

LA-UR-18-21934 (Accepted Manuscript)

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Provided by the author(s) and the Los Alamos National Laboratory (2019-02-05).

**To be published in:** Journal of Physics Communications

**DOI to publisher's version:** 10.1088/2399-6528/aae2a4

**Permalink to record:** <http://permalink.lanl.gov/object/view?what=info:lanl-repo/lareport/LA-UR-18-21934>

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To cite this article: Scott D Ramsey *et al* 2018 *J. Phys. Commun.* **2** 105009

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## PAPER

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## OPEN ACCESS

RECEIVED  
22 May 2018REVISED  
26 July 2018ACCEPTED FOR PUBLICATION  
19 September 2018PUBLISHED  
10 October 2018

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Scott D Ramsey<sup>1</sup> , Jacob A Tellez<sup>2</sup>, Eric J Riewski<sup>3</sup> and Brian A Temple<sup>1,4</sup><sup>1</sup> Applied Physics, Los Alamos National Laboratory, United States of America<sup>2</sup> Department of Nuclear, Plasma, and Radiological Engineering, University of Illinois, United States of America<sup>3</sup> Department of Engineering Physics, University of Wisconsin, United States of America<sup>4</sup> Author to whom any correspondence should be addressed.E-mail: [ramsey@lanl.gov](mailto:ramsey@lanl.gov), [jatelle2@illinois.edu](mailto:jatelle2@illinois.edu), [reiwski@wisc.edu](mailto:reiwski@wisc.edu) and [temple@lanl.gov](mailto:temple@lanl.gov)**Keywords:** neutron diffusion, symmetry analysis, separation of variables**Abstract**

Separation of variables is one of the oldest techniques for solving certain classes of partial differential equations (PDEs). As is the case with many other solution techniques for differential equations, separation of variables may be codified within the broader framework of symmetry analysis. Though the separation of variables technique is frequently used in the nuclear engineering context with various equations describing neutron transport, its connection to the symmetries of those equations has not yet been thoroughly established. It is thus the purpose of this work to establish that connection using neutron diffusion as both an initial step toward analysis of more generally applicable equations, and as a connection to previous results in related problems. Using Lie group analysis, it is found that the traditional space-time separable solution of the neutron diffusion equation (featuring a single  $\alpha$ -eigenvalue) corresponds to time translation and flux scaling symmetries. Additional solutions of this equation are also constructed using its broader symmetry set.

**1. Introduction**

Separation of variables is one of the oldest techniques for solving certain classes of partial differential equations (PDEs). Like many other such methods, it is often employed as a specific element out of a ‘bag of tricks’, so that given a certain equation the technique can either be expected to apply, or not. In this more general sense, it was not until the late 19th century that S. Lie began his investigation into a ‘unification theory’ of solution techniques for differential equations, inspired in part by the preceding successful application of group theory to algebraic equations. Lie was successful in his aim, and indeed it has since been found that a wide variety of solution techniques for differential equations can be integrated into the broader framework of symmetry analysis (as now set forth in rigorous detail by Ovsiannikov [1], Bluman and Anco [2], Ibragimov [3], Hydon [4], Olver [5], Cantwell [6], Stephani [7], and many others).

Beginning in the 1970s, C Boyer, E Kalnins, and W Miller embarked upon a thorough program of analyzing the connection between Lie symmetries, separation of variables, and special functions in a variety of contexts, including Schrödinger equations, Helmholtz and Laplace equations, harmonic oscillator equations, Hamilton-Jacobi equations, wave equations, and multitudinous other structures in numerous coordinate systems and spaces. This effort is largely summarized in two books and a survey by Miller [8–10]. Following this work, a generalized attempt at demonstrating the connection in question was provided by Gegelia and Markovski [11]. Additional studies have been performed by Chou and Qu [12] for nonlinear diffusion-convection equations and Estevez *et al* [13] for a porous medium equation, and Polyanin and Zhurov [14] for the axisymmetric unsteady boundary-layer equations, to name a few. Modern developments intended to reconcile (or not) separation of variables within the symmetry analysis framework appear to be ongoing, with much work currently emphasizing the use of Lie-Bäcklund and other non-point symmetries.

As in many other mathematics, physics, engineering, and computational fields, the separation of variables technique has concurrently found broad use with various equations describing neutron transport. For the linear neutron transport equation, separability, eigenvalues, and eigenfunctions are discussed at length by Bell and

Glasstone [15], Weinberg and Wigner [16], Lewins [17], and briefly by Keepin [18]. The same concepts in the context of neutron diffusion equations are motivated by Duderstadt and Hamilton [19], Hetrick [20], and many other texts that include a derivation of the point kinetics equations from diffusion theory. A time-separable assumption is key in these developments, as among other concepts it may be used to construct definitions of criticality. It can also be used in various schemes for the numerical solution of the neutron transport equation or its surrogates.

Symmetry analysis of the neutron transport equation or its surrogates has been conducted by, for example, Tsyfra and Czyzycki [21] and Giron *et al* [22]. Otherwise, the existing literature on the subject appears to be sparse, and aside from some commentary by Case and Zweifel [23] no specific connection appears to have been established between these results and those obtainable via the classical separation of variables technique. This outcome is the basis of the primary motivations of this work, which are twofold:

- (1) The explicit connection between the separation of variables ansatz commonly employed in the analysis of the neutron transport equation (or its surrogates) and the outcome of invariance under symmetry groups,
- (2) As a corollary to the above, an explicit characterization of the transformation properties of separable solutions of the neutron transport equation (or its surrogates), and the associated physical implications.

Along these lines, exhaustive symmetry analyses have been conducted on the linear heat equation (see, for example, Bluman and Anco [2] or Olver [5]), and categorization as outcomes of symmetry properties of its separable solutions in a variety of coordinate systems accomplished by Miller [8–10]. Given this existing body of work, a neutron diffusion equation—which is reducible to the linear heat equation by a point transformation—is chosen as a natural starting point for the broader investigation. Given the nature of this equation, emphasis will be placed throughout on space-time (multiplicative) separability.

The results of this work are novel and expected to be useful in that

- (1) They will provide an explicit understanding of the Lie symmetries that result in the usual space-time separable solution of the neutron diffusion equation or, conversely, the symmetries that solution should be expected to possess (e.g., whether or not they are scale-invariant, for the motivation of scaled experiments or the assessment of counterpart computational results),
- (2) Scenarios will also be identified where the usual space-time separable solutions do not exist, but others may be available based on the symmetry properties of the underlying PDEs. While a variety of such solutions have been demonstrated to exist in the context of the linear heat equation, they have not been widely disseminated in the context of neutron diffusion. ‘Unusual’ solutions may thus prove useful as test or benchmark problems for the verification of codes used to simulate the underlying equations,
- (3) Finally, this work is intended to lay the groundwork for future studies of more complicated (and realistic) mathematical models, which may manifest similar properties to those considered here (given that neutron diffusion may be viewed as a limit of more complicated neutron transport processes).

Though these results are rightfully expected to bear much resemblance to similar outcomes derived in the context of the linear heat equation, they represent a new, site-specific application of general ideas in the spirit of the studies performed by Chou and Qu [12], Estevez *et al* [13], Polyanin and Zhurov [14], and many other researchers who have investigated the fit of separation of variables techniques within the broader symmetry analysis framework.

In support of the above, a brief review of the relevant mathematical model is provided in section 2 (included a self-contained primer in differential geometry, which facilitates some of the analysis which follows; this is also an original contribution of this work). Section 3 includes the necessary background and outcomes of symmetry analysis, further substantiated by the examples provided in section 4. Conclusions and recommendations for future study are provided in section 5. Some expanded solutions of the underlying mathematical model are provided in two appendices.

## 2. Mathematical model

Of principal interest to this work is the one-dimensional (1D) time-dependent neutron diffusion equation as discussed by Duderstadt and Hamilton [19], Hetrick [20], and many other authors,

$$l_{\infty} \frac{\partial \varphi}{\partial t} = L^2 \frac{1}{r^n} \frac{\partial}{\partial r} \left( r^n \frac{\partial \varphi}{\partial r} \right) + \Delta k_{\infty} \varphi, \quad (1)$$

where

$r$  = generalized 1D space coordinate,  
 $t$  = time coordinate,  
 $\varphi(r, t)$  = neutron scalar flux,  
 $n = 0, 1, 2$  for 1D planar, cylindrical, or spherical symmetries,  
 $l_\infty$  = infinite medium neutron lifetime,  
 $L^2$  = diffusion area,  
 $\Delta k_\infty$  = excess infinite medium multiplication factor.

The parameters  $l_\infty$  and  $L^2$  will be taken as constants throughout this work. In section 4, two cases will be considered where  $\Delta k_\infty$  is allowed to vary as a function of time (consistent with similar work done in the point kinetics context by Hetrick [20], Keepin [18], and others).

For a well-posed problem, equation (1) must be supplemented by one initial condition and two boundary conditions. In general, these may be written as, respectively,

$$\varphi|_{t=t_0} = \varphi_0(r), \quad (2)$$

$$c_1 \left. \frac{\partial \varphi}{\partial r} + c_2 \varphi \right|_{r=r_1} = \varphi_1(t) \quad \text{and} \quad c_3 \left. \frac{\partial \varphi}{\partial r} + c_4 \varphi \right|_{r=r_2} = \varphi_2(t), \quad (3)$$

where  $t_0$  is an arbitrary reference time where  $\varphi_0(r)$  specifies the initial  $r$ -dependence of the flux, and  $r_1$  and  $r_2$  are arbitrary reference positions where  $\varphi_1(t)$  and  $\varphi_2(t)$  specify the  $t$ -dependence of linear combinations of the fluxes and their first derivatives, respectively (i.e.,  $c_1$ – $c_4$  are arbitrary constants).

As written, equations (1)–(3) represent a ‘monoenergetic’ model that is regarded as the purely isotropic scattering and flux limit of the more complicated neutron transport equation. The material properties appearing in equation (1) are implicitly taken as averages with respect to a prescribed neutron energy spectrum, and the energy dependence of  $\varphi$  is not explicitly represented; the scalar flux is assumed to be evaluated at either a discrete or averaged value of the neutron energy. While equation (1) may be of limited practical utility, the forthcoming analysis conducted in its context will reveal many concepts that may eventually be extended to more complicated and realistic models (e.g., multi-group  $P_N$  or  $S_N$  approximations).

## 2.1. Differential forms

Equation (1) will be subjected to symmetry analysis, so as to determine the correspondence between its space-time separable solutions and invariance under Lie groups of transformations. The generalized framework for conducting symmetry analysis is discussed in full detail by Ovsiannikov [1], Bluman and Anco [2], Ibragimov [3], Hydon [4], Olver [5], Cantwell [6], Stephani [7], and many others.

In 1971, using the language of differential forms, Harrison and Estabrook [24] formulated an alternative approach to traditional symmetry analysis methods. In this so-called ‘isovector’ context, the usual infinitesimal Lie symmetry generators are interpreted as Lie derivative or dragging operators in a manifold spanned by all independent and dependent variables native to a problem under investigation. Any differential equations being analyzed are re-cast as exterior differential systems (EDS), thus allowing for the leveraging of the commutability between the Lie and exterior derivative operations (among other advantages).

The isovector method has since been developed in rigorous detail and applied to multitudinous problems in science and engineering by authors including Harrison [25, 26], Edelen [27], and Suhubi [28]. It has been found that the principal advantages of the isovector method include (but are not necessarily limited to) a natural geometric setting from which to initiate symmetry analysis, and enabling a more elegant concurrent evaluation of extended infinitesimal symmetry generators in the analysis of differential structures.

In the spirit of this previous work: to write equation (1) in the language of differential forms, it must first be reduced to a first order system of PDEs. With

$$w = \frac{\partial \varphi}{\partial r}, \quad (4)$$

Equation (1) becomes

$$l_\infty \frac{\partial \varphi}{\partial t} = L^2 \left( \frac{\partial w}{\partial r} + \frac{nw}{r} \right) + \Delta k_\infty \varphi. \quad (5)$$

Equations (4) and (5)<sup>5</sup> may then be multiplied by the differential volume element  $dr \wedge dt$  to yield

$$\mu_1 = w dr \wedge dt - d\varphi \wedge dt, \quad (6)$$

<sup>5</sup> The procedure of recasting equation (1) as equations (4) and (5) is also widely used in the identification of potential symmetries of differential equations (see, for example, Bluman and Kumei [29]). However, the focus of the current work will be on the determination of point symmetries.

$$\mu_2 = l_\infty d\varphi \wedge dr + L^2 \left( dw \wedge dt + \frac{nw}{r} dr \wedge dt \right) + \Delta k_\infty \varphi dr \wedge dt. \quad (7)$$

Equations (6) and (7) are referred to as a system of 2-forms, where the operator  $\wedge$  used to multiply differentials is known as the wedge product. Its relevant properties are

$$dx \wedge dy = -dy \wedge dx, \quad (8)$$

$$dx \wedge dx = 0, \quad (9)$$

for any variables  $x$  and  $y$ . As written, equations (6) and (7) represent a system of differential objects on a higher-order manifold where  $r$ ,  $t$ ,  $\varphi$ , and  $w$  are regarded as entirely independent. To recover equations (4) and (5) from equations (6) and (7), let  $\varphi = \varphi(r, t)$  and  $w = w(r, t)$  (i.e., restricting to the sub-manifold or graph space in which equations (4) and (5) are defined). Expanding the total differentials and setting  $\mu_1 = 0$  and  $\mu_2 = 0$ , equations (6) and (7) become

$$w dr \wedge dt - \left( \frac{\partial \varphi}{\partial r} dr + \frac{\partial \varphi}{\partial t} dt \right) \wedge dt = 0, \quad (10)$$

$$l_\infty \left( \frac{\partial \varphi}{\partial r} dr + \frac{\partial \varphi}{\partial t} dt \right) \wedge dr + L^2 \left[ \left( \frac{\partial w}{\partial r} dr + \frac{\partial w}{\partial t} dt \right) \wedge dt + \frac{nw}{r} dr \wedge dt \right] + \Delta k_\infty \varphi dr \wedge dt = 0, \quad (11)$$

or, with equations (8) and (9),

$$\left( w - \frac{\partial \varphi}{\partial r} \right) dr \wedge dt = 0, \quad (12)$$

$$\left[ -l_\infty \frac{\partial \varphi}{\partial t} + L^2 \left( \frac{\partial w}{\partial r} + \frac{nw}{r} \right) + \Delta k_\infty \varphi \right] dr \wedge dt = 0, \quad (13)$$

the nontrivial solution of which is given by equations (4) and (5). Thus, equations (6) and (7) will be used for the symmetry analysis studies to follow.

### 3. Symmetry analysis

Symmetry analysis techniques as applied to the study of differential equations are in a sense a unification theory for disparate, otherwise ad hoc methods for the solution of those equations. The key point surrounding these techniques is that if a differential equation possesses symmetries, they enable a change of coordinates through which the original equation may either be reduced to a simpler structure [e.g., from a PDE to an ordinary differential equation (ODE)] or solved outright. As discussed in section 1, of principal interest to this work is the correspondence between symmetry reduction and separation of variables (which effectively reduces PDEs to systems of ODEs), in the context of equation (1). This exercise represents an application of the general ideas laid out by Miller [8–10] to a specific mathematical model, in the spirit of previous studies performed by Chou and Qu [12], Estevez *et al* [13], and Polyanin and Zhurov [14] for a variety of physically-motivated PDEs.

Put simply, a differential equation (or corresponding collection of objects such as an EDS) has a symmetry when there exists a coordinate transformation that leaves it invariant, or unchanged in structure when written in terms of the new variables. For example, in the context of section 2, for a generalized set of point transformations parameterized in terms of a parameter  $a$  given by

$$r_{\text{new}} = f_1(r, t, \varphi; a), \quad (14)$$

$$t_{\text{new}} = f_2(r, t, \varphi; a), \quad (15)$$

$$\varphi_{\text{new}} = f_3(r, t, \varphi; a), \quad (16)$$

$$w_{\text{new}} = f_4(r, t, \varphi, w; a), \quad (17)$$

a generalized EDS of the form

$$\mu_i = F_i(r, t, \varphi, w, dr, dt, d\varphi, dw), \quad i = 1, 2, 3, \dots, \quad (18)$$

is said to be invariant under equations (14)–(17) when

$$F_i(r_{\text{new}}, t_{\text{new}}, \varphi_{\text{new}}, w_{\text{new}}, dr_{\text{new}}, dt_{\text{new}}, d\varphi_{\text{new}}, dw_{\text{new}}) = F_i(r, t, \varphi, w, dr, dt, d\varphi, dw), \quad (19)$$

for all  $i$ . For simple coordinate transformations of the form indicated by equations (14)–(17) (e.g., translations or scaling), equation (19) often proves straightforward to evaluate directly. This will obviously not be the case when

equations (14)–(17) assume either nonlinear or generalized forms. In these cases, it often proves advantageous to formulate an infinitesimal representation of equation (19).

If the transformations given by equations (14)–(17) possess an identity element  $a_0$  such that  $f_1(r, t, \varphi; a_0) = r$ ,  $f_2(r, t, \varphi; a_0) = t$ , etc, then the left side of equation (19) may be expanded in a Taylor series about that element:

$$F_{i,\text{new}} = F_i + (a - a_0) \left. \frac{\partial F_i}{\partial a} \right|_{a=a_0} + \frac{1}{2} (a - a_0)^2 \left. \frac{\partial}{\partial a} \left( \left. \frac{\partial F_i}{\partial a} \right|_{a=a_0} \right) \right|_{a=a_0} + \dots, \tag{20}$$

for all  $i$ , and  $F_{\text{new}}$  taken as shorthand for the left side of equation (19). Using the chain rule,

$$F_{i,\text{new}} = F_i + (a - a_0) X^{(\text{pr})} F_i + \frac{1}{2} (a - a_0)^2 X^{(\text{pr})} (X^{(\text{pr})} F_i) + \dots, \tag{21}$$

where the operator  $X^{(\text{pr})}$  is

$$X^{(\text{pr})} = \xi_1 \frac{\partial}{\partial r} + \xi_2 \frac{\partial}{\partial t} + \xi_3 \frac{\partial}{\partial \varphi} + \xi_4 \frac{\partial}{\partial w}, \tag{22}$$

and the functions  $\xi_i$  are referred to as coordinate functions, defined by

$$\xi_1(r, t, \varphi) = \left. \frac{\partial r_{\text{new}}}{\partial a} \right|_{a=a_0}, \tag{23}$$

$$\xi_2(r, t, \varphi) = \left. \frac{\partial t_{\text{new}}}{\partial a} \right|_{a=a_0}, \tag{24}$$

$$\xi_3(r, t, \varphi) = \left. \frac{\partial \varphi_{\text{new}}}{\partial a} \right|_{a=a_0}, \tag{25}$$

$$\xi_4(r, t, \varphi, w) = \left. \frac{\partial w_{\text{new}}}{\partial a} \right|_{a=a_0}. \tag{26}$$

With equation (21), the invariance condition given by equation (19) becomes

$$(a - a_0) X^{(\text{pr})} F_i + \frac{1}{2} (a - a_0)^2 X^{(\text{pr})} (X^{(\text{pr})} F_i) + \dots = 0, \tag{27}$$

the nontrivial solution of which is

$$X^{(\text{pr})} F_i = 0, \tag{28}$$

for all  $i$ . Equations (27) and (28) show that the linear infinitesimal representation of invariance is sufficient for the determination of the global invariance properties of the functions  $F_i$ . If it is further required that  $F_i = 0$  for all  $i$  when equation (28) is satisfied, the symmetries arising from the evaluation of equation (28) will correspond to those possessed by the differential equation structure or equivalent EDS under investigation.

The operator  $X^{(\text{pr})}$  is variously referred to as the prolonged vector field generated by the group of transformations given by equations (14)–(17) (or, more compactly, the ‘prolonged group generator’). The terminology ‘prolonged’ refers to the presence of the basis element in the  $w$ -derivative; as  $w$  is interpreted as a derivative of  $\varphi$  via equation (4), its presence in equation (22) represents the extension of a ‘kernel’ group generator

$$X = \xi_1 \frac{\partial}{\partial r} + \xi_2 \frac{\partial}{\partial t} + \xi_3 \frac{\partial}{\partial \varphi}, \tag{29}$$

to a higher-order manifold containing derivative information.

As discussed by Olver [5], Harrison and Estabrook [24], Harrison [25, 26], Edelen [27], and Suhubi [28], given the suggestive vector basis notation in which the group generators  $X$  and  $X^{(\text{pr})}$  are written, they may also be interpreted as Lie derivative or dragging operators. This interpretation is convenient when coupled to an EDS structure, as exterior derivatives  $d$  and Lie derivatives  $X$  (prolonged or otherwise) commute:

$$XdF = dXF, \tag{30}$$

where  $F$  is any smooth function. As a result, the evaluation of structures such as equations (6) and (7) for invariance becomes straightforward: all original independent and dependent variables appearing in these relations are regarded as entirely independent of one another in the appropriate manifold, and any differential terms appearing within these expressions are treated using equation (30).

### 3.1. Similarity variables

Given a prolonged group generator of the form given by equation (22), a change of variables may be constructed from its kernel given by equation (29). The invariants of the group generated by equation (29) may be regarded as

similarity variables in terms of which the governing differential equation structure may be reformulated. Assuming the group generated by equation (29) is nontrivial (i.e., at least two of the  $\xi_i$  are nonzero), the change of variables arising from it leads to a reduction in order of the original differential equation.

The invariants of equation (29) are determined by evaluating the condition

$$XF(r, t, \varphi) = 0, \quad (31)$$

where  $F$  is an arbitrary smooth function of the indicated arguments. As equation (31) is a quasi-linear first-order PDE, it may be solved using the Method of Characteristics. The characteristic equations associated with equation (31) are

$$\frac{dr}{\xi_1} = \frac{dt}{\xi_2} = \frac{d\varphi}{\xi_3}. \quad (32)$$

For physical problems it is often the case that  $\xi_1 = \xi_1(r, t)$ ,  $\xi_2 = \xi_2(r, t)$ , and  $\xi_3 = \xi_3(r, t, \varphi)$ . It is then convenient to solve the first two members of equation (32), and either the first and third or second and third members as a simultaneous system. Assuming this system can be solved in closed form (which depends on the exact form of the coordinate functions), that solution can be represented as

$$g_1(r, t) = \text{const.} \quad \text{and} \quad g_2(r, t, \varphi) = \text{const.}, \quad (33)$$

where either the  $r$  or  $t$  dependence of the second member of equation (33) can be eliminated using the first member. The two constants of integration appearing in equation (33) are the invariants of the group generated by equation (29), and the functions  $g_1$  and  $g_2$  are in turn interpreted as a change of variables in terms of which the original differential equation may be reformulated.

In the current example, the first member of equation (33) leads to a new independent variable that is written in terms of the two original independent variables  $r$  and  $t$ . The second member of equation (33) defines a new dependent variable that is written in terms of the original dependent variable and one of the original independent variables.

### 3.2. Separation of variables

The classical (multiplicative; assumed throughout the remainder of this work) space-time separation of variables technique as applied to equation (1) assumes

$$\varphi(r, t) = R(r)T(t). \quad (34)$$

For certain forms of the material parameters  $l_\infty$ ,  $L^2$ , and  $\Delta k_\infty$ , the assumption of equation (34) allows for the decoupling of equation (1) into two ODEs: one each for  $R(r)$  and  $T(t)$ . The reduction of the PDE in this manner is similar in outcome to the symmetry analysis procedure discussed in sections 3 and 3.1.

The group generator corresponding to the general group of point transformations admitted by equation (1) is given by equation (29). The goal is to determine the specific forms of the coordinate functions  $\xi_1$ ,  $\xi_2$ , and  $\xi_3$  (and their global/physical interpretations) such that a structure equivalent to equation (34) is recovered as the corresponding change of variables. In this case, it may then be shown that equation (34) follows from a subset of the full range of symmetries represented by equation (29).

To begin, equation (34) indicates that combinations of  $r$  and  $t$  (or functions of those combinations; e.g.,  $r - ct$ , where  $c$  is a constant) do not appear in space-time separable solutions. However, since the invariant  $g_1$  appearing in equation (33) must be a function of at least one original independent variable, it then follows that either  $g_1 = r$  or  $g_1 = t$ . With equation (32), the only possible cases giving rise to these solutions are

$$\frac{dr}{0} = \frac{dt}{\xi_2} = \frac{d\varphi}{\xi_3} \quad \text{or} \quad \frac{dr}{\xi_1} = \frac{dt}{0} = \frac{d\varphi}{\xi_3}. \quad (35)$$

Following the discussion surrounding equation (33), the first member of equation (35) leaves  $r$  as the new independent similarity variable, while the second member correspondingly leaves  $t$ . As a result the function  $g_1$  appearing in equation (35) is trivial; the only achievable reduction using the symmetry analysis formalism is through the definition of a new dependent variable arising from  $g_2$  (obtained using the nontrivial solutions for  $\varphi$  of equation (35)).

From equation (35), the corresponding nontrivial ODEs for the new dependent similarity variables are

$$\frac{d\varphi}{dt} = \frac{\xi_3}{\xi_2} \quad \text{or} \quad \frac{d\varphi}{dr} = \frac{\xi_3}{\xi_1}, \quad (36)$$



respectively. However, with equation (34), these expressions become

$$\varphi \frac{d \ln T}{dt} = \frac{\xi_3}{\xi_2} \quad \text{or} \quad \varphi \frac{d \ln R}{dr} = \frac{\xi_3}{\xi_1}, \tag{37}$$

or, without loss of generality,

$$\varphi \tau(t) = \frac{\xi_3}{\xi_2} \quad \text{or} \quad \varphi \rho(r) = \frac{\xi_3}{\xi_1}, \tag{38}$$

where  $\tau$  and  $\rho$  are arbitrary functions of the indicated arguments. The representation of equation (29) associated with equation (38) is not unique, but two physically intuitive possibilities are

$$X = \tau(t) \frac{\partial}{\partial t} + \varphi \frac{\partial}{\partial \varphi} \quad \text{or} \quad X = \rho(r) \frac{\partial}{\partial r} + \varphi \frac{\partial}{\partial \varphi}. \tag{39}$$

Equation (39) demonstrates that space-time separable solutions of equation (1) feature a nonzero coordinate function in only one of the original independent variables; in fact, the nonzero coordinate function depends only on the variable associated with the vector basis function against which it is multiplied (e.g.,  $\rho$  multiplied against the  $r$ -derivative is a function of  $r$  only). Equation (39) indicates that space-time separable solutions of equation (1) also feature scaling in the dependent variable  $\varphi$ . This result is expected in light of the fact that equation (1) is both linear and homogeneous.

### 4. Examples

Following the developments of section 3, a general prolonged group generator from which space-time separable solutions of equation (1) may be constructed is

$$X^{(pr)} = \rho(r) \frac{\partial}{\partial r} + \tau(t) \frac{\partial}{\partial t} + \varphi \frac{\partial}{\partial \varphi} + \xi_4 \frac{\partial}{\partial w}, \tag{40}$$

provided that, in the course of the analysis, one and only one of  $\rho(r)$  and  $\tau(t)$  vanish.

Utilizing the differential form representation, the condition that equations (6) and (7) are invariant under the transformation group generated by equation (40) is

$$X^{(pr)} \mu_1 = 0 \quad \text{where} \quad \mu_1 = 0, \tag{41}$$

$$X^{(pr)} \mu_2 = 0 \quad \text{where} \quad \mu_2 = 0. \tag{42}$$

Assuming  $\Delta k_\infty = \Delta k_\infty(t)$  and using equation (30), equations (41) and (42) become, respectively,

$$X^{(pr)} w \, dr \wedge dt + w \, dX^{(pr)} r \wedge dt + w \, dr \wedge dX^{(pr)} t - dX^{(pr)} \varphi \wedge dt - d\varphi \wedge dX^{(pr)} t = 0, \tag{43}$$

$$\begin{aligned} & l_\infty(dX^{(pr)} \varphi \wedge dr + d\varphi \wedge dX^{(pr)} r) + L^2(dX^{(pr)} w \wedge dt + dw \wedge dX^{(pr)} t) \\ & + \frac{nL^2}{r} \left( X^{(pr)} w \, dr \wedge dt - \frac{wX^{(pr)} r}{r} \, dr \wedge dt + w \, dX^{(pr)} r \wedge dt + w \, dr \wedge dX^{(pr)} t \right) \\ & + X^{(pr)} \Delta k_\infty(t) \varphi \, dr \wedge dt + \Delta k_\infty(t) X^{(pr)} \varphi \, dr \wedge dt \\ & + \Delta k_\infty(t) \varphi \, dX^{(pr)} r \wedge dt + \Delta k_\infty(t) \varphi \, dr \wedge dX^{(pr)} t = 0. \end{aligned} \tag{44}$$

Evaluating the various Lie derivatives, equations (43) and (44) reduce to

$$\left[ \xi_4 + w \left( \frac{dw}{dr} - 1 \right) \right] dr \wedge dt = 0, \tag{45}$$

$$\begin{aligned} & -L^2 \left( 1 + \frac{d\varphi}{dr} \right) \left( dw \wedge dt + \frac{nw}{r} \, dr \wedge dt \right) - \left( 1 + \frac{d\varphi}{dr} \right) \Delta k_\infty(t) \varphi \, dr \wedge dt \\ & + L^2 \left[ \left( \frac{\partial \xi_4}{\partial r} \, dr + \frac{\partial \xi_4}{\partial \varphi} \, d\varphi + \frac{\partial \xi_4}{\partial w} \, dw \right) \wedge dt + \frac{d\tau}{dt} \, dw \wedge dt \right] \\ & + \frac{nL^2}{r} \left( \xi_4 - \frac{w\rho}{r} + w \frac{d\varphi}{dr} + w \frac{d\tau}{dt} \right) dr \wedge dt \\ & + \left[ \tau \frac{d\Delta k_\infty(t)}{dt} \varphi + \Delta k_\infty(t) \varphi + \Delta k_\infty(t) \varphi \frac{d\varphi}{dr} + \Delta k_\infty(t) \varphi \frac{d\tau}{dt} \right] dr \wedge dt = 0, \end{aligned} \tag{46}$$

where the conditions  $\mu_1 = 0$  and  $\mu_2 = 0$  have been used to eliminate one of the basis 2-forms appearing in each of equations (43) and (44), respectively.

For the case of point symmetries, equations (45) and (46) are nontrivially satisfied when they are identities in each unique basis 2-form appearing within them. From equation (45),

$$dr \wedge dt: \quad \xi_4 + w \left( \frac{d\rho}{dr} - 1 \right) = 0, \quad (47)$$

and from equation (46),

$$dw \wedge dt: \quad -1 - \frac{d\rho}{dr} + \frac{\partial \xi_4}{\partial w} + \frac{d\tau}{dt} = 0. \quad (48)$$

Together, these constraints yield

$$2 \frac{d\rho}{dr} = \frac{d\tau}{dt}. \quad (49)$$

Given that  $\rho$  is only a function of  $r$  and  $\tau$  is only a function of  $t$ , equation (49) is satisfied only when

$$\rho = \frac{a_1 r}{2} + a_2, \quad (50)$$

$$\tau = a_1 t + a_3, \quad (51)$$

where  $a_1 - a_3$  are constants to be determined. With equation (50), equation (47) yields

$$\xi_4 = \left( 1 - \frac{a_1}{2} \right) w. \quad (52)$$

With equations (50)–(52), equation (46) collapses to

$$\begin{aligned} & -\frac{nL^2 a_1}{2} \frac{w}{r} dr \wedge dt - nL^2 a_2 \frac{w}{r^2} dr \wedge dt \\ & + \left[ (a_1 t + a_3) \frac{d\Delta k_\infty(t)}{dt} + a_1 \Delta k_\infty(t) \right] \varphi dr \wedge dt = 0, \end{aligned} \quad (53)$$

therefore enabling

$$\frac{w}{r} dr \wedge dt: \quad n a_1 = 0, \quad (54)$$

$$\frac{w}{r^2} dr \wedge dt: \quad n a_2 = 0, \quad (55)$$

$$\varphi dr \wedge dt: \quad (a_1 t + a_3) \frac{d\Delta k_\infty(t)}{dt} + a_1 \Delta k_\infty(t) = 0. \quad (56)$$

The solution of equation (56) is given by

$$\Delta k_\infty(t) = \frac{a_4}{a_1 t + a_3}, \quad (57)$$

where  $a_4$  is an arbitrary constant of integration. Equation (57) represents the functional form in time the excess reactivity must take so as to enable either space/time scaling ( $a_1$ ) or time translation ( $a_3$ ) symmetries. If  $\Delta k_\infty(t)$  assumes any other functional form, it can still be made to satisfy the constraint given by equation (56) if  $a_1 = a_3 = 0$ .

In general, with the above equation (40) becomes

$$X^{(\text{pr})} = \left( \frac{a_1}{2} r + a_2 \right) \frac{\partial}{\partial r} + (a_1 t + a_3) \frac{\partial}{\partial t} + \varphi \frac{\partial}{\partial \varphi} + \left( 1 - \frac{a_1}{2} \right) w \frac{\partial}{\partial w}, \quad (58)$$

where the constants  $a_1 - a_3$  are controlled by the problem geometry  $n$  via equations (54) and (55), and the form of the excess multiplication via equation (56).

#### 4.1. Constant material property general solution

For a constant excess multiplication factor

$$\Delta k_\infty(t) = \text{const.} \neq 0, \quad (59)$$

$a_1 = 0$  as appearing in equation (57), and  $\Delta k_\infty = a_4/a_3$ . The group generator  $X$  becomes

$$X = a_2 \frac{\partial}{\partial r} + a_3 \frac{\partial}{\partial t} + \varphi \frac{\partial}{\partial \varphi}, \quad (60)$$

indicating the presence of space and time translations ( $a_2$  and  $a_3$ , respectively) in addition to flux scaling. Equation (60) holds for planar symmetry ( $n = 0$ ); for cylindrical or spherical symmetries ( $n = 1$  or  $2$ , respectively)  $a_2 = 0$  from equation (55). However, following from the developments of section 3.2, in order for

equation (60) to produce similarity variables that result in a space-time separable solution, one of  $a_2$  or  $a_3$  must be zero (see appendix A for a brief discussion of the non-separable case where both  $a_2$  and  $a_3$  are non-zero).

For example, let  $a_2 = 0$  (generalizing the cylindrical/spherical cases to planar geometry). Following the developments of section 3.1, the similarity variables associated with equation (60) are determined by solving the characteristic equations

$$\frac{dr}{0} = \frac{dt}{a_3} = \frac{d\varphi}{\varphi}, \quad (61)$$

a solution of which is

$$r = \text{const.} \quad \text{and} \quad \varphi \exp(-\alpha t) = \text{const.}, \quad (62)$$

where  $\alpha = 1/a_3$ . The integration constants appearing in equation (62) are the invariant functions of the group generated by equation (60); these constants may in turn be interpreted as the change of variables

$$r = \eta_1 \quad \text{and} \quad \varphi = \eta_2(\eta_1) \exp(\alpha t), \quad (63)$$

that in this case enable a substitution for  $\varphi$  that may be used in conjunction with equation (1). This substitution is the classical space-time separable assumption, resulting in the solution to equation (1) derived by Duderstadt and Hamilton [19], Hetrick [20] and many other authors.

Moreover, with equation (60), this result indicates that the classical space-time separable solution associated with equation (1) corresponds to time translation and flux scaling groups. This result would perhaps have been intuitively obvious by making the appropriate substitutions in the separable solution of equation (1), namely

$$t_{\text{new}} = t + t_0, \quad (64)$$

$$\varphi_{\text{new}} = s_0 \varphi, \quad (65)$$

and the resulting expression checked for invariance under the presence of the arbitrary constants  $t_0$  and  $s_0$ . However, the current result indicates the symmetry group given by equation (60) and  $a_2 = 0$  is entirely responsible for the classical space-time separable solution of equation (1).

#### 4.2. Linear transient general solution

A linear excess multiplication transient is given by

$$\Delta k_{\infty}(t) = bt, \quad (66)$$

for  $b = \text{const.} > 0$ . Given that this form of  $\Delta k_{\infty}(t)$  does not correspond to equation (57), its consistency with equation (56) requires  $a_1 = a_3 = 0$ . The associated group generator  $X$  becomes

$$X = a_2 \frac{\partial}{\partial r} + \varphi \frac{\partial}{\partial \varphi}, \quad (67)$$

which with equation (55) indicates  $n = 0$  for the presence of nontrivial space translation and flux scaling symmetries.

As in section 4.1, the similarity variables associated with equation (67) are determined by solving the characteristic equations

$$\frac{dr}{a_2} = \frac{dt}{0} = \frac{d\varphi}{\varphi}, \quad (68)$$

a solution of which is

$$t = \text{const.} \quad \text{and} \quad \varphi \exp(-\beta r) = \text{const.}, \quad (69)$$

where  $\beta = 1/a_2$ . The integration constants appearing in equation (69) are the invariant functions of the group generated by equation (67); these constants may in turn be interpreted as the change of variables

$$t = \eta_1 \quad \text{and} \quad \varphi = \eta_2(\eta_1) \exp(\beta r), \quad (70)$$

that in this case enable a substitution for  $\varphi$  that may be used in conjunction with equation (1). The associated similarity solution of equation (1) does not appear to have been widely disseminated, and is discussed in more detail in appendix B.

With equation (67), this result indicates that this space-time separable solution associated with equation (1) corresponds to space translation and flux scaling groups:

$$r_{\text{new}} = r + r_0, \quad (71)$$

$$\varphi_{\text{new}} = s_0 \varphi. \quad (72)$$

### 4.3. Invariant initial and boundary conditions

The initial and boundary conditions associated with equation (1) must also be invariant under equation (58) for its space-time separable solution to remain available. Consistent with the developments of section 3, applying equation (58) to equation (2) (under the assumption that equation (2) holds) yields

$$\varphi_0 - \left( \frac{a_1}{2}r + a_2 \right) \frac{\partial \varphi_0}{\partial r} = 0, \quad (73)$$

the solution of which for  $\varphi_0$  is given by

$$\varphi_0 = c(2a_2 + a_1 r)^{2/a_1}, \quad (74)$$

where  $c$  is an arbitrary constant. The initial space-dependence of  $\varphi$  must be of the form indicated by equation (74) for the space-time separable solution generated by equation (58) to exist.

Similarly, applying equation (58) to equation (3) (under the assumption that equation (3) holds) yields

$$\begin{aligned} \left(1 - \frac{a_1}{2}\right)\varphi_1 + \frac{a_1 c_2}{2}\varphi \Big|_{r=r_1} & \quad \text{and} \quad \left(1 - \frac{a_1}{2}\right)\varphi_2 + \frac{a_1 c_4}{2}\varphi \Big|_{r=r_2} \\ & = (a_1 + a_3 t) \frac{d\varphi_1}{dt} \quad \quad \quad = (a_1 + a_3 t) \frac{d\varphi_2}{dt}, \end{aligned} \quad (75)$$

where equation (4) has also been used to express the  $r$ -derivative of  $\varphi$  as  $w$ . For both members of equation (75) to collapse to ODEs for the functions  $\varphi_1(t)$  and  $\varphi_2(t)$ , it is immediately evident that either  $a_1 = 0$ , or  $c_2 = 0$  and  $c_4 = 0$ . In either case, the solutions of equation (75) are then given by

$$\varphi_1 = \varphi_2 = c(a_1 + a_3 t)^{\frac{2-a_1}{2a_3}}, \quad (76)$$

where  $c$  is an arbitrary constant. The boundary time-dependence of  $\varphi$  must be of the form indicated by equation (76) for the space-time separable solution generated by equation (60) to exist.

As noted in section 4, for a space-time separable solution to exist, either  $\rho(r)$  or  $\tau(t)$  as appearing in equation (40) must vanish. As a result, either  $a_1 r/2 + a_2 = 0$  or  $a_1 t + a_3 = 0$  in equations (73)–(76), depending on both the underlying constraints of the specific problem being investigated, or the character of the desired space-time separable solution. These conditions more strongly constrain the allowable forms of  $\varphi_0$ ,  $\varphi_1$ , and  $\varphi_2$  appearing in equations (74) and (76).

Appendices A and B include examples of application of these boundary conditions to some problems of interest.

## 5. Conclusions

The space-time separable solution of the neutron diffusion equation (1) is pervasive in nuclear engineering texts, but its connection to the broader outcomes of symmetry analysis of that equation is heretofore unexplored directly, well-established connections to the linear heat equation notwithstanding. In the current work, space-time separable solutions of equation (1) are found to correspond to an infinitesimal symmetry generator of the form given by equation (40). The symmetries that generate space-time separable solutions include scaling in the dependent flux variable plus

- (1) A transformation in the independent space variable ( $r$ ) that depends only on  $r$ , but is otherwise arbitrary, or,
- (2) A transformation in the independent time variable ( $t$ ) that depends only on  $t$ , but is otherwise arbitrary.

The correspondences noted above are constructed by viewing the usual space-time separable ansatz of equation (1) as a change of variables that reduces the original PDE to two ODEs. This outcome is similar to that of symmetry analysis, which also results in the reduction of more ‘complicated’ structures to ostensibly simpler forms. By interpreting the separability assumption given by equation (34) as a similarity transformation, it may be viewed as originating from a group generator structure that is itself a representation of a symmetry.

Two examples of this phenomenon are provided in sections 4.1 and 4.2, indicating that the usual space-time separable solution of equation (1) follows from Case (2) above, with the transformation in time corresponding to translation. However, neither this solution nor its feature of space-time separability is unique. Though this outcome has not been widely disseminated, it is not entirely unexpected in light of the multitudinous solutions of the linear heat equation as derived, for example, by Bluman and Anco [2] and Olver [5]. The practical relevance or computational utility of these additional solutions remains to be investigated.

### 5.1. Recommendations for future work

The practical utility of the mathematical model represented by equation (1) is limited, as discussed in section 2. However, the results of this work may likely be extended to a variety of scenarios, including those most accurately modeled by the linear neutron transport equation:

$$\begin{aligned} & \frac{1}{v(E)} \frac{\partial \Psi(r, t, \Omega, E)}{\partial t} + \Omega \cdot \nabla_r \Psi(r, t, \Omega, E) + \Sigma_T(r, t, E) \Psi(r, t, \Omega, E) \\ & = \int dE' \int d\Omega' [\chi(E') \nu(E') \Sigma_F(r, t, E') + \Sigma_S(r, t, \Omega' \rightarrow \Omega, E' \rightarrow E)] \Psi(r, t, \Omega', E'), \end{aligned} \quad (77)$$

where

- $r$  = space coordinate vector,
- $t$  = time coordinate,
- $\Omega$  = neutron direction vector,
- $E$  = neutron energy,
- $\Psi$  = neutron angular flux,
- $v$  = neutron speed,
- $\Sigma_T$  = macroscopic total interaction cross section,
- $\Sigma_F$  = macroscopic fission cross section,
- $\Sigma_S$  = macroscopic scattering cross section,
- $\chi$  = fission neutron energy spectrum,
- $\nu$  = mean number of neutrons released per fission.

As discussed by Bell and Glasstone [15], some methods used for the solution of equation (1) may also be used with equation (77), including the separation of variables technique. Indeed, solutions of the form

$$\Psi(r, t, \Omega, E) = \psi(r, \Omega, E) \exp(\alpha t), \quad (78)$$

are often employed in the analytical treatment or numerical solution of equation (77). As seen in section 4.1, the existence of a time-separable solution of this form is connected with the invariance of equation (77) under time translation and flux scaling symmetries:

$$X = a_3 \frac{\partial}{\partial t} + \Psi \frac{\partial}{\partial \Psi}. \quad (79)$$

If the material parameters appearing in equation (77) ( $v$ ,  $\chi$ ,  $\nu$ , and the various  $\Sigma$ 's) do not depend on  $t$ , it is clear by inspection that equation (77) admits these symmetries (equation (77) also admits space translation symmetries if its material properties do not depend on  $r$ ). A more rigorous investigation of this outcome in the context of the broader symmetry set admitted by equation (77) constitutes a logical course of future study.

Moreover, while spherical harmonics expansions given by

$$\Psi(r, t, \Omega, E) = \sum_{i=0}^{\infty} \sum_{j=-i}^i \psi_i^j(r, t, E) Y_i^j(\Omega), \quad (80)$$

are also often employed in the solution of equation (77), their explicit connection to a symmetry reduction has yet to be investigated. The same is true of  $S_N$  methods or multi-group methods for handling the angular and energy-dependence of equation (77), respectively; given that these methods are approximate, their connection to exact symmetries may not be as straightforward to apprehend.

## Acknowledgments

This work has been authored by employees of Los Alamos National Security, LLC, operator of the Los Alamos National Laboratory under Contract No. DE-AC52-06NA25396 with the U.S. Department of Energy. The United States Government retains and the publisher, by accepting this work for publication, acknowledges that the United States Government retains a nonexclusive, paid-up, irrevocable, world-wide license to publish or reproduce this work, or allow others to do so for United States Government purposes. The authors thank EJ Albright, R Baty, Z Boyd, J Giron, P Jaegers, E Schmidt, and J Schmidt for valuable insights on these topics. SDR also acknowledges the long and expert tutelage of Prof. R Axford on the subject of symmetry analysis methods.

## Appendix A. A non-separable solution of PLANAR equation (1) with constant material properties

For equation (1) with  $n = 0$  and constant material properties,  $\Delta k_{\infty} = a_4/a_3$  as discussed in section 4.1. A symmetry generator associated with this case is given by equation (60). The similarity variables associated with

equation (60) are determined by solving the characteristic equations

$$\frac{dr}{a_2} = \frac{dt}{a_3} = \frac{d\varphi}{\varphi}, \quad (\text{A.1})$$

a solution of which is given by

$$r - a_2\alpha t = \text{const.} \quad \text{and} \quad \varphi \exp(-\alpha t) = \text{const.} \quad (\text{A.2})$$

where  $\alpha = 1/a_3$ . The integration constants appearing in equation (A.2) are the invariant functions of the group generated by equation (60); these constants may in turn be interpreted as the change of variables

$$\eta_1 = r - a_2\alpha t \quad \text{and} \quad \eta_2(\eta_1) = \varphi \exp(-\alpha t), \quad (\text{A.3})$$

that in this case enable a substitution for  $\varphi$  that may be used in conjunction with equation (1). With equation (A.3), the derivatives appearing in equation (1) may be written as

$$\frac{\partial \varphi}{\partial t} = \left( \alpha \eta_2 - a_2 \alpha \frac{d\eta_2}{d\eta_1} \right) \exp(\alpha t), \quad (\text{A.4})$$

$$\frac{\partial^2 \varphi}{\partial r^2} = \frac{d^2 \eta_2}{d\eta_1^2} \exp(\alpha t), \quad (\text{A.5})$$

so that equation (1) becomes

$$\frac{d^2 \eta_2}{d\eta_1^2} + 2\delta \frac{d\eta_2}{d\eta_1} + \omega^2 \eta_2 = 0, \quad (\text{A.6})$$

where

$$\delta = \frac{l_\infty a_2 \alpha}{2L^2}, \quad (\text{A.7})$$

$$\omega^2 = \frac{(a_4 - l_\infty) \alpha}{L^2}. \quad (\text{A.8})$$

The solution of equation (A.6) is given by

$$\eta_2 = C_1 \exp[(-\delta + \sqrt{\delta^2 - \omega^2})\eta_1] + C_2 \exp[(-\delta - \sqrt{\delta^2 - \omega^2})\eta_1], \quad (\text{A.9})$$

where  $C_1$  and  $C_2$  are arbitrary constants. Transforming back to physical variables using equation (A.3), equation (A.9) becomes

$$\varphi = \{ C_1 \exp[(-\delta + \sqrt{\delta^2 - \omega^2})(r - a_2\alpha t)] + C_2 \exp[(-\delta - \sqrt{\delta^2 - \omega^2})(r - a_2\alpha t)] \} \exp(\alpha t). \quad (\text{A.10})$$

The solution indicated by equation (A.10) is not space-time separable in general. It instead includes terms of the form  $\exp(r \pm ct)$ , where the propagation speed  $c$  of these traveling wave type solutions is comprised of the various material constant data appearing in the problem formulation. In equation (A.10), these traveling wave components are multiplied by the global scale factor  $\exp(\alpha t)$ , which is typical of parabolic PDEs. These types of solutions have been shown to exist for diffusion-type equations by Olver [5], and their physical interpretation has been discussed by Moore [30]. In short, it appears solutions of this type feature not only the usual global time dependence, but some traveling wave-like structure.

The traditional separable solution of equation (1) can be recovered by setting  $a_2 = 0$  [whence,  $\delta = 0$  via equation (A.7)] in equation (A.10) to yield

$$\varphi = [C_1 \exp(i\omega r) + C_2 \exp(-i\omega r)] \exp(\alpha t), \quad (\text{A.11})$$

where  $i$  is the imaginary unit.

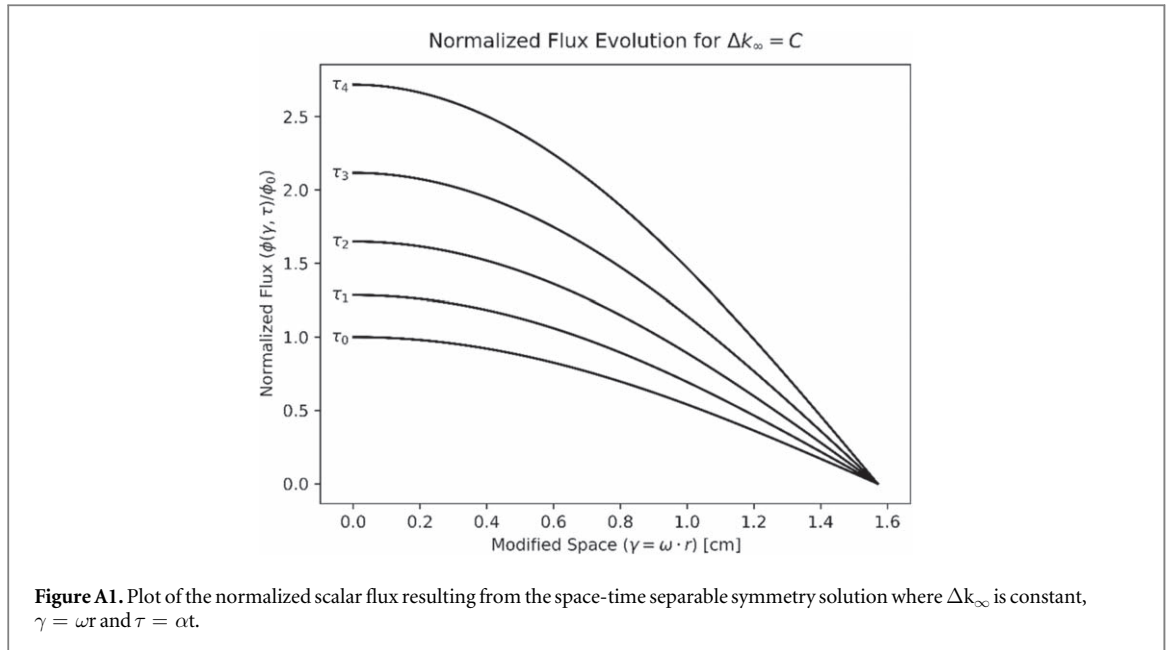
Naturally, equation (A.10) may also be found using classical methods for solving linear PDEs. Perhaps the only advantage to constructing such solutions using symmetry analysis methods lies in the ability to *a priori* build solutions with desired properties (in the current case, the translation and scaling symmetries from which the solution follows).

Numerical results from equation (A.11) are readily obtainable after application of the correct boundary conditions. Specifically, consider the alternative representation of equation (A.11):

$$\varphi(\gamma, \tau) = \varphi_0 \cos(\gamma) \exp(\tau), \quad (\text{A.12})$$

where  $\gamma = \omega r$  and  $\tau = \alpha t$ , with the group-invariant boundary conditions

$$\frac{\varphi(\gamma, \tau_0 = 0)}{\varphi_0} = \cos(\gamma), \quad (\text{A.13})$$



$$\frac{\varphi(\gamma_L = 0, \tau)}{\varphi_0} = \exp(\tau), \quad (\text{A.14})$$

$$\frac{\varphi(\gamma_R = \pi/2, \tau)}{\varphi_0} = 0. \quad (\text{A.15})$$

Comparison of equations (A.13)–(A.15) with the group invariant diffusion solution found in equation (A.12) reveals that the boundary conditions preserve the translation symmetries emitted by equation (60). With this, the time evolution scheme of the scalar flux on  $0 \leq \gamma \leq \pi/2$  and  $0 \leq \tau \leq 1$  results in figure A1.

## Appendix B. Separable solution of planar equation (1) with a linear transient

For equation (1) with  $n = 0$  and the linear excess multiplication transient given by equation (66), the admissible group generator is given by equation (67). The similarity variables associated with equation (67) are given by equation (70).

With equation (70), the derivatives appearing in equation (1) may be written as

$$\frac{\partial \varphi}{\partial t} = \exp(\beta r) \frac{d\eta_2}{d\eta_1}, \quad (\text{B.1})$$

$$\frac{\partial^2 \varphi}{\partial r^2} = \beta^2 \exp(\beta r) \eta_2, \quad (\text{B.2})$$

so that equation (1) becomes the first order ODE

$$l_\infty \frac{d\eta_2}{d\eta_1} = (L^2 \beta^2 + b \eta_1) \eta_2, \quad (\text{B.3})$$

the solution of which is given by

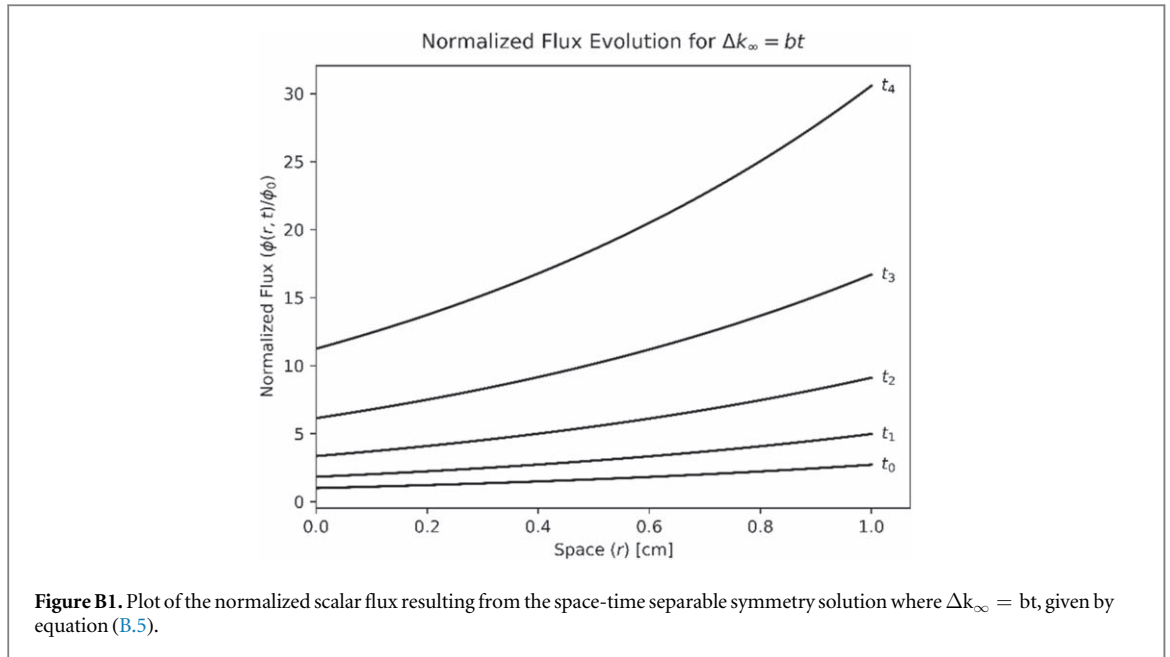
$$\eta_2 = C \exp\left(\frac{L^2 \beta^2}{l_\infty} \eta_1 + \frac{b}{2l_\infty} \eta_1^2\right), \quad (\text{B.4})$$

where  $C$  is an arbitrary constant of integration. Using equation (70), equation (B.4) may be transformed back to physical variables:

$$\varphi = C \exp\left(\frac{L^2 \beta^2}{l_\infty} t + \frac{b}{2l_\infty} t^2 + \beta r\right), \quad (\text{B.5})$$

which is a space-time separable solution as expected.

The space-time separability indicated by equation (B.5) is different than that demonstrated in section 4 and appendix A: in addition to the usual multiplicative time factor of the order  $\exp(t)$ , an additional multiplicative



factor of the order  $\exp(t^2)$  is present. This ‘faster’ rate of rise in  $\varphi$  owes to the time dependence imposed on  $\Delta k_\infty$ . Since the excess multiplication is increasing linearly in time, the associated flux rises at a concordantly greater rate.

As was the case with the solution derived in appendix A, equation (B.5) may also be found using classical methods for solving linear PDEs. The potential advantage in using symmetry analysis techniques is the same as discussed previously.

Finally, as demonstrated in appendix A, the invariant solution given by equation (B.5) can be expressed numerically given the correct boundary conditions. For example, for an insertion rate given by  $b = 0.0001 \text{ s}^{-1}$ ,  $\nu = 2.35$  neutrons per fission,  $l_\infty = 10^{-6} \text{ s}$ , diffusion coefficient  $D = 1.1 \text{ cm}$ ,  $\Sigma_f = 0.073 \text{ cm}^{-1}$ , and  $\beta = 1$  at thermal neutron energies, it follows that  $\Sigma_a = 1/(vl_\infty)$ , and  $L^2 = D/\Sigma_a$  so that

$$\frac{L^2\beta^2}{l_\infty} = 2.42 \times 10^5 \text{ s}^{-1} \quad (\text{B.6})$$

$$\frac{b}{2l_\infty} = 50.0 \text{ s}^{-2} \quad (\text{B.7})$$

$$\beta = 1.0 \text{ cm}^{-1}. \quad (\text{B.8})$$

The above parameters give a scalar flux that increases exponentially in both time and space, as equation (B.5) describes.

Again, invariant boundary conditions must be implemented to preserve the symmetries emitted by equation (67). Similar to the method found in appendix A, the group-invariant initial and boundary conditions are

$$\frac{\varphi(r, t_0 = 0)}{\varphi_0} = \exp(\beta r), \quad (\text{B.9})$$

$$\frac{\varphi(r_L = 0, t)}{\varphi_0} = \exp\left(\frac{L^2\beta^2}{l_\infty}t + \frac{b}{2l_\infty}t^2\right), \quad (\text{B.10})$$

$$\frac{\varphi_r(r_L = 0, t)}{\varphi_0} = \beta \exp\left(\frac{L^2\beta^2}{l_\infty}t + \frac{b}{2l_\infty}t^2\right). \quad (\text{B.11})$$

Note that the boundary conditions found in equations (B.9)–(B.11) and equations (A.13)–(A.15) are not unique. Several different boundary conditions can be implemented as long as each satisfies the requirements outlined in section 4.3. A plot using the above parameters and the invariant solution, equation (B.5), is given in figure B1.

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