a temperature that increases as $T \approx T_0/a$, where T_0 is the temperature at $t = t_0$. Similarly, the number density of plasma particles decreases with the volume of the Universe, $n = n_0/a^3$ [14], with n_0 as the number density at time $t = t_0$. In the early Universe, for very high temperatures, the function f can be approximated by $f \approx 4T/m + m/(2T)$. Hence Eq. (37) for hot plasmas in the radiation–dominated era becomes

$$\frac{\partial^2 \mathcal{T}}{\partial \tau^2} + \omega_p^2 \left(\frac{\beta f_0}{4} - \frac{\beta^3 f_0}{128\, t_0^2} \tau^2 - 1 \right) \mathcal{T} = 0 \,, \qquad (40)$$

where $\beta = m/T_0$, $f_0 = K_3(\beta)/K_2(\beta)$, and $\tau = \sqrt{t_0 t}/2 = t_0 a/\sqrt{2}$. An exact solution can be found in terms of the convergent series

$$\begin{aligned}
\mathcal{T} &= \sum_{n=1}^{\infty} \zeta_{2n-1}\, \tau^{2n-1} \,, \\
\zeta_1 &= \omega_p \sqrt{\frac{\beta f_0}{4} - 1} \,, \\
\zeta_3 &= -\frac{\omega_p^3}{6} \left(\frac{\beta f_0}{4} - 1 \right)^{3/2} \,, \\
(2n+3)(2n+2)\zeta_{2n+3} &= -\omega_p^2 \left(\frac{\beta f_0}{4} - 1 \right) \zeta_{2n+1} \\
&\quad + \left(\frac{\omega_p^2 \beta^3 f_0}{128\, t_0^2} \right) \zeta_{2n-1} \,. \qquad (41)
\end{aligned}$$

as $\beta f_0 > 4$. Since the era of interest for the current enquiry belongs to relatively smaller times ($\tau \ll 1$), the approximate solution $\mathcal{T} \approx 2\sqrt{t_0 t}\,(\zeta_1 + 4\zeta_3 t_0 t)$ is more instructive. Notice that the temporal evolution of the magnetic field is dependent on the plasma frequency (39), and thus, this solution cannot be obtained in vacuum.

The magnetic field, described by Eqs. (36), (40), and the solution (41), though characterizing a CPD state, does not describe a strictly equilibrium state; it evolves with the expansion of the universe. However, the spatial configuration is, as usual, described by the London equation. Physically, the solution

$$\boldsymbol{B}(\boldsymbol{x}, t) \approx \frac{\boldsymbol{b}(\boldsymbol{x}) t_0^2 \omega_p}{t} \sqrt{\frac{\beta f_0}{4} - 1} \,, \qquad (42)$$

for the radiation-dominated era in the early–Universe, describes a perfect diamagnetic state of the primordial plasma that expells the magnetic field from its inner spatial region.

V. DISCUSSION ON THE MAGNETIC FIELD

The magnetic field (42) in the CPD state is the cosmological version (in an expanding universe) of the self–consistent magnetic plus velocity fields that can exist even in plasmas with vanishing generalized vorticities. Such "superconducting" macroscopic states do not depend on an external input like a broclinic Biermann battery [7, 8, 15, 47] or its generalizations.

Let us compare the solution (42) with a standard temporal decay of a cosmological magnetic field in the radiation–dominated era [35]

$$\boldsymbol{B}_0(t) = \frac{B_0}{a^3} = B_0 \left(\frac{t_0}{t} \right)^{3/2} \,, \qquad (43)$$

for a given constant magnitude of the magnetic field at $t = t_0$. In contrast to this decay $\sim (t_0/t)^{3/2}$, the CPD solution in the early–Universe shows a temporal behavior of the form $\sim t_0^2 \omega_p / t$ that depends on the plasma characteristics. As the Universe expands with time, we see that it is possible for the CPD magnetic field state to decay faster in certain plasma region for times less than the characteristic time

$$t^* = \frac{1}{t_0 \omega_p^2} \left(\frac{\beta f_0}{4} - 1 \right)^{-1} \,. \qquad (44)$$

For times larger than t^*, however, the decay is slower than the one expected (without plasma effects) from cosmological expansion.

Thus, the temporal growth of the magnetic field in the CPD state ($\propto \tau \sim \sqrt{t} \sim a$) tends to compete with the dilution ($\propto a^{-3}$) caused by the cosmological expansion. These new states with initially growing fields (whose detailed nature is yet to be explored) add a brand new element towards advancing our understanding of the cosmological magnetic fields and flows.

In order to evaluate explicitly the impact of the above classical perfect diamagnetic state, let us consider an early–Universe stage of the radiation–dominated era. At $t_0 = 1$ sec., $n_0 \sim 4 \times 10^{31}\mathrm{cm}^{-3}$, $T_0 \sim 10^6$eV, and $B_0 \sim 10^{13}$G [35]. For these values, $\beta \approx 0.511$, $f_0 \approx 8.05$, leading to: plasma frequency $\omega_p \approx 1.26 \times 10^{20}\mathrm{seg}^{-1}$, the collisionless skin depth $\lambda_p \approx 2.39 \times 10^{-12}$m, and the characteristic time $t^* \approx 3.74 \times 10^{-40}$sec.

The spatial diamagnetic behavior (36) of the magnetic field allow us to estimate that $|\boldsymbol{b}(\boldsymbol{x})| \sim B_0 \exp(-x/\lambda_p)$, where x represent the scale length of variation of the magnetic field. Then, we can compare the CPD solution (42) with the usual one (43) due to the cosmological expansion as

$$\left| \frac{\boldsymbol{B}}{\boldsymbol{B}_0} \right| \sim \exp\left(-\frac{x}{\lambda_p} \right) \sqrt{\frac{t}{t^*}} \,. \qquad (45)$$

The CPD magnetic field is larger than its usual counterpart when

$$\exp\left(-\frac{2x}{\lambda_p} \right) t > t^* \,. \qquad (46)$$

For example, consider the time $t = 10^{-2}$sec., with the redshift $z = 1/a - 1 = 9$, where $|\boldsymbol{B}_0| = 10^{16}$G [35]. For times $t > t^*$, it is expected that the temporal decaying will be slower than the one for $\boldsymbol{B}_0$. Also, from (46) we obtain that for regions as large as $x < 43\,\lambda_p$, the CPD magnetic field $\boldsymbol{B}$ is always larger in magnitude than $\boldsymbol{B}_0$.

For $t = 10^{-2}$sec., we find from (42), that the magnetic field is about 10^{30}G in a region around $x \sim 10\lambda_p$. Similarly, for smaller times, as $t \approx 10^{-20}$sec. ($z = 10^{10}$), we have magnetic fields of the order $\boldsymbol{B}/\boldsymbol{B}_0 \approx 10^5$.

To sum up, the solution (42) represents a class of self–consistent (growing) magnetic and vorticity fields accessible to a "perfectly diamagnetic" relativistic plasma in an expanding universe.

The generation and amplification of large scale magnetic fields is an open issue; there are several conventional approaches. For example, dynamo mechanisms (like α–Ω dynamo or turbulence) are invoked to explain galactic magnetic fields [15, 18, 48–52]. On the other hand, it is not clear how cluster magnetic fields can grow [18]. For a dynamo mechanism to work, there has to be an initial nonzero magnetic field -often created by an external mechanism driven by a specific thermodynamics. For cosmological magnetic fields, this kind of seed can be formed in the early–Universe, in the inflation era [18, 53]; there is, though, no satisfactory solution to this problem [15]. Often, Biermann battery is the model used to give the initial seed [15], that can be used as the mechanism to generate magnetic field of very small scales [15, 54].

Our approach, on the other hand, is based on the demonstration that the ideal plasma dynamics is perfectly capable of sustaining a self–consistent field structure (where electromagnetic, kinematic and thermal forces come to a balance) for arbitrary thermodynamics, clearly obviating the need for mechanisms like (and similar to) the Biermann battery. So even if the Biermann term were to vanish for the initial thermodynamics, a magnetic field can arise as a necessary consequence of the dynamics which connects thermal and kinematic, and electromagnetic energies. The electromagnetic, and fluid–thermal vorticities are always interacting, as it is the generalized vorticity that remains null along the evolution of the system. Such solutions are possible for any general thermodynamics. It is important to remark that the above model could be readily extended to study the fate of generalized vorticity in response to dissipative processes [55, 56].

The main contribution of the current paper is to obtain particular solutions of the self–consistent magnetic field that are pertinent to a plasma embedded in an expanding universe. One finds that, in an expanding universe, plasma effects slow down (as compared to the expected rates) the decay of the field intensities with time. Thus the overall dilution of the magnetic field is partially stemmed. We believe that this newly established characteristic can and does bring additional insights into the story of the origin and evolution of the primordial magnetic fields.

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