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Design of Continuously Graded Elastic Acoustic Cloaks

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In this report, we present optimized designs of graded linear viscoelastic, isotropic materials that are cloaked from harmonic acoustic excitation. We show neutrally-buoyant disks featuring constant density and radially varying isotropic, elastic bulk moduli effectively minimize reflections from incident acoustic pressure. Cloaking behavior is achieved for both individual frequencies and extended frequency bandwidths. The design task is approached as a PDE-constrained optimization problem: the complex, frequency-dependent constitutive parameters of a generalized viscoelastic solid are selected to minimize the residual of a spreading wave pressure field in a fluid and the scattered field produced by the solid inclusion. The distributions of elastic properties change according to the loading frequency, while viscous properties are minimized. The presented design framework may be extended to other generalized geometries and physically realized with novel functionally graded elastic materials.

I. INTRODUCTION

A wealth of research aims at the design and development of acoustic cloaks, materials or structures rendered undetectable to acoustic detection. Interest in acoustic cloaks has risen in the confluence of improved material manufacturing, high performance computing, and developments in analogous designs for electromagnetic (EM) radiation cloaks. Exploratory studies of acoustic cloaks have drawn inspiration from coordinate transformation based methods used in EM formulations. Ideal cloaks derived in this class of solutions feature radially varying materials with anisotropic effective mass density and bulk modulus that prevent acoustic energy from entering a solid scatterer[3,6]. Additional schemes include scattering cancellation techniques, in which cloaking layers produce non-resonant scattering cancellation around an object, and anomalous resonance, for which the proper spatial arrangement of scatterers with active or negative material properties can allow a region to be cloaked[4].

However, physical realization of proposed acoustic cloaking schemes remains a hurdle. The prescribed radial mass-anisotropy of transformation-type solutions is atypical in natural materials and difficult to manufacture synthetically[4]. Metamaterials typically use local imbedded resonators to achieve subwavelength control and wave reflection or absorption. Materials featuring subwavelength scale resonators are susceptible to local resonances in the inclusions, which can lead to wave absorption. The required anisotropy or negative material properties of ideal acoustic cloaking strategies has prompted research in metamaterials, engineered materials whose microstructure or composition endow responses not typically found in nature. Acoustic metamaterials include passive media, like pentamodal lattices, and active materials, including tunable piezoelectric materials, but are yet to be produced at feasible scales or work for extended bandwidths[2].

Elastic metamaterials represent an expanding frontier for acoustic wave cloaking development. Elastic metamaterials typically consist of periodically varying layers of extremal elastic materials or periodic arrangement of resonators suspended in an elastic solid[1,2]. Guild et al. explored the ideal radial variation in density and bulk modulus for elastic acoustic cloaking layers, proposing even a single isotropic cloaking layer may significantly reduce scattering from wavelength scale structures[4]. Shen et. al derived analytical solutions for multilayered elastic spherical cloaks in a two-dimensional waveguide, showing periodic variation in elastic properties along the cloak radius approximates the radial effective mass anisotropy of transformation-type solutions[9]. Chen et. al. proposed multifunctional composite materials, arranged as multilayered viscoelastic continuum media, that can effectively attenuate transient and blast loading energy[1].

The optimal distribution and properties of heterogeneous elastic materials has been pursued for a variety of structural and material design problems. Typically, topology optimization of two-phase or multiphase composites is used to design materials that manage energy propagation through a structure. Nakshratala et. al used a topological optimization scheme to design rate-independent elastoplastic material composites that dissipate or focus bulk wave energy[7]. In such cases, minimum compliance or maximum energy dispersion typically mea-

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sure design performance.

In a similar vein, we employ material-identification inverse methods[11] to select the elastic material properties magnitudes for a given disk geometry to minimize its acoustic reflections. In our approach, we optimize the frequency-dependent complex moduli of graded viscoelastic materials such that the scattered field produced by the solid inclusion matches the steady-state propagation of a bulk wave through fluid. We show that continuously graded, linear-elastic isotropic media can achieve both frequency-specific and broadband acoustic cloaking. The report is organized as follows: we present the system of partial differential equations that govern the acoustic-structural interaction of the material and the surrounding fluid domain. We then present the objective for our design problem and formulate abstract optimization operators which are implemented computationally. We finally present numerical results for an example application, in which we design a graded disk that minimally reflects acoustic loadings over a range of frequencies.

II. THEORETICAL FORMULATION

A. Governing System of Partial Differential Equations

We will first derive the coupled system of differential equations that govern the interaction of the structural and acoustic domains. The domain of interest Ω is partitioned into solid and fluid subdomains, Ω_S and Ω_F. The equations of motion of a solid body in three dimensions under steady-state conditions are given by

$$-\omega^2 \rho_s \boldsymbol{u} - \Delta : \boldsymbol{\sigma} = \boldsymbol{0} \text{ in } \Omega_S \tag{1}$$

where $\boldsymbol{u}(\boldsymbol{x}, \omega)$ is the complex vector of displacements, ω is angular frequency, ρ_s is the solid density, and $\boldsymbol{\sigma}$ is the stress tensor. The boundary of the considered domain Ω_S is defined as a wet boundary Γ_W with the fluid domain.

Motion in the compressible, inviscid fluid domain is described by the steady-state Helmholtz equation:

$$\begin{aligned} -k^2\psi - \Delta\psi &= 0 \text{ in } \Omega_F \\ \nabla\psi \cdot \boldsymbol{n} &= -\rho_f v_n \text{ on } \Gamma_{NF} \\ \nabla\psi \cdot \boldsymbol{n} &= -ik\psi \quad \text{on } \Gamma_C \end{aligned} \tag{2}$$

where $\psi(\boldsymbol{x})$ is the complex scalar velocity potential, the wave number $k = \omega/c$, where c is the speed of sound in the fluid medium, Γ_{NF} is the Neumann portion of the boundary with a prescribed normal particle velocity v_n, and Γ_C is the portion of the boundary with absorbing Sommerfeld conditions[8].

The fluid and structural domains are coupled through the continuity of velocity potential and stress at the boundary Γ_W between the fluid and solid domains. (Note, the acoustic pressure p is related to the time-derivative of the velocity potential: $p = \dot{\psi}$). By convention, the normal to Γ_W points from the solid into the fluid. The boundary conditions placed on Γ_W for the fluid and solid domains are expressed respectively:

$$\frac{\partial\psi}{\partial n} = -\rho_f v_n \tag{3}$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = -\psi \boldsymbol{n} \tag{4}$$

We derive the weak form from the coupled acoustic-elastodynamic strong form system in Equations 1 through 4 and apply a finite-space approximation to yield a discretized system of equations:

$$\begin{aligned} &\boldsymbol{g}\left(\begin{bmatrix} \boldsymbol{u} \\ \psi \end{bmatrix}\right) = \\ &= \left(\begin{bmatrix} \boldsymbol{K}_s & 0 \\ 0 & -\boldsymbol{K}_f/\rho_f \end{bmatrix} + i\omega \begin{bmatrix} \boldsymbol{C}_s & \boldsymbol{L} \\ \boldsymbol{L}^T & -\boldsymbol{C}_f/\rho_f \end{bmatrix} \right. \\ &\quad \left. -\omega^2 \begin{bmatrix} \boldsymbol{M}_s & 0 \\ 0 & \boldsymbol{M}_f/\rho_f \end{bmatrix} \right) \begin{bmatrix} \boldsymbol{u} \\ \psi \end{bmatrix} \\ &= \hat{\boldsymbol{0}} \end{aligned} \tag{5}$$

where $\boldsymbol{K}$ denotes the stiffness matrices; $\boldsymbol{C}$, the damping matrices; and M, the mass matrices. The block $\boldsymbol{L}$ represents coupling equations between the domains.

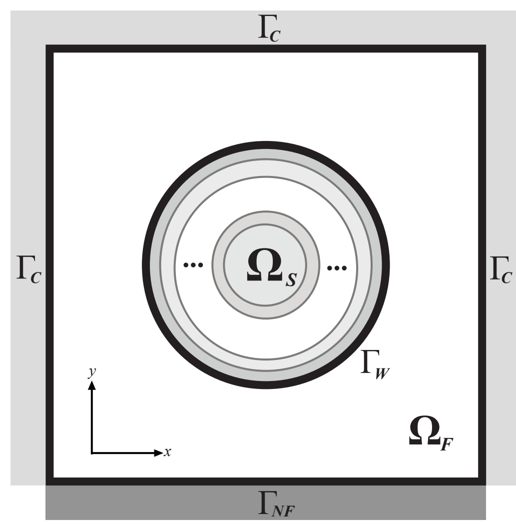


FIG. 1: Coupled fluid-structural domain, with absorbing Sommerfeld conditions Γ_C and pressure-loading Γ_{NF} applied to fluid domain

B. Viscoelastic Material Model

We generalize the materials in the solid domain as linear viscoelastic. In frequency domain analysis, the constitutive law for linear viscoelastic materials is expressed:

$$\boldsymbol{\sigma}(\omega) = \boldsymbol{D}(\omega)\boldsymbol{\epsilon} = (b(\omega)\boldsymbol{D}_b + G(\omega)\boldsymbol{D}_G)\boldsymbol{\epsilon}(\omega) \tag{6}$$

where $\boldsymbol{\epsilon}$ is the strain tensor, $\boldsymbol{D}_b$ and $\boldsymbol{D}_G$ are the bulk and shear components of the fourth-order stiffness tensor, and the bulk modulus b and shear modulus G are complex functions of frequency and space:

$$b(\omega, \boldsymbol{x}) = b_R(\omega, \boldsymbol{x}) + b_I(\omega, \boldsymbol{x}) \tag{7}$$

$$G(\omega, \boldsymbol{x}) = G_R(\omega, \boldsymbol{x}) + G_I(\omega, \boldsymbol{x}) \tag{8}$$

The real components of the moduli (b_R, G_R) here represent elastic storage, whereas the imaginary components (b_I, G_I) represent lossy behavior. The complex moduli are constant at a given frequency, though are treated as spatially variable in the solid.

C. Abstract Optimization Formulation

We pose the design of viscoelastic material parameters as a PDE-constrained optimization problem. We may define the optimization problem abstractly as:

$$\begin{aligned} \underset{\boldsymbol{u},\boldsymbol{p}}{\text{minimize}} \quad & J(\boldsymbol{u}, \boldsymbol{p}) \\ \text{subject to} \quad & \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{p}) = \boldsymbol{0} \end{aligned}$$

where $\boldsymbol{u} \in \Re^n$ is the state vector (a concatenation of the real and imaginary components of the structural displacements and fluid velocity potential), $\boldsymbol{p} \in \Re^m$ is the design vector, $J : \Re^n \times \Re^m \rightarrow \Re$ is the objective function and $\boldsymbol{g} : \Re^n \times \Re^m \rightarrow \Re^n$ represents the set of discretized PDE constraints from Equation 5.

In the design problem, we desire the velocity potential field produced by the optimized structure to match a desired velocity potential distribution in its surrounding fluid domain. We choose a least-squares residual cost functional $J : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ to measure the misfit between the desired state field $\boldsymbol{u}^d$ and the measured field $\boldsymbol{u}^h$:

$$J(\boldsymbol{u}, \boldsymbol{p}) := \frac{\kappa}{2} \left(\boldsymbol{u}^d - \boldsymbol{u}^h\right)^T [\boldsymbol{Q}] \left(\boldsymbol{u}^d - \boldsymbol{u}^h\right) + \mathcal{R}(\boldsymbol{p}) \tag{9}$$

Here, $R(\boldsymbol{p}) : \mathbb{R}^m \rightarrow \mathbb{R}$ is a regularization term, a functional of the parameter vector, and κ is a positive scaling constant. A boolean weight matrix $[Q]$ isolates the degrees of freedom at which velocity potential is compared to the desired state field.

The parameter vector $\boldsymbol{p}(\boldsymbol{x}, \omega)$ represents the frequency-dependent complex bulk and shear moduli for the viscoelastic solid. In discretized form, the complex moduli are defined uniquely and treated as piecewise constant for blocks of elements. For a given frequency, the parameter vector is a concatenation of the real and imaginary components of the bulk and shear moduli for the different blocks in the structural domain.

We use a gradient-based optimization scheme, with adjoint-based gradient computations, to solve the nonlinear optimization problem. Refer to Walsh et. al[11] for an elaborated explanation of the optimization operators employed. A modified Newton Method with a line search algorithm is employed to update the design vector[5].

III. RESULTS

In a demonstrative application, the distribution of material properties in a radially graded elastic disk was optimized so that the disk would minimize reflections from acoustically energy in its surrounding fluid medium. By tuning the radial distribution of the complex viscoelastic moduli, we aimed to design a dispersive solid which minimally reflected incident acoustic pressure. We arrived at frequency dependent distributions for the disk's constitutive properties, characterized by graded bulk moduli and minimal loss moduli components.

The computational model featured a disk composed of fifty concentric rings of unique viscoelastic properties immersed in a fluid waveguide. The considered domain, as shown in Figure 1, included a 1m × 1m rectangular fluid region, modeled as water, with the solid inclusion of radius 0.25m at its center. A harmonic acoustic pressure loading, modeled as a plane wave, was applied to one side of the domain. The remaining three sides were treated with absorbing boundary conditions, approximating a semi-infinite domain. The circular inclusion was considered to be neutrally buoyant ($\rho_s = \rho_f = 1000\ kg/m^3$). The finite element (FE) mesh was assembled with linear elements and constructed one element thick to approximate a two-dimensional disk. Plane strain conditions were also enforced in the inclusion to approximate two-dimensional behavior. The FE model featured sufficient fineness to accommodate at least ten elements per bulk wave length, in order to resolve wave variation.

Forward analyses on the model were run to generate a target velocity potential field for the design problem. In the forward run, the entire domain was defined as a fluid. The direct frequency response of a harmonic pressure loading was calculated and the velocity potential was recorded at nodes in the fluid domain surrounding the inclusion. This forward distribution became the target distribution for the optimization step; we desired to minimize the difference between the forward distribution and the scattered field produced by the structural response, entailing the minimization of reflections off the inclusion.

Design of the disk was performed through the computational implementation of the described optimization algorithms. The structural dynamics finite element computations were performed with Sierra-SD, which then interfaced with optimization algorithms from the Rapid Optimization Library[5]. Optimization stopping criteria were based on the reduction of both the objective function and the gradient, the derived optimality conditions requiring a zero gradient for a solution[11]. The optimization formulation used bounds on the real and imaginary moduli values to restrict designs within realistic damping ratios[10]. We supplied the initial guesses and bounds for the material moduli presented in Table I.

Designs were computed for the graded viscoelastic disc at a variety of loading frequencies. Optimization was performed independently for each considered frequency to

TABLE I: Initial Values & Bounds for Design Parameters (Pa)

Modulus	Initial	Lower Bound	Upper Bound
b_R	5e+08	1e+08	1e+10
b_I	1e+03	1e+02	1e+05
G_R	1e+06	1e+05	1e+07
G_I	1e+03	1e+02	1e+05

capture the frequency dependence of the viscoelastic material moduli. For the considered set of frequencies–400, 800, 1200, 1600, and 2000 Hz— material properties were successfully chosen to yield minimizers to the objective function, which allowed the disk to be cloaked from incident acoustic pressure. Figure 2 compares the pressure scattered fields for the model with the initial guess and the optimized material distribution to the target field for the loading of 1200 Hz.

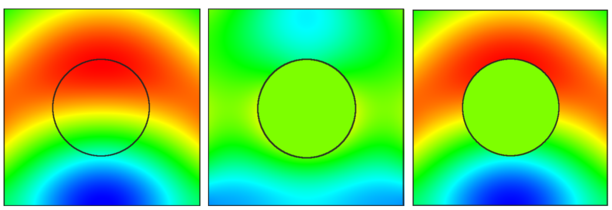


FIG. 2: The target pressure field, left, for a plane wave with frequency 1200 Hz, is distorted by the scattered field from the inclusion with initial material guesses, center, but is recovered in the fluid domain with the optimized material distribution, right

Inspection of the material designs suggests that the dispersive cloaking behavior was most affected by the radial variation of the elastic bulk modulus. In the designed material distributions, the material properties varied along the radius of the disk in patterns unique to the loading frequency. In Figure 3, the value of the real bulk modulus component is shown for the considered frequencies. The bulk modulus distributions share generally smooth variations which decay towards the disk center. However, the designs, non-unique and affected by initial material guess and selected bounds, differ in the the character and number of the peaks along the radius. These variations are nontrivial: averaging the bulk moduli or eliminating the inner peak in the 1600 and 2000 Hz solutions, for example, causes the disk to reflect pressure, quantified by multiple order of magnitude increases in the objective function.

The radial variations in the elastic shear modulus affected the cloaking behavior less significantly. Optimized shear values tended to have highly-oscillatory values layer by layer. Averaging all shear modulus values along the radius was shown to marginally increase the objective. Furthermore, maintaining a constant ratio to the elastic

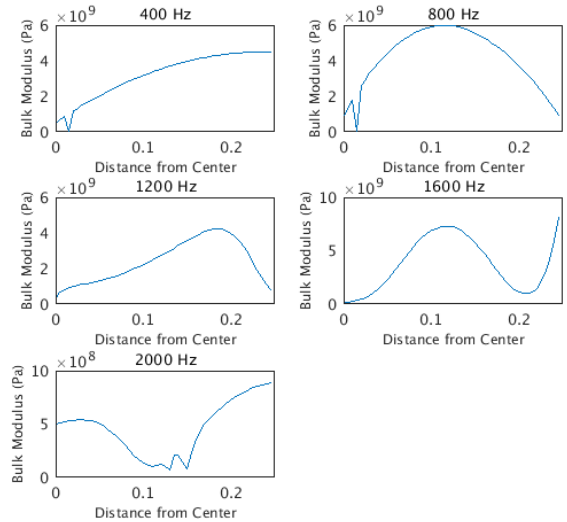


FIG. 3: Solutions for the radial variation in the real bulk modulus for five considered load frequencies

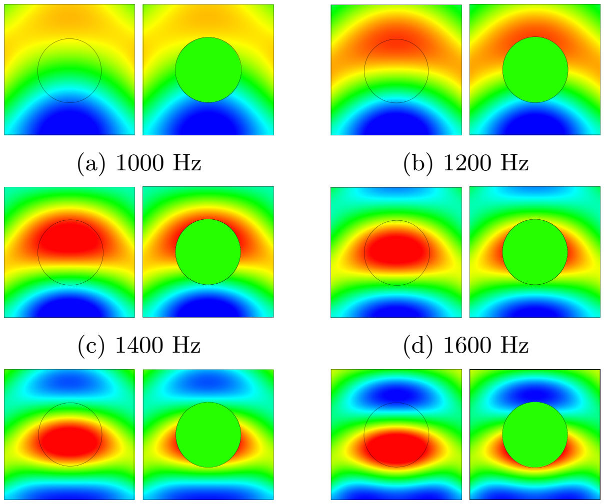
TABLE II: Comparison of Objective Function Values for Post-Processed Modulus Distributions (1200 Hz Example)

Case	**Objective Value**
Initial Guess	6.1041e-01
Optimized Solution	8.6274e-05
Averaged G_R	8.9483e-05
Constant G_R/b_R = 1e-3	8.9732e-05
$G_I = b_I = 0$	8.5905e-05

bulk modulus, or constant Poisson ratio, likewise maintains the cloaking behavior with minimal change in the objective value. Table II compares the objective function value evaluated with the output optimized material distribution to objective values evaluated for models with transformed shear distributions. Relatively minor changes in the objective suggest only variation in the elastic bulk modulus is fundamental to demonstrated cloaking behavior; a shear modulus which varies in magnitude along with the bulk maintains this behavior.

The loss moduli deviated little from the supplied initial guesses (≈1kPa), maintaining low damping ratios for both the shear and bulk moduli. Though the supplied optimization bounds allowed latitude for increase, the loss moduli showed little sensitivity to change. In post-processing, driving the loss moduli components to zero either maintained or improved the performance of the cloaking disk, as shown in Table II. High loss moduli would increase energy dissipation, affecting the scattered field behind the inclusion, and thus increases in the loss moduli were undesired.

We also show that computed minimizers for selected



(a) 1000 Hz (b) 1200 Hz (c) 1400 Hz (d) 1600 Hz (e) 1800 Hz (f) 2000 Hz

FIG. 4: Target and solution acoustic pressure distributions for varying frequencies with a single elastic distribution design

frequencies have extended bandwidths of frequencies for which they do not produce reflections. The elastic distributions for the solution at 1200 Hz were input as initial guesses for the optimization of purely elastic materials for the frequency range 1000 Hz to 2000 Hz. A single elastic distribution was determined which cloaked the disk over the range of frequencies, as shown in Figure 4.

In summary, numerical designs suggest that variation of purely elastic, but frequency dependent, material moduli may allow a neutrally-buoyant solid medium to be acoustically reflection-less in a surrounding fluid. Furthermore, numerical optimization can yield the ideal patterns for material property variation in elastic dispersive materials. Finally, solutions exhibit some extended bandwidth of multiple frequency support. The manufacturability of these materials remains a challenge to the realization of these material schemes. Developments in functionally graded materials, with continuously varying bulk and shear moduli, and in actively-tunable elastic metamaterials, could prove as material candidates for these acoustic cloaking schemes[9].

IV. REFERENCES

[1] Chen, Y., Barnhart, M., Chen, J., Hu, G., Sun, C., and Huang, G., Composite Structures **136**, 358 (2016).

[2] Cheng, Y., Yang, F., Xu, J. Y., and Liu, X. J., Applied Physics Letters **92**, 151913 (2008).

[3] Cummer, S. A., Popa, B.-I., Schurig, D., Smith, D. R., Pendry, J., Rahm, M., and Starr, A., Physical review letters **100**, 024301 (2008).

[4] Guild, M. D., Alu, A., and Haberman, M. R., The Journal of the Acoustical Society of America **129**, 1355 (2011).

[5] Heroux, M. A., Bartlett, R. A., Howle, V. E., Hoekstra, R. J., Hu, J. J., Kolda, T. G., Lehoucq, R. B., Long, K. R., Pawlowski, R. P., Phipps, E. T., *et al.*, ACM Transactions on Mathematical Software (TOMS) **31**, 397 (2005).

[6] Milton, G. W., Briane, M., and Willis, J. R., New Journal of Physics **8**, 248 (2006).

[7] Nakshatrala, P. and Tortorelli, D., Computer Methods in Applied Mechanics and Engineering **295**, 305 (2015).

[8] Reese, G., Bhardwaj, M. K., and Walsh, T., Sandia National Laboratories, Albuquerque, New Mexico SAND2011-8272 (2011).

[9] Shen, H., Wen, J., Yu, D., Cai, L., Wen, X., *et al.*, Modelling and Simulation in Materials Science and Engineering **21**, 065011 (2013).

[10] Walsh, T., Aquino, W., Ridzal, D., and Kouri, D., "Inversion for eigenvalues and modes using sierra-sd and rol." Tech. Rep. (Sandia National Laboratories (SNL-NM), Albuquerque, NM (United States), 2015).

[11] Walsh, T., Aquino, W., Ridzal, D., Kouri, D., van Bloemen Waanders, B., and Urbina, A., "Viscoelastic material inversion using sierra-sd and rol." Tech. Rep. (Sandia National Laboratories (SNL-NM), Albuquerque, NM (United States); Duke University,, Durham, NC, 2014).