Three-body Faddeev equations in two-particle Alt-Grassberger-Sandhas form with Distorted-Wave-Born-approximation amplitudes as effective potentials

A. M. Mukhamedzhanov[1]

[1] *Cyclotron Institute, Texas A&M University, College Station, TX 77843, USA*

$A(d,p)B$ reactions on heavier nuclei are peripheral at sub-Coulomb energies and can be peripheral even at energies above the Coulomb barrier due to the presence of the distorted waves in the initial and final channels. Usually, to analyze such reactions the distorted-wave-Born-approximation (DWBA) is used. The DWBA amplitude for peripheral reactions is parametrized in terms of the asymptotic normalization coefficient (ANC) of the bound state $B = (n\,A)$. However, in the DWBA, the coupling of the different channels is not taken into account explicitly. The three-body Faddeev equations written as the two-particle Alt-Grassberger-Sandhas (AGS) equations are very suitable for the analysis of the $A(d,p)B$ reactions because they take into account explicitly the coupling of the different channels. It is well known that the overall normalization of the DWBA amplitude or even of the more advanced continuum-discretized coupled channels (CDCC) amplitude for peripheral reactions (both sub-Coulomb and above the Coulomb barrier) is determined by the ANC. However, it is not apparent that it is the case for the AGS solution due to the coupling of the (d,p) channel to other channels. In this paper, it is proved that the sub-Coulomb $A(d,p)B$ reaction amplitude, which is a solution of the two-body AGS equations, is peripheral and is parametrized in terms of the ANC of the bound state $B = (nA)$ if the corresponding DWBA amplitude is peripheral. Both nonlocal separable and local nuclear interaction potentials between the constituent particles are considered. To prove the peripheral character of the AGS amplitude for the sub-Coulomb $A(d,p)B$ reactions the effective potentials are expressed in terms of the corresponding sub-Coulomb DWBA amplitudes of the different channels. The analysis of the $A(d,p)B$ reactions above the Coulomb barrier requires the inclusion of the optical potentials. Hence, to analyze such reactions, the AGS equations are generalized by including the optical nuclear potentials in the same manner as it is done in the DWBA. The obtained AGS equations contain the DWBA effective potentials with distorted waves generated by the sum of the nuclear optical and the channel Coulomb potentials. It is shown that if the DWBA amplitude is peripheral than it is also the case for the AGS amplitude, which is also parametrized in terms of the ANC of the bound state $B = (nA)$. The inclusion of the coupling of the different channels in the AGS formalism allows one to improve the treatment of the peripheral sub-Coulomb and above the Coulomb (d,p) reactions compared to the DWBA and CDCC method.

PACS numbers: 21.45.v, 24.10.i, 25.45.z,21.10.Jx

I. INTRODUCTION

Peripheral $A(d,p)B$ reactions, asymptotic normalization coefficients (ANCs) of the final nuclei bound states, which play an important role in nuclear astrophysics [1]. Moreover, the deuteron stripping reactions on unstable nuclei A in the inverse kinematics provide a unique tool to obtain the spectroscopic information about the $(n\,A)$ bound states and resonances.

Usually, for analysis of such reactions the standard distorted-wave-Born-approximation (DWBA) is used. For the energies above the Coulomb barrier also a more advanced, the continuum-discretized-coupled-channel (CDCC) method or its simplified version, adiabatic distorted wave (ADWA) method, are used [2, 3]. While in the DWBA the explicit coupling of the (d,p) channel to other channels is not taken into account, the CDCC includes coupling of the (d,p) channel to the deuteron breakup one. Despite of this coupling, the CDCC amplitude is still parametrized in terms of the ANC of the bound state $B = (nA)$.

The way to find the transfer reaction amplitude unambiguously was suggested by Faddeev [4] by using the coupled Faddeev integro-differential equations in the three-body problem. This seminal work by Faddeev showed how to solve exactly the three-body quantum-mechanical problem and opened a new field in physics: few-body physics. In his original work Faddeev considered $3\,particles \to 3\,particles$ case. Later on in [5] Alt, Grassberger and Sandhas (AGS) modified the Faddeev equations by transforming them into two-particle ones describing $2\,particles \to 2\,particles$ processes, which are called the Faddeev equations in the AGS form. An important advantage of the AGS formalism is that it reduces the three-body Faddeev equations to the two-particle form when the separable potentials are used. In [6] the AGS equations were modified by including the Coulomb interaction for the processes involving two charged particles and a neutron.

The AGS equations offer a very effective way to check the accuracy of the DWBA and CDCC for the peripheral $A(d,p)B$ reactions. One of the most challenging problem is that whether the AGS equations for peripheral reactions can be parametrized in terms of the ANCs. This is not a trivial question because a solution of the AGS equations for the $A(d,p)B$ reactions is coupled to the other channels whose amplitudes do not contain the bound-state wave function of the final nucleus B.

In this paper the formalism of the two-body AGS equations is used to investigate whether their solution for the $A(d,p)B$ peripheral reactions in the DWBA remains also peripheral, that is, the AGS solution for the $A(d,p)B$ channel can be parametrized in terms of ANC of the bound state $B=(nA)$. For the sub-Coulomb processes both nonlocal separable and local nuclear interaction potentials between the constituent particles are used. For the reactions above the Coulomb barrier only the local nuclear potentials are considered. To compare with the DWBA formalism for the energies above the Coulomb barrier the two-body AGS formalism is generalized by including the optical potentials in the same manner as it is done in the DWBA. The effective potentials in the new generalized AGS equations are themselves the reaction amplitudes and are expressed in terms of the DWBA amplitudes of the involved channels. It is shown that the AGS solution in the $A(d,p)B$ channel is peripheral if the corresponding DWBA amplitude is peripheral. Hence, the AGS solution for the $A(d,p)B$ channel can be parametrized in terms of the ANC for the bound state $B=(nA)$ although its dependence on the ANC, in contrast to the DWBA, not necessarily is linear.

The system of units in which $\hbar = c = 1$ is used throughout the paper.

II. STANDARD APPROACH

Let us consider the transfer reaction in the three-body model of three non-identical constituent structureless particles:

$$\alpha + (\beta\,\gamma) \to \beta + (\alpha\,\gamma), \tag{1}$$

where $(\beta\,\gamma)$ is the bound state of particles β and γ. The general expression for the reaction amplitude in the center-off-mass (c.m.) of reaction (1) in the three-body model for screened Coulomb potentials is

$$\mathcal{T}_{\beta\,\alpha}(\mathbf{q}_\beta,\,\mathbf{q}_\alpha;z) =< \psi^{(0)}_{\mathbf{q}_\beta}\,\varphi_\beta\big|U^{(+)}_{\beta\,\alpha}(z)\big|\varphi_\alpha\,\psi^{(0)}_{\mathbf{q}_\alpha}>, \tag{2}$$

$$U^{(+)}_{\beta\,\alpha}(z) = \overline{V}_\beta + \overline{V}_\beta\,G(z)\,\overline{V}_\alpha \tag{3}$$

is the transition operator, $V = \sum\limits_{\nu=\alpha,\,\beta\,\gamma} V_\nu$,

$$G(z) = \frac{1}{z-K-V} \tag{4}$$

is the three-body Green function resolvent, E and K are the total energy and kinetic energy operator of the three-body system. Note that Eq. (2) for the reaction amplitude with the plane-wave $\psi^{(0)}_{\mathbf{q}_\beta}$ in the bra-state is valid only for the screen Coulomb potentials. In what follows only the screened Coulomb potentials are used.

Also the following supplemental notations usually accepted in few-body papers are used: for a one-body quantity an index α characterizes the particle α, for a two-body quantity the pair of particles $(\beta+\gamma)$, with $\beta,\,\gamma\neq\alpha$ and finally for a three-body quantity the two-fragment partition $\alpha+(\beta\,\gamma)$ describing free particle α and the bound state $(\beta\,\gamma)$. $\psi^{(0)}_{\mathbf{q}_\alpha}$ is the plane wave describing the relative motion of particles α and the bound state $(\beta\,\gamma)$ of pair α with the relative momentum $\mathbf{q}_\alpha$, φ_α is the bound state of particles of the pair α.

$$\begin{aligned}&\overline{V}_\alpha = V - V_\alpha,\\ &V_\alpha \equiv V_{\beta\,\gamma} = V^N_\alpha + V^C_\alpha,\end{aligned} \tag{5}$$

where $V^N_\alpha \equiv V^N_{\beta\,\gamma}$ and $V^C_\alpha \equiv V^C_{\beta\,\gamma}$ are the nuclear and screened Coulomb interaction potentials of particles of pair α. Note that the plane waves in Eq. (2) appear only for the screened Coulomb potentials.

Taking into account that

$$\Psi^{(+)}_\alpha = \big(1+G(E)\,\overline{V}_\alpha\big)\,\varphi_\alpha\,\psi^{(0)}_{\mathbf{q}_\alpha} \tag{6}$$

one gets for the reaction amplitude

$$\mathcal{T}_{\beta\,\alpha}(\mathbf{q}_\beta,\,\mathbf{q}_\alpha;E) =< \psi^{(0)}_{\mathbf{q}_\beta}\,\varphi_\beta\big|\overline{V}_\beta\big|\Psi^{(+)}_\alpha>, \tag{7}$$

where $\overline{V}_\beta = V_\alpha + V_\gamma$.

Using the two-potential formula I can rewrite Eq. (7) as

$$\mathcal{T}_{\beta\,\alpha}(\mathbf{q}_\beta,\,\mathbf{q}_\alpha;E) =< \psi_{\gamma,\beta}\,\big|V_\alpha\big|\Psi^{(+)}_\alpha>, \tag{8}$$

where $<\psi_{\gamma,\beta}$ is a solution of the Schrödinger equation

$$<\psi_{\gamma,\beta}|(E-K-\overline{V}_\alpha) = 0. \tag{9}$$

Specifically for the $A(d,p)B$ reaction Eq. (10) in the standard notations takes the form:

$$\mathcal{T}_{p\,d}(\mathbf{q}_{pB},\,\mathbf{q}_{dA};\,E) =< \psi_{n,B}\,\big|V_{pn}\big|\Psi^{(+)}_{dA}>, \tag{10}$$

Here, $\mathbf{q}_{ij}$ is the relative momentum of particles i and j, $B=(n\,A)$,

$$<\psi_{n,B}|(E-K-V_{pA}-V_{nA}) = 0. \tag{11}$$

Equation (11) was derived in [7].

To calculate $\mathcal{T}_{\beta\,\alpha}$ one needs to find two three-body wave functions, $\Psi^{(+)}_\alpha$ and $\psi_{\gamma,\beta}$. For example, the exact scattering wave function in the initial channel $\Psi^{(+)}_\alpha$ is a solution of the equation

$$\Psi^{(+)}_\alpha = \varphi_\alpha\,\psi^{(0)}_\alpha + G_\alpha(E)\,\overline{V}_\alpha\,\Psi^{(+)}_\alpha, \tag{12}$$

where

$$G_\alpha(z) = \frac{1}{z-K-V_\alpha}. \tag{13}$$

This equation does not have a unique solution because one can add to $\Psi_\alpha^{(+)}$ a linear combination of solutions of the homogeneous equations

$$\Psi_\nu^{(+)} = G_\alpha(E)\,\overline{V}_\alpha\,\Psi_\nu, \quad \nu = \beta,\,\gamma \neq \alpha. \tag{14}$$

Faddeev equations provide a tool to find the three-body wave functions. Below the Faddeev equations for the $A(d,p)B$ reaction amplitude are considered in the AGS form.

III. AGS EQUATIONS WITH SEPARABLE POTENTIALS

Let me consider the system of three distinguishable constituent particles, 1, 2, 3 with masses m_ν, $\nu = 1, 2, 3$. Moreover, it is assumed that particles 1 and 2 are charged with charges $Z_1 e$ and $Z_2 e$ satisfying $Z_1\,Z_2 > 0$. In this case only one Coulomb potential $V_3^C \equiv V_{12}^C$ enters the AGS equations. For simplicity, the nuclear interaction potential between the particles of the pair α is taken in the form of the rank one separable potential:

$$V_\alpha^N = |g_\alpha > \lambda_\alpha < g_\alpha|, \qquad \alpha \neq 3, \tag{15}$$

$$V_3 = |g_3\,\lambda_3 < g_3| + V_3^C, \tag{16}$$

where g_α is the form factor of the pair α and λ_α is the strength parameter, V_3^C is the screened Coulomb interaction potential between particles 1 and 2. Extension for the arbitrary separable rank potential is straightforward [8].

Then the Faddeev equations for the transition operators take the form

$$U_{\beta\,\alpha}^{(+)}(z) = \mathcal{V}_{\beta\,\alpha} + \sum_\nu \mathcal{V}_{\beta\,\nu}\,S_\nu(z)\overline{G}_\nu(z+\varepsilon_\nu)\,U_{\nu\,\alpha}^{(+)}(z), \tag{17}$$

where

$$\overline{G}_\alpha(z_\alpha) = \frac{1}{z + \varepsilon_\alpha - \overline{K}_\alpha}. \tag{18}$$

is the two-body free Green function describing the propagation of the system $\alpha + (\beta\,\gamma)$. $\overline{K}_\alpha$ is the two-body kinetic energy operator of the relative motion of particles α and $(\beta\,\gamma)$. The factor $S_\alpha(z)$ is determined in [5, 6].

Taking into account the matrix elements from both sides of Eq. (17) and the definition (2) of the reaction amplitude one gets the two-particle AGS equations. The main advantage of the AGS equations is that they reduce the three-body Faddeev equations to the two-body ones. This us achieved by using the separable potentials what allows one to single out explicitly the bound-state poles. The three-body Green functions in this approach are absorbed into the effective potentials.

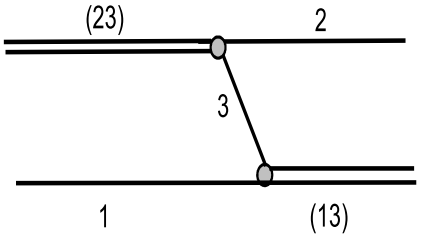

FIG. 1: Pole diagram describing the $(23) + 1 \to 2 + (13)$ transfer reaction.

A. AGS equation with separable potentials for $A(d,p)B$ reaction amplitude

Here the AGS equations with separable potentials are written down explicitly:

$$\mathcal{T}_{p\,A}^{NC}(\mathbf{q}_p^{C(-)}, \mathbf{q}_A^{C(+)};\,E_+) = \mathcal{V}_{p\,A}^{NC}(\mathbf{q}_p^{C(-)}, \mathbf{q}_A^{C(+)}; E_+) + \sum_{i=A,\,p,\,n} \int \frac{\mathrm{d}\mathbf{p}_i}{(2\,\pi)^3}\,\tilde{\mathcal{V}}_{p\,i}^{NC}(\mathbf{q}_p^{C(-)}, \mathbf{p}_i^{C(-)}; E_+)\,\frac{1}{E_+ + \varepsilon_i - p_i^2/(2\,\mathcal{M}_i)}\,\mathcal{T}_{i\,A}^{NC}(\mathbf{p}_i^{C(-)}, \mathbf{q}_A^{C(+)};\,E_+). \tag{19}$$

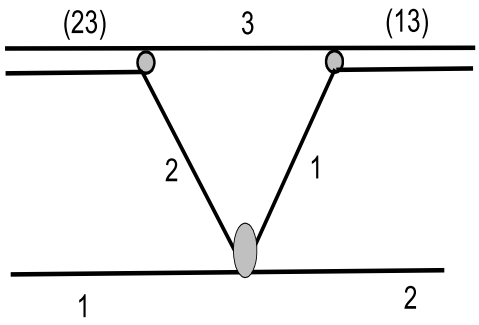

FIG. 2: Exchange triangle diagram describing the $(23)+1 \to (13)+2$ transfer reaction. Dashed bulb is the $1+2$ is the Coulomb elastic scattering amplitude.

The channel p stands for $p+B,\ B=(nA)$ and the channel A stands for $d+A$. Also the channel n stands for $n+F,\ F=(pA)$. $\mathbf{q}_i^{C(\pm)} \equiv \psi_{\mathbf{q}_i}^{C\pm}$ is the Coulomb scattering wave function in the channel i. The amplitude $\mathcal{T}_{p\,A}^{NC}$ corresponds to the reaction $d+A \to p+B$. In Eq. (19) the reaction amplitude $\mathcal{T}_{p\,A}^{NC}(\mathbf{q}_p^{C(-)}, \mathbf{q}_A^{C(+)};\, E_+)$ is on-the energy-shell (ONES) and $E_+ = E + i0$. Note that the initial channel $d(p\,n)+A$ is denoted by the free particle A, while the final channel $p+B(n\,A)$ by p. Also in the c.m. $\mathbf{p}_d = -\mathbf{p}_A$ and $\mathbf{p}_p = -\mathbf{p}_B$, $\varepsilon_p \equiv \varepsilon_{nA}$, $\varepsilon_n \equiv \varepsilon_{pA}$ and $\varepsilon_A \equiv \varepsilon_{pn}$, $\mathcal{M}_p \equiv \mathcal{M}_{pB} = m_p\, m_{nA}/\mathcal{M}$, $\mathcal{M}_n \equiv \mathcal{M}_{nF} = m_n\, m_{pA}/\mathcal{M}$ and $\mathcal{M}_A \equiv \mathcal{M}_{dA} = m_A\, m_{pn}/\mathcal{M},\ \mathcal{M} = m_p + m_n + m_A$. Also the form factor $g_p \equiv g_{nA}$, $g_A \equiv g_{pn}$ and $g_n \equiv g_{pF}$ are the form factors,

The ONES effective potential $\mathcal{V}_{p\,A}^{NC}(\mathbf{q}_p^{C(-)}, \mathbf{q}_A^{C(+)};\, E_+)$ in Eq. (19) (the first term on the right-hand-side) is

$$\mathcal{V}_{p\,A}^{NC}(\mathbf{q}_p^{C(-)}, \mathbf{q}_A^{C(+)};\, E_+) =< \psi_{\mathbf{q}_p}^{C(-)}\, g_p \big| G_0(E_+) + G_0(E_+)\, T_3^C(E_+)\, G_0(E_+) \big| g_A\, \psi_{\mathbf{q}_A}^{C(+)} > . \qquad (20)$$

The other three effective potentials under the integral sign are:

$$\tilde{\mathcal{V}}_{p\,A}^{NC}(\mathbf{q}_p^{C(-)}, \mathbf{p}_A^{C(-)};\, E_+) =< \psi_{\mathbf{q}_p}^{C(-)}\, g_p \big| \big[G_0(E_+) + G_0(E_+)\, T_3^C(E_+)\, G_0(E_+) \big]\, S_A(E_+ - \overline{K}_A) \big| g_A\, \psi_{\mathbf{p}_A}^{C(-)} >, \qquad (21)$$

$$\tilde{\mathcal{V}}_{p\,p}^{NC}(\mathbf{q}_p^{C(-)}, \mathbf{p}_p^{C(-)};\, E_+) =< \psi_{\mathbf{q}_p}^{C(-)}\, g_p \big| G_0(E_+) \big(T_n^C(E_+) S_p(E_+ - \overline{K}_p) - U_n^C \big) G_0(E_+) \big| g_p\, \psi_{\mathbf{p}_p}^{C(-)} >, \qquad (22)$$

$$\tilde{\mathcal{V}}_{p\,n}^{NC}(\mathbf{q}_p^{C(-)}, \mathbf{p}_n;\, E_+) =< \psi_{\mathbf{q}_p}^{C(-)}\, g_p \big| \big[G_0(E_+) + G_0(E_+) T_3^C G_0(E_+) \big] \times S_n(E_+ - \overline{K}_n) \big| g_n\, \psi_{\mathbf{p}_n} >, \qquad (23)$$

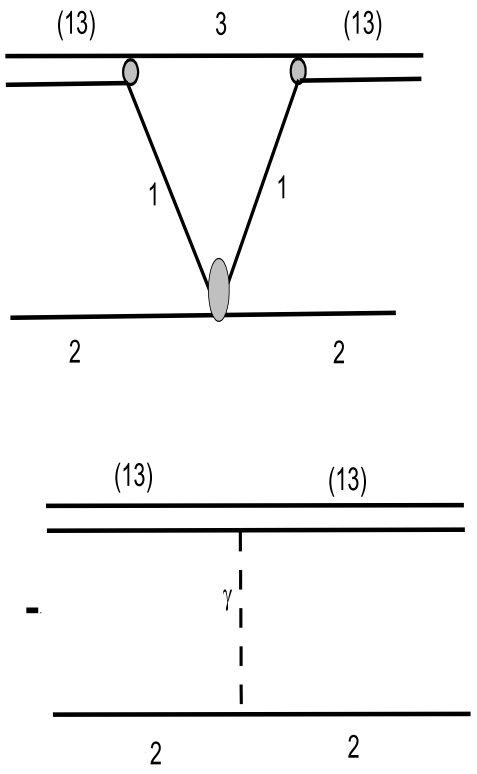

FIG. 3: Triangle diagram describing the $(13)+2 \to (13)+2$ elastic scattering via the $1+2$ Coulomb scattering. Dashed bulb is the $1+2$ Coulomb elastic scattering amplitude. Also is shown the subtracted diagram corresponding to the photon exchange between particle 2 and the c.m. of the bound state (13)

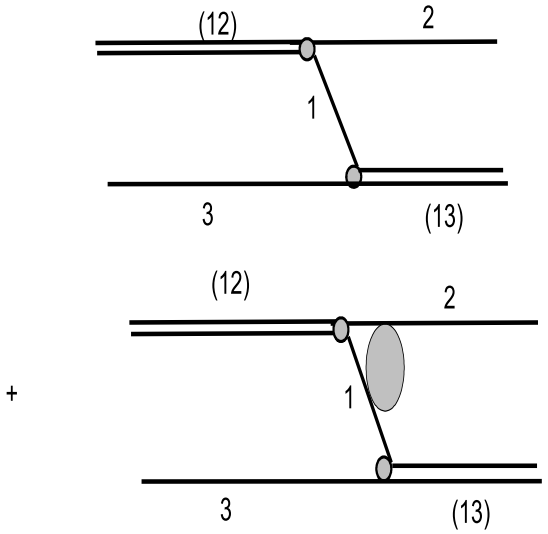

FIG. 4: Sum of the diagrams describing the transfer reaction $(12)+3 \to 2+(13)$. The dashed bulb is the $1+2$ Coulomb scattering amplitude.

$\overline{K}_A$ is the kinetic energy operator of the relative motion of A and d, $S_A(E_+ - \overline{K}_A)\,|\mathbf{p}_A> = |\mathbf{p}_A > S_A(E_+ - \frac{p_A^2}{2\mathcal{M}_A})$.

Each ONES effective potential is described by the corresponding nonrelativistic Feynman diagrams. The ONES effective potential $\tilde{\mathcal{V}}^{NC}_{p\,A}(\mathbf{q}_p^{C(-)}, \mathbf{q}_A^{C(+)}; E_+) = \mathcal{V}^{NC}_{p\,A}(\mathbf{q}_p^{C(-)}, \mathbf{q}_A^{C(+)}; E_+)$ and is given by the sum of the diagrams in Figs. 1 and 2 sandwiched by the Coulomb distorted waves. The ONES effective potential $\tilde{\mathcal{V}}^{NC}_{p\,p}(\mathbf{q}_p^{C(-)}, \mathbf{q}_p^{C(-)}; E_+) = \mathcal{V}^{NC}_{p\,p}(\mathbf{q}_p^{C(-)}, \mathbf{q}_p^{C(-)}; E_+)$ and is given by the difference of the diagrams in Fig. 3 sandwiched by the Coulomb distorted waves. The screened Coulomb-Born $d-A$ potential U^C_{dA} is subtracted from the triangular amplitude to compensate for the most singular term coming from the Born term of the $p-A$ scattering T-matrix. Finally, the ONES effective potential $\tilde{\mathcal{V}}^{NC}_{p\,n}(\mathbf{q}_p^{C(-)}, \mathbf{q}_n; E) = \mathcal{V}^{NC}_{p\,n}(\mathbf{q}_p^{C(-)}, \mathbf{q}_n; E)$ is described by the sum of the diagram in Fig. 4 sandwiched by the Coulomb distorted waves. $T_3^C \equiv T^C_{pA}$ is the off-shell Coulomb $p-A$ T-matrix generated by the screened $p-A$ Coulomb potential. I note that in all these diagrams $1 = A$, $2 = p$ and $3 = n$.

B. AGS equations for the sub-Coulomb (d,p) reactions on heavier nuclei

Let me consider now the application of the AGS equations for the sub-Coulomb (d,p) reactions on heavier targets for which the Coulomb penetrability factors are very small. Because all the amplitudes and effective potentials are sandwiched by the Coulomb scattering wave functions containing the penetrability factors each effective potential and amplitude in Eq. (19) also becomes very small. Hence, one can replace in Eq. (19) the reaction amplitude $\mathcal{T}^{NC}_{p\,A}(\mathbf{p}_p^{C(-)}, \mathbf{q}_A^{C(+)}; E_+)$ under the integral sign with $i = p$ on the right-hand-side by the effective potential $\mathcal{V}^{NC}_{p\,A}(\mathbf{p}_p^{C(-)}, \mathbf{q}_A^{C(+)}; E_+)$:

$$\mathcal{V}^{NC}_{p\,A}(\mathbf{p}^{C(-)}_p,\mathbf{q}^{C(+)}_A;E_+) =< \psi^{C(-)}_{\mathbf{p}_p}\, g_p\big|G^C_{pA}(E_+)\big|g_A\,\psi^{C(+)}_{\mathbf{q}_A}> \\ =< \psi^{C(-)}_{\mathbf{p}_p}\, g_p\big|\big[G^C_{pA}(E_+)\,\Delta V^C_{dA}\big]\big]\big|\varphi_A\,\psi^{C(+)}_{\mathbf{q}_A}> + < \psi^{C(-)}_{\mathbf{p}_p}\,\varphi_p\big|V_p\big|\varphi_A\,\psi^{C(+)}_{\mathbf{q}_A}>, \tag{24}$$

where $\Delta V^C_{dA} = V^C_{pA} - U^C_{dA}$ and $V_p \equiv V_{nA}$. Taking into account that

$$< g_p|G^C_{pA}\,\Delta V^C_{dA} =< g_p|G^C_{pB}[1+\Delta V^C_{pB}G^C_{pA}]\,\Delta V^C_{dA} \\ =< g_p|G^{(0)}_{nA}[1+\Delta E_{pB}\,G^C_{pB}]\,\Delta V^C_{dA} \\ =< \varphi_p|[1+\Delta E_{pB}\,G^C_{pB}]\,\Delta V^C_{dA} \tag{25}$$

I can rewrite

$$\mathcal{V}^{NC}_{p\,A}(\mathbf{p}^{C(-)}_p,\mathbf{q}^{C(+)}_A;E_+) =< \psi^{C(-)}_{\mathbf{p}_p}\, g_p\big|G^C_{pA}(E_+)\big|g_A\,\psi^{C(+)}_{\mathbf{q}_A}> \\ = M^{DWHOES}(p^{C(-)}_p,q^{C(+)}_A) + < \psi^{C(-)}_{\mathbf{p}_p}\,\varphi_p\big|\Delta E_{pB}\,G^C_{pB}\,\Delta V^C_{dA} + G^C_{pB}\,\Delta V^C_{pB}\,G^C_{pA}\Delta V^C_{dA}\big|\varphi_A\,\psi^{C(+)}_{\mathbf{q}_A}> . \tag{26}$$

Here, the half-off-energy-shell (HOES) prior-form of the DWBA amplitude is

$$M^{DWHOES}(p^{C(-)}_p,q^{C(+)}_A) =< \psi^{C(-)}_{\mathbf{p}_p}\, g_p\big|\big|V_{nA} \\ + \Delta V^C_{dA}\big|\varphi_A\,\psi^{C(+)}_{\mathbf{q}_A}> . \tag{27}$$

$\Delta E_{pB} = E_{pB} - \frac{p_p^2}{2\,\mu_{pB}}$, $p_p \equiv p_{pB}$ is the off-shell $p-B$ relative momentum.

The second term on the right-hand-side of Eq. (26) is significantly smaller than the DWBA amplitude $M^{DWHOES}(p^{C(-)}_p,q^{C(+)}_A)$ because after the spectral decomposition of the Coulomb Green functions it contains four penetrabilities factors.

Similarly, the reaction amplitude $\mathcal{T}^{NC}_{n\,A}(\mathbf{p}^{(0)}_n,\mathbf{q}^{C(+)}_A;\,E_+)$ can be replaced by the effective potential $\mathcal{V}^{NC}_{n\,A}(\mathbf{p}^0_n,\mathbf{q}^{C(+)}_A;E_+)$:

$$\mathcal{V}^{NC}_{n\,A}(\mathbf{p}^{(0)}_n,\mathbf{q}^{C(+)}_A;E_+) =< \psi^{(0)}_{\mathbf{p}_n}\, g_n\big|G^C_{pA}(E_+)\big|g_A\,\psi^{C(+)}_{\mathbf{q}_A}> . \tag{28}$$

Note that $g_A \equiv g_{pn}$ and $g_n \equiv g_{pA}$. In the channel $n+F$, where $F=(nA)$, the channel Coulomb interaction is absent. That is why in the bra state one has the plane wave rather than the Coulomb distorted wave.

For the sub-Coulomb (d,p) reactions on heavier targets the elastic $d-A$ scattering is dominated by the Coulomb one. To simplify Eq. (19) the elastic scattering amplitude $\overline{T}^{NC}_{A\,A}(\mathbf{p}^{C(-)}_A,\mathbf{q}^{C(+)}_A;E_+)$ in the term with $i=A$, describing the $d+A$ elastic scattering, is replaced by the HOES pure Coulomb $d-A$ elastic scattering amplitude $\tilde{T}^C_{A\,A}(\mathbf{p}^{C(-)}_A,\mathbf{q}^{C(+)}_A;E_+)$ generated by the channel Coulomb potential U^C_{dA} from which the Born Coulomb term is subtracted.

The amplitude $\overline{\mathcal{T}}^{(C)}_{A\,A}(\mathbf{p}^{C(-)}_A,\mathbf{q}^{C(+)}_A;E_+)$ is given by the integral term in Eq. (B.10) [9]. Its operator takes the form

$$\overline{\mathcal{T}}^C_{A\,A}(z) = U^C_{dA}\,\hat{G}^C_{dA}(z)\,U^C_{dA}. \tag{29}$$

Here, $\overline{G}^C_{dA}(z) = (z+\varepsilon_{pn}-\overline{K}_{dA}-U^C_{dA})^{-1}$ is the two-body $d-A$ Coulomb Green function, $\overline{K}_{dA}$ is the kinetic energy operator of the $d-A$ relative motion.

Then Eq. (19) reduces to

$$\mathcal{T}^{NC}_{p\,A}(\mathbf{q}^{C(-)}_p,\mathbf{q}^{C(+)}_A;\,E_+) = \mathcal{V}^{NC}_{p\,A}(\mathbf{q}^{C(-)}_p,\mathbf{q}^{C(+)}_A;E_+) \\ + \int \frac{\mathrm{d}\mathbf{p}_A}{(2\,\pi)^3}\tilde{\mathcal{V}}^{NC}_{p\,A}(\mathbf{q}^{C(-)}_p,\mathbf{p}^{C(-)}_A;E_+)\,\frac{1}{E_+ + \varepsilon_A - p_A^2/(2\,\mathcal{M}_A)+i0}\,\overline{\mathcal{T}}^C_{A\,A}(\mathbf{p}^{C(-)}_A,\mathbf{q}^{C(+)}_A;E_+) \\ + \int \frac{\mathrm{d}\mathbf{p}_p}{(2\,\pi)^3}\tilde{\mathcal{V}}^{NC}_{p\,p}(\mathbf{q}^{C(-)}_p,\mathbf{p}^{C(-)}_p;E_+)\,\frac{1}{E_+ + \varepsilon_p - p_p^2/(2\,\mathcal{M}_p)+i0}\,\mathcal{V}^{NC}_{p\,A}(\mathbf{p}^{C(-)}_p,\mathbf{q}^{C(+)}_A;E_+) \\ + \int \frac{\mathrm{d}\mathbf{p}_n}{(2\,\pi)^3}\tilde{\mathcal{V}}^{NC}_{p\,n}(\mathbf{q}^{C(-)}_p,\mathbf{p}_n;E_+)\,\frac{1}{E_+ + \varepsilon_n - p_n^2/(2\,\mathcal{M}_n)+i0}\,\mathcal{V}^{NC}_{n\,A}(\mathbf{p}_n,\mathbf{q}^{C(+)}_A;E_+). \tag{30}$$

Thus for the sub-Coulomb $A(d,p)B$ reactions on the heavier nuclei the AGS coupled equations are reduced to one equation (30) for the ONES reaction amplitude $\mathcal{T}^{NC}_{pA}(\mathbf{q}_p^{C(-)}, \mathbf{q}_A^{C(+)}; E_+)$. The goal is to analyze the peripheral character of the expression (30) for the sub-Coulomb (d,p) reactions rather than solving it.

At sub-Coulomb energies, due to the presence of the Coulomb scattering wave functions in the $d-A$ and $p-B$ channels, the (d,p) reactions are peripheral and are contributed by a few smallest partial waves. Peripheral character in the momentum space means that in the intermediate states the integration momenta p_i do not deviate much from the on-shell values q_i. For the peripheral (d,p) reactions the dominant contribution comes from $r_{nA} > 1/\kappa_{nA}$; $\mathbf{r}_{ij}$ is the radius-vector connecting particles i and j, and κ_{ij} is the bound-state wave number of the bound state (ij). In the momentum space it is equivalent to the dominant contribution of the momenta $p_{ij} < \kappa_{ij}$, where $\mathbf{p}_{ij}$ is the momentum conjugated to $\mathbf{r}_{ij}$.

In the DWBA for the peripheral $A(d,p)B$ reaction the $B = (nA)$ bound-state wave function can be replaced by its asymptotic tail whose amplitude is the asymptotic normalization coefficient (ANC) C_{nA} (it is assumed that the spectroscopic factor is one). Then the DWBA cross section is proportional to the C^2_{nA}. The ANC can be determined by normalization of the DWBA differential cross section to the experimental one. It constitutes the ANC method [1, 10]. The question is whether the amplitude of the sub-Coulomb $A(d,p)B$ reaction calculated using the AGS equation (30) is peripheral and can be parametrized in terms of the ANC C_{nA}.

Let me begin with the effective potential $\mathcal{V}^{NC}_{pA}(\mathbf{q}_p^{C(-)}, \mathbf{q}_A^{C(+)}; E_+)$, which is the first term on the right-hand-side of Eq. (30). Repeating the transformations done for the HOES effective potential $\mathcal{V}^{NC}_{pA}(\mathbf{p}_p^{C(-)}, \mathbf{q}_A^{C(-)}; E_+)$ in Eq. (24), it can be shown that the ONES effective potential $\mathcal{V}^{NC}_{pA}(\mathbf{q}_p^{C(-)}, \mathbf{q}_A^{C(+)}; E_+)$ can be expressed in terms of the sub-Coulomb ONES DWBA amplitude plus next order term.

$$\mathcal{V}^{NC}_{pA}(\mathbf{q}_p^{C(-)}, \mathbf{q}_A^{C(+)}; E_+) =< \psi^{C(-)}_{\mathbf{q}_p}\, g_p \big| G_0(E_+) + G_0(E_+)\, T_3^C(E_+)\, G_0(E_+) \big| g_A\, \psi^{C(+)}_{\mathbf{q}_A} > =< \psi^{C(-)}_{\mathbf{q}_p}\, g_p \big| G^C_{pA}(E_+) \big| g_A\, \psi^{C(+)}_{\mathbf{q}_A} > . \quad (31)$$

For the Coulomb Green function $G^C_{pA}(z)$ of the particles p and A interacting via the screened Coulomb potential $V^C_n \equiv V^C_{pA}$ in the three-body space the post-transformation is used:

$$G^C_{pA}(z) = \frac{1}{z - K - V^C_{pA}} = \frac{1}{z - K - \Delta V^C_{pB} - U^C_p} = G^C_{pB}(z)\Big(1 + \Delta V^C_{pB}\, G^C_{pA}(z)\Big). \quad (32)$$

Here, $G^C_{pB}(z) = (z - K - U^C_{pB})^{-1}$, $U^C_p \equiv U^C_{pB}$, $\Delta V^C_{pB} = V^C_{pA} - U^C_{pB}$. Then

$$\mathcal{V}^{NC}_{pA}(\mathbf{q}_p^{C(-)}, \mathbf{q}_A^{C(+)}; E_+) =< \psi^{C(-)}_{\mathbf{q}_p}\, g_p \big| G^C_{pB}(E_+)\Big(1 + \Delta V^C_{pB}\, G^C_{pA}(E_+)\Big) \big| g_A\, \psi^{C(+)}_{\mathbf{q}_A} > =< \psi^{C(-)}_{\mathbf{q}_p}\, \varphi_p \big| 1 + \Delta V^C_{pB}\, G^C_{pA}(E_+) \big| g_A\, \psi^{C(+)}_{\mathbf{q}_A} > . \quad (33)$$

Here the equation

$$< \psi^{C(-)}_{\mathbf{q}_p}\, g_p \big| G^C_{pB}(E_+) = - < \psi^{C(-)}_{\mathbf{q}_p}\, g_p \big| \frac{1}{\varepsilon_p + \overline{K}_p} =< \psi^{C(-)}_{\mathbf{q}_p}\, \varphi_p \big| \quad (34)$$

is used. The bound-state wave function $\varphi_p \equiv \varphi_{nA}$ is the $B = (nA)$, $\varepsilon_p \equiv \varepsilon_{nA}$ is the binding energy of the bound-state (nA), $\overline{K}_p \equiv \overline{K}_{nA}$ is the $n - A$ free Green function at energy $-\varepsilon_{nA}$.

Now, instead of the post transformation, one can use the prior transformation of $G^C_{pA}(z)$ is used in Eq. (33) :

$$G^C_{pA}(z) = \frac{1}{z - K - \Delta V^C_{dA} - U^C_{dA}} = \Big(1 + G^C_{pA}(z)\, \Delta V^C_{dA}\Big)\, G^C_{dA}(z), \quad (35)$$

where $G^C_{dA}(z) = (z - K - U^C_{dA})^{-1}$, $\Delta V^C_{dA} = V^C_{pA} - U^C_{dA}$. Taking into account the equation

$$G^C_{dA}(E_+) \big| g_A \psi^{C(+)}_{\mathbf{q}_A} > = -\frac{1}{\varepsilon_A + \overline{K}_A} < g_A\, \psi^{C(+)}_{\mathbf{q}_A} \big| =< \varphi_A\, \psi^{C(-)}_{\mathbf{q}_A} \big|, \quad (36)$$

$\overline{K}_A \equiv \overline{K}_{pn}$ is $p - n$ two-body Green function at energy $-\varepsilon_A = -\varepsilon_{pn}$.

Then

$$\mathcal{V}^{NC}_{p\,A}(\mathbf{q}^{C(-)}_{p},\mathbf{q}^{C(+)}_{A};E_{+})=\mathcal{T}^{DW}_{pA}(\mathbf{q}^{C(-)}_{p},\mathbf{q}^{C(+)}_{A})+<\psi^{C(-)}_{\mathbf{q}_p}\,\varphi_p\big|\Delta V^{C}_{pB}\,G^{C}_{pA}(E_{+})\,\Delta V^{C}_{dA}\,\big|\varphi_A\,\psi^{C(+)}_{\mathbf{q}_A}>, \quad (37)$$

where the post-form of the sub-Coulomb DWBA amplitude is

$$\mathcal{T}^{DW}_{pA}(\mathbf{q}^{C(-)}_{p},\mathbf{q}^{C(+)}_{A};E_{+}) =<\psi^{C(-)}_{\mathbf{q}_p}\,\varphi_{nA}\big|V_{pn}+V^{C}_{pA}-U^{C}_{pB}\big|\varphi_{pn}\,\psi^{C(-)}_{\mathbf{q}_A}>. \quad (38)$$

If one would change the order of application of the prior and post transformations of $G^{C}_{pA}(E_{+})$, then the effective potential $\mathcal{V}^{NC}_{p\,A}(\mathbf{q}^{C(-)}_{p},\mathbf{q}^{C(+)}_{A};E_{+})$ can be expressed in terms of the prior-form DWBA amplitude. The second term in Eq. (38) is small compared to the DWBA amplitude because, after the spectral decomposition of the Green function it contains four penetrability factors at sub-Coulomb energies.

It is well known that the sub-Coulomb (d,p) reactions are peripheral [11, 12]. Nevertheless, for the general reader it will be useful to see directly that the amplitude $\mathcal{T}^{DW}_{pA}(\mathbf{q}^{C(-)}_{p},\mathbf{q}^{C(+)}_{A};E_{+})$ is peripheral for the sub-Coulomb $A(d,p)B$ on heavier targets for which the Coulomb parameters in the initial and final states $\eta_{q_A},\,\eta_{q_p}>>1$, where $\eta_{q_A}=(Z_d\,Z_A/137)\mu_{dA}/q_A,\;\;q_A\equiv q_{dA}$, and $\eta_{q_p}=(Z_p\,Z_B/137)\mu_{pB}/q_p,\;\;q_p\equiv q_{pB},\;Z_p=Z_d=1$ and $Z_B=Z_A$.

First, one should introduce the partial wave decomposition of the DWBA amplitude which can be schematically written as

$$\mathcal{T}^{DW}_{pA}(q^{C}_{p\,l_p},\,q^{C}_{A\,l_A};E_{+})=<\psi^{C}_{q_p\,lp}\big|O(r)\big|\psi^{C}_{q_A\,l_A}>, \quad (39)$$

where $\psi^{C}_{q_A\,l_A}$ and $\psi^{C}_{q_p\,lp}$ are the Coulomb scattering wave functions in the initial and final states; $l_A\equiv l_{dA}$ ($l_p\equiv l_{pB}$) is the relative orbital angular momentum in the initial $d+A$ (final $p+B$) channel. All other functions of the matrix elements, except for the partial Coulomb distorted waves, are absorbed into $O(r)$. Now it is convenient to use the quasiclassical approach for the Coulomb partial waves [11]:

$$\psi_{q\,l}\sim\sqrt{\Big[\frac{f(r)}{q^2}\Big]^{-1/4}}\,\sin\,\phi, \quad (40)$$

$$f(r)=q^2-\frac{2\,q\,\eta}{r}-\frac{l(l+1)}{r^2}, \quad (41)$$

$$\phi=\frac{\pi}{4}+\int\limits_{r_0}^{r}\mathrm{d}r\,\sqrt{[f(r)]^{1/2}}. \quad (42)$$

r_0 is the classical turning point determined by the condition: $f(r_0)=0$. r_0 increases with increasing of the Coulomb parameter η. Thus from the classical approach, which is valid at large Coulomb parameter η, follows that the dominant contribution to the Coulomb partial wave give distances $r>r_0$, while the internal distances $r<r_0$, which are located in the classically forbidden region, give negligible contribution. Hence, any matrix element sandwiched by the partial Coulomb distorted waves, is peripheral. For example, for the ^{208}Pb$(d,p)^{209}$Pb reaction at $E_{dA}=5$ MeV (the Coulomb barrier is $V_{CB}=12.2$ MeV) and the head-on collision $l_{dA}=0$ in the initial channel $r_0=23.6$ fm. Such a large r_0 makes the reaction amplitude both peripheral and small. Head-on collision is dominant because for $l_{dA}>0$ r_0 increases decreasing the reaction amplitude. The Rutherford trajectory at head on-collisions is peaked backward. Hence the proton differential cross section generated by the amplitude $\mathcal{T}^{DW}_{pA}(\mathbf{q}^{C(-)}_{p},\mathbf{q}^{C(+)}_{A};E_{+})$ is backward peaked.

To demonstrate it in Fig. 5 is shown the proton's angular distribution in the direct ^{208}Pb$(d,p)^{209}$Pb reaction at $E_{dA}=5$ MeV calculated using the DWBA FRESCO code [13]. It is a sub-Coulomb reaction because the Coulomb barrier is $V_{CB}\approx 12.2$ MeV and the Coulomb parameter in the initial state is $\eta_{dA}=8.16$. Thus this process demonstrates a perfect example of the sub-Coulomb reaction with large Coulomb parameters. The proton's angular distribution, as explained, has a pronounced backward peak. In the calculations the Reid soft-core potential for the deuteron bound state and standard Woods-Saxon for the neutron $2g_{9/2}$ bound state in ^{209}Pb are used. However, the details of the adopted potentials are not important because the backward peak is an universal pattern of the angular distribution of sub-Coulomb direct transfer reactions on nuclei with higher charges.

In summarizing the analysis of the effective potential $\mathcal{V}_{pA}(\mathbf{q}^{C(-)}_{p},\mathbf{q}^{C(+)}_{A};E_{+})$, it is useful to remind the proved essential results: for the sub-Coulomb (d,p) reactions the effective potential $\mathcal{V}_{pA}(\mathbf{q}^{C(-)}_{p},\mathbf{q}^{C(+)}_{A};E_{+})$ whose mechanism is described by the sum of the pole and triangular exchange diagrams in Figs. 1 and 2, correspondingly, is dominantly contributed by the DWBA amplitude $\mathcal{T}^{DW}_{pA}(\mathbf{q}^{C(-)}_{p},\mathbf{q}^{C(+)}_{A};E_{+})$. The second term in Eq. (37) is significantly smaller then the DWBA amplitude because, after the spectral decomposition of the Green function, it contains four penetrability factors versus two in the DWBA amplitude. If the energies in the initial and final states are well below the Coulomb barrier then the amplitude $\mathcal{T}^{DW}_{pA}(\mathbf{q}^{C(-)}_{p},\mathbf{q}^{C(+)}_{A};E_{+})$ of the $A(d,p)B$ is peripheral and parametrized in terms of the ANC C_{nA}. The differential cross section generated by $\mathcal{T}^{DW}_{pA}(\mathbf{q}^{C(-)}_{p},\mathbf{q}^{C(+)}_{A};E_{+})$ is backward peaked at sub-

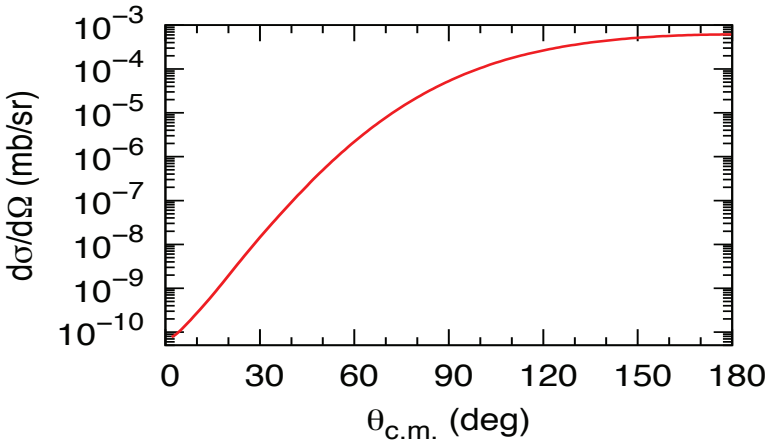

FIG. 5: Proton's angular distribution for the ^{208}Pb$(d,p)^{209}$Pb reaction at $E_{dA} = 5$ MeV populating the $2g_{9/2}$ ground state in ^{209}Pb. $\theta_{c.m.}$ is the proton scattering angle in the c.m. of the reaction.

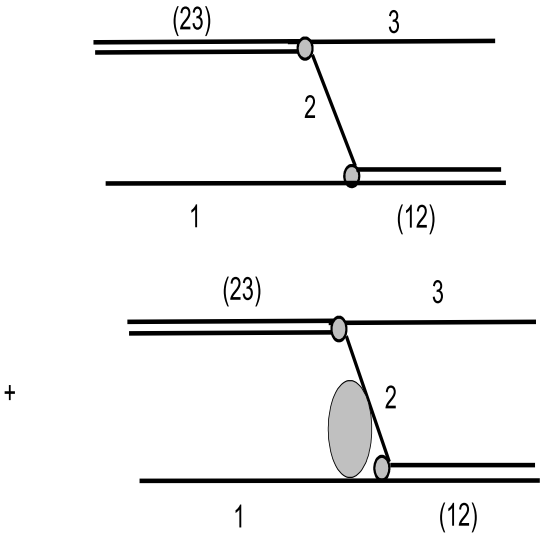

FIG. 6: Sum of the diagrams describing the $(23)+1 \to 3+(12)$ reaction. The dashed bulb is the $1+2$ Coulomb scattering amplitude.

Coulomb energies on heavier targets [12].

Let us return to Eq. (30). The integrand of the second term on the right-hand-side of this equation contains the effective potential $\tilde{\mathcal{V}}^{NC}_{pA}(\mathbf{q}_p^{C(-)}, \mathbf{p}_A^{C(-)}; E_+)$, which is expressed in terms of the HOES DWBA $\mathcal{T}^{DW}_{pA}(\mathbf{q}_p^{C(-)}, \mathbf{p}_A^{C(+)}; E_+)$. The matrix element of the partial wave HOES DWBA amplitude written in the quasiclassical approach is peripheral and contains the factor $e^{-|\zeta|\,\pi/2}$ [11], where $\zeta = \eta_{q_p} - \eta_{p_A}$. Hence, at large Coulomb parameter η_{q_p} the dominant contribution in the integral over p_A comes from minimal ζ. From the previous discussion it is evident that $\tilde{\mathcal{V}}^{NC}_{pA}(\mathbf{q}_p^{C(-)}, \mathbf{p}_A^{C(-)}; E_+)$ is peripheral with regard to the bound-state wave function $\varphi_p \equiv \varphi_{nA}$ and is parametrized in terms of the ANC C_{nA}. Hence, the second term of Eq. (30) is also peripheral and is parametrized in terms of the ANC C_{nA}.

The same is true for the third term on the right-hand-side of Eq. (30), which contains $\mathcal{V}^{NC}_{pA}(\mathbf{p}_p^{C(-)}, \mathbf{q}_A^{C(+)}; E_+)$. This amplitude has been discussed above and is dominantly contributed by the HOES DWBA amplitude $\mathcal{T}^{DW}_{pA}(\mathbf{p}_p^{C(-)}, \mathbf{q}_A^{C(+)}; E_+)$. Evidently that the DWBA amplitude is peripheral and parametrized in terms of the ANC C_{nA} what is also true for the whole third term.

Now let me consider the fourth term. This term contains only two Coulomb penetrability factors corresponding to the initial and final states because there is no channel Coulomb interaction in the intermediate $n+F$ channel. The fourth team describes the two-step reaction $d + A \to n + F \to B + p$. The integrand of the fourth term contains two amplitudes, $\mathcal{V}^{NC}_{nA}(\mathbf{p}_n, \mathbf{q}_A^{C(+)}; E_+)$ and $\tilde{\mathcal{V}}^{NC}_{pn}(\mathbf{q}_p^{C(-)}, \mathbf{p}_n; E_+)$ describing the first and second steps, correspondingly. The effective potential $\mathcal{V}^{NC}_{nA}(\mathbf{p}_n, \mathbf{q}_A^{C(+)}; E)$ described by the sum of the diagrams shown in Fig. 6. The notations of the particles are the same as in the previous cases. This sum of the diagrams can be replaced by one diagram shown in Fig. 7.

In this diagram is introduced the Coulomb-modified form factor of the pair 3, which takes into account the Coulomb interaction between the particles 1 and 2:

$$|\phi_3(z_3)> = \left[1 + T_3^{C(R)}(z_3)\,\hat{G}_0(z_3)\right]|g_3>, \qquad (43)$$

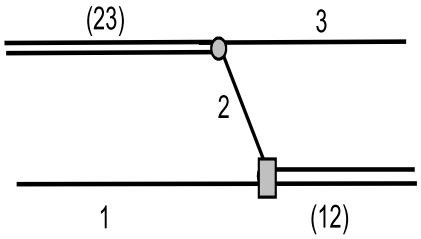

FIG. 7: Diagram describing the $(23)+1 \to 3+(12)$ transfer reaction in which the vertex $1+2 \to (12)$ is described by the form factor $(|\phi_3>)^*$.

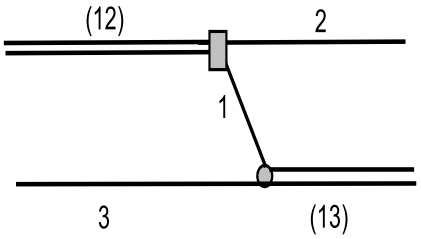

FIG. 8: Diagram describing the $(12)+3 \to 2+(13)$ transfer reaction in which the vertex $(12) \to 1+2$ is described by the form factor $|\phi_3>$.

$$\big|\mathbf{q}_3> \big|\phi_3(z-\frac{q_3^2}{2\,\mathcal{M}_3})> \\ = \big[1+T_3^{C(R)}(z)\,G_0(z)\big]\big|g_3(z)> \big|\mathbf{q}_3>\,. \qquad (44)$$

Here $\mathbf{q}_3$ is the relative momentum of particle 3 and the bound state (12). $\mathcal{M}_3$ is the reduced mass of the particle 3 and the bound state (12). The properties of $|\phi_3(z_3)>$ were discussed in details in [14].

The effective potential $\tilde{\mathcal{V}}_{p\,n}^{NC}(\mathbf{q}_p^{C(-)}, \mathbf{p}_n; E_+)$ is the amplitude of the proton transfer reaction $A(d,n)F$, where $F=(p\,A)$ and ONES is given by the sum of diagrams in Fig 4 which can be replaced by the diagram in Fig 8.

Combining the diagrams in Figs 7 and 8 one obtains the rectangular diagram shown in Fig. 9 describing the fourth term (without the Coulomb distorted waves in the initial and final states of the reaction, which do not affect the location of the singularities of the diagram).

To find its nearest to the physical region singularity in the $\cos(\mathbf{q}_p\cdot\mathbf{q}_d)$ plane ($\mathbf{q}_d=-\mathbf{q}_A$), which governs the angular distribution of the cross section generated by this diagram, one can contract the line F in the rectangular diagram in Fig. 9 reducing it to the triangular diagram in Fig. 10 , which is the skeleton diagram of the rectangular diagram. The nearest to the physical region singularity of the ONES triangular diagram, and, hence, of the rectangular diagram, generated by the propagators (all the vertices are taken to be constant) is located in the $\cos(\mathbf{q}_p\cdot\mathbf{q}_d)$ plane at

$$z_t=-\frac{m_d\,m_B}{2\,m_n^2}\,\frac{(\frac{m_n}{m_d})^2q_d^2+(\frac{m_n}{m_B})^2q_p^2+(\kappa_{pn}+\kappa_{nA})^2}{q_d\,q_p}<-1. \qquad (45)$$

$\mathbf{q}_p$ is the $p-B(nA)$ relative momentum and $\mathbf{q}_d$ is the $d-A$ relative momentum.

This singularity is located quite far away from the border of the physical region $\cos(\mathbf{q}_p\cdot\mathbf{q}_d)=-1$. The nearest to the physical region singularity of the ONES amplitude of the pole diagram in Fig. 1 (the notations of the particles are the same as in the previous cases) is

$$z_p=\frac{m_d}{2\,m_p}\,\frac{q_p^2+(\frac{m_p}{m_d})^2q_d^2+\kappa_{pn}^2}{q_d\,q_p}>1. \qquad (46)$$

It is located on the opposite site of the unphysical region but much closer to the border of the physical region $\cos(\mathbf{q}_p\cdot\mathbf{q}_d)=1$ than the singularity of the triangular diagram.

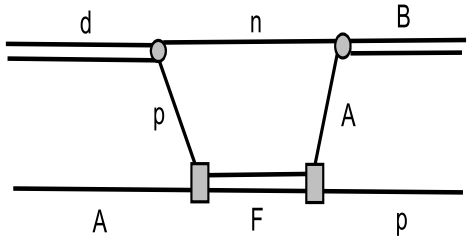

FIG. 9: Rectangular diagram describing the two-step process $d + A \to B + p$.

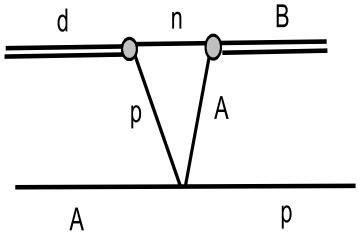

FIG. 10: Triangular diagram describing the reaction $d + A \to B + p$ obtained from the rectangular diagram in Fig. 9 by contracting the line F.

As an example, is considered the sub-Coulomb ^{208}Pb$(d, p)^{209}$Pb reaction at $E_{dA} = 5$ MeV. For this case one gets $z_t = -432.048$ and $z_p = 1.11$. These singularities govern the angular distributions generated by the corresponding diagrams. In Fig. 11 are shown the angular distributions generated by $(\cos\theta - z_t)^{-2}$ and $(\cos\theta - z_p)^{-2}$. As one can see, the angular distribution generated by the pole singularity has pronounced forward peak while the triangular singularity produces absolutely flat angular distribution. The folding of the amplitude of the pole diagram with the Coulomb distorted waves in the initial and final states converts the forward peak into the backward one because of the dominant head-on collision while the angular distribution generated by the rectangular diagram sandwiched with the Coulomb distorted waves remains flat.

Therefore, one can neglect the contribution of the fourth term in Eq. (30) at the backward proton angles compared to the first three terms on the right-hand-side of Eq. (30).

Because the second and third terms contain four penetrability factors each, they are smaller than the first term, $\mathcal{V}^{NC}_{p\,A}(\mathbf{q}_p^{C(-)}, \mathbf{q}_A^{C(+)}; E_+)$. Thus it is shown that for the sub-Coulomb $A(d, p)B$ reactions on heavier nuclei the AGS amplitude $\mathcal{T}^{NC}_{p\,A}(\mathbf{q}_p^{C(-)}, \mathbf{q}_A^{C(+)}; E_+)$ with separable potentials is well approximated by the post form of the sub-Coulomb DWBA amplitude:

$$\begin{aligned}
&\mathcal{T}^{NC}_{p\,A}(\mathbf{q}_p^{C(-)}, \mathbf{q}_A^{C(+)}; E_+) \\
&\approx \mathcal{V}^{NC}_{p\,A}(\mathbf{q}_p^{C(-)}, \mathbf{q}_A^{C(+)}; E_+) \\
&= \mathcal{T}^{DW}_{p\,A}(\mathbf{q}_p^{C(-)}, \mathbf{q}_A^{C(+)}; E_+). \qquad (47)
\end{aligned}$$

Since the sub-Coulomb DWBA amplitude is peripheral and parametrized in terms of the ANC C_{nA} of the bound state (nA), the same is also the case for the AGS $A(d, p)B$ reaction amplitude $\mathcal{T}^{NC}_{p\,A}(\mathbf{q}_p^{C(-)}, \mathbf{q}_A^{C(+)}; E_+)$. For better accuracy one can add to the DWBA amplitude the second and third terms of the right-hand-side of Eq. (30), which can become important when energy E increases but still below the Coulomb barrier.

IV. AGS EQUATIONS WITH GENERAL LOCAL POTENTIALS

In this section the AGS equations are written for general forms of the two-body local potentials rather than for nonlocal separable potentials. I briefly describe the

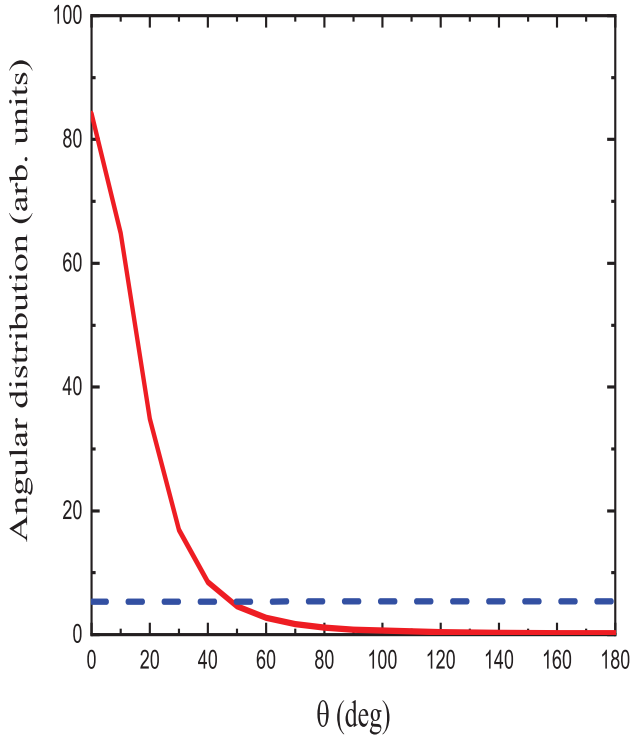

FIG. 11: Angular distributions of the protons from the $d + A \to B + p$ reaction at $E_{dA} = 5$ MeV generated by $(\cos\theta - z_t)^{-2}$, dashed blue line, and by $(\cos\theta - z_p)^{-2}$, solid red line.

derivation of these equations because it will be used in the next section where the AGS equations are modified by including the optical potentials. The AGS equations can be derived directly from the equations for the transition operator:

$$
\begin{aligned}
U^{NC(+)}_{\beta\,\alpha}(z) &= \Delta\overline{V}_\beta + \Delta\overline{V}_\beta\, G(z)\, \Delta\overline{V}_\alpha \\
&= \Delta\overline{V}_\beta + \left[V^N_\alpha\, G(z) + V^N_\gamma\, G(z) + \Delta\overline{V}^C_\beta\, G(z)\right] \Delta\overline{V}_\alpha \\
&= \Delta\overline{V}_\beta + \left[V^N_\alpha\, \overline{G}_\alpha(z)\left(\Delta\overline{V}_\alpha\, G(z) + 1\right) + V^N_\gamma\, \overline{G}_\gamma(z)\left(\Delta\overline{V}_\gamma\, G(z) + 1\right)\right. \\
&\left. + \Delta\overline{V}^C_\beta\, \overline{G}_\beta(z)\left(\Delta\overline{V}_\beta\, G(z) + 1\right)\right] \Delta\overline{V}_\alpha \\
&= \Delta\overline{V}_\beta + \sum_\nu \left[\overline{\delta}_{\beta\,\nu}\, V^N_\nu + \delta_{\beta\,\nu}\, \Delta\overline{V}^C_\nu\right] \overline{G}_\nu(z)\, U^{NC(+)}_{\nu\,\alpha}(z),
\end{aligned}
\tag{48}
$$

$$
\overline{G}_\alpha(z) = \frac{1}{z - K - V_\alpha - U^C_\alpha}. \tag{49}
$$

$$
\Delta\overline{V}_\alpha = \overline{V}_\alpha - U^C_\alpha = \overline{V}^N_\alpha + \Delta\overline{V}^C_\alpha, \tag{50}
$$

$$
\Delta\overline{V}^C_\alpha = \overline{V}^C_\alpha - U^C_\alpha. \tag{51}
$$

Here, $\overline{V}_\alpha$ is determined in Eq. (5), $\overline{V}^N_\alpha = V^N_\beta + V^N_\gamma$, $\overline{V}^C_\alpha = V^C_\beta + V^C_\gamma$, U^C_α is the channel Coulomb potential describing the interaction between particle α and the c.m. of the bound state $(\beta\,\gamma)$.

To derive the coupled equations for the transition operator the potential $\Delta\overline{V}_\alpha$ has been split into three terms: two nuclear potentials V^N_β and V^N_γ, and one Coulomb term $\Delta\overline{V}^C_\alpha$. This allows one to express $U^{NC(+)}_{\beta\,\alpha}(z)$ in terms of the three components, $U^{NC(+)}_{\nu\,\alpha}(z)$, $\nu = \beta,\, \gamma,\, \alpha$.

The ONES reaction amplitude is given by the matrix element from $U^{NC(+)}_{\beta\,\alpha}(z)$ taken between the initial and

final physical states:

$$\mathcal{T}^{NC}_{\beta\,\alpha}(\mathbf{q}^{C(-)}_{\beta},\,\mathbf{q}^{C(+)}_{\alpha};E_+) =< \psi^{C(-)}_{\mathbf{q}_\beta}\,\varphi_\beta\Big|U^{NC(+)}_{\beta\,\alpha}(E_+)\Big|\varphi_\alpha\,\psi^{C(+)}_{\mathbf{q}_\alpha} >, \tag{52}$$

where $\psi^{C(+)}_{\mathbf{q}_\alpha}$ and $\psi^{C(-)}_{\mathbf{q}_\beta}$ are the Coulomb scattering wave functions in the initial and final states calculated for the screened channel Coulomb potentials U^C_α and U^C_β, correspondingly.

Instead of the transition operator $U^{NC(+)}_{\beta\,\alpha}(z)$ one may consider the transition operator

$$U^{NC}_{\beta\,\alpha}(z) = \overline{\delta}_{\beta\,\alpha}\,[\overline{G}_\alpha(z)]^{-1} + U^{NC(+)}_{\beta\,\alpha}(z). \tag{53}$$

Then $U^{NC}_{\beta\,\alpha}(z)$ satisfies the equation

$$U^{NC}_{\beta\,\alpha}(z) = \overline{\delta}_{\beta\,\alpha}\left([\overline{G}_\alpha(z)]^{-1} + V^N_\alpha\right) + \delta_{\beta\,\alpha}\,\Delta\overline{V}^C_\beta + \sum_\nu\left[\overline{\delta}_{\beta\,\nu}\,V^N_\nu + \delta_{\beta\,\nu}\,\Delta\overline{V}^C_\nu\right]\overline{G}_\nu(z)\,U^{NC}_{\nu\,\alpha}(z). \tag{54}$$

Equations (54) were derived in [15]. Note that the ONES matrix elements from $U^{NC(+)}_{\beta\,\alpha}(z)$ and $U^{NC}_{\beta\,\alpha}(z)$, in which the initial state is physical, coincide:

$$< \psi^{C(-)}_{\mathbf{p}_\beta}\,\varphi_\beta\Big|U^{NC(+)}_{\beta\,\alpha}(E_+)\Big|\varphi_\alpha\,\psi^{C(+)}_{\mathbf{q}_\alpha} > =< \psi^{C(-)}_{\mathbf{p}_\beta}\,\varphi_\beta\Big|U^{NC}_{\beta\,\alpha}(E_+)\Big|\varphi_\alpha\,\psi^{C(+)}_{\mathbf{q}_\alpha} > \tag{55}$$

because

$$[\overline{G}_\alpha(E_+)]^{-1}\,\big|\varphi_\alpha\,\psi^{C(+)}_{\mathbf{q}_\alpha} >= 0. \tag{56}$$

After having derived the Faddeev equations for the transition operators $U^{NC(+)}_{\beta\,\alpha}(z)$ one can write down the Faddeev equations in the AGS form for the reaction amplitude. For the separable potentials the Faddeev three-body equations are reduced to the two-body AGS equations. For general potentials it is not the case. When writing the AGS equations for the general potentials Eq. (48) is used in which one needs to introduce the spectral decomposition of the Green functions $\overline{G}_\nu(z)$. This spectral decomposition contains both two-body and three-body terms. Here, when deriving the AGS equations for the reaction amplitudes, the three-body terms in the spectral decomposition of $\overline{G}_\nu(z)$ are neglected, that is, the contribution from the three-body continuum in the intermediate states is neglected. Thus the spectral decomposition of the Green functions $\overline{G}_\nu(z)$ is used:

$$\overline{G}_\nu(z) = \int\frac{\mathrm{d}\mathbf{p}_\nu}{(2\,\pi)^3}\frac{|\,\mathbf{p}^{C(-)}_\nu\,\varphi_\nu >< \varphi_\nu\,\mathbf{p}^{C(-)}_\nu|}{z+\varepsilon_\nu-\frac{p^2_\nu}{2\,\mathcal{M}_\nu}}. \tag{57}$$

Neglecting the contribution from the three-body continuum in the spectral decomposition of the channel Green functions $\overline{G}_\nu(z)$ allows one to derive the two-particle Faddeev equations in the AGS form in which the effective potentials are expressed in terms of the DWBA amplitudes for the sub-Coulomb transfer reactions. Also only one bound state is taken into account in each channel. The extension for a few bound states is straightforward.

Taking the matrix elements from the left- and right-hand-sides of Eq. (48) and using the spectral decomposition (57) one can get the desired coupled Faddeev equations.

A. Sub-Coulomb (d,p) reactions

Here the described above formalism is applied for the analysis of the peripheral character of the sub-Coulomb $A(d,p)B$ reactions because they contain the Coulomb distorted waves in the initial and final states of the matrix elements, which have crucial importance for the sub-Coulomb reactions. The transition operator in this case satisfies equation

$$U^{NC(+)}_{p\,A}(z) = \Delta\overline{V}_p + \sum_{i=A,p,n}\left[\overline{\delta}_{p\,i}\,V^N_i + \delta_{p\,i}\,\Delta\overline{V}^C_i\right]\overline{G}_i(z)\,U^{NC(+)}_{i\,A}(z). \tag{58}$$

The channel indexes p, A, n correspond to the channels $p+B(nA)$, $d(pn)+A$, $n+F(pA)$, correspondingly, while the potential $V_p \equiv V_{nA},\ V_A \equiv V_{pn}$ and $V_n \equiv V_{pA}$, $\Delta\overline{V}_p \equiv \overline{V}_p - U^C_p,\ \overline{V}_p = V_{pn} + V_{pA},\ U^C_p \equiv U^C_{pB},\ \Delta\overline{V}^C_p \equiv V^C_{pA} - U^C_{pB}$.

Then the two-particle AGS equation for the $A(d,p)B$

reaction amplitude takes the form

$$\mathcal{T}^{NC}_{p\,A}(\mathbf{q}^{C(-)}_p,\,\mathbf{q}^{C(+)}_A;E_+)=\mathcal{T}^{DW}_{p\,A}(\mathbf{q}^{C(-)}_p,\,\mathbf{q}^{C(+)}_A;E_+)$$
$$+\sum_{i=A,p,n}\int\frac{\mathrm{d}\mathbf{p}_i}{(2\,\pi)^3}\frac{\tilde{\mathcal{T}}^{DW}_{p\,i}(\mathbf{q}^{C(-)}_p,\,\mathbf{p}^{C(-)}_i;E_+)\,\mathcal{T}^{NC}_{i\,A}(\mathbf{p}^{C(-)}_i,\,\mathbf{q}^{C(+)}_A;E_+)}{E_+ +\varepsilon_i-\frac{p_i^2}{2\,\mathcal{M}_i}}. \tag{59}$$

Thus one of the important goals of the paper is achieved: the Faddeev equations in the two-particle AGS form with local potentials for the sub-Coulomb $A(d,p)B$ reactions has been derived. The next goal is to demonstrate that this equation is peripheral. Note that on the left-hand-side of Eq. (59) one has the ONES reaction amplitude $\mathcal{T}^{NC}_{p\,A}(\mathbf{q}^{C(-)}_p,\,\mathbf{q}^{C(+)}_A;E_+)$, while under the integral sign the same reaction amplitude (at $i=p$) is the HOES because the momentum $\mathbf{p}_p$ is the integration variable.

The first term on the right-hand-side of Eq. (59) is the effective potential

$$\mathcal{T}^{DW}_{p\,A}(\mathbf{q}^{C(-)}_p,\,\mathbf{q}^{C(+)}_A;E_+)=<\psi^{C(-)}_{\mathbf{q}_p}\,\varphi_{nA}\Big|V_{pn}$$
$$+V^N_{pA}+V^C_{pA}-U^C_{pB}\Big|\varphi_{pn}\,\psi^{C(+)}_{\mathbf{q}_A}>, \tag{60}$$

which is the ONES sub-Coulomb post-form of the DWBA $A(d,p)B$ reaction amplitude. The effective potential in the second term on-the right-hand-side ($i=A$)

$$\tilde{\mathcal{T}}^{DW}_{p\,A}(\mathbf{q}^{C(-)}_p,\,\mathbf{p}^{C(-)}_A;E_+)$$
$$=<\psi^{C(-)}_{\mathbf{q}_p}\,\varphi_{nA}\big|V_{pn}\big|\varphi_{pn}\,\psi^{C(-)}_{\mathbf{p}_A}>, \tag{61}$$

is the HOES post-form of the DWBA $A(d,p)B$ reaction amplitude with V_{pn} as the transition operator. The HOES elastic scattering amplitude $\mathcal{T}^{NC}_{A\,A}(\mathbf{p}^{C(-)}_A,\,\mathbf{q}^{C(+)}_A;E_+)$ under the integral in the second term at the sub-Coulomb energies can be replaced by the HOES pure Coulomb $d-A$ elastic scattering amplitude $\tilde{\mathcal{T}}^{C}_{A\,A}(\mathbf{p}^{C(-)}_A,\mathbf{q}^{C(+)}_A;E_+)$ generated by the channel Coulomb potential U^C_{dA} from which the Born Coulomb term is subtracted.

The effective potential in the third term ($i=p$) on the right-hand-side

$$\tilde{\mathcal{T}}^{DW}_{p\,p}(\mathbf{q}^{C(-)}_p,\,\mathbf{p}^{C(-)}_p;E_+)$$
$$=<\psi^{C(-)}_{\mathbf{q}_p}\,\varphi_{nA}\Big|V^C_{pA}-U^C_{pB}\Big|\varphi_{nA}\,\psi^{C(-)}_{\mathbf{p}_p}> \tag{62}$$

is the $p+B\to p+B$ HOES DWBA elastic scattering amplitude with the pure Coulombic transition operator $V^C_{pA}-U^C_{pB}$. The reaction amplitude $\mathcal{T}^{NC}_{p\,A}(\mathbf{p}^{C(-)}_p,\,\mathbf{q}^{C(+)}_A;E_+)$ in the third term at sub-Coulomb energies is small and in the leading order can be replaced by the HOES DWBA amplitude $\mathcal{T}^{DW}_{p\,A}(\mathbf{p}^{C(-)}_p,\,\mathbf{q}^{C(+)}_A;E_+)$.

Finally, the effective potential in the fourth term ($i=n$) is the HOES DWBA amplitude of the $n+F\to B+p$ reaction:

$$\tilde{\mathcal{T}}^{DW}_{p\,n}(\mathbf{q}^{C(-)}_p,\,\mathbf{p}_n;E_+)$$
$$=<\psi^{C(-)}_{\mathbf{q}_p}\,\varphi_{nA}\Big|V^N_{pA}\Big|\varphi_{pA}\,\psi^{(0)}_{\mathbf{p}_n}>. \tag{63}$$

The reaction amplitude $\mathcal{T}^{NC}_{n\,A}(\mathbf{p}_n,\,\mathbf{q}^{C(+)}_A;E_+)$ at the sub-Coulomb energies in the leading order can be replaced by the HOES DWBA reaction amplitude $\mathcal{T}^{DW}_{n\,A}(\mathbf{p}_n,\,\mathbf{q}^{C(+)}_A;E_+)$.

Then for the sub-Coulomb $A(d,p)B$ reaction the AGS Eq. (59) reduces to the expression for the AGS reaction amplitude:

$$\mathcal{T}^{NC}_{p\,A}(\mathbf{q}^{C(-)}_p,\,\mathbf{q}^{C(+)}_A;E_+)=\mathcal{T}^{DW}_{p\,A}(\mathbf{q}^{C(-)}_p,\,\mathbf{q}^{C(+)}_A;E_+)$$
$$+\int\frac{\mathrm{d}\mathbf{p}_A}{(2\,\pi)^3}\frac{\tilde{\mathcal{T}}^{DW}_{p\,A}(\mathbf{q}^{C(-)}_p,\,\mathbf{p}^{C(-)}_A;E_+)\,\tilde{\mathcal{T}}^{C}_{A\,A}(\mathbf{p}^{C(-)}_A,\,\mathbf{q}^{C(+)}_A;E_+)}{E_+ +\varepsilon_A-\frac{p_A^2}{2\,\mathcal{M}_A}}$$
$$+\int\frac{\mathrm{d}\mathbf{p}_p}{(2\,\pi)^3}\frac{\tilde{\mathcal{T}}^{DW}_{p\,p}(\mathbf{q}^{C(-)}_p,\,\mathbf{p}^{C(-)}_p;E_+)\,\mathcal{T}^{DW}_{p\,A}(\mathbf{p}^{C(-)}_p,\,\mathbf{q}^{C(+)}_A;E_+)}{E_+ +\varepsilon_p-\frac{p_p^2}{2\,\mathcal{M}_p}}$$
$$+\int\frac{\mathrm{d}\mathbf{p}_n}{(2\,\pi)^3}\frac{\tilde{\mathcal{T}}^{DW}_{p\,n}(\mathbf{q}^{C(-)}_p,\,\mathbf{p}^{C(-)}_n;E_+)\,\mathcal{T}^{DW}_{n\,A}(\mathbf{p}_n,\,\mathbf{q}^{C(+)}_A;E_+)}{E_+ +\varepsilon_n-\frac{p_n^2}{2\,\mathcal{M}_n}}. \tag{64}$$

Now, word by word one can repeat the end of subsection III B. The proof that for the sub-Coulomb (d,p) reaction the AGS amplitude determined by expression (64) is peripheral is the same as in section III B for the AGS equation with separable potentials. Thus the sub-Coulomb $A(d,p)B$ reaction amplitude on heavier nuclei $\mathcal{T}^{NC}_{pA}(\mathbf{q}_p^{C(-)}, \mathbf{q}_A^{C(+)}; E_+)$ with local potentials is peripheral and its normalization is determined by the ANC C_{nA} of the bound state (nA).

V. AGS EQUATIONS WITH INCLUDED OPTICAL POTENTIALS

In this section the Faddeev equations in the AGS form are generalized by including the optical potentials. Usually the Faddeev equations were derived for real V_α, $\alpha = 1,2,3$ potentials. For the first time the optical potentials in the AGS formalism were introduced in [16] and in practice were used in [17] in the calculations of the ^{12}C$(d,p)^{13}$C reactions using the AGS equations with separable potentials. The optical potential appeared because the excitation of the target ^{12}C was taken into account. In [18, 19] the V_{pB} optical potential was used when solving the AGS equations for the $A(d,p)B$ reactions.

In this paper a generalization of the Faddeev equations in the AGS form is achieved by including the optical potentials in addition to the basic real nuclear potentials V^N_α, $\alpha = 1, 2, 3$, which describe the interaction between the constituent particles 1, 2 and 3. The optical potentials introduced in a way which is similar to the procedure used in the DWBA. The inclusion of the optical potentials in the transition operators will generate the optical model distorted waves in the initial and final channels of the reaction. These distorted waves are the solutions of the Schrödinger equation with the optical potentials, which are given by the sum of the nuclear optical and Coulomb channel potentials. Until now only the Coulomb distorted waves generated by the corresponding channel Coulomb potentials have been used. Introducing the optical potentials allows one to express the effective potentials in the AGS equations in terms of the DWBA amplitudes. The goal is to derive the Faddeev equations in the two-particle AGS form with optical potentials.

One can start from the modified equation for the transition operator

$$\tilde{U}^{ONC(+)}_{\beta\,\alpha}(z) = \Delta\overline{V}^{ONC}_\beta + \Delta\overline{V}^{ONC}_\beta\, G(z)\, \Delta\overline{V}^{ONC}_\alpha. \tag{65}$$

Here

$$\Delta\overline{V}^{ONC}_\alpha = \Delta\overline{V}^{ON}_\alpha + \Delta\overline{V}^{C}_\alpha, \tag{66}$$

$$\Delta\overline{V}^{ON}_\alpha = V^N_\beta + V^N_\gamma - U^{ON}_\alpha, \tag{67}$$

where U^{ON}_α is the α-channel nuclear optical potential describing the interaction between particle α and the c.m. of the bound state $(\beta\,\gamma)$. $\Delta\overline{V}^{C}_\alpha$ is given by Eq. (51). Superscript ON means the channel optical nuclear potential, superscript C stands for the screened Coulomb potential.

To obtain the Faddeev equations in the AGS form one needs to write down the Faddeev equations for the operator $\tilde{U}^{ON(+)}_{\beta\,\alpha}(z)$:

$$\begin{aligned}
&\tilde{U}^{ONC(+)}_{\beta\,\alpha}(z) = \Delta\overline{V}^{ONC}_\beta + \Delta\overline{V}^{ONC}_\beta\, G(z)\, \Delta\overline{V}^{ONC}_\alpha \\
&= \Delta\overline{V}^{ONC}_\beta + \Big[\big(V^N_\alpha - c^\alpha_\beta\, U^{ON}_\beta\big)\, G(z) + \big(V^N_\gamma - c^\gamma_\beta\, U^{ON}_\beta\big)\, G(z) + \Delta\overline{V}^{C}_\beta\, G(z)\Big]\, \Delta\overline{V}^{ONC}_\alpha \\
&= \Delta\overline{V}^{ONC}_\beta + \Big[\big(V^N_\alpha - c^\alpha_\beta\, U^{ON}_\beta\big)\overline{G}^{ONC}_\alpha(z)\, \big(\Delta\overline{V}^{ONC}_\alpha\, G(z) + 1\big) \\
&+ \big(V^N_\gamma - c^\gamma_\beta\, U^{ON}_\beta\big)\, \overline{G}^{ONC}_\gamma(z)\, \big(\Delta\overline{V}^{ONC}_\gamma\, G(z) + 1\big) + \Delta\overline{V}^{C}_\beta\, \overline{G}^{ONC}_\beta(z)\big(\Delta\overline{V}^{ONC}_\beta\, G(z) + 1\big)\Big]\Delta\overline{V}^{ONC}_\alpha \\
&= \Delta\overline{V}^{ONC}_\beta + \sum_\nu \Big[\overline{\delta}_{\beta\,\nu}\,(V^N_\nu - c^\nu_\beta\, U^{(ON)}_\beta) + \delta_{\beta\,\nu}\,\Delta\overline{V}^{C}_\nu\Big]\, \overline{G}^{ONC}_\nu(z)\, \tilde{U}^{(+)}_{\nu\,\alpha}(z),
\end{aligned} \tag{68}$$

$$\overline{G}^{ONC}_\alpha(z) = \frac{1}{z - T - V_\alpha - U^{(ON)}_\alpha - U^C_\alpha}, \tag{69}$$

$V_\alpha = V^N_\alpha + V^C_\alpha,\ c^\alpha_\beta + c^\gamma_\beta = 1,\ \alpha \neq \beta \neq \gamma.$

For the diagonal transition one gets from Eq. (68)

$$\tilde{U}^{ONC(+)}_{\alpha\,\alpha}(z) = \Delta\overline{V}^{ONC}_\alpha + \sum_\nu \Big[\overline{\delta}_{\alpha\,\nu}\,(V^N_\nu - c^\nu_\alpha\, U^{(ON)}_\alpha) + \delta_{\alpha\,\nu}\,\Delta\overline{V}^{C}_\nu\Big]\, \overline{G}^{ONC}_\nu(z)\, \tilde{U}^{(+)}_{\nu\,\alpha}(z). \tag{70}$$

Equations (68) and (70) are exact three-body Faddeev equations for the transition operators $\tilde{U}^{ONC(+)}_{\beta\,\alpha}(z)$. The

AGS equations are the Faddeev equations for the transition operators written for the matrix elements from the transition operators taken between the coupled two-body channels.

A. AGS equations with optical potentials for $A(d,p)B$ reaction

In this section the developed in the previous section formalism is applied for the $A(d,p)B$ reaction. To obtain the AGS equations for the reaction amplitudes one needs to use the spectral decomposition for the intermediate Green functions $\overline{G}_\nu^{ONC}(z)$ in terms of the distorted waves in the channel ν generated by the channel potential $U_\nu^{(ON)}+U_\nu^C$. The channel potential now contains the complex optical potential $U_\nu^{(ON)}$. Hence the distorted waves are not orthogonal. To derive the AGS equations from the Faddeev equations for the transition operator I use the following approximation. First, the AGS equations are derived for the real "optical" potentials and then in the obtained equations the distorted waves generated by the real channel nuclear plus Coulomb potentials are replaced by the optical model distorted waves which are solutions of the Schrödinger equations for the complex channel potentials $U_\nu^{(ON)}+U_\nu^C$.

The final AGS equations with the optical potentials for the reaction amplitude of the $A(d,p)B$ reaction are:

$$\mathcal{T}_{p\,A}(\mathbf{q}_p^{ONC(-)},\,\mathbf{q}_A^{ONC(+)};E_+)=\mathcal{T}_{p\,A}^{DW}(\mathbf{q}_p^{ONC(-)},\,\mathbf{q}_A^{ONC(+)};E_+)$$
$$+\sum_{i=A,p,n}\int\frac{\mathrm{d}\mathbf{p}_i}{(2\,\pi)^3}\frac{\tilde{\mathcal{T}}_{p\,i}^{DW}(\mathbf{q}_p^{ONC(-)},\,\mathbf{p}_i^{ONC(-)};E_+)\,\mathcal{T}_{i\,A}(\tilde{\mathbf{p}}_i^{ONC(-)},\,\mathbf{q}_A^{ONC(+)};E_+)}{E_+ +\varepsilon_i-\frac{p_i^2}{2\,\mathcal{M}_i}}. \tag{71}$$

$\mathbf{q}_i^{ONC(+)}$ is the distorted wave, which is the solution of the Schrödinger equation with the nuclear optical plus channel Coulomb potentials in the channel $i=A,p$, $\tilde{\mathbf{p}}_i^{ONC(-)}$ is the distorted wave from the dual space to $\mathbf{p}_i^{ONC(-)}$. Note that for the channel n $\mathbf{p}_n^{ON(-)}$ is the pure nuclear distorted wave because the channel Coulomb interaction $n-F$ is absent.

The effective potential

$$\mathcal{T}_{p\,A}^{DW}(\mathbf{q}_p^{ONC(-)},\,\mathbf{q}_A^{ONC(+)};E_+)$$
$$=<\psi_{\mathbf{q}_p}^{ONC(-)}\,\varphi_{nA}|V_{pn}+V_{pA}-U_{pB}^{ON}$$
$$-U_{pB}^C|\varphi_{pn}\,\psi_{\mathbf{q}_A}^{ONC(+)}> \tag{72}$$

is the post form of the ONES DWBA amplitude for the $A(d,p)B$ reaction. The other DWBA amplitudes in Eqs. (71) are:

$$\tilde{\mathcal{T}}_{p\,A}^{DW}(\mathbf{q}_p^{ONC(-)},\,\mathbf{p}_A^{ONC(-)};E_+)$$
$$=<\psi_{\mathbf{q}_p}^{ONC(-)}\,\varphi_{nA}\Big|V_{pn}\Big|\varphi_{pn}\,\psi_{\mathbf{p}_A}^{ONC(-)}> \tag{73}$$

is the post form of the HOES DWBA amplitude for the $A(d,p)B$ reaction with the transition operator V_{pn},

$$\tilde{\mathcal{T}}_{p\,p}^{DW}(\mathbf{q}_p^{ONC(-)},\,\mathbf{p}_p^{ONC(-)};E_+)$$
$$=<\psi_{\mathbf{q}_p}^{ONC(-)}\,\varphi_{nA}\Big|V_{pA}^C-U_{pB}^C\Big|\varphi_{nA}\,\psi_{\mathbf{p}_p}^{ONC(-)}> \tag{74}$$

is the HOES DWBA $p+(nA)$ elastic scattering amplitude,

$$\tilde{\mathcal{T}}_{p\,n}^{DW}(\mathbf{q}_p^{ONC(-)},\,\mathbf{p}_n^{ON(-)};E_+)$$
$$=<\psi_{\mathbf{q}_p}^{ONC(-)}\,\varphi_{nA}\Big|V_{pA}^N-U_{pB}^{ON}\Big|\varphi_{pA}\,\psi_{\mathbf{p}_n}^{ON(-)}> \tag{75}$$

is the HOES DWBA $n+(pA)\to p+(nA)$ reaction amplitude.

It is taken into account that $\Delta\overline{V}_p^C\equiv\Delta\overline{V}_{nA}^C=V_{pA}^C-U_{pB}^C$ and that $c_\alpha^\beta+c_\alpha^\gamma=1$. Because there is no optical potential in the $p-n$ channel one can adopt $c_p^A=0$ and $c_p^n=1$.

Now Eq. (71) can be analyzed. For the sub-Coulomb case the Coulomb distortion in the initial and final states is dominant and the proof of the peripheral character of Eq. (71) is the same as in section III B, that is, the AGS amplitude of the $A(d,p)B$ reaction is well approximated by the corresponding DWBA amplitude:

$$\mathcal{T}_{p\,A}(\mathbf{q}_p^{ONC(-)},\,\mathbf{q}_A^{ONC(+)};E_+)$$
$$\approx\mathcal{T}_{p\,A}^{DW}(\mathbf{q}_p^{ONC(-)},\,\mathbf{q}_A^{ONC(+)};E_+). \tag{76}$$

One important thing to note. The sub-Coulomb reaction amplitudes in subsections III B and IV A are well approximated by the sub-Coulomb DWBA amplitudes when the energies are so low that the nuclear optical potentials can be neglected and the distorted waves in the initial and final states can be approximated by the Coulomb ones. However, when the energy, still being sub-Coulomb, increases, the approximation of the AGS reaction amplitude by the sub-Coulomb DWBA one fails. Meantime, approximation (76) works practically at all sub-Coulomb energies because the DWBA amplitude determined by Eq. (72) contains the distorted waves generated by the sum of the channel Coulomb and nuclear optical potentials. It also contains the optical potential in the transition operator.

Now one can consider the $A(d,p)B$ reaction at the energies above the Coulomb barrier on heavier nuclei. Owing to the presence of the distorted waves in the initial and final channels the DWBA amplitude can be peripheral and dominantly contributed by the tail of the (nA) bound-state wave function. It can be easily checked using the FRESCO code [13] Assume that it is the case and let me analyze the AGS Eq. (71).

The first term on the right-hand-side of this equation is the post-form of the ONES DWBA amplitude. Assume that the amplitude $\mathcal{T}^{DW}_{p\,A}(\mathbf{q}_p^{\,ONC(-)}, \mathbf{q}_A^{\,ONC(+)}; E_+)$ is peripheral. Its peripheral character means that it is contributed by the tail of the (nA) bound-state wave function and, hence, is parametrized in terms of the ANC C_{nA} of this bound state.

The second term contains $\tilde{\mathcal{T}}^{DW}_{p\,A}(\mathbf{q}_p^{\,ONC(-)}, \mathbf{p}_A^{\,ONC(+)}; E_+)$, which is the DWBA $A(d,p)B$ reaction amplitude with the V_{pn} as the transition operator. It is also peripheral and can be easily estimated because one can use the zero-range approximation for the V_{pn}. Then the radial integration in this amplitude is carried over r_{nA}. Since it is assumed that $\mathcal{T}^{DW}_{p\,A}(\mathbf{q}_p^{\,ONC(-)}, \mathbf{q}_A^{\,ONC(+)}; E_+)$ is peripheral, it is also true for $\tilde{\mathcal{T}}^{DW}_{p\,A}(\mathbf{q}_p^{\,ONC(-)}, \mathbf{p}_A^{\,ONC(+)}; E_+)$, which is aslo is parameterized in terms of the ANC C_{nA} of the $(n\,A)$ bound state.

The third terms contains the HOES amplitude $T_{p\,A}(\tilde{\mathbf{p}}_p^{\,ONC(-)}, \mathbf{q}_A^{\,ONC(+)})$, which is the same reaction amplitude as the one on the left-hand-side but the HOES. All three first terms on the right-hand-side of Eq. (71) provide forward peaked proton's angular distribution. The fourth term, as in all the previous considerations, has a flat angular distribution and can be neglected compared to the first three terms when considering the angular distributions near the stripping peak.

To further simplify the AGS equation the $d+A$ elastic scattering amplitude $\mathcal{T}_{A\,A}(\tilde{\mathbf{p}}_A^{\,ONC(-)}, \mathbf{q}_A^{\,ONC(+)}; E_+)$ is replaced by the DWBA elastic scattering amplitude in which the Born term is subtracted:

$$\mathcal{T}^{DW}_{A\,A}(\tilde{\mathbf{p}}_A^{\,ONC(-)}, \mathbf{q}_A^{\,ONC(+)}; E_+) = < \psi^{\,ONC(-)}_{\tilde{\mathbf{p}}_A}\,\varphi_{pn}\Big|\Delta\overline{V}^{\,ONC}_A\Big|\varphi_{pn}\,\psi^{\,ONC(+)}_{\mathbf{q}_A} > . \qquad (77)$$

Here $\Delta\overline{V}^{\,ONC}_A = \Delta\overline{V}^{\,ONC}_{pn} = V_{pA} + V_{nA} - U^{ON}_{dA} - U^{C}_{dA}$.

Then AGS Eq. (71) reduces to the equation

$$\mathcal{T}_{p\,A}(\mathbf{q}_p^{\,ONC(-)}, \mathbf{q}_A^{\,ONC(+)}; E_+) = \mathcal{T}^{DW}_{p\,A}(\mathbf{q}_p^{\,ONC(-)}, \mathbf{q}_A^{\,ONC(+)}; E_+) + \int \frac{\mathrm{d}\mathbf{p}_A}{(2\,\pi)^3}\,\frac{\tilde{\mathcal{T}}^{DW}_{p\,A}(\mathbf{q}_p^{\,ONC(-)}, \mathbf{p}_A^{\,ONC(-)}; E_+)\,\mathcal{T}^{DW}_{A\,A}(\tilde{\mathbf{p}}_A^{\,ONC(-)}, \mathbf{q}_A^{\,ONC(+)}; E_+)}{E_+ + \varepsilon_A - \frac{p_A^2}{2\,\mathcal{M}_A}} + \int \frac{\mathrm{d}\mathbf{p}_p}{(2\,\pi)^3}\,\frac{\tilde{\mathcal{T}}^{DW}_{p\,p}(\mathbf{q}_p^{\,ONC(-)}, \mathbf{p}_p^{\,ONC(-)}; E_+)\,\mathcal{T}_{p\,A}(\tilde{\mathbf{p}}_p^{\,ONC(-)}, \mathbf{q}_p^{\,ONC(+)}; E_+)}{E_+ + \varepsilon_p - \frac{p_p^2}{2\,\mathcal{M}_p}}. \qquad (78)$$

This is an integral equation for the $A(d,p)B$ reaction amplitude $\mathcal{T}_{p\,A}(\mathbf{q}_p^{\,ONC(-)}, \mathbf{q}_A^{\,ONC(+)}; E_+)$ for the energies above the Coulomb barrier. It is assumed that the DWBA reaction amplitude is peripheral, that is, parametrized in terms of the ANC C_{nA}. Hence two amplitudes on the right-hand-side of Eq. (78) are paramtrized in terms of the ANC. Then a solution of this equation is also parametrized in terms of the ANC C_{NA} although its dependence on the ANC may be complicated. The more dominant contribution of the first term on the right-hand-side of Eq. (78) the closer to the linear the dependence on the ANC of its solution.

VI. SUMMARY

Usually, for the analysis of the (d,p) reactions the DWBA, ADWA or CDCC methods [2, 3, 20] are used. In these last two approaches the coupling of the neutron transfer channel with the deuteron breakup channel is taken effectively into account, while the explicit coupling to the proton and heavy-particle transfer channels and elastic scattering is neglected. Meantime, the Faddeev equations take into account the coupling of all the transfer, elastic and breakup channels simultaneously. In this paper the formalism of the three-body Faddeev equations for the (d,p) reactions is formulated using the two-body AGS equations. For separable potentials these equations are exact and can be used for the analysis of the direct $A(d,p)B$ reactions on heavier nuclei at sub-Coulomb energies. The advantage of the AGS equations with separable potentials is that the effective potentials are given by a few simple diagrams. The sum of the pole and triangle exchange diagrams can be expressed in terms of the DWBA amplitude for the sub-Coulomb (d,p) reactions. To obtain the two-body AGS equations for local potentials the contribution from the deuteron breakup channel is neglected but the coupling to the transfer and elastic scattering channels is taken into account explicitly.

For low-energy reactions, especially for the sub-Coluomb ones, the contribution from the breakup channel is small and the developed formalism is well suited for the direct sub-Coulomb $A(d,p)B$ reactions on heavier nuclei. It is shown that the AGS equation for the sub-Coulomb $A(d,p)B$ reactions is peripheral and dominated by the post-form of the DWBA amplitude, which is peripheral. Hence, the AGS amplitude is also parametrized in terms of the ANC.

In this paper the two-body AGS equations are also generalized by including the optical potentials in the same manner as it is done in the DWBA. Naturally, the effective potentials in the obtained AGS equations are the DWBA amplitudes. Although it is shown that the AGS $A(d,p)B$ reaction amplitude can be parametrized in terms of the ANC C_{nA} of the bound state (nA), there is a conceptual problem of determination of the ANC from comparison of the AGS cross section with experimental data. The problem is that the AGS equations are based on the three-body model. Hence the AGS amplitude contains only the single-particle (nA) bound-state wave function rather than the overlap integral, which includes the spectroscopic factor. This issue will be addressed in another paper.

VII. ACKNOWLEDGMENTS

This work was supported by the U.S. DOE Grant No. DE-FG02-93ER40773, NNSA Grant No. DE-NA0003841 and U.S. NSF Award No. PHY-1415656.

[1] R. E. Tribble, C. A. Bertulani, M. La Cognata, A. M. Mukhamedzhanov and C. Spitaleri, Rep. Prog. Phys. **77**, 106901 (2014).
[2] N. Austern *et al.* **154**, 125 (1987).
[3] R. C. Johnson and P. J. R. Soper, Phys. Rev. **C 1**, 976 (1970).
[4] L. D. Faddeev, Sov. Phys. JETP 12, 1014 1961; Mathematical Aspects of the Three-Body Problem in the Quantum Scattering Theory Israel Program for Scientific Translations, Jerusalem, 1965.
[5] E. O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. B2, 167 1967.
[6] E. O. Alt, W. Sandhas and H. Ziegelmann, Phys. Rev. **C 17**, 1981 (1978).
[7] R. C. Johnson, Phys. Rev. **C 80**, 044616 (2009).
[8] E. O. Alt, A. M. Mukhamedzhanov, M. M. Nishonov, and A. I. Sattarov, Phys Rev **C 65**, 064613 (2002).
[9] A. M. Mukhamedzhanov, V. Eremenko, and A. I. Sattarov, Phys. Rev. **C 86**, 034001 (2012).
[10] A. M. Mukhamedzhanov and N. K. Timofeyuk, Pis'ma Zh. Eksp. Teor. Fiz. **51**, 247 (1990) [JETP Lett. **51**, 282 (1990)].
[11] K. Alder *er al.*, Rev. Mod. Phys. **28**, 432 (1956).
[12] L. D. Knutson, Ann. Phys. **106**, 1 (1977).
[13] I. Thompson, Computer Phys. Reports **7**, 167 (1988).
[14] A. M. Mukhamedzhanov, E. O. Alt and G. V. Avakov, Phys. Rev. **C 61**, 064006 (2000).
[15] A. Deltuva, A. C. Fonseca, and P. U. Sauer, Phys. Rev. **C 71**, 054005 (2005).
[16] P. Grassberger and W. Sandhas, Z. Physik **220**, 29 (1969).
[17] E. O. Alt, L. D. Blokhintsev, A. M. Mukhamedzhanov, and A. I. Sattarov, Phys. Rev **C 75**, 054003 (2007).
[18] A. Deltuva, A. M. Moro, E. Cravo, F. M. Nunes, and A. C. Fonseca, Phys. Rev. **C 76**, 064602 (2007).
[19] A. Deltuva, Phys. Rev. **C 79**, 021602 (2009).
[20] I. J. Thompson and F. M. Nunes, Nuclear reactions for Astrophysics, Cambridge University Press, 2009.