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Chance Constrained Economic Dispatch Problem with Renewable Energy and Energy Storage

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***Abstract*—Increased penetration of renewables, along with uncertainties associated with them, have transformed how power systems are operated. High levels of uncertainty means that it is not longer possible to guarantee operational feasibility with certainty, instead constraints are required to be satisfied with high probability. We present a chance-constrained economic dispatch model that efficiently integrates energy storage and high renewable penetration to satisfy renewable portfolio requirements. Specifically, it is required that wind energy contributes at least a prespecified ratio of the total demand and that the scheduled wind energy is dispatchable with high probability. We develop an approximated partial sample average approximation (PSAA) framework to enable efficient solution of large-scale chance-constrained economic dispatch problems. Computational experiments on the IEEE-24 bus system show that the proposed PSAA approach is more accurate, closer to the prescribed tolerance, and about 100 times faster than sample average approximation. Improved efficiency of our PSAA approach enables solution of WECC-240 system in minutes.**

***Index Terms*—Economical dispatch, chance constraint, sample average approximation, renewable energy, energy storage**

NOMENCLATURE

Sets and indices

$\mathcal{I}$	Set of buses. Indexed by i for individual buses. i and j for pairs of buses.
$\mathcal{I}_r$	Set of buses with renewable generators.
I	Number of buses. $I = \|\mathcal{I}\|$.
I_r	Number of renewal buses. $I_r = \|\mathcal{I}_r\|$.
$\mathcal{G}$	Set of generation units. Indexed by g.
G	Number of generation units. $G = \|\mathcal{G}\|$.
$\mathcal{G}_i$	Set of generation units located at bus $i \in \mathcal{I}$.
$\mathcal{E}$	Set of directed transmission lines connecting pairs of buses. Indexed by $e = (i, j)$.
E	Number of directed transmission lines. $E = \|\mathcal{E}\|$.
$\mathcal{E}_i^+$	Set of transmission lines oriented into bus $i \in \mathcal{I}$.
$\mathcal{E}_i^-$	Set of transmission lines oriented out of bus $i \in \mathcal{I}$.
$\mathcal{T}$	Set of time periods in the planning horizon. Indexed by $t \in \{1, 2, \cdots, T\}$.

Parameters

B_e	Susceptance of transmission line e.
F_e	Power flow limit of transmission line e.
D_i^t	Demand (load) at bus $i \in \mathcal{I}$ in time period t.
$P_g^{\min}$	Lower limits on power output for generation unit g.
$P_g^{\max}$	Upper limits on power output for generation unit g.
R_g^d	Maximum ramp-down rate for generation unit g between adjacent time periods.
R_g^u	Maximum ramp-up rate for generation unit g between adjacent time periods.
$R_s^d(i)$	Maximum discharging energy in one time period at bus i
$R_s^u(i)$	Maximum charging energy in one time period at bus i adjacent time periods.
$Q_i^{\min}$	Lower limits on storage level for storage device unit i.
$Q_i^{\max}$	Upper limits on storage level for storage device unit i.
$C_g(\cdot)$	Production cost function for generation unit g.
$\tilde{p}_{w_i}^t$	Wind power available at bus $i \in \mathcal{I}$ in time period t.

Decision variables

p_g^t	Power output by generation unit g in time period t.
f_e^t	Power flow on transmission line e in time period t.
θ_i^t	Phase angle at bus i in time period t.
$\boldsymbol{p}$	Length $G \times T$ power output vector.
$\boldsymbol{f}$	Length $E \times T$ power flow vector.
$\boldsymbol{\theta}$	Length $E \times T$ voltage phase angle vector.
$p_{w_i}^t$	Amount of wind power dispatched at bus w_i in time period t.
q_i^t	Energy storage level at bus i in time t.

I. INTRODUCTION

High penetration levels of renewable generation is no longer a goal, but a requirement. For example, the U.S. Department of Energy recently targeted that wind energy provide 20% of U.S. electricity needs by 2030 [9] and California's Renewable Portfolio Standard (RPS) requires that renewable generation sources comprise 33% of energy supply by 2020 [3]. Because of these aggressive targets, high penetration levels of renewable energy will pose significant challenges for power system reliability, due to significant intermittency and limited predictability of renewable energy production [27]. To address these challenges, advanced operations models and solution methods are required to ensure system reliability and maximize cost-effectiveness.

We propose an *approximate* version of *partial sample average approximation* [8] to solve a large-scale chance-constrained optimization problems that arise in electricity grid operations. Specifically, we introduce a novel stochastic economic dispatch model that integrates significant renewable energy penetration, energy storage, and a joint chance constraint to ensure that RPS requirements are met with high probability.

The unit commitment problem (UC), which deals with scheduling power generation to meet anticipated demand, is a

fundamental problem in power systems operations. Economic dispatch (ED), closely related to UC, minimizes production costs while satisfying demand and transmission constraints subject to a set of committed, active thermal generators; in UC, the commitments are additionally determined. UC and ED are traditionally formulated as deterministic optimization problems, but the high levels of uncertainty that are inherent in renewable energy production require alternate formulations that effectively and directly account for such uncertainty, avoiding strict reliance on deterministic reserve margins. For more detailed information about UC and ED, we refer the reader to [17] and [29].

Approaches to handle uncertainty in UC or ED can be grouped into two categories: robust optimization ([1], [11], [13]) and stochastic programming ([21], [28], [30]). Robust optimization does not rely on explicit characterizations of probability distributions of random variables; rather, the focus is on the ranges of these variables. The goal of robust optimization is to provide solutions that are feasible in any realization of the random variables, subject to the specified interval bounds. Thus, commitment and dispatch decisions are made considering the worst-case under the uncertainty set. Construction of uncertainty sets and connection between interval bounds and cost has drawn recent attention [1], [13].

In contrast to robust optimization, stochastic programming approaches assume probability distributions of random variables are known. Stochastic programming variants of UC are typically formulated as two-stage decision models and solved via decomposition methods such as Benders' decomposition [31] and Progressive Hedging [6], [22]. An important variant of stochastic programming is chance-constrained programming, which requires that constraints involving uncertain parameters be satisfied with high probability. Wang *et al.* [28] introduced a chance constraint at the first decision stage of UC to ensure that, with high probability, a large ratio of generated wind power was utilized. Chance constraints were also integrated into stochastic ED by [21] and [30]. Wu *et al.* [30] proposed a day-ahead model where reserve requirements and line capacity constraints were satisfied with a high probability, while Qiu and Wang [21] introduced a chance-constraint model that ensured, with high probability, that the total amount of wind energy utilized was at least a specified fraction of the total wind energy.

We consider a novel chance constrained economic dispatch model, that efficiently integrates a large amount of renewable resources (e.g. wind) and energy storage to ensure that a significant amount of the total electricity demand can be satisfied by renewable output with a prescribed high probability. Energy storage can help mitigate the effects of renewable energy variability by storing excess wind energy for use in subsequent periods, and consequently has drawn significant interest from both industry and academia in recent years ([5], [10], [26]). For more information on energy storage, we refer the reader to recent surveys [4], [25]. In our formulation, we require that, with high probability, the scheduled wind energy is at least β percent of the total demand and the scheduled wind energy is dispatchable. The difference between our chance constraints and the chance constraints in [21] is that we require that the dispatched wind is at least β percent of total *demand* with high probability, which is congruent with states' RPS mandates, as opposed to total *wind production* which is irrespective of the actual demand as in [21].

While chance constraints present a natural language to model operational problems under uncertainty, they come at the cost of increased computational complexity. There are two main challenges. First, feasible sets of the chance constrained problems are generally not convex. Therefore, it is difficult to find a globally optimal solution. Secondly, even for a convex problem, given a candidate solution, it is generally hard to check its feasibility since this requires a multi-dimensional integration. For instance, Nemirovski and Shapiro [15] stated that evaluating the distribution of a weighted sum of uniformly distributed independent random variables is NP-hard. Subsequently, most work in the literature apply approximation techniques to achieve near-optimal solutions. One approach is to employ convex (or tractable) approximations (see, e.g., [15]). Other approaches use sample based techniques, such as Sample Average Approximation (SAA) [14], [18]. The basic idea of SAA is to approximate the true distribution of random variables by its empirical distribution obtained from Monte Carlo (MC) samples. SAA can provide a high-quality solution when the number of scenarios is large enough. Given the convergence rate of MC methods, the required number of samples increase quadratically with required accuracy. Especially for chance constrained problems, the "big-M" reformulations ([14]) of SAA problems have the same number of binary decision variables as the number of scenarios. It is well known that mixed-integer problems are generally hard to solve. However, under some conditions on the underlying uncertainty, a modified SAA, named Partial Sample Average Approximation (PSAA) [8], can be leveraged to solve chance constraint problems without introducing binary variables. In this paper, we present a PSAA approach to the ED problem with uncertain generation and energy storage, compare our results to SAA, and show that PSAA can enable solution of larger instances in a fraction of the time and with better handling of the uncertainties.

Our contributions can be summarized as follows.

1) We introduce a new chance constrained economic dispatch model with energy storage. With the new model, users can explicitly control the ratio of renewable energy's contribution to the total demand (production), which is coherent with the RPS mandates that required increase production of energy from renewable sources.
2) We present an approximate PSAA approach to effectively solve the chance-constrained ED model and demonstrate its computational efficiency. Since an exact PSAA reformulation yields a non-linear program, we apply piecewise approximation schemes to derive a final linear program. This is in sharp contrasts to the SAA approach, which results in a mixed-integer linear program. Lastly, the proposed approximate PSAA approach is generally applicable to other forms of chance-constrained ED models, e.g, that introduced by [21].

The remainder of the paper is organized as follows. § II reviews chance-constrained programming, along with SAA

and PSAA. In § III, and § IV we describe, respectively, our chance-constrained ED model and solution approaches (SAA and PSAA). We perform computational experiments and analyze the results in § V. Finally, we conclude in § VI.

II. Preliminaries

A. Chance Constrained Optimization Models

A general chance constrained model is defined as follows:

$$\begin{aligned} \min_{x} \quad & f(x) \\ s.t. \quad & \mathbb{P}\{g(x,\xi) \geq 0\} \geq 1-\alpha; \quad x \in X \end{aligned} \tag{1}$$

where ξ denotes a random vector, $\mathbb{P}\{E\}$ denotes the probability that the constraints E are satisfied, and the set X represents additional deterministic constraints. The input parameter $\alpha \in [0,1]$ is a parameter chosen by decision makers, and represents the tolerance for risk, or more explicitly, the acceptable probability for violating the chance constraints.

Chance constrained optimization was first introduced by Charnes *et al.* [7]. Since then, it has attracted significant attention due to its wide applicability in engineering, telecommunication, and finance. For a review of the theory and applications of chance-constrained optimization, we defer to [19], [24].

B. Sample average approximation

Chance constrained optimization models are generally difficult to solve, even at small scales, which limited their practical implementations to date despite their intuitive appeal. Consequently, various approximation methods were introduced. Most commonly used among these methods is SAA. The idea of SAA is to approximate the expectation of a random variable by its sample mean. We begin by formulating the general chance constrained optimization model as an expectation

$$\mathbb{P}\{g(x,\xi) \geq 0\} = \mathbb{E}[\mathbb{I}(g(x,\xi)], \tag{2}$$

where $\mathbb{I}(\cdot)$ denotes a standard indicator function. We then approximate the expectation via a sample mean, i.e., $\frac{1}{N}\sum_{t=1}^{N}\mathbb{I}(g(x,\hat{\xi}^t))$ where scenarios $\hat{\xi}^1,\ldots,\hat{\xi}^N$ are independent samples drawn from the distribution of ξ. The SAA formulation of problem (1) can then be written as:

$$\begin{aligned} \min_{x} \quad & f(x) \\ s.t. \quad & \frac{1}{N}\sum_{k=1}^{N}\mathbb{I}(g(x,\hat{\xi}^k)) \geq 1-\alpha; \quad x \in X. \end{aligned} \tag{3}$$

Problem (3) can be reformulated as

$$\begin{aligned} \min \quad & f(x) \\ s.t. \quad & g(x,\hat{\xi}^k) + My_k \geq 0, \ k=1,...,N \\ & \frac{\sum_{k=1}^{N} y_k}{N} \leq \alpha \\ & y_k \in \{0,1\}, \ k=1,...,N; \quad x \in X \end{aligned} \tag{4}$$

where the binary indicator variables y_k denote whether constraints $g(x,\hat{\xi}^k)$ are satisfied and M denotes a sufficiently large constant such that $g(x,\hat{\xi}^k) + M \geq 0$ is satisfied for all $x \in X$ and $k \in \{1,\ldots,N\}$. For additional details on SAA approaches for solving chance constrained optimization models, we refer to [14], [18].

C. Partial sample average approximation

SAA can in theory provide near-optimal solutions when the number of samples, N, is sufficiently large [14]. However, because the number of binary variables y_k is equal to the number of scenarios, solving SAA model (4) pose significant computational challenges. Recently, Cheng *et al.* [8] introduced a modified SAA approach called PSAA, which only samples a portion of random variables, in contrast to SAA which sample all of the random variables. The PSAA approach makes uses of the independence of some of the random variables for stepwise evaluation of the expectation. Specifically, $\mathbb{E}_{(\xi_1,\xi_2)}g(\xi_1,\xi_2) = \mathbb{E}_{\xi_1}\mathbb{E}_{\xi_2}g(\xi_1,\xi_2)$ when random vectors ξ_1 and ξ_2 are independent. Then it applies SAA to approximate the inner (or outer) expectation, i.e., it only samples ξ_2 (or ξ_1) to approximate the inner (or outer) expectation.

PSAA assumes that $\xi = (\xi_1,\xi_2)$ such that ξ_1 is independent of ξ_2. Then the probability of satisfying the joint constraints is formulated as an expectation: $\mathbb{P}\{g(x,\xi) \geq 0\} = \mathbb{E}[\mathbb{I}(g(x,\xi))] = \mathbb{E}_{\xi_1,\xi_2}[\mathbb{I}(g(x,\xi))]$. As ξ_1 and ξ_2 are independent, then the expectation can be calculated stepwisely on ξ_1 and ξ_2, precisely, $\mathbb{E}_{\xi_1,\xi_2}[\mathbb{I}(g(x,\xi_1,\xi_2))] = \mathbb{E}_{\xi_1}\mathbb{E}_{\xi_2}[\mathbb{I}(g(x,\xi_1,\xi_2))]$ where PSAA now replaces one of two expectation by its sample mean. For instance, we replace the inner expectation by a sample (where $\hat{\xi_2}^1,\ldots,\hat{\xi_2}^N$ are N independent samples of ξ_2), then the PSAA formulation of (1) is as follows:

$$\begin{aligned} \min \quad & f(x) \\ & \frac{1}{N}\sum_{k=1}^{N}\mathbb{E}_{\xi_2}[\mathbb{I}(g(x,\xi_1,\hat{\xi}_2^k))] \geq 1-\alpha; \quad x \in X \end{aligned} \tag{5}$$

Problem (5) is equivalent to the following problem:

$$\begin{aligned} \min \quad & f(x) \\ & \frac{1}{N}\sum_{k=1}^{N}\mathbb{P}\{g(x,\xi_1,\hat{\xi}_2^k) \geq 0\} \geq 1-\alpha; \quad x \in X \end{aligned} \tag{6}$$

PSAA retains the general idea of SAA, in that the expectation is replaced by a sample mean. However, as we demonstrate below, Eq. (6) has a convex reformulation for certain problem classes. For example, Eq. (6) can be linearly approximated under mild assumptions State assumptions. For additional details we refer to [8].

III. Mathematical Formulation

Given a fixed unit commitment vector $\boldsymbol{x}$, the objective in the deterministic (multi-period) economic dispatch problem is to minimize total production cost while meeting demand, subject to constraints associated with linearized (DC) approximations

of AC power flow and unit ramping limitations, as follows:

$$\min_{\boldsymbol{f},\boldsymbol{p},\boldsymbol{\theta}} \quad \sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}} C_g(p_g^t) \tag{7a}$$

$$\text{s.t.} \quad \sum_{g\in\mathcal{G}_i} p_g^t + \sum_{e\in\mathcal{E}(i)^+} f_e^t - \sum_{e\in\mathcal{E}(i)^-} f_e^t = D_i^t,\ \forall i\in\mathcal{I},\ \forall t\in\mathcal{T} \tag{7b}$$

$$B_e(\theta_i^t-\theta_j^t) - f_e^t = 0\ \forall e=(i,j)\in\mathcal{E},\forall t\in\mathcal{T} \tag{7c}$$

$$-F_e \le f_e^t \le F_e \quad \forall e\in\mathcal{E},\forall t\in\mathcal{T} \tag{7d}$$

$$P_g^{\min}x_g^t \le p_g^t \le P_g^{\max}x_g^t \quad \forall g\in\mathcal{G},\forall t\in\mathcal{T} \tag{7e}$$

$$-R_g^d \le p_g^t - p_g^{t-1} \le R_g^u \quad \forall g\in\mathcal{G},\forall t\in\mathcal{T} \tag{7f}$$

The generation cost $C_g(p_g^t)$ is typically represented by a convex quadratic function. Constraints (7b)–(7f) require (in order): power balance at each bus; power flow on lines is proportional to the difference in voltage phase angles at terminal buses; transmission line capacity limits are satisfied; lower and upper bounds for generation output levels are met; and generation ramp-up/ramp-down limits are satisfied.

Incorporating wind resources (and renewable resources in general) and energy storage devices requires some changes to this basic formulation. First, because wind generation is uncertain, the formulation must be stochastic. Consequently, we introduce dispatch variables $p_{w_i}^t$ for wind resources. Secondly, we need to take into account the amount of energy retained in storage devices. Here, we let q_i^t denote the energy level of the storage unit at bus i at time t. Leveraging these additional variables, our (multi-period) chance constrained economic dispatch model is given as follows:

$$\min_{\boldsymbol{f},\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{q}} \quad \sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}} C_g(p_g^t)$$

$$\text{s.t.} \quad p_{w_i}^t + \sum_{g\in\mathcal{G}_i} p_g^t + \sum_{e\in\mathcal{E}(i)^+} f_e^t - \sum_{e\in\mathcal{E}(i)^-} f_e^t = D_i^t + q_i^t - q_i^{t-1}, \forall i\in\mathcal{I}_r, \forall t\in\mathcal{T} \tag{8a}$$

$$\sum_{g\in\mathcal{G}_i} p_g^t + \sum_{e\in\mathcal{E}_{\cdot i}} f_e^t - \sum_{e\in\mathcal{E}_{i\cdot}} f_e^t = D_i^t, \quad \forall i\in\mathcal{I}\setminus\mathcal{I}_r, \forall t\in\mathcal{T} \tag{8b}$$

Constraints (7c) – (7f)

$$-R_i^d \le q_i^t - q_i^{t-1} \le R_i^u,\ \forall i\in\mathcal{I}_r, \forall t\in\mathcal{T} \tag{8c}$$

$$Q_i^{\min} \le q_i^t \le Q_i^{\max}, t=1,\dots T-1,\ \forall i\in\mathcal{I}_r \tag{8d}$$

$$\sum_{i\in\mathcal{I}_r}\sum_{t=1}^{T} p_{w_i}^t \ge \beta \sum_{i\in\mathcal{I}}\sum_{t=1}^{T} D_i^t, \tag{8e}$$

$$\mathbb{P}\{p_{w_i}^t \le \tilde{p}_{w_i}^t,\ \forall t\in\mathcal{T},\ \forall i\in\mathcal{I}_r\} \ge 1-\alpha \tag{8f}$$

In this formulation, we assumed that storage devices are installed at the same buses as wind generators but this assumption can be easily relaxed without structurally changing the problem. The parameter $q_i^0, i\in\mathcal{I}_r$, denotes the initial storage level at bus i. The standard power balance constraint (8a) is augmented with wind dispatch variables $p_{w_i}^t$ and nodal energy charge/discharge as given by the difference between battery storage levels at successive time periods $q_i^t - q_i^{t-1}$. Constraint (8c) ensures that charge and discharge limits on storage devices are satisfied. Constraint (8d) imposes upper and lower bounds on battery storage levels. Note that battery minimum levels must be at least $Q_i^{\min}$ to prevent end of horizon behaviors in which batteries are completely discharged to minimize production cost. Constraints (8e-8f) together enforce the RPS requirement that at least β fraction of the total demand is met using wind output with probability at least $1-\alpha$. Specifically, Eq. (8e) dictates that wind dispatch must satisfy at least β fraction of the total demand over the planning horizon. However, (8e) alone is not sufficient to guarantee that the amount of wind dispatched $p_{w_i}^t$ is feasible with high probability. Chance constraint (8f) ensures that the joint probability that the chosen wind dispatched $p_{w_i}^t$ for all $t\in\mathcal{T}, \forall i\in\mathcal{I}_r$ is *jointly* feasible with probability at least $1-\alpha$. In other words, the amount of wind dispatched at each wind site at each time period is less than the actual wind power available, which is upper bounded by $\tilde{p}_{w_i}^t$, is at least $1-\alpha$.

Remark 1. For clarity, we present a simple model with a single storage device at each candidate bus. However, our model can be easily extended to the multiple storage devices where capacities and charging/discharging profiles vary.

Remark 2. The parameter β in Eq. (8e) should be chosen dynamically for each planning horizon as the availability of wind power can vary significantly during different periods of a year. Accordingly, different β values should be selected for different planning periods.

Given β^*, the actual required RPS level, and probability threshold α, system operators are faced with the problem of how to choose β dynamically for day-to-day operations such that the required RPS requirements are satisfied in the long term with high probability.

Theorem III.1. *Let $p_{w_i}^{t*}$ be an optimal solution of Eq. (8). Given actual RPS mandate β^*, if β is selected such that $\beta \ge \beta^*/(1-\alpha)$ and the resulting dispatch of wind power $\hat{p}_{w_i}^t = \min\{p_{w_i}^{t*}, \tilde{p}_{w_i}^t\}$ is executable, then the expected wind energy dispatch is at least β^* of the total demand, i.e., $\sum_{i\in\mathcal{I}_r}\sum_{t=1}^{T}\mathbb{E}(\hat{p}_{w_i}^t) \ge \beta^*\sum_{i\in\mathcal{I}}\sum_{t=1}^{T} D_i^t$.*

Proof. First, as $p_{w_i}^{t*}$ is an optimal solution of problem (8), we have $\sum_{i\in\mathcal{I}_r}\sum_{t=1}^{T} p_{w_i}^{t*} \ge \beta\sum_{i\in\mathcal{I}}\sum_{t=1}^{T}\tilde{D}_i^t$ immediately due to (8e). Second, (8f) implies that $\mathbb{P}\{\hat{p}_{w_i}^t \ge p_{w_i}^{t*},\ \forall t\in\mathcal{T},\ \forall i\in\mathcal{I}_r\} \ge 1-\alpha$. Further as $\hat{p}_{w_i}^t \ge 0$, we have $\mathbb{E}(\hat{p}_{w_i}^t) \ge (1-\alpha)p_{w_i}^{t*}$. Given the above, we have

$$\mathbb{E}(\sum_{i\in\mathcal{I}_r}\sum_{t=1}^{T}\hat{p}_{w_i}^t) \ge (1-\alpha)\sum_{i\in\mathcal{I}_r}\sum_{t=1}^{T} p_{w_i}^{t*} \ge \beta(1-\alpha)\sum_{i\in\mathcal{I}}\sum_{t=1}^{T} D_i^t.$$

The conclusion that $\mathbb{E}(\sum_{i\in\mathcal{I}_r}\sum_{t=1}^{T}\hat{p}_{w_i}^t) \ge \beta^*\sum_{i\in\mathcal{I}}\sum_{t=1}^{T} D_i^t$ follows. □

Theorem III.1 provides a viable method to choose the parameter β for (8) given a static wind output distribution. However, by Remark 2, the output of wind power varies during different time periods, which leads to changes in wind power profiles over time. As such, the parameter β must be chosen dynamically per planning period throughout the entire planning horizon.

Corollary III.2. *Given RPS required β^* and T' days with different β values $(\beta_1, \ldots, \beta_{T'})$, let $p^{t*}_{w_i}(\beta_1), \ldots, p^{t*}_{w_i}(\beta_{T'})$ be the corresponding optimal solutions of (8). If $\sum_{\tau=1}^{T'} \beta_\tau / T' \geq \beta^*/(1-\alpha)$ and the following dispatch of wind power $\hat{p}^t_{w_i}(\beta_\tau) = \min\{p^{t*}_{w_i}(\beta_\tau), \tilde{p}^t_{w_i}\}, \tau = 1, \ldots, T'$ is executable, then the expected wind energy dispatch is at least β^* of the total demand over T' days, i.e., $\sum_{\tau=1}^{T'} \sum_{i \in \mathcal{I}_r} \sum_{t=1}^{T} \mathbb{E}(\hat{p}^t_{w_i}) \geq \beta^* \sum_{\tau=1}^{T'} \sum_{i \in \mathcal{I}} \sum_{t=1}^{T} D^t_i$.*

IV. Solution Methods

The main difficulty in solving (8) lies in the computation of the chance constraint (8f). Using SAA, the chance constraint is linearized via new binary variables and big-M constraints, which results in a hard-to-solve large-scale mixed-integer linear program. In the present paper, we develop a variant of PSAA to *approximate* chance constraint (8f) by employing a piecewise linear approximation of the cumulative distribution function of ξ_2. In stark contrast to SAA, the approximate PSAA reformulation only introduces new continuous variables and constraints and is thus computationally tractable and scalable to practical size systems.

A. Sample average approximation of chance constraint

The chance constraint (8f), repeated below for clarity,

$$\mathbb{P}\{p^t_{w_i} \leq \tilde{p}^t_{w_i}, \ \forall t \in \mathcal{T}, \ \forall i \in \mathcal{I}_r\} \geq 1 - \alpha,$$

can be reformulated using SAA as follows:

$$p^t_{w_i} - z_k M \leq \tilde{p}^{t,k}_{w_i}, \ \forall t \in \mathcal{T}, \ \forall i \in \mathcal{I}_r \tag{9}$$

$$\frac{\sum_{k=1}^{N} z_k}{N} \leq \alpha, z_k \in \{0,1\} \tag{10}$$

where parameter M is chosen to be sufficiently large such that constraints (9) is satisfied when $z_k = 1$ and $\tilde{p}^{t,k}_{w_i}, k = 1, \ldots, N$ are independent scenarios of $\tilde{p}^t_{w_i}$. It is well known that the particular value of M plays an important role in computational tractability. Typically, as M increases, the linear programming relaxation weakens and fractionality of the resulting MILP increases (i.e., number of branch-and-bound nodes that must be explored increases). In computational experiments presented in §V, M is set to the α-quantile of wind power output.

B. Partial sample average approximation

PSAA leverages the same idea as SAA in that the true distribution of random variables is replaced by an empirical distribution. However, PSAA approximates the distribution of only a subset of random variables rather than all the random variables. [8] proposed the general idea of PSAA where only a subset of the random variables are sampled but did not provide guidance on how to partition the random variables into sampled and not-sampled sets. In this paper, we apply SAA to approximates the distribution of all but a single random variable, in other words only a single variable will not be sampled. We then provide guidance and intuition on how to select the single not-sampled random variable and present a piecewise linearization scheme to embed an approximation of the cumulative distribution function of the not-sampled random variable directly into the resulting PSAA formulation. Before presenting our method for solving Eq. (8), we state our main assumption.

Assumption $\mathcal{A}_1$: Random vector $(\tilde{p}^t_{w_i})_{t \in \mathcal{T} i \in \mathcal{I}_r}$ is normally distributed with mean μ and covariance matrix Σ.

Remark 3. It is widely assumed that the output of renewable energy is normally distributed (see e.g., [2], [28]). Additionally, the distribution of wind power can be estimated using historical information.

Under assumption $\mathcal{A}_1$, we can reformulate the random vector $(\tilde{p}^t_{w_i})_{t \in \mathcal{T} i \in \mathcal{I}_r}$ as follows:

$$(\tilde{p}^t_{w_i})_{t \in \mathcal{T} i \in \mathcal{I}_r} = V\xi + \mu \tag{11}$$

where $V = \bar{V}(\Lambda)^{\frac{1}{2}}$ and $\Sigma = \bar{V}\Lambda(\bar{V})^T$ is the eigen-decomposition of Σ. Without loss of generality, it is assumed that the elements of the diagonal of Λ are ordered in non-increasing order. $\xi \in R^{TI_r}$ is standard normal vector whose components are independent. In what follows, ξ_1 which has the most variance is not sampled while the remainder of ξ, i.e., $(\xi_2, \ldots, \xi_{TI_r})$, are sampled.

Remark 4. We assume that the elements of diagonal of Λ are ordered in a decreasing manner. Analogous to the idea of principal component analysis, we select the random variable with the biggest eigenvalue (i.e., variance) to be the one not sampled (ξ_1). The main motivation for doing so is that the convergence of SAA is an increasing function of the variance of the estimator. Thus, from a convergence perspective it is preferable to select $\xi_2, \ldots, \xi_{TI_r}$, the set of sampled random variables, to be those with the smaller variances.

With ξ_1 not be sampled, we reconsider constraint (8f). Given (11), constraint (8f) can be reformulated as follows:

$$\begin{aligned} p^t_{w_i} \ &\leq \ \tilde{p}^t_{w_i} = \mathbf{e}^T_{(t-1)I_r+i}(V\xi + \mu) \\ &= (\mathbf{e}^T_{(t-1)I_r+i}V)_1\xi_1 + \sum_{l=2}^{TI_r}(\mathbf{e}^T_{(t-1)I_r+i}V)_l\xi_l + \mu_{(t-1)I_r+i}, \\ &\quad \forall t \in \mathcal{T} \text{ and } \forall i \in \mathcal{I}_r \end{aligned}$$

where $\mathbf{e}_l \in R^{TI_r}, t \in \mathcal{T}$ is a vector where its l-th element is one and others are zero, and $(\mathbf{e}^T_{(t-1)I_r+i}V)_l$ is the l-th element of the row vector $\mathbf{e}^T_{(t-1)I_r+i}V$.

Given assumption $\mathcal{A}_1$, the approximate PSAA reformulation of chance constraints (8f) is as follows:

$$\sum_{k=1}^{N} \frac{\mathbb{P}\{p^t_{w_i} - H^t_i(k) \leq (\mathbf{e}^T_{(t-1)I_r+i}V)_1\xi_1, \forall t \in \mathcal{T}; i \in \mathcal{I}_r\}}{N} \geq 1-\alpha \tag{12}$$

where

$$H^t_i(k) = \sum_{l=2}^{TI_r}(\mathbf{e}^T_{(t-1)I_r+i}V)_l\xi^k_l + \mu_{(t-1)I_r+i} \tag{13}$$

and $\xi^k_l, k = 1, \ldots, N$ are independent samples of ξ_l. It is easy to see that $H^t_i(k)$ is a constant. Leveraging the SAA approximation scheme by sampling random variables $(\xi_2, \ldots, \xi_{TI_r})$, constraint (12) may still be computationally

intractable. However, constraint (12) can be further approximated using a piecewise linear approximation approach to obtain a set of linearized constraints as follows. There are three cases related to the value of $(\mathbf{e}^T_{(t-1)I_r+i}V)_1$, being positive,zero, and negative which we denote by$\mathcal{I}^+$, $\mathcal{I}^-$ and $\mathcal{I}^0$, respectively. More precisely,

$$\begin{aligned}
&\xi_1 \geq \tfrac{p^t_{w_i}-H^t_i(k)}{(\mathbf{e}^T_{(t-1)I_r+i}V)_1} && \text{if } (\mathbf{e}^T_{(t-1)I_r+i}V)_1 > 0,\\
&\xi_1 \leq \tfrac{p^t_{w_i}-H^t_i(k)}{(\mathbf{e}^T_{(t-1)I_r+i}V)_1} && \text{if } (\mathbf{e}^T_{(t-1)I_r+i}V)_1 < 0,\\
&0 \geq p^t_{w_i} - H^t_i(k) && \text{if } (\mathbf{e}^T_{(t-1)I_r+i}V)_1 = 0.
\end{aligned}$$

Note that the three inequalities above correspond to conditions $(t,i) \in \mathcal{I}^+$, $(t,i) \in \mathcal{I}^-$, and $(t,i) \in \mathcal{I}^0$, respectively.

As ξ_1 is a standard normal random variable, constraint (12) has a deterministic formulation as follows:

$$\sum_{k=1}^{N} \frac{\Phi(U^k) - \Phi(L^k)}{N} \geq 1-\alpha, p^t_{w_i} - H^t_i(k) \leq 0, (t,i) \in \mathcal{I}^0 \tag{14}$$

where $U^k = \min_{(t,i)\in\mathcal{I}^+} \frac{p^t_{w_i}-H^t_i(k)}{(\mathbf{e}^T_{(t-1)I_r+i}V)_1}$, $L^k = \max_{(t,i)\in\mathcal{I}^-} \frac{p^t_{w_i}-H^t_i(k)}{(\mathbf{e}^T_{(t-1)I_r+i}V)_1}$ and $\Phi(\cdot)$ is the cumulative distribution function (CDF) of the standard normal distribution.

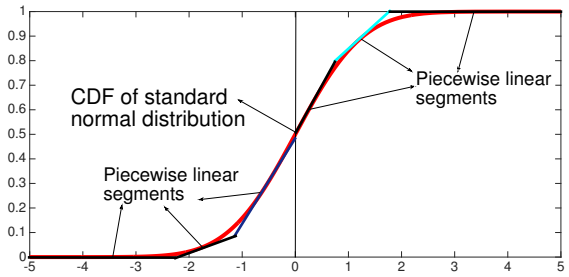


Fig. 1: Piecewise linear approximation of CDF of standard normal distribution

As α is usually close to zero (e.g., 0.05), $\Phi(U^k)$ is usually greater than 0.5, while $\Phi(L^k)$ is usually less than 0.5. Figure 1 provides an illustration of the function $\Phi(\cdot)$. It also illustrates how this function can be upper bounded by a piecewise linear concave function which is tangent to the actual function at a finite number of points when $U^k \geq 0$ while the CDF can be lower bounded by a piecewise linear convex function when $L^k \geq 0$. To this end, we approximate $\Phi(U^k)$ with a piecewise linear concave function

$$\Phi(U^k) \approx \min_{s\in\{1,\ldots,S\}} \hat{a}_s U^k + \hat{b}_s, \tag{15}$$

while $\Phi(L^k)$ is approximated by a piecewise linear convex function

$$\Phi(L^k) \approx \max_{s\in\{1,\ldots,S\}} \bar{a}_s L^k + \bar{b}_s. \tag{16}$$

In the latter numerical tests, we choose 25 tangent points on interval $[-3,3]$ for the tangent approximation. It is easy to find that the more tangent points we choose, the more accurate the approximation will be, whereas at the same time it will increase the size of problem.

With the piecewise linear approximation of $\Phi(U^k)$ and $\Phi(L^k)$ respectively, we obtain the following approximation problem of PSAA problem by introducing auxiliary variables $\bar{U}^k$ and $\bar{L}^k$ to represent $\Phi(U^k)$ and $\Phi(L^k)$ respectively:

$$\begin{aligned}
\min\quad & \sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}} C^p_g(p^t_g)\\
\text{s.t.}\quad & (7c)-(7f),\ (8a)-(8e)\\
& \sum_{k=1}^{N} \frac{(\bar{U}^k - \bar{L}^k)}{N} \geq 1-\alpha\\
& U^k \leq \min_{(t,i)\in\mathcal{I}^+} \frac{p^t_{w_i}-H^t_i(k)}{(\mathbf{e}^T_{(t-1)I_r+i}V)_1},\ \forall k=1,\ldots,N\\
& L^k \geq \max_{(t,i)\in\mathcal{I}^-} \frac{p^t_{w_i}-H^t_i(k)}{(\mathbf{e}^T_{(t-1)I_r+i}V)_1},\ \forall k=1,\ldots,N\\
& p^t_{w_i} - H^t_i(k) \leq 0, (t,i) \in \mathcal{I}^0,\ \forall k=1,\ldots,N\\
& \bar{U}^k \leq \min_{s\in\{1,\ldots,S\}} \hat{a}_s U^k + \hat{b}_s, \forall k=1,\ldots,N\\
& \bar{L}^k \geq \max_{s\in\{1,\ldots,S\}} \bar{a}_s L^k + \bar{b}_s, \forall k=1,\ldots,N\\
& \bar{U}^k \geq \bar{L}^k \geq 0,\ \forall k=1,\ldots,N
\end{aligned} \tag{17}$$

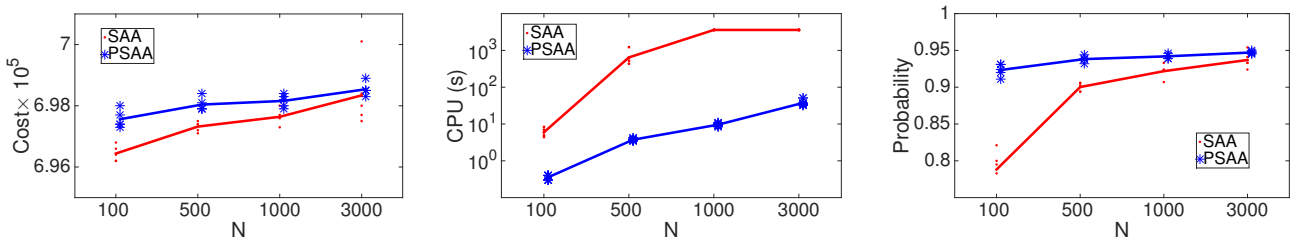
This model employs a convex quadratic objective function and linear constraints.

V. Computational Experiments

We now conduct numerical experiments to test the SAA and PSAA approaches to economy dispatch on a revised IEEE 24-bus system [32] and the WECC 240-bus system (representing the US Western Interconnection [20]). We added 3 wind farms at buses 7, 13, and 15 in the IEEE 24-bus system and 10 wind farms at buses 22, 23, 42, 44, 54, 64, 77, 129, 174, and 240 in the WECC 240-bus system. We assume we also have storage capacity at the same buses as the wind farms. We compare the relative performances of SAA and PSAA on the IEEE 24-bus system. Due to scalability limitations of SAA, only the PSAA approach is applied to WECC 240-bus system. All optimization models are implemented in Matlab and solved using Gurobi 6.0.4. We impose a 3600 second time limit and an optimality gap of 0.1% as stopping criteria for all runs. All experiments were performed on a Linux workstation with an Intel Quad Core 2.2 GHz processor and 16 GB memory.

A. Comparison of SAA and PSAA on the 24-bus System

The IEEE 24-bus system consists of 32 conventional generators, 34 transmission lines, 17 loads, and three wind farms. Data for the three wind farms are based on NREL's Western Wind Dataset [16]. We selected two geographically close wind farms located in Wyoming – site #15414 $(41.48N, 105.14W)$ and site #16238 $(41.61N, 105.11W)$ – and a single site in California – site #3560 $(35.08N, 118.41W)$ in California. We model wind speeds as random fields and represent time dependent uncertainties via Karhunen-Loeve expansions. A detailed description of this approach is presented in [23]. We



(a) Solution Cost (b) Runtime (c) Robustness

Fig. 2: Comparison of SAA and PSAA the IEEE 24-bus system

set the total installed wind capacity to be 1500 MW in order to meet high RPS requirements. The remaining parameters are set as follows: $\alpha = 0.05$, $\beta = 33\%$ and $T = 24$ hours. The storage capacity is set to be $1/5$ of installed hourly wind capacity while the initial storage level is set to $1/4$ of storage capacity. For simplicity, we assume that charge and discharge rates in one hour are identical to the capacity. We consider four settings for the number of scenarios used: $N \in \{100, 500, 1000, 3000\}$. Both SAA and PSAA solutions are functions of the input scenarios. Thus, in order to compare the performance of the two approaches, five sets of scenarios for each N are considered. We show the performances of both SAA and PSAA in Figure 2, which shows how cost, CPU time, and posterior probability of solution change with creating number of samples. Here, posterior probability is defined as the probability of solution satisfying the chance constraint (8f). These results presented in summarized form in Table I.

Figure 2 and Table I show that there is no big difference between the SAA and PSAA approaches in terms of cost. The cost of the PSAA approach is a little higher than the cost of SAA when $N \leq 1000$ because the PSAA's solution has higher posterior probability. As for $N = 1000$ or $N = 3000$, the SAA approach hits the stopping criteria 3600 seconds and thus the solutions we collect are incumbent solutions (the best feasible solutions so far) rather than the optimal solutions of SAA problems. Therefore, the incumbent solutions are more conservative than the optimal solution which can explain why the cost of SAA approach is quite higher than others when $N = 3000$.

Figure 2b compares the runtime performances of SAA and PSAA. As evident in this figure, the PSAA approach leads to two orders of magnitude reduction in computational time relative to the SAA approach for $N \geq 500$. In particular, Table I shows that it takes only 38 seconds for the PSAA approach whereas the SAA approach failed to provide an optimal solution at the end of one hour.

The quality of the solutions by the two approaches in terms of posterior probability are presented in Figure 2c. PSAA provides better solution quality compared to SAA in terms of posterior probability, as the posterior probabilities of the PSAA solutions are closer to $1 - \alpha = 0.95$ in all cases compared to the SAA solutions. As an example, when $N = 100$, the posterior probability of the SAA solution is less than 0.8 which is far from the 0.95 chance-constraint requirement, whereas it is 0.923 for the PSAA approach. In other words, the PSAA requires much fewer scenarios to obtain a high-quality solution compared to the SAA approach.

TABLE I: Comparisons of SAA and PSAAA on average statistics on 5 instances

$\alpha = 0.05$	SAA			PSAA		
N	Cost($10K)	CPU(s)	Prob	Cost($10K)	CPU(s)	Prob
100	6.964	6.1	0.788	6.976	0.4	0.923
500	6.973	689.9	0.900	6.980	3.8	0.938
1000	6.976	3600	0.922	6.982	9.6	0.942
3000	6.983	3600	0.937	6.985	38.0	0.947

From the numerical results, it can be concluded that PSAA provides an efficient method for solving this class of chance-constrained optimization problems given its computational tractability and rapid convergence to high-quality solutions.

B. *Sensitivity of storage capacity*

We now test the sensitivities of the total cost to variations in installed wind and storage capacities. We tested combinations of 6 storage capacities (0, 200, 400, 600, 800, and 1000 MWh) and 4 installed wind capacities (1350, 1500, 1650, 1800 MW). The results are summarized in Figure 3. First, we observe that total cost decreases as the storage capacity and/or installed wind capacity increases. Second, we observe that marginal improvements in costs are more evident for small increases of either the storage capacity or the installed wind capacity. For instance, when the installed wind capacity is 1500 MW, increasing storage capacity from 0 to 200 results in cost reductions of more than $2000 while an increased storage capacity, from 200 to 400, only reduces the cost by $500. Moreover, when installed wind capacities are large enough, increasing storage capacities do not provide further cost reductions.

C. *240-bus System*

The 240-bus system consists of 85 conventional generators, 348 transmission lines, 142 loads and 10 wind farms. The uncertainties in the wind power generation are modeled similarly to the approach employed in Section V-A (for more details see [23]). We set the installed wind capacity as 510 MW. The remaining parameters (e.g., $\alpha = 0.05$, $\beta = 33\%$ and $T = 24$) are identical to those of the 24-bus system. As our focus here is to demonstrate the tractability of PSAA, we selected a large number of scenario samples to test ($N = 1000$, 3000, and 5000). Additionally, timing results for these large

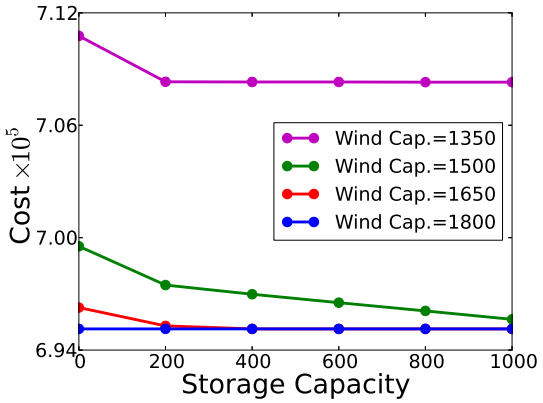


Fig. 3: Cost with different sizes of wind and storage capacities.

instances using a linear cost function are also presented. The results in Table II demonstrate the tractability of PSAA and the intractability of SAA, which failed to solve any of these larger instances to optimality within the given time limit. Although expected, increased computational times are observed for these larger instances, the largest instance with 5000 scenario samples can still be solved in about 700 seconds using a quadratic objective function. When a linear approximation of the cost function is employed, the runtime reduces to approximately 60 seconds.

VI. Conclusion

We presented a new formulation of the economic dispatch (ED) problem which integrates renewable energy and energy storage. We also proposed a solution based on partial sample average approximation (PSAA) to enable solving large instances. Our formulation uses chance constraints to ensure that the scheduled wind energy is dispatchable and the portion of demand met with wind resources is above a specified ratio with high probability. This later constraint is consistent with recent regulations; e.g.,, DOE requires that 20% of electricity needs to be provided by wind. As the sample average approximation leads to mixed-integer problems which are generally hard to solve, we adopt the PSAA to get a high-quality solution, which results in a convex program and thus, is tractable for large instances. The effectiveness of the PSAA is illustrated by experiments on the 24-bus and WECC 240-bus test systems. In this model, we follow convention and assume that wind output is normally distributed. A natural line of further research lies in generalizing the approximate PSAA approach to other distributions (e.g. Weibull distribution). Integrating our method to higher level problems, such as unit commitment, is also another promising direction.

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References

[1] Bertsimas, Dimitris, Eugene Litvinov, Xu Andy Sun, Jinye Zhao, and Tongxin Zheng, "Adaptive robust optimization for the security constrained unit commitment problem." IEEE T. Power Systems, 28, no. 1 (2013): 52-63.

[2] Bienstock, Daniel, Michael Chertkov, and Sean Harnett. "Chance-constrained optimal power flow: Risk-aware network control under uncertainty." SIAM Review 56.3 (2014): 461-495.

[3] California Renewables Portfolio Standard (RPS), http://www.cpuc.ca.gov/PUC/energy/Renewables/.

[4] Castillo, Anya, and Dennice F. Gayme. "Grid-scale energy storage applications in renewable energy integration: A survey." Energy Conversion and Management 87 (2014): 885-894.

[5] Chandy, K. Mani, Steven H. Low, Ufuk Topcu, and Huan Xu. "A simple optimal power flow model with energy storage." In Decision and Control (CDC), 2010 49th IEEE Conference on, pp. 1051-1057. IEEE, 2010.

[6] Cheung, K., D. Gade, C. S. Monroy, S. M. Ryan, J. P. Watson, R. J. B. Wets, and D. L. Woodruff. "Toward scalable stochastic unit commitment-part 2: Assessing solver performance." Submitted (preprint available online) (2014): 1-8.

[7] Charnes, Abraham, William W. Cooper, and Gifford H. Symonds. "Cost horizons and certainty equivalents: an approach to stochastic programming of heating oil." Management Science 4, no. 3 (1958): 235-263.

[8] Cheng, Jianqiang, Céline Gicquel, and Abdel Lisser. "A New Partial Sample Average Approximation Method for Chance Constrained Problems.", http://www.optimization-online.org/DB_FILE/2014/11/4622.pdf.

[9] "20% Wind Energy by 2030: Increasing Wind Energy's Contribution to US Electricity Supply," DOE/GO-102008-2567, US Department of Energy, May. http://www.nrel.gov/docs/fy08osti/41869.pdf, 2008.

[10] Gayme, Dennice, and Ufuk Topcu. "Optimal power flow with large-scale storage integration." IEEE T. Power Systems, 28, no. 2 (2013): 709-717.

[11] Jiang, Ruiwei, Jianhui Wang, and Yongpei Guan. "Robust unit commitment with wind power and pumped storage hydro." IEEE T. Power Systems, 27, no. 2 (2012): 800-810.

[12] Lew, Debra, and R. Piwko. "Western wind and solar integration study." National Renewable Energy Laboratories, Technical Report No. NREL/SR-550-47781 (2010).

[13] Lorca, Alvaro, and Xu Andy Sun. "Adaptive robust optimization with dynamic uncertainty sets for multi-period economic dispatch under significant wind." IEEE T. Power Systems, 30.4 (2015): 1702-1713.

[14] Luedtke, James, and Shabbir Ahmed. "A sample approximation approach for optimization with probabilistic constraints." SIAM J. Optimization 19.2 (2008): 674-699.

[15] Nemirovski, Arkadi, and Alexander Shapiro. "Convex approximations of chance constrained programs." SIAM J. Optimization 17.4 (2006): 969-996.

[16] NREL: Transmission Grid Integration - Western Wind Dataset, http://www.nrel.gov/electricity/transmission/wind_integration_dataset.html.

[17] Padhy, Narayana Prasad. "Unit commitment-a bibliographical survey." IEEE T. Power Systems, 19, no. 2 (2004): 1196-1205.

[18] Pagnoncelli, Bernardo K., Shabbir Ahmed, and Alexander Shapiro. "Sample average approximation method for chance constrained programming: theory and applications." J. optimization theory and applications 142.2 (2009): 399-416.

[19] Prékopa, András. "Probabilistic programming." Handbooks in operations research and management science 10 (2003): 267-351.

[20] Price, James E., and John Goodin. "Reduced network modeling of WECC as a market design prototype." IEEE Power and Energy Society General Meeting, pp. 1-6. IEEE, 2011.

[21] Qiu, Feng, and Jianhui Wang. "Chance-constrained transmission switching with guaranteed wind power utilization." IEEE T. Power Systems, 30, no. 3 (2015): 1270-1278.

[22] Rockafellar, R. Tyrrell, and Roger J-B. Wets. "Scenarios and policy aggregation in optimization under uncertainty." Mathematics of operations research 16, no. 1 (1991): 119-147.

[23] Safta, Cosmin, Richard Chen, Habib N. Najm, Ali Pinar, and Jean-Paul Watson. "Efficient Representation of Uncertainty for Stochastic Economic Dispatch." submitted to IEEE T. Power Systems, (2015).

[24] Shapiro, Alexander, Darinka Dentcheva, and Andrzej Ruszczyński . Lectures on stochastic programming: modeling and theory. Vol. 16. SIAM, 2014.

[25] Silva-Monroy, Cesar, and Jean-Paul Watson. "Integrating Energy Storage Devices Into Market Management Systems." Proceedings of the IEEE 102, no. 7 (2014): 1084-1093.

TABLE II: Comparison of SAA and PSAAA for WECC-240. "–" indicates that no solution found in one hour

	Quadratic cost function						Linear cost function					
$\alpha = 0.05$	SAA			PSAA			SAA			PSAA		
N	Cost(10^6\$)	CPU(s)	Prob	Cost(10^6\$)	CPU(s)	Prob	Cost(10^6\$)	CPU(s)	Prob	Cost(10^6\$)	CPU(s)	Prob
1000	2.122	3600	0.945	1.850	39.3	0.928	1.987	3600	0.917	1.899	6.1	0.929
3000	–	–	–	1.953	259.3	0.945	2.380	3600	0.958	1.996	30.4	0.944
5000	–	–	–	2.064	735.3	0.947	3.547	3600	0.986	2.107	65.3	0.947

[26] Su, Han-I., and Abbas El Gamal. "Modeling and analysis of the role of energy storage for renewable integration: Power balancing." IEEE T. Power Systems, 28, no. 4 (2013): 4109-4117.

[27] Ummels, Bart C., Madeleine Gibescu, Engbert Pelgrum, Wil L. Kling, and Arno J. Brand. "Impacts of wind power on thermal generation unit commitment and dispatch." IEEE T. Energy Conversion, 22, no. 1 (2007): 44-51.

[28] Wang, Qianfan, Yongpei Guan, and Jianhui Wang. "A chance-constrained two-stage stochastic program for unit commitment with uncertain wind power output." IEEE T. Power Systems, 27.1 (2012): 206-215.

[29] Wood, Allen J., and Bruce F. Wollenberg. Power generation, operation, and control. John Wiley & Sons, 2012.

[30] Wu, Hongyu, Mohammad Shahidehpour, Zuyi Li, and Wei Tian. "Chance-constrained day-ahead scheduling in stochastic power system operation." IEEE T. Power Systems, 29, no. 4 (2014): 1583-1591.

[31] Van Slyke, Richard M., and Roger Wets. "L-shaped linear programs with applications to optimal control and stochastic programming." SIAM J. Applied Mathematics 17, no. 4 (1969): 638-663.

[32] Zimmerman, Ray Daniel, Carlos Edmundo Murillo-Sánchez, and Robert John Thomas. "MATPOWER: Steady-state operations, planning, and analysis tools for power systems research and education." Power Systems, IEEE T.26, no. 1 (2011): 12-19.