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Phys. Rev. D **99**, 116015 — Published 20 June 2019

DOI: 10.1103/PhysRevD.99.116015

Diffusive dynamics of critical fluctuations near the QCD critical point

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A quantitatively reliable theoretical description of the dynamics of fluctuations in non-equilibrium is indispensable in the experimental search for the QCD critical point by means of ultra-relativistic heavy-ion collisions. In this letter we consider the fluctuations of the net-baryon density which becomes the slow, critical mode near the critical point. Due to net-baryon number conservation the dynamics is described by the fluid dynamical diffusion equation, which we extend to contain a white noise stochastic current. Including nonlinear couplings from the 3d Ising model universality class in the free energy functional, we solve the fully interacting theory in a finite size system. We observe that purely Gaussian white noise generates non-Gaussian fluctuations, but finite size effects and exact net-baryon number conservation lead to significant deviations from the expected behavior in equilibrated systems. In particular the skewness shows a qualitative deviation from infinite volume expectations. With this benchmark established we study the real-time dynamics of the fluctuations. We recover the expected dynamical scaling behavior and observe retardation effects and the impact of critical slowing down near the pseudo-critical temperature.

I. INTRODUCTION

In local thermal equilibrium the physics of long-wavelength phenomena can be described effectively by fluid dynamics. Conventional fluid dynamics specifies how energy and momentum as well as charges such as the net-baryon number are conserved on average. However, in reality fluids experience noise and dissipation [1]. Intrinsic thermal fluctuations in the fluid dynamical quantities drive the system constantly out of equilibrium. Dissipative processes balance the impact of the fluctuations such that local equilibrium emerges as a consequence of this balance. Near phase transitions, in particular near a critical point, the influence of fluctuations is drastically enhanced and cannot be considered as a small perturbation. Thus, a direct and full propagation of fluctuations in the fluid dynamical evolution is essential to understand fluctuation-driven dynamical effects in systems at a critical point.

The search for the critical point in the QCD phase diagram has attracted special attention in recent years. It is at the heart of many heavy-ion collision experiments performed at GSI/SIS18, the CERN-SPS and in the beam energy scan at RHIC. As a signature of its existence a non-monotonic behavior in observables related to event-by-event fluctuations of conserved charges is expected with varying beam energy [2]. Indeed, recent data of net-proton number fluctuations reveal signs of non-monoticity [3, 4]. The global picture, however, remains inconclusive, mostly due to the lack of quantitative theoretical predictions.

Near a critical point, fluctuations of the critical mode grow as they scale with the correlation length ξ. Importantly, higher-order cumulants are sensitive to higher powers of ξ [5, 6]. The slow, critical mode associated with the divergence of the correlation length at the QCD critical point is the net-baryon density n_B [7]. Enhanced fluctuations in the conserved net-baryon number are therefore expected if the matter passes through the critical region.

These expectations are based on the assumption that the critical mode is in equilibrium with the rest of the matter. However, the system created in a heavy-ion collision is small and short-lived, and due to its violent expansion dynamics spends only a short time in the critical region. Thus, non-equilibrium effects are essential [8–13]. The time for ξ to reach the equilibrium value scales as ξ^z. For a purely diffusive dynamics near the critical point, $z \simeq 4$ according to the dynamical universality class of model B [14]. Coupling the dynamics with the evolution of the energy and momentum density fields, notably the transverse velocity field, changes that expectation to model H with $z \simeq 3$ [14]. In both cases the actual growth of ξ is already dynamically limited by the lifetime of the system [8]. In the search for the QCD critical point, the development of dynamical models which capture the non-equilibrium evolution is therefore crucial. This letter reports first, important results from such a dynamical model.

II. DIFFUSIVE DYNAMICS OF CRITICAL FLUCTUATIONS

The evolution of the regular bulk matter created in a heavy-ion collision can successfully be described by conventional, relativistic fluid dynamics with a surprisingly small shear viscosity over entropy density ratio, see [15] for a recent review. Starting from the equations of rel-

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ativistic fluid dynamics describing the conservation of charges, energy and momentum

$$\partial_\mu N^\mu = 0\,, \qquad \partial_\mu T^{\mu\nu} = 0\,, \tag{1}$$

we focus in this work on the conservative evolution of the net-baryon number N_B decoupled from the fields of energy and momentum density. The diffusive dynamics occurs such as to minimize the free energy of the system. Since we are interested in the dynamics of intrinsic fluctuations near the QCD critical point we include a stochastic current and obtain the following stochastic partial differential equation (similar to [14, 16])

$$\partial_t n_B(\mathbf{x},t) = \Gamma\nabla^2\left(\mathcal{F}'[n_B]\right) + \nabla\cdot\mathbf{J}(\mathbf{x},t)\,. \tag{2}$$

Here, Γ is the mobility coefficient, $\mathcal{F}[n_B]$ denotes the free energy functional, $\mathcal{F}'$ its variation and $\mathbf{J}(\mathbf{x},t)$ is the stochastic current given by

$$\mathbf{J}(\mathbf{x},t) = \sqrt{2T\Gamma}\boldsymbol{\zeta}(\mathbf{x},t) \tag{3}$$

for temperature T. The spatio-temporal white noise field $\boldsymbol{\zeta}(\mathbf{x},t)$ is Gaussian with zero mean and covariance $\langle\zeta_i(\mathbf{x},t)\zeta_j(\mathbf{x}',t')\rangle = \delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\delta_{ij}$. Eq. (3) guarantees that the fluctuation-dissipation balance is fulfilled and the equilibrium distribution $P_{\rm eq}[n_B]$ is given by the known statistical expression.

Near the critical point we approximate the free energy functional in Ginzburg-Landau form as

$$\mathcal{F}[n_B] = T\int \mathrm{d}^3x \left(\frac{m^2}{2n_c^2}\left(\Delta n_B\right)^2 + \frac{K}{2n_c^2}(\nabla n_B)^2 + \frac{\lambda_3}{3n_c^3}\left(\Delta n_B\right)^3 + \frac{\lambda_4}{4n_c^4}\left(\Delta n_B\right)^4 + \frac{\lambda_6}{6n_c^6}\left(\Delta n_B\right)^6\right) \tag{4}$$

with $\Delta n_B = n_B - n_c$ and n_c is the critical net-baryon density. Apart from the typical Gaussian mass term which gives rise to the standard diffusion equation, Eq. (4) contains a kinetic term modified by the surface tension K as a measure for the range of the interaction as well as nonlinear interaction terms.

To make contact with the static universality class of QCD we define $K = \tilde{K}/\xi_0$, $m^2 = \tilde{m}^2/\xi_0^3$, $\lambda_3 = n_c\tilde{\lambda}_3\tilde{m}^{3/2}$, $\lambda_4 = n_c\tilde{\lambda}_4\tilde{m}$ and $\lambda_6 = n_c\tilde{\lambda}_6$. Here, the dimensionless mass $\tilde{m}$ vanishes at the critical point and can be associated with the equilibrium correlation length in the noninteracting Gaussian limit. Finally, $\xi_0 = 0.48$ fm is a characteristic length scale of the system far away from criticality, e.g. the scale of noncritical correlations. These definitions ensure that the coupling coefficients exhibit the universal critical scaling with $\tilde{m}$ [17, 18] known from the 3-dimensional Ising model. In contrast to previous considerations we include a term proportional to $\left(\Delta n_B\right)^6$ in the free energy functional, Eq. (4), because this turned out to be important for a quantitative description of lattice Monte Carlo results for the probability distribution of the average magnetization in the Ising model [17, 18].

We thus obtain the nonlinear stochastic diffusion equation for a system with homogeneous temperature

$$\partial_t n_B(\mathbf{x},t) = \frac{D}{n_c}\left(m^2\nabla^2 n_B - K\nabla^4 n_B\right) + D\nabla^2\left(\frac{\lambda_3}{n_c^2}\left(\Delta n_B\right)^2 + \frac{\lambda_4}{n_c^3}\left(\Delta n_B\right)^3 + \frac{\lambda_6}{n_c^5}\left(\Delta n_B\right)^5\right) + \sqrt{2Dn_c}\nabla\cdot\boldsymbol{\zeta}(\mathbf{x},t) \tag{5}$$

with the diffusion coefficient $D = \Gamma T/n_c$.

The exact value of the critical net-baryon density is sensitive to the location of the critical point in the QCD phase diagram and to the equation of state. We assume $n_c = 1/3\ \mathrm{fm}^{-3}$, which we also choose as the average value of n_B independent of T. On the crossover side of the critical point we consider the temperature dependence of $\xi_0/\tilde{m}$ as shown in Fig. 1 (black solid line in the lower inlay). This is determined by a matching to the susceptibility of the Ising model scaling equation of state [19] near the critical point. The dimensionless mass $\tilde{m}$ is minimal near $T_c = 0.15$ GeV and approaches 1 for $T \to 0.5$ GeV. Values for the dimensionless universal couplings $\tilde{\lambda}_3$, $\tilde{\lambda}_4$ and $\tilde{\lambda}_6$ can similarly be deduced from the scaling equation of state, see [20]. Here, we consider the constant values $\tilde{\lambda}_3 = 1$, $\tilde{\lambda}_4 = 10$ and $\tilde{\lambda}_6 = 3$ which are also comparable with the lattice Ising model studies in [17, 18]. The values are such that near T_c the nonlinear couplings λ_4 and λ_6 are large compared to $\tilde{m}^2$ and, thus, interesting nonlinear effects can be expected in this temperature region. In the following we use $\tilde{K} = 1$, for which the simple relation $\xi = \xi_0/\tilde{m}$ is recovered in the Gaussian limit [21].

Equation (5) is solved by a semi-implicit predictor-corrector scheme. For all our results in the remainder of this work we consider fully dynamical fluctuations in one spatial dimension. In the context of a heavy-ion collision this means that we treat only fluctuations in the longitudinal direction using the full nonlinear theory. This is motivated by the fact that the dynamics of a heavy-ion collision is highly anisotropic and fluctuations are measured as a function of rapidity. In a future study, we plan to include the full coupling to the evolution in the transverse direction. This will allow us to determine to which extent longitudinal and transverse fluctuations equilibrate in a dynamical setting.

III. FLUCTUATION OBSERVABLES IN EQUILIBRIUM

Let us first evaluate the fluctuation observables in an equilibrated system and thus establish a benchmark for the dynamical calculations. At each temperature we fix $D = 1$ fm and take the results from the late-time limit. The equilibrium correlation length ξ of the fully interacting theory can be read off from the exponential decay behavior of the equal-time spatial correlator $\langle\Delta n_B(r)\Delta n_B(0)\rangle$. The result is shown in the inlay of

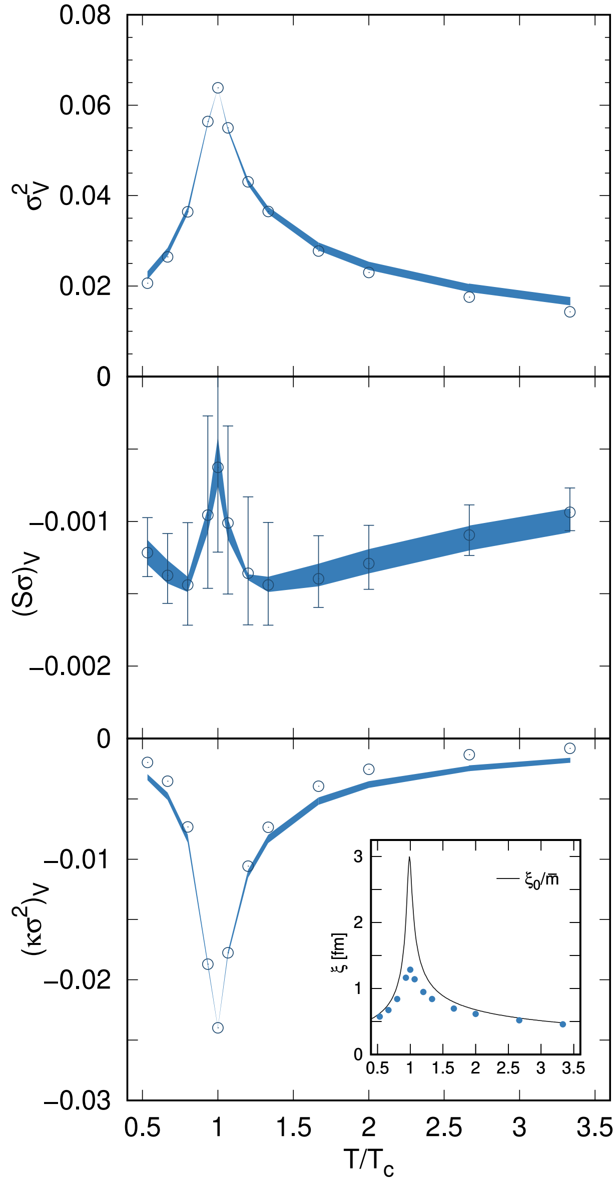

FIG. 1. Variance σ_V^2, skewness $(S\sigma)_V$ and kurtosis $(\kappa\sigma^2)_V$ of the net-baryon number (open circles) in the observation region $V \simeq 1.95$ fm as a function of scaled temperature T/T_c. For variance and kurtosis the statistical errors are smaller than the symbols. The colored bands demonstrate the scaling behavior with the equilibrium correlation length ξ as discussed in the text. The inlay in the lower panel shows ξ for the fully interacting theory (circles) and for the Gaussian limit (solid line).

Fig. 1 (blue circles). The impact of the nonlinear interaction terms and finite size effects manifests itself as a significant reduction of the correlation length near T_c compared to the Gaussian limit.

For various temperatures we calculate the equilibrium values of the fluctuation observables of the net-baryon number in a given volume, i.e. the variance σ_V^2, the skewness $(S\sigma)_V$ and the kurtosis $(\kappa\sigma^2)_V$, in the fully interacting theory. Here, $\langle N_B\rangle_V = \int_V n_B(x)\mathrm{d}x$ can take continuous values in each event, and $(S\sigma^3)_V$ and $(\kappa\sigma^4)_V$ denote the third- and fourth-order cumulants of the event-by-event distribution of $\langle N_B\rangle_V$. The volume V corresponds to a subregion of observation smaller than the system size, $V < L = 20$ fm. This is analogous to the experimental situation in which N_B is measured in a given kinematic region, e. g. in a given rapidity window for heavy-ion collision experiments. Our lattice spacing is $\Delta x = L/256$.

In Fig. 1 we show our results for an observation volume $V \simeq 1.95$ fm (open symbols). In all three quantities, we clearly observe a prominent signal around T_c, where the equilibrium correlation length is the largest. Due to the nonlinear interactions the non-Gaussian fluctuations $(S\sigma)_V$ and $(\kappa\sigma^2)_V$ are created from purely Gaussian white noise in Eq. (5). The colored bands show fits for the temperature range near T_c measuring the scaling behavior of the fluctuation observables with ξ. We find an optimal description for $\sigma_V^2 \propto \xi^{1.30\pm0.05}$ and $(\kappa\sigma^2)_V \propto \xi^{2.5\pm0.1}$. These results demonstrate a clear effective reduction of the scaling behavior compared to the leading-order expectations $\sigma_V^2 \sim \xi^2$ and $(\kappa\sigma^2)_V \sim \xi^5$ from [5] where $\xi/V \ll 1$ is assumed, see [10]. Solving the fully interacting theory for a finite size system leads to a complicated interplay between nonlinear mode couplings, finite ξ/V effects and exact net-baryon number conservation. For $\xi/V < 1$ this can be studied systematically [22]. We expect, for example, that the nonlinear mode couplings $\lambda_3\lambda_4$ and $\lambda_3\lambda_6$ can affect the skewness near T_c. In Fig. 1 we observe indeed a competition between different scaling behaviors in the skewness whose infinite volume expectation [5] is $(S\sigma)_V \sim \xi^{2.5}$. We find instead an optimal description with a term $\propto \xi^{1.47\pm0.05}$ and a competing contribution of opposite sign $\propto \xi^{2.40\pm0.05}$. We conclude that already in equilibrium for the fully interacting theory and including net-baryon number conservation in a finite size system the scaling behavior can be drastically different from what has been expected so far.

IV. DYNAMICS OF FLUCTUATION OBSERVABLES

Having settled the equilibrium benchmark for net-baryon fluctuations in a finite system, we can now turn to the dynamics of the critical fluctuations. In the following we consider a diffusion coefficient D that depends on temperature as $D = D_0 T/T_0$ with $D_0 = 1$ fm at $T_0 = 0.5$ GeV. As a first quantity we study the dynamical structure factor and the dynamical scaling behavior near T_c. For this purpose, we analyze the correlator $\langle \Delta n_B(k, t_0 + t)\Delta n_B(-k, t_0)\rangle$ for different wavevectors k. It is found to decay over time $\propto e^{-t/\tau_k}$. As one would expect, the relaxation time τ_k decreases with increasing k and becomes larger for fixed mode k as $T \to T_c$.

Considering the modes with $k^* = 1/\xi$ for the correla-

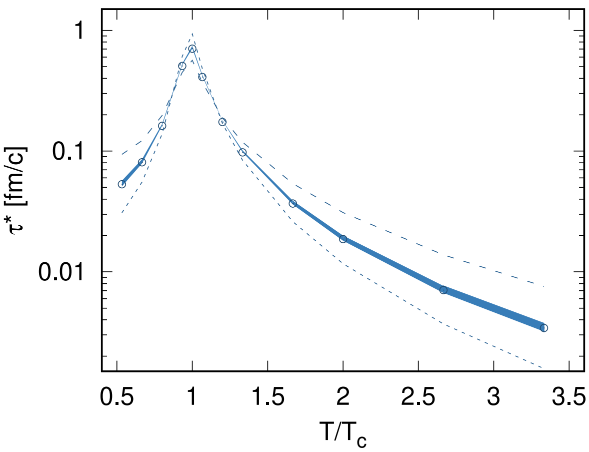

FIG. 2. Scaling behavior of the relaxation time τ^* (circles) with ξ for modes $k^* = 1/\xi$ as a function of T/T_c. The colored band shows the scaling $\propto \xi^z$ with $z = 4 \pm 0.1$.

tion length realized in the fully interacting theory at a given T, we find that the corresponding relaxation time τ^* scales as $\tau^* \propto \xi^z$ with $z \simeq 4$. This is exhibited in Fig. 2 where we also contrast the scaling behavior with the scaling exponents $z = 3$ (dashed line) and $z = 5$ (dotted line), which give a poor description and are clearly excluded. The excellent agreement with $z \simeq 4$ demonstrates that the expected dynamical critical scaling of model B is reproduced.

Next, we consider a dynamical evolution of the background temperature of the system according to

$$T(\tau) = T_0 \left(\frac{\tau_0}{\tau} \right) . \tag{6}$$

Now, the coupling constants and the diffusion coefficient are time dependent. At $\tau_0 = 1$ fm/c we start with an equilibrated system at $T_0 = 0.5$ GeV. For the given temperature evolution T_c is reached at around $\tau_c = \tau - \tau_0 = 2.3$ fm/c.

In Fig. 3, we present the results of the real-time dynamics for the volume-integrated variance, skewness and kurtosis. The time at which T_c is reached is indicated by the vertical line. In comparison with the equilibrium values we can see that the variance and kurtosis, which have a simple structure as a function of temperature (see Fig. 1), have smaller extremal values, which are about 75% of the equilibrium variance and about 50% of the equilibrium kurtosis. This is an effect of the long relaxation times for modes which are of the order of the inverse correlation length, see Fig. 2, and the non-equilibrium situation of a rapidly cooling system. In addition we can observe a dynamical retardation effect, which shifts these extrema (a slightly stronger shift is observed for $(\kappa\sigma^2)_V$) to times larger than τ_c. The skewness follows qualitatively this behavior, but quantitative statements are difficult given the larger uncertainties. The observed effects, the decrease of extremal values and the retardation effect, are found to be stronger for slower diffusion and/or faster cooling.

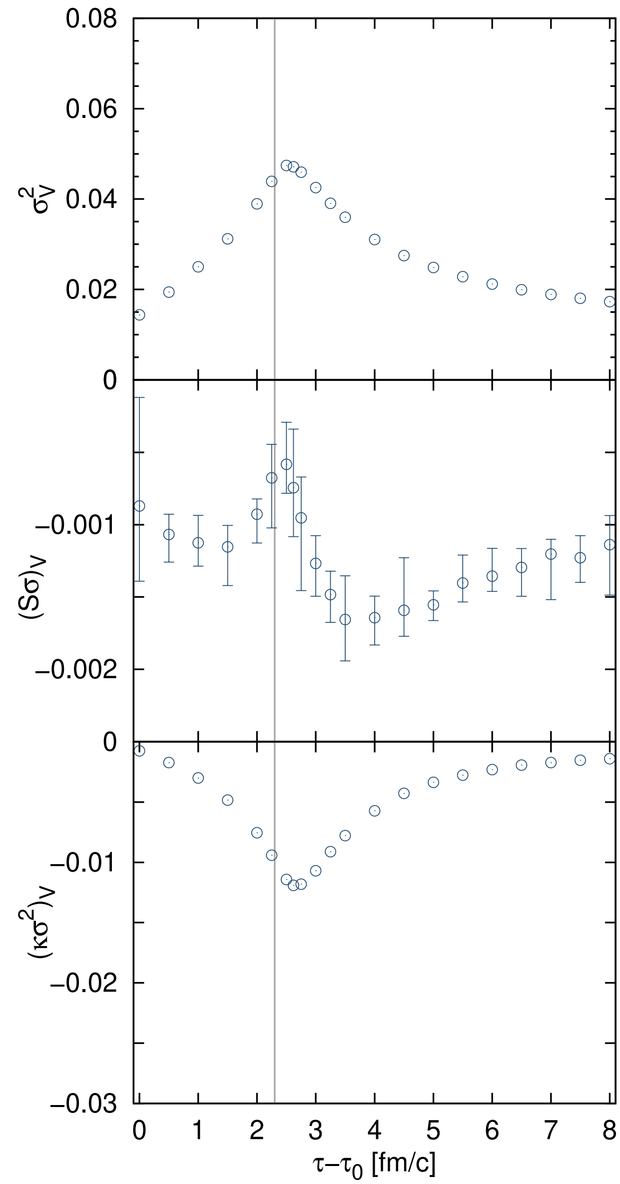

FIG. 3. Dynamical evolution of the variance σ_V^2, the skewness $(S\sigma)_V$ and the kurtosis $(\kappa\sigma^2)_V$ as a function of time. The pseudo-critical temperature T_c is reached at $\tau_c = 2.3$ fm/c.

V. CONCLUSIONS

In this letter we presented first results of a fully dynamical treatment of the diffusive behavior of fluctuations near the QCD critical point. Our study takes into account nonlinear mode couplings, the finite size of the system, and exact net-baryon number conservation. In equilibrium we find that these effects limit the growth of the correlation length ξ near T_c. We observe that the nonlinear couplings generate non-Gaussian fluctuations from the Gaussian stochastic noise. The scaling behavior of the variance and the higher-order cumulants with the correlation length is affected by a non-trivial interplay between nonlinear mode couplings, the finite ratio of ξ over the observation volume V, and the impact of net-baryon number conservation. These effects can be studied systematically as an expansion in ξ/V [22].

From the scaling behavior of the relaxation time of the critical modes we demonstrate that the dynamical scaling of model B is realized in the presented framework. In the dynamics of the fluctuations we see clear signatures of retardation and non-equilibrium effects. Critical slowing down leads to a visible reduction of the magnitude of the fluctuations, which is stronger for the higher-order cumulants. Our work marks an important step in the theoretical development of dynamical models guiding the search for the QCD critical point.

ACKNOWLEDGMENTS

M. Nahrgang and M. Bluhm acknowledge the support of the program "Etoiles montantes en Pays de la Loire 2017". The work of M. Bluhm is funded by the European Unions Horizon 2020 research and innovation program under the Marie Skłodowska Curie grant agreement No 665778 via the National Science Center, Poland, under grant Polonez UMO-2016/21/P/ST2/04035. T. Schäfer was supported in part by the U.S. Department of Energy under grant DE-FG02-03ER41260 and by the DoE Beam Energy Scan Theory (BEST) Topical Collaboration. S.A. Bass has been supported by the U.S. Department of Energy under grant DE-FG02-05ER41367. This research was supported in part by the ExtreMe Matter Institute (EMMI) at the GSI Helmholtzzentrum für Schwerionenforschung, Darmstadt, Germany. The authors acknowledge fruitful discussions within the framework of the BEST Topical Collaboration.

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