A lattice Maxwell system with discrete space-time symmetry and local energy-momentum conservation

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Abstract

A lattice Maxwell system is developed with gauge-symmetry, symplectic structure and discrete space-time symmetry. Noether's theorem for Lie group symmetries is generalized to discrete group symmetries for the lattice Maxwell system. As a result, the lattice Maxwell system is shown to admit a discrete local energy-momentum conservation law corresponding to the discrete space-time symmetry. A lattice model that respects all local conservation laws and geometric structures is as good as and probably more preferable than standard models on continuous space-time. It can also be viewed as an effective algorithm for the governing differential equations on continuous space-time.

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1. Introduction

Maxwell's equations are partial differential equations (PDEs) governing the dynamics of electromagnetic field on space-time. In this paper, we propose a lattice Maxwell system with gauge-symmetry, symplectic structure, discrete space-time symmetry and discrete local energy-momentum conservation. The electromagnetic field is only defined on a space-time lattice with a set of discrete rules governing its dynamics. A lattice Maxwell system can be interpreted in two ways. First, it can be viewed as a model for the electromagnetic field by itself. Second, it can be treated as a numerical algorithm of the continuous Maxwell's equations. For both purposes, it is desirable for the lattice Maxwell system to conserve fundamental physics quantities, such as energy-momentum, and preserve important structures, such as space-time symmetry, gauge-symmetry and symplectic structure. Especially for the first purpose, the conservation and structure preserving properties are indispensable. One can argue that a lattice Maxwell system which respects many important conservation laws and structures is a valid model for the electromagnetic field. The model is as good as the continuous electromagnetic model, but with an added advantage that it can be easily calculated. For the second purpose, i.e., using the lattice Maxwell system as an algorithm, the conservative and structure-preserving properties of the algorithm, will render more reliable numerical solutions of the continuous Maxwell's equations. This is especially true for simulations of long-term multi-scale dynamics. A well-known example in this respect is Yee's algorithm [1] and its generalizations using discrete exterior calculus (DEC). The outstanding performance of Yee's algorithm is attributed to the fact that it preserves the differential form structure of the Maxwell system [2, 3, 4]. Under the simulation hypothesis by Bostrom [5] and in consistent with the discussion on lattice QCD by Beane et al. [6], the two interpretations of physics laws on space-time lattice

could be essentially the same.

How to design a lattice Maxwell system with these desirable conservative and structure-preserving properties? It is natural and probably necessary to adopt a variational approach. In this paper, the lattice Maxwell system with gauge symmetry, symplectic structure, discrete space-time symmetry and discrete local energy-momentum conservation is constructed by discretizing the Lagrangian of the electromagnetic field on a space-time lattice.

For continuous systems, Noether's theorem [7] connects Lie group symmetries and conservation laws. For example, energy-momentum conservation is the consequence of space-time symmetry. However, for discrete system, such a connection has not been fully understood. Even though Noether's theorem has been applied to discrete systems [8, 9, 10, 11, 12, 13, 14, 15], the symmetries in these applications are continuous Lie groups, as required by the Noether's theorem. In this paper, we demonstrate a generalized version of Noether's theorem which establishes connections between discrete group symmetries and discrete conservation laws for the lattice Maxwell system. As far as we know, such a generalized Noether's theorem for discrete group symmetries has not been discussed in the existing literature. We then show that the lattice Maxwell system admits a discrete space-time symmetry, and more importantly that the discrete space-time symmetry induces a discrete local energy-momentum conservation law.

We should mention that there are other methods to construct numerical schemes with conservation properties for partial differential equations, e.g., the finite volume method and the discontinuous Galerkin method. In particular, Poynting's theorem, i.e., the local energy conservation law, for Yee's algorithm has been discussed [16, 17], and several applications have been developed using this result [18, 19, 20, 21, 22, 23]. However, the momentum conservation law for Yee's algorithm has not been established. In addition, these existing studies are only on the level of numerical algorithms, and are not related to the fundamental properties of discrete space-time symmetries.

We start our discussion from the action of the electromagnetic field on the

continuous space-time,

$$\mathcal{A}[\mathbf{A}] = \int \mathrm{d}t\mathrm{d}\mathbf{x}\mathcal{L}\left(\mathbf{A},\phi\right) , \tag{1}$$

$$\mathcal{L}\left(\mathbf{A},\phi\right) = \frac{1}{2}\left(\left(-\nabla\phi-\dot{\mathbf{A}}\right)^2-\left(\nabla\times\mathbf{A}\right)^2\right) , \tag{2}$$

where $\mathcal{L}$ is the Lagrangian density. For simplicity, permeability μ_0 and permittivity ε_0 are set to 1. The Euler-Lagrange equations for $\mathbf{A},\phi$ are

$$\frac{\partial}{\partial t}\left(\dot{\mathbf{A}}+\nabla\phi\right)-\nabla\times\left(\nabla\times\mathbf{A}\right) = 0 , \tag{3}$$

$$\nabla\cdot\left(\dot{\mathbf{A}}+\nabla\phi\right) = 0 . \tag{4}$$

We select a "cubic" lattice in space-time. The discrete Lagrangian density is chosen to be

$$\mathcal{L}_{J,l} = \frac{1}{2}\left(\mathrm{Dt}\left(\mathbf{A}_{J,l-1}\right)-\nabla_d\phi_{J,l}\right)^2-\frac{1}{2}\left(\mathrm{curl_d}\mathbf{A}_{J,l}\right)^2 , \tag{5}$$

$$\mathrm{Dt}\left(f_l\right) \equiv \frac{f_{l+1}-f_l}{\Delta t} , \tag{6}$$

where J is the spatial grid indices $[i,j,k]$, and l is the temporal index, and ∇_d and $\mathrm{curl_d}$ are discrete gradient and discrete curl operators, respectively. They are defined by Eqs.(A1) and (A2) in Ref. [24]. The discrete action is the sum of the discrete Lagrangian density,

$$\mathcal{A}_d[\mathbf{A},\phi] = \sum_{J,l}\mathcal{L}_{J,l}\Delta t\Delta x^3 . \tag{7}$$

The Lagrangian density $\mathcal{L}_{J,l}$ admits a gauge symmetry, meaning that the following transformation

$$\mathbf{A}_{J,l}\to\mathbf{A}_{J,l}+\nabla_d\psi_{J,l} , \ \phi_{J,l}\to\phi_{J,l}-\mathrm{Dt}^*\left(\psi_{J,l}\right) , \tag{8}$$

will leave $\mathcal{L}_{J,l}$ invariant. Here, Dt^* is the dual operator of of Dt,

$$\mathrm{Dt}^*\left(f_l\right) \equiv \frac{f_l-f_{l-1}}{\Delta t} .$$

We note that this symmetry is defined on the space-time lattice. However, the gauge symmetry group itself is continuous and forms a Lie group. Due to the

gauge symmetry, we can choose a gauge to simplify the calculation. In the present study, the temporal gauge is adopted, i.e., $\phi_{J,l} = 0$. In this gauge, the discrete Euler-Lagrangian equation for $\mathbf{A}_{J,l}$ is

$$\mathrm{Dt}^* \left(\mathrm{Dt} \left(\mathbf{A}_{J,l}\right)\right) + \mathrm{curl_d}^* \mathrm{curl_d} \mathbf{A}_{J,l} \quad = \quad 0 \ . \tag{9}$$

where $\mathrm{curl_d}^*$ is defined in Eq. (A5) in Ref. [24]. For a detailed discussion on the discrete Euler-Lagrangian equation, Ref. [10] is recommended.

Now we introduce the concept of discrete group symmetry. A discrete transformation of the lattice field $\mathbf{A}_{J,l}$,

$$\mathbf{A}_{J,l} \to \mathbf{A}'_{J,l} \ , \tag{10}$$

is a discrete group symmetry, if the resulting variation of the Lagrangian density is at most a discrete 4-divergence of a discrete 4-vector field $(\mathcal{L}_{t,J,l}, \mathcal{L}_{\mathbf{x},J,l})$, i.e.,

$$\Delta\mathcal{L}_{J,l} = \mathcal{L}_{J,l} - \mathcal{L}'_{J,l} = \mathrm{Dt}^* \left(\mathcal{L}_{t,J,l}\right) + \mathrm{div_d}^* \mathcal{L}_{\mathbf{x},J,l} \ . \tag{11}$$

Here, $\mathcal{L}_{t,J,l}$ and $\mathcal{L}_{\mathbf{x},J,l}$ are the temporal and spatial components of the 4-vector field, which only depends on the values of discrete vector potential $\mathbf{A}$ near the grid point J, l, e.g., $\mathbf{A}_{J,l}, \mathbf{A}_{J,l+1}, \mathbf{A}_{i,j-1,k,l}, \ldots$. The operator $\mathrm{div_d}^*$ is defined in Eq. (A6) in Ref. [24]. This definition of discrete group symmetry is similar but different to that of continuous (Lie group) symmetry in Noether's theorem [25].

We emphasize again that the gauge symmetry of Eq. (8) is a Lie group symmetry, instead of a discrete group symmetry. This may be a little confusing since that the gauge field $\psi_{J,l}$ itself is defined on the discrete spacetime lattice. However, the difference between continuous Lie group symmetry and discrete symmetry is significant. For a Lie group symmetry, every symmetry transformation is an element of the Lie group, and the symmetry transformation can be changed "continuously". In other words, for any two different symmetry transformation g_1 and g_2, we can always find a transformation between g_1 and g_2, e.g, the transformation defined by $g_3 = (g_1 + g_2)/2$. On the contrary, for a discrete group symmetry, every symmetry transformation is an element of the

discrete group, and between two elements of the discrete symmetry group, there is no other element of the group in general. In this sense, the gauge symmetry Eq. (8) is a continuous Lie group symmetry, because for any two gauge fields $\psi_{1,J,l}$ and $\psi_{2,J,l}$, we can always find a new gauge field $\psi_{3,J,l} = (\psi_{1,J,l} + \psi_{2,J,l})/2$. Note that when the gauge filed varies, the grid points of the spacetime lattice are fixed. At each grid point, the value of $\psi_{J,l}$ varies smoothly in R, and the variation of $\psi_{J,l}$ is in the vertical direction relative to the spacetime. Therefore, the gauge symmetry is always a continuous Lie group, regardless whether the spacetime is discrete or continuous. The relationship between continuous Lie group symmetries and conservation laws have been investigated intensively [7, 8, 25, 9, 10, 11, 12, 13, 14, 15]. As studied in Refs. [26, 27], the conservation law corresponding to the continuous gauge symmetry over a discrete spacetime lattice is the charge conservation over the discrete spacetime lattice.

On the other hand, the spacetime translation symmetry in the present study is a discrete symmetry. Because the spacetime is discrete, we can only translate the spacetime by integer number of grids. Translations of a half and a quarter grids is not allowed. Thus, the standard Noether's theorem, which requires the symmetry group to be a continuous Lie group, does not apply. We now show that for the lattice Maxwell system proposed here, a discrete group symmetry will induce a discrete local conservation law. This result can be viewed as a discrete generalization of Noether's theorem for the electrodynamics. Again, the fundamental difference is that the symmetry group in the current context is a discrete group, instead of a continuous Lie group for the standard Noether's theorem [25, 8, 9, 10]. As far as we known, such connection between discrete group symmetries and conservation laws have not yet been discovered.

To prove this fact, we first calculate the change of the Lagrangian density due to the transformation Eq. (10),

$$\begin{aligned}
\Delta\mathcal{L}_{J,l} &= \frac{1}{2}\mathrm{Dt}\left(\mathbf{A}_{J,l-1}\right)^2 - \frac{1}{2}\left(\mathrm{curl_d}\mathbf{A}_{J,l}\right)^2 - \frac{1}{2}\mathrm{Dt}\left(\mathbf{A}'_{J,l-1}\right)^2 + \frac{1}{2}\left(\mathrm{curl_d}\mathbf{A}'_{J,l}\right)^2 \\
&= \mathrm{Dt}\left(\mathbf{A}^{\dagger}_{J,l-1}\right)\cdot\mathrm{Dt}\left(\left(\mathbf{A}_{J,l-1} - \mathbf{A}'_{J,l-1}\right)\right) - \\
&\quad \mathrm{curl_d}\mathbf{A}^{\dagger}_{J,l}\cdot\mathrm{curl_d}\left(\mathbf{A}_{J,l} - \mathbf{A}'_{J,l}\right) \ , \qquad (12)
\end{aligned}$$

where $\mathbf{A}^{\dagger}_{J,l} \equiv \frac{1}{2}\left(\mathbf{A}_{J,l} + \mathbf{A}'_{J,l}\right)$. Similar to Eq. (9), the dynamics of $\mathbf{A}'_{J,l}$ is governed by

$$\mathrm{Dt}^*\left(\mathrm{Dt}\left(\mathbf{A}'_{J,l}\right)\right) + \mathrm{curl_d}^*\mathrm{curl_d}\mathbf{A}'_{J,l} \quad = \quad 0 \ . \tag{13}$$

The combination of Eq. (12) + (Eq. (9) + Eq. (13)) $\cdot \frac{1}{2}\left(\mathbf{A}_{J,l} - \mathbf{A}'_{J,l}\right)$ gives

$$\begin{aligned}
\Delta\mathcal{L}_{J,l} \quad &= \quad \mathrm{Dt}\left(\mathbf{A}^{\dagger}_{J,l-1}\right)\cdot\mathrm{Dt}\left(\left(\mathbf{A}_{J,l-1} - \mathbf{A}'_{J,l-1}\right)\right) - \mathrm{curl_d}\mathbf{A}^{\dagger}_{J,l}\cdot\mathrm{curl_d}\left(\mathbf{A}_{J,l} - \mathbf{A}'_{J,l}\right) \\
&\quad +\mathrm{Dt}^*\left(\mathrm{Dt}\left(\mathbf{A}^{\dagger}_{J,l}\right)\right)\cdot\left(\mathbf{A}_{J,l} - \mathbf{A}'_{J,l}\right) + \left(\mathrm{curl_d}^*\mathrm{curl_d}\mathbf{A}^{\dagger}_{J,l}\right)\cdot\left(\mathbf{A}_{J,l} - \mathbf{A}'_{J,l}\right) \\
&= \quad \mathrm{Dt}^*\left(\mathrm{Dt}\left(\mathbf{A}^{\dagger}_{J,l}\right)\cdot\left(\mathbf{A}_{J,l} - \mathbf{A}'_{J,l}\right)\right) - \\
&\quad \mathrm{div_d}^*\left(\left(\mathbf{A}_{J,l} - \mathbf{A}'_{J,l}\right) * \times\mathrm{curl_d}\mathbf{A}^{\dagger}_{J,l}\right) \ .
\end{aligned} \tag{14}$$

In deriving Eq. (14), use has been made of the operator identity Eq. (A18) in Ref. [24]. Now, if the discrete transformation $\mathbf{A}_{J,l} \to \mathbf{A}'_{J,l}$ is a discrete group symmetry, then by definition $\Delta\mathcal{L}_{J,l}$ is the discrete 4-divergence of a discrete 4-vector $(\mathcal{L}_{t,J,l}, \mathcal{L}_{\mathbf{x},J,l})$,

$$\Delta\mathcal{L}_{J,l} \quad = \quad \mathrm{Dt}^*\left(\mathcal{L}_{t,J,l}\right) + \mathrm{div_d}^*\mathcal{L}_{\mathbf{x},J,l} \ . \tag{15}$$

Then, Eqs. (14) and (15) can be combined to give a discrete local conservation law,

$$\begin{aligned}
&\mathrm{Dt}^*\left(\mathrm{Dt}\left(\mathbf{A}^{\dagger}_{J,l}\right)\cdot\left(\mathbf{A}_{J,l} - \mathbf{A}'_{J,l}\right) - \mathcal{L}_{t,J,l}\right) - \\
&\mathrm{div_d}^*\left(\left(\mathbf{A}_{J,l} - \mathbf{A}'_{J,l}\right) * \times\mathrm{curl_d}\mathbf{A}^{\dagger}_{J,l} + \mathcal{L}_{\mathbf{x},J,l}\right) \quad = \quad 0 \ .
\end{aligned} \tag{16}$$

This completes the proof of the generalized Noether's theorem for discrete group symmetries of the lattice Maxwell system.

In the present study, to unveil the discrete energy-momentum conservation law, we will consider the following spatial and the temporal translation symmetries,

$$\mathbf{A}_{J,l} \to \mathbf{A}_{J-x,l} \ , \tag{17}$$

$$\mathbf{A}_{J,l} \to \mathbf{A}_{J-y,l} \ , \tag{18}$$

$$\mathbf{A}_{J,l} \to \mathbf{A}_{J-z,l} \ , \tag{19}$$

$$\mathbf{A}_{J,l} \to \mathbf{A}_{J,l+1} \ , \tag{20}$$

where $J-x$, $J-y$, $J-z$ denote $[i-1,j,k]$, $[i,j-1,k]$, $[i,j,k-1]$, respectively. First, let's look at the symmetry in the Eq. (17), which is the discrete translation symmetry in the $\mathbf{e}_x$-direction. In this case, the change of Lagrangian density is

$$\begin{aligned}\Delta\mathcal{L}_{J,l} &= \frac{1}{2}\Big(\mathrm{Dt}\left(\mathbf{A}_{J,l-1}\right)^2-\mathrm{Dt}\left(\mathbf{A}_{J-x,l-1}\right)^2-\left(\mathrm{curl_d}\mathbf{A}_{J,l}\right)^2+\\ &\quad\left(\mathrm{curl_d}\mathbf{A}_{J-x,l}\right)^2\Big)\ ,\end{aligned}\tag{21}$$

which is the discrete 4-divergence of the following discrete 4-vector

$$\mathcal{L}_{t,J,l}=0\ ,\ \mathcal{L}_{\mathbf{x},J,l}=[\frac{1}{2}\mathrm{Dt}\left(\mathbf{A}_{J,l-1}\right)^2-\frac{1}{2}\left(\mathrm{curl_d}\mathbf{A}_{J,l}\right)^2,0,0]\ .\tag{22}$$

This verifies that Eq. (17) is indeed a discrete group symmetry. Therefore, according to Eq. (16), we obtain the following discrete conservation law,

$$\begin{aligned}\mathrm{Dt}^*\left(\mathrm{Dt}\left(\mathbf{A}_{J,l}^{x/2}\right)\cdot\left(\mathbf{A}_{J,l}-\mathbf{A}_{J-x,l}\right)\right)-\mathrm{div_d}^*\Big(&\left(\mathbf{A}_{J,l}-\mathbf{A}_{J-x,l}\right)*\times\mathrm{curl_d}\mathbf{A}_{J,l}^{x/2}\\ &+[\frac{1}{2}\mathrm{Dt}\left(\mathbf{A}_{J,l-1}\right)^2-\frac{1}{2}\left(\mathrm{curl_d}\mathbf{A}_{J,l}\right)^2,0,0]\Big)=0\ .\end{aligned}\tag{23}$$

Here,

$$\mathbf{A}_{J,l}^{x/2}\equiv\frac{\mathbf{A}_{J,l}+\mathbf{A}_{J-x,l}}{2}\ .\tag{24}$$

Eq. (23) is the discrete local momentum conservation law in the $\mathbf{e}_x$-direction. It can be transformed into a familiar form expressed in terms of $\mathbf{E}$ and $\mathbf{B}$. Taking a discrete divergence $\mathrm{div_d}^*$ of Eq. (9) and using the discrete operator identity Eq. (A16) in Ref. [24], we can see that $\mathrm{Dt}^*\left(\mathrm{div_d}^*\mathrm{Dt}\left(\mathbf{A}_{J,l}\right)\right)=0$. If initially $\mathrm{div_d}^*\mathrm{Dt}\left(\mathbf{A}_{J,0}\right)$ is zero, which will be automatically satisfied when there is no charge in the space, then it remains zero all time. Therefore,

$$\mathrm{div_d}^*\left(\mathrm{Dt}\left(\mathbf{A}_{J,l}\right)\right)\ =\ 0\ .\tag{25}$$

The electromagnetic fields on the lattice are defined as

$$\mathbf{E}_{J,l}=-\mathrm{Dt}\left(\mathbf{A}_{J,l}\right)\ ,\ \mathbf{B}_{J,l}=\left(\mathrm{curl_d}\mathbf{A}\right)_{J,l}\ .\tag{26}$$

In terms of $\mathbf{E}_{J,l}$ and $\mathbf{B}_{J,l}$, the discrete Maxwell equations are

$$\mathrm{Dt}^*\left(\mathbf{E}_{J,l}\right)=\left(\mathrm{curl_d}^*\mathbf{B}\right)_{J,l}\ ,\ \mathrm{Dt}\left(\mathbf{B}_{J,l}\right)=-\left(\mathrm{curl_d}\mathbf{E}\right)_{J,l}\ .\tag{27}$$

With the help of Eq. (25) and discrete operator identities Eq. (A19) and (A20) in Ref. [24], Eq. (23) can be transformed into

$$\mathrm{Dt}^*\left(\mathbf{E}^{x/2}_{J,l}\times *\mathbf{B}_{J,l}\right)_x+\mathrm{div_d}^*\left(\mathrm{Dt}^*\left(\mathbf{E}^{x/2}_{J,l}\right)\otimes A_{x,J,l}-\left(\mathbf{E}^{x/2}_{J,l-1}\otimes E_{x,J,l-1}\right)\right)- \\ \mathrm{div_d}^*\Big(-(\mathbf{A}_{J,l}-\mathbf{A}_{J-x,l})*\times\mathrm{curl_d}\mathbf{A}^{x/2}_{J,l} \\ -[\frac{1}{2}\mathbf{E}^2_{J,l-1}-\frac{1}{2}\left(\mathrm{curl_d}\mathbf{A}_{J,l}\right)^2,0,0]\Big)=0\ . \quad (28)$$

In deriving Eq. (28), use is also made of

$$\mathrm{Dt}^*\left(\mathrm{div_d}^*\left(\mathbf{E}^{x/2}_{J,l}\otimes A_{x,J,l}\right)\right)=\mathrm{div_d}^*\left(\mathrm{Dt}^*\left(\mathbf{E}^{x/2}_{J,l}\right)\otimes A_{x,J,l}-\left(\mathbf{E}^{x/2}_{J,l-1}\otimes E_{x,J,l-1}\right)\right). \quad (29)$$

The second term in Eq. (28) can be simplified by identities Eq. (A21) and (A16) in Ref. [24],

$$\mathrm{div_d}^*\left(\mathrm{Dt}^*\left(\mathbf{E}^{x/2}_{J,l}\right)\otimes A_{x,J,l}\right)=\mathrm{div_d}^*\left(-\nabla_d A_{x,J,l}\times_2\mathbf{B}^{x/2}_{J,l}\right)\ . \quad (30)$$

Finally, using identity Eq. (A22) in Ref. [24], the discrete local momentum conservation law in the $\mathbf{e}_x$-direction in terms of $\mathbf{E}_{J,l}$ and $\mathbf{B}_{J,l}$ is

$$\mathrm{Dt}^*\left(\mathbf{E}^{x/2}_{J,l}\times *\mathbf{B}_{J,l}\right)_x+\mathrm{div_d}^*[\frac{1}{2}\mathbf{E}_{J,l-1}\cdot\mathbf{E}_{J,l-1},0,0]-\mathrm{div_d}^*\left(\mathbf{E}^{x/2}\otimes E_x\right)_{J,l-1}+ \\ \mathrm{div_d}^*[\frac{\mathbf{B}_{J-x,l}\cdot\mathbf{B}_{J,l}}{2},0,0]-\mathrm{div_d}^*\left(\mathbf{B}\otimes^* B^{x/2}_x\right)_{J,l}=0\ . \quad (31)$$

It is straightforward to verify that it recovers the familiar momentum conservation law in the continuous space-time when the grid-size goes to zero. The discrete local momentum conservation law in the $\mathbf{e}_y$ or $\mathbf{e}_z$ direction can be obtained in a similar way.

Next, we look at the discrete local energy conservation due to the discrete temporal symmetry specified by Eq. (20). In this case, the change of the Lagrangian density in terms of $\mathbf{E}_{J,l}$ and $\mathbf{B}_{J,l}$ is

$$\Delta\mathcal{L}_{J,l} \ = \ \frac{1}{2}\left(\mathbf{E}^2_{J,l-1}-\mathbf{E}^2_{J,l-2}-\mathbf{B}^2_{J,l}+\mathbf{B}^2_{J,l-1}\right)\ , \quad (32)$$

and it is the discrete 4-divergence of the discrete 4-vector field

$$\mathcal{L}_{t,J,l}=\frac{1}{2}\left(\mathbf{E}^2_{J,l-1}-\mathbf{B}^2_{J,l}\right)\ ,\ \mathcal{L}_{\mathbf{x},J,l}=[0,0,0]\ . \quad (33)$$

This verifies that $\mathbf{A}_{J,l} \to \mathbf{A}_{J,l+1}$ is a discrete group symmetry. According to Eq. (16), the local discrete conservation law in terms of $\mathbf{E}_{J,l}$ and $\mathbf{B}_{J,l}$ for this discrete group symmetry is

$$\mathrm{Dt}^* \left(\frac{1}{2} \mathbf{E}_{J,l} \cdot \mathbf{E}_{J,l-1} + \frac{1}{2} \mathbf{B}_{J,l}^2 \right) + \mathrm{div_d}^* \left(\mathbf{E}_{J,l-1} * \times \frac{\mathbf{B}_{J,l} + \mathbf{B}_{J,l-1}}{2} \right) \quad = \quad 0 \tag{34}$$

Of course, this is the discrete local energy conservation law. It recovers the well-known continuous local energy conservation law, a.k.a. Ponyting's theorem, when the grid-size approaches zero.

Another interesting physics brought up by the present research is the conservation laws associated with global discrete group symmetries. As mentioned previously, Noether's theorem established the connections between local conservation laws with continuous Lie groups of symmetries. What established in this paper is the connection between discrete local conservation laws and discrete local symmetries. However, from the derivation of Eq. (16), if the locality condition of the discrete group symmetry is removed, a corresponding conservation law still exists. But the conservation law is a now a global one. Specifically, if $L_{t,J,l}$ and $L_{\mathbf{x},J,l}$ in Eq. (11) depend on the values of discrete vector potential $\mathbf{A}$ near as well as far away from the grid point J, l , then the conservation law Eq. (16) still holds, except that the second term involves global operations. For example, a parity (or mirror) symmetry is not a local symmetry, nevertheless it is a global symmetry and a corresponding global conservation law exists. For the parity transformation in the x-direction, i.e. $\mathbf{A}'_{J,l} = \mathbf{A}_{-i,j,k,l}$, we can find that the corresponding conservation laws is

$$\mathrm{Dt}^* \left(\sum_J \mathrm{Dt} \left(\mathbf{A}^\dagger_{J,l} \right) \cdot \left(\mathbf{A}_{J,l} - \mathbf{A}'_{J,l} \right) \right) \quad = \quad 0 \ . \tag{35}$$

Furthermore, this type of analysis of global discrete group symmetries for the lattice system is also valid in the continuous case. If $\mathbf{A}'$ is a discrete global symmetry transformation of $\mathbf{A}$ under which the Lagrangian does not change, then we have a global conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\int \mathrm{d}\mathbf{x} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\mathbf{A} + \mathbf{A}'}{2} \right) \cdot \left(\mathbf{A} - \mathbf{A}' \right) \right) \quad = \quad 0 \ . \tag{36}$$

More generalizations of the discrete group symmetry-conservation will be explored in future studies.

In this paper, only cubic mesh is discussed. For other meshes, we believe that discrete group symmetries and corresponding conservation laws for discrete Maxwell equation should exist if the meshes themselves admit discrete symmetries. As for general discrete Hamiltonian field systems, the existence of discrete translation symmetries depends on the forms of the systems, and needs to be studied case by case. Lorentz covariance for discrete field systems is another issue to be addressed. Detailed discussions on these topics are beyond the scope of the present paper. We will report relevant results in future publications.

In conclusion, we reported three important advances in the study of lattice model and structure-preserving geometric algorithm for the Maxwell system. (i) A lattice Maxwell system is developed with gauge symmetry, symplectic structure, and discrete space-time symmetry. (ii) Noether's theorem is generalized to the case of discrete group symmetry for the lattice Maxwell system, which establishes the correspondence between discrete group symmetries and discrete local conservation laws. (iii) Applying the discrete Noether's theorem, the lattice Maxwell system is shown to admit a discrete local energy-momentum conservation law.

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