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Numeric Invariants from Multidimensional Persistence

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Abstract Topological data analysis is the study of data using techniques from algebraic topology. Often, one begins with a finite set of points representing data and a "filter" function which assigns a real number to each datum. Using both the data and the filter function, one can construct a filtered complex for further analysis. For example, applying the homology functor to the filtered complex produces an algebraic object known as a "one-dimensional persistence module", which can often be interpreted as finite set of intervals representing various geometric features in the data.

If one runs the above process incorporating multiple filter functions simultaneously, one instead obtains a multidimensional persistence module. Unfortunately, these are much more difficult to interpret. In this article, we analyze the space of multidimensional persistence modules from the perspective of algebraic geometry. We first build a moduli space of a certain subclass of easily analyzed multidimensional persistence modules, which we construct specifically to capture much of the information which can be gained by using multidimensional persistence instead of one-dimensional persistence. We argue that the global sections of this space provide interesting numeric invariants when evaluated against our subclass of multidimensional persistence modules. Finally, we extend these global sections to the space of all multidimensional persistence modules and discuss how the resulting numeric invariants might be used to study data. This paper extends the results of Adcock, Carlsson, and Carlsson ([2]) by constructing numeric invariants from the computation of a multidimensional persistence module as given by Carlsson, Singh, and Zomorodian ([6]).

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1 Introduction

The use of topology to study point cloud data has been well established ([4], [5]). Given a finite metric space (e.g., a finite subset of $\mathbb{R}^n$), one first constructs a filtered complex which attempts to approximate the shape of the underlying data. The *persistent homology* of a filtered complex is an abstract algebraic entity which combines information about the homology of the levelwise complexes of a filtered simplicial complex and the maps on homology induced by the filtration maps of the complex. The process of constructing a filtered complex and taking persistent homology provides abstract algebraic information about the original point cloud data.

It is difficult to interpret raw calculations of persistent homology from a geometric and intuitive standpoint. This is partially remedied by Adcock, Carlsson, and Carlsson, who have successfully studied ways of interpreting persistent homology geometrically through the construction of numeric invariants [2]. They produce an infinite family of algebraic functions, each of which takes as input any one-dimensional persistence module (the most notable of such objects being the persistent homology of a one-dimensional filtered complex) and outputs a nonnegative number which has a concrete interpretation in terms of the geometry of the filtered complex. If the filtered complex is constructed from point cloud data, these values provide information about the size and density of prominent geometric features of the point cloud. More importantly, these values can then be used as features in machine learning algorithms.

We note that multiple other researchers have successfully addressed the problem of preprocessing the output of persistent homology for machine learning algorithms. For example, Adams et al. view one-dimensional persistence modules as images and use convolutional filters to ensure that their technique is robust to noise [1]. Bubenik applies tools from the probability theory of Banach spaces to study one-dimensional persistence modules [3]. Reininghaus et al. bypass feature extraction altogether by designing a multi-scale kernel for one-dimensional persistence modules for kernel-based machine learning techniques [9]. Although we focus on generalizing the results of [2], it would be very interesting to generalize any of the approaches mentioned in this paragraph to the setting of multidimensional persistence.

The method of [2] does not generalize nicely to multidimensional persistence modules. The construction of the algebraic functions in [2] relies heavily on the classification theorem of finitely generated modules over a PID and the categorical equivalence between one-dimensional persistence modules and finitely generated modules over a PID. Multidimensional persistence modules are categorically equivalent to finitely presented multigraded $\mathbb{R}[x_1,...,x_n]$-modules, for which there is no analogous classification theorem. Furthermore, the results of [7] show that there is no complete discrete invariant for multidimensional persistence; any numeric invariants that we provide must necessarily be incomplete.

It is our goal to provide functions which generalize those of [2]. Although our functions are identical to those of [2] in the case of one-dimensional persistence, we

also provide an alternate measure-theoretic construction of our functions. As such, our functions can be defined just as easily to the class of multidimensional persistence modules as on the space of one-dimensional persistence modules.

In Section 2, we review multidimensional persistence. In Section 3, we calculate the ring of graded-finite functions on a convenient set of multidimensional persistence modules, which we then extend to all multidimensional persistence modules in Section 4. In Section 5, we present results pertaining to the power of the invariants discussed in Sections 3 and 4, and we also propose an alternative way to calculate these invariants. Finally, in Section 6, we show how to recover information about a multidimensional persistence module from the functions defined in Section 4.

Notation. In this paper, k denotes a field of arbitrary characteristic. We represent elements of k^n using bold letters (i.e., $\mathbf{a} \in k^n$), but we will represent the components of such elements using italic letters (i.e., $a_j \in k$). For $\mathbf{a} \in k^n$ and $\mathbf{m} \in \mathbb{N}^n$, we will denote the sum $\sum_{j=1}^n a_j$ by $\sum(\mathbf{a})$ and the product $\prod_{i=1}^n a_i^{m_i}$ by $\mathbf{a}^{\mathbf{m}}$. Additionally, if $c \in k$, we denote by $\mathbf{a} \pm c$ the vector obtained from $\mathbf{a}$ by adding or subtracting c from each coordinate. For a variable or constant $\mathbf{x} \in \mathbb{R}^n$, denote by $\lfloor \mathbf{x} \rfloor$ (respectively, $\lceil \mathbf{x} \rceil$) the elements of $\mathbb{R}^n$ obtained by taking the floor (respectively, ceiling) of each of the components x_j of $\mathbf{x}$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we say that $\mathbf{x} \leq \mathbf{y}$ if $x_j \leq y_j$ for all j. Additionally, we will use $\leq_\ell$ to denote the lexicographic order on $\mathbb{R}^n$.

For a scheme X, we denote the ring of global sections of X by $A[X]$. We emphasize that virtually all schemes that appear in this paper will be affine. Let $\mathbb{AA}_n^m$ denote the affine $2mn$ $\mathbb{R}$-space

$$\mathbb{AA}_n^m = \operatorname{Spec}\left(\mathbb{R}[x_{ij},\, y_{ij}]_{1\leq i\leq m,\, 1\leq j\leq n}\right).$$

We consider $\mathbb{AA}_n^m$ as having coordinates $(\mathbf{x}_1, \mathbf{y}_1, ..., \mathbf{x}_m, \mathbf{y}_m)$, where each $\mathbf{x}_i$ (resp. $\mathbf{y}_i$) is a vector with components $x_{i1}, ..., x_{in}$ (resp. $y_{i1}, ..., y_{in}$). The symmetric group S_m acts on $A[\mathbb{AA}_n^m]$ by simultaneously permuting the $\mathbf{x}_i$ and $\mathbf{y}_i$. Denote by $A[\mathbb{AA}_n^m]^{S_m}$ the elements of $A[\mathbb{AA}_n^m]$ invariant under this S_m-action.

Finally, unless otherwise stated, sets in this paper will be multisets (i.e., sets with repetition).

2 Multidimensional persistence

We begin by reviewing the concept of persistence.

Definition 1 A persistence module M indexed by the partially ordered set V is a family of k-modules $\{M_{\mathbf{v}}\}_{\mathbf{v}\in V}$ together with homomorphisms $\phi_{\mathbf{u},\mathbf{v}} : M_{\mathbf{u}} \to M_{\mathbf{v}}$ for all $\mathbf{u} \leq \mathbf{v}$, such that $\phi_{\mathbf{v},\mathbf{w}} \circ \phi_{\mathbf{u},\mathbf{v}} = \phi_{\mathbf{u},\mathbf{w}}$ whenever $\mathbf{u} \leq \mathbf{v} \leq \mathbf{w}$.

In this paper, the indexing set V will either be $\mathbb{N}^n$ or $\mathbb{R}^n$ for some $n \in \mathbb{N}_{>0}$. When the indexing set V is $\mathbb{N}^1$ or $\mathbb{R}^1$, we say that M is one-dimensional. When V is $\mathbb{N}^n$ or $\mathbb{R}^n$ for $n > 1$, we say that M is multidimensional. Furthermore, when V is $\mathbb{N}^n$, we refer to M as an integral persistence module; when V is $\mathbb{R}^n$, we refer to M as a real persistence module.

Definition 2 Given a persistence module M indexed over $\mathbb{N}^n$, we can define an n-graded module $\alpha(M)$ over $k[x_1,...,x_n]$ via the following:

$$\alpha(M) = \bigoplus_{\mathbf{v}\in\mathbb{N}^n} M_{\mathbf{v}},$$

where the $k[x_1,...,x_n]$-module structure is given by

$$\mathbf{x}^{\mathbf{v}-\mathbf{u}}\, m_{\mathbf{u}} = \phi_{\mathbf{u},\mathbf{v}}(m_{\mathbf{u}})$$

for $m_{\mathbf{u}} \in M_{\mathbf{u}}$ whenever $\mathbf{u} \leq \mathbf{v}$.

Furthermore, we have the following theorem:

Theorem 1 ([7]) *The correspondence α defines an equivalence of categories between the category of finite persistence modules over k and the category of finitely presented n-graded modules over $k[x_1,...,x_n]$.*

Theorem 1 allows us to interpret persistence modules indexed by $\mathbb{N}^n$ as finitely presented graded $k[x_1,...,x_n]$-modules. This correspondence allows us to study persistence modules using the well-developed theory of graded modules.

Definition 3 Define $\mathcal{M}(k,n)$ to be the category of all finite persistence k-modules indexed by $\mathbb{N}^n$. For an element $M \in \mathcal{M}(k,n)$, we will often consider M simultaneously as a persistence module and as a finitely presented n-graded $k[x_1,...,x_n]$-module.

Remark 1 In most applications of applied topology to data, the field k is almost always taken to be either $\mathbb{R}$ or a finite field of small characteristic. We are most concerned in this paper with $k = \mathbb{R}$.

3 The ring of graded-finite algebraic functions on persistent cubes

In this section, we calculate the ring of graded-finite algebraic functions on a simple and easily studied subset $\mathcal{R}(k,n)$ of $\mathcal{M}(k,n)$. Because $\mathcal{R}(k,n)$ is very easily parameterized, we may analyze it from the standpoint of algebra and algebraic geometry. This section extends a result of [2].

3.1 Defining the reduced symmetric product $\widetilde{Sp}$.

Let $\mathcal{R}^m(k,n) \subseteq \mathcal{M}(k,n)$ consist of all multidimensional persistence modules isomorphic to those of the form

$$\bigoplus_{i=1}^{m'} \left(\frac{k[x_1,...,x_n]}{\left(x_1^{d_{i1}},\cdots,x_n^{d_{in}}\right)} \right)_{\mathbf{v}_i},$$

where $m' \leq m$, $\mathbf{v}_i$ represents a grading and $0 < d_{ij} < \infty$. Define $\mathcal{R}(k,n) \subseteq \mathcal{M}(k,n)$ by

$$\mathcal{R}(k,n) = \bigcup_{m=1}^{\infty} \mathcal{R}^m(k,n).$$

Each summand of an element of $\mathcal{R}(k,n)$ can be represented by an element in $\mathbb{N}^{2n}$: $(v_{i1},...,v_{in},v_{i1}+d_{i1},...,v_{in}+d_{in})$. The first n coordinates can be viewed as "birth" times of summands, while the last n coordinates contain information about when they "die." Note additionally that the ordering of each summand within the direct sum decomposition given above is irrelevant. We now formulate this intuition algebraically.

Definition 4 Let $\mathcal{J}$ denote any object in a category. Define $Sp^m(\mathcal{J})$ to be the colimit, if it exists, of the diagram

$$\mathcal{J}^{\times m} \underset{\sigma}{\overset{id}{\rightrightarrows}} \mathcal{J}^{\times m},$$

where σ varies over all elements in the symmetric group S_m.

Remark 2 When $\mathcal{J}$ is a set, then $Sp^m(\mathcal{J})$ is simply the collection of all multisets of cardinality m of objects in $\mathcal{J}$.

We have natural maps

$$+ : Sp^m(\mathcal{J}) \times Sp^{m'}(\mathcal{J}) \to Sp^{m+m'}(\mathcal{J}).$$

Fixing some basepoint $j_0 : * \to \mathcal{J}$, we have natural inclusions

$$\iota_m : Sp^m(\mathcal{J}) \hookrightarrow Sp^{m+1}(\mathcal{J}),$$

where ι_m is defined as the composite

$$Sp^m(\mathcal{J}) \longrightarrow Sp^m(\mathcal{J}) \times * \xrightarrow{id \otimes j_0} Sp^m(\mathcal{J}) \times Sp^1(\mathcal{J}) \xrightarrow{+} Sp^{m+1}(\mathcal{J}).$$

Define $Sp^\infty(\mathcal{J})$ as the limit

$$Sp^\infty(\mathcal{J}) = \varinjlim_m Sp^m(\mathcal{J}).$$

Note that we have canonical inclusions $Sp^m(\mathcal{J}) \hookrightarrow Sp^\infty(\mathcal{J})$ and that $Sp^\infty(\mathcal{J})$ is a commutative monoid generated by $Sp^1(\mathcal{J})$ with monoid operation given by $+$.

If $\mathcal{J}$ is a scheme, then the inclusions ι_m induce maps

$$\iota_m^* : A[Sp^{m+1}(\mathcal{J})] \to A[Sp^m(\mathcal{J})].$$

In this case, we also have

$$A[Sp^\infty(\mathcal{J})] = \varprojlim_m A[Sp^m(\mathcal{J})].$$

We fix as the basepoint of $Sp^1(\mathbb{AA}^1_n)$ the map

$$j_0^* : * \to \mathbb{AA}^1_n = Sp^1(\mathbb{AA}^1_n)$$

defined as the dual of the "evaluation at 0" map

$$j_0 : \mathbb{R}[\mathbf{x},\mathbf{y}] \to \mathbb{R} \qquad \text{defined by} \qquad x_i \mapsto 0, \quad y_i \mapsto 0.$$

Letting $J(n) = \{(\mathbf{x},\mathbf{y}) \in \mathbb{Z}^n \times \mathbb{Z}^n \mid \mathbf{x} < \mathbf{y}\} \cup \{(\mathbf{0},\mathbf{0})\}$, we see that our former intuition about $\mathcal{R}^m(k,n)$ translates into the following statement.

Lemma 1 *There is a set-isomorphism between $\mathcal{R}^m(k,n)$ and $Sp^m(J(n))$:*

$$\bigoplus_{i=1}^{m'} \left(\frac{k[x_1,...,x_n]}{\left(x_1^{d_{i1}}, \cdots, x_n^{d_{in}}\right)} \right)_{\mathbf{v}_i} \longmapsto \begin{array}{c} \{(v_{i1},...,v_{in},v_{i1}+d_{i1},...,v_{in}+d_{in})\}_{1\leq i\leq m'} \\ \cup \\ \{(\mathbf{0},\mathbf{0})\}_{m'<i\leq m}. \end{array}$$

This set-isomorphism extends to a set-isomorphism between $\mathcal{R}(k,n)$ and $Sp^\infty(J(n))$.

We now discuss how we might approach our study of $Sp^m(J(n))$ from an algebraic geometric viewpoint.

Lemma 2 *There is a set-isomorphism between $Sp^m(J(n))$ and a well-chosen subset of the $\mathbb{R}$-points of $Sp^m(\mathbb{AA}^1_n)$. This set-isomorphism extends to a set-isomorphism between $Sp^\infty(J(n))$ and a well-chosen subset of the $\mathbb{R}$-points of $Sp^\infty(\mathbb{AA}^1_n)$.*

Proof By symmetrizing the identification of $\mathbb{R}^n \times \mathbb{R}^n$ with the set of $\mathbb{R}$-points of $\mathbb{AA}^1_n$, we may identify $Sp^m(\mathbb{R}^n \times \mathbb{R}^n)$ with a subset of the $\mathbb{R}$-points of $Sp^m(\mathbb{AA}^1_n)$. In particular, we identify $Sp^m(\mathbb{R}^n \times \mathbb{R}^n)$ with the subset of $\mathbb{R}$-points of $Sp^m(\mathbb{AA}^1_n)$ which factor through $\mathrm{Spec}(\mathbb{R}[\mathbf{x}_i,\mathbf{y}_i])$ (where the map $\mathrm{Spec}(\mathbb{R}[\mathbf{x}_i,\mathbf{y}_i]) \to Sp^m(\mathbb{AA}^1_n)$ is induced by the canonical inclusion of the ring of multi-symmetric polynomials into its ambient polynomial ring). Hence,

$$Sp^m(J(n)) \subseteq Sp^m(\mathbb{R}^n \times \mathbb{R}^n) \subseteq \left(\mathbb{R}\text{-points of } Sp^m(\mathbb{AA}^1_n)\right).$$

Remark 3 We emphasize that $Sp^m(\mathbb{R}^n \times \mathbb{R}^n)$ is identified with a *subset* of the $\mathbb{R}$-points of $Sp^m(\mathbb{AA}^1_n)$. The fact that we work over the non-algebraically closed field $\mathbb{R}$ is crucial here. For example, in the case $m = 2$ and $n = 1$, the $\mathbb{R}$-point of $Sp^2(\mathbb{AA}^1_1)$ induced by the homomorphism $A[Sp^2(\mathbb{AA}^1_1)] \to \mathbb{R}$ defined by

$$y_1 \mapsto 0 \qquad y_2 \mapsto 0 \qquad x_1 + x_2 \mapsto 0 \qquad x_1^2 + x_2^2 \mapsto -1$$

is not identified with any element of $Sp^m(\mathbb{R}^n \times \mathbb{R}^n)$.

The previous two lemmas combine to yield the following:

Corollary 1 *There is a set-isomorphism between $\mathcal{R}(k,n)$ and the subset of the $\mathbb{R}$-points of $Sp^\infty(\mathbb{AA}^1_n)$ induced by homomorphisms*

$$\varphi : \mathbb{R}[\mathbf{x}_i,\mathbf{y}_i]_{1\leq i\leq m} \to \mathbb{R}$$

such that $(\varphi(\mathbf{x}_i),\varphi(\mathbf{y}_i)) \in J(n)$ for all i.

Remark 4 In a real (rather than integral) formulation of multidimensional persistence, one would instead define $J(n) = \{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \mathbf{x} < \mathbf{y}\} \cup \{(\mathbf{0},\mathbf{0})\}$. In either case, we view $J(n)$ as a subset of $\mathbb{R}^n \times \mathbb{R}^n$; all results discussed above are thus valid in either setting.

In the definition of $\mathcal{R}(k,n)$ given above, we required that the d_{ij} be strictly positive, and we require a strict inequality in our definition of $J(n)$. This inequality is lost if we work with $\mathbb{R}^n \times \mathbb{R}^n$ rather than with $J(n)$. Nevertheless, we still wish to encode algebraically that we want to disregard any summand of any element of $Sp^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ of the form $(\mathbf{x},\mathbf{y})$ where one coordinate x_i of $\mathbf{x}$ is equal to the corresponding coordinate y_i of $\mathbf{y}$. Put differently, we intuitively think of $(\mathbf{x},\mathbf{y})$ as the opposite vertices of an n-dimensional cube, and we wish to disregard any cubes of volume 0.

To this end, define a reduced symmetric product

$$\widetilde{Sp}(\mathbb{R}^n \times \mathbb{R}^n) = \frac{\coprod_m Sp^m(\mathbb{R}^n \times \mathbb{R}^n)}{\simeq},$$

where $\simeq$ is the equivalence relation generated by all relations of the form

$$\{(\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x}_2,\mathbf{y}_2),...,(\mathbf{x}_m,\mathbf{y}_m),(\mathbf{z},\mathbf{z}')\} \simeq \{(\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x}_2,\mathbf{y}_2),...,(\mathbf{x}_m,\mathbf{y}_m)\},$$

where one of the coordinates of $\mathbf{z}$ equals one of the coordinates of $\mathbf{z}'$.

We now generalize our definition of the reduced symmetric product $\widetilde{Sp}$ to the algebraic geometric setting. Define $\widetilde{Sp}(\mathbb{AA}^1_n)$ as the colimit of the diagram

$$Sp^\infty(\mathbb{AA}^1_n) \underset{\widetilde{j}}{\overset{id}{\rightrightarrows}} Sp^\infty(\mathbb{AA}^1_n),$$

where $\widetilde{j}$ runs over all maps induced on $Sp^\infty(\mathbb{AA}^1_n)$ by maps of the form

$$Sp^m(\mathbb{AA}^1_n) \longrightarrow Sp^m(\mathbb{AA}^1_n) \times * \xrightarrow{id \otimes j^*} Sp^m(\mathbb{AA}^1_n) \times Sp^1(\mathbb{AA}^1_n) \xrightarrow{+} Sp^{m+1}(\mathbb{AA}^1_n),$$

where j^* is the dual of a map $j : \mathbb{R}[\mathbf{x},\mathbf{y}] \to \mathbb{R}$ such that $j(x_i) = j(y_i)$ for some i.

Since $\widetilde{Sp}(\mathbb{AA}^1_n)$ is a quotient of $Sp^\infty(\mathbb{AA}^1_n)$, we have that

$$A\left[\widetilde{Sp}(\mathbb{AA}^1_n)\right] \subseteq A\left[Sp^\infty(\mathbb{AA}^1_n)\right].$$

It is our goal to investigate $A\left[\widetilde{Sp}(\mathbb{AA}^1_n)\right]$ and gain insight into its structure. Let us say, perhaps somewhat preemptively, that the reader may identify $\widetilde{Sp}(\mathbb{R}^n \times \mathbb{R}^n)$ with the "finite" $\mathbb{R}$-points of $\widetilde{Sp}(\mathbb{AA}^1_n)$ – those which can be induced by homomorphisms

$$\mathbb{R}[\mathbf{x}_i,\mathbf{y}_i]_{1 \le i \le m} \to \mathbb{R}.$$

The next section will justify this identification. For $X \in Sp^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, we denote the associated $\mathbb{R}$-point of $Sp^\infty(\mathbb{AA}^1_n)$ (or $\widetilde{Sp}(\mathbb{AA}^1_n)$) by φ^*_X.

3.2 Multisymmetric and invariant polynomials.

We review some facts about $A[Sp^\infty(\mathbb{AA}^1_n)]$ from [8].

Lemma 3 ([8]) *$Sp^m(\mathbb{AA}^1_n)$ and $Sp^\infty(\mathbb{AA}^1_n)$ are affine schemes. Furthermore, the ring $A[Sp^m(\mathbb{AA}^1_n)] = A[\mathbb{AA}^m_n]^{S_m}$ is generated (with relations) as an $\mathbb{R}$-algebra by the multi-symmetric power sums*

$$p_{\mathbf{a},\mathbf{b},m} = \sum_{i=1}^{m} x_{i1}^{a_1} x_{i2}^{a_2} \cdots x_{in}^{a_n} y_{i1}^{b_1} y_{i2}^{b_2} \cdots y_{in}^{b_n},$$

where $a_j, b_j \in \mathbb{N}$. Additionally, $A\left[Sp^\infty(\mathbb{AA}^1_n)\right]$ is the subset of the ring of power series $\mathbb{R}[\![p_{\mathbf{a},\mathbf{b}}]\!]$, where

$$p_{\mathbf{a},\mathbf{b}} = \sum_{i=1}^{\infty} x_{i1}^{a_1} x_{i2}^{a_2} \cdots x_{in}^{a_n} y_{i1}^{b_1} y_{i2}^{b_2} \cdots y_{in}^{b_n},$$

consisting of power series p with the property that, given any finite subset S of $\{p_{\mathbf{a},\mathbf{b}}\}$, there are finitely many terms in p involving only elements of S. In particular, there are no relations among the $p_{\mathbf{a},\mathbf{b}}$ in $A\left[Sp^\infty(\mathbb{AA}^1_n)\right]$.

Unfortunately, the ring $A\left[Sp^\infty(\mathbb{AA}^1_n)\right]$ is too infinite for our purposes. We wish to isolate a tractable yet still suitably large subring $A_{fin}\left[Sp^\infty(\mathbb{AA}^1_n)\right]$ of $A\left[Sp^\infty(\mathbb{AA}^1_n)\right]$ so that if we are given $X \in Sp^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and $f \in A_{fin}\left[Sp^\infty(\mathbb{AA}^1_n)\right]$, the quantity $\varphi_X(f)$ is guaranteed to be finite.

To achieve this goal, it will be necessary to work in the category of graded rings. We give all variables x_{ij} and y_{ij} the grading 1. Define

$$A_{fin}\left[Sp^\infty(\mathbb{AA}^1_n)\right] = \varprojlim_m A\left[Sp^m(\mathbb{AA}^1_n)\right],$$

where this inverse limit is taken in the category of graded rings. If we forget the gradings, we see that $A_{fin}\left[Sp^\infty(\mathbb{AA}^1_n)\right] \subseteq A\left[Sp^\infty(\mathbb{AA}^1_n)\right]$ is simply the polynomial ring $\mathbb{R}[p_{\mathbf{a},\mathbf{b}}]$ (because $A\left[Sp^\infty(\mathbb{AA}^1_n)\right]$ has only finitely many generators in each degree). Due to the interpretation of $A_{fin}\left[Sp^\infty(\mathbb{AA}^1_n)\right]$ as an inverse limit in the category of graded rings, we call $A_{fin}\left[Sp^\infty(\mathbb{AA}^1_n)\right]$ the graded-finite elements of $A\left[Sp^\infty(\mathbb{AA}^1_n)\right]$.

Our goal is to isolate the elements f of $A_{fin}\left[Sp^\infty(\mathbb{AA}^1_n)\right]$ which "disregard" any cubes with volume 0 in the following sense: for all $X_1, X_2 \in Sp^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that X_1 and X_2 are identified in $\widetilde{Sp}(\mathbb{R}^n \times \mathbb{R}^n)$, we require that $\varphi_{X_1}(f) = \varphi_{X_2}(f)$ (where $\varphi^*_{X_1}$ and $\varphi^*_{X_2}$ are the $\mathbb{R}$-points of $\widetilde{Sp}(\mathbb{AA}^1_n)$ associated with X_1 and X_2). That is, the collection of such f is given by $A_{fin}\left[\widetilde{Sp}(\mathbb{AA}^1_n)\right]$.

Let $W_i \subseteq (\mathbb{R}^n \times \mathbb{R}^n)^m$ be the closed subset of all points which satisfy the equation

$$\prod_{k=1}^{n} (y_{ik} - x_{ik}) = 0.$$

Note that if $m = 1$, then W_i is the union of n hyperplanes in $\mathbb{R}^n \times \mathbb{R}^n$. To any point $X \in W_i$, we can associate an $\mathbb{R}$-point φ_X^* of $\mathbb{AA}_n^m$. Consider the subring R_n^m of $A[\mathbb{AA}_n^m]$ consisting of all $f \in A[\mathbb{AA}_n^m]$ such that for all i and for all $X_1, X_2 \in W_i$, $\varphi_{X_1}(f) = \varphi_{X_2}(f)$. Put differently, R_n^m is the subring of $A[\mathbb{AA}_n^m]$ consisting of all polynomials

$$f \in \mathbb{R}[x_{ij},\ y_{ij}]_{1\le i\le m,\ 1\le j\le n}$$

such that for all i and j, the restriction of f to W_i is independent of x_{ij} and y_{ij}. Let $R_n^\infty = \varprojlim_m R_n^m$ (where the inverse limit is taken in the category of graded rings).

To summarize the exposition of the previous paragraphs, we have the following proposition:

Proposition 1 *Taking inverse limits in the category of graded rings,*

$$\begin{aligned} A_{fin}\left[\widetilde{Sp}(\mathbb{AA}_n^1)\right] &= R_n^\infty \cap A_{fin}\left[Sp^\infty(\mathbb{AA}_n^1)\right] \\ &= \left(\varprojlim_m R_n^m\right) \cap \left(\varprojlim_m A\left[Sp^m(\mathbb{AA}_n^1)\right]\right) \\ &\cong \varprojlim_m \left(R_n^m \cap A\left[Sp^m(\mathbb{AA}_n^1)\right]\right). \end{aligned}$$

3.3 The explicit calculation of $A_{fin}\left[\widetilde{Sp}(\mathbb{AA}_n^1)\right]$.

We now examine the structure and properties of R_n^m and $A_{fin}\left[\widetilde{Sp}(\mathbb{AA}_n^1)\right]$. It will be convenient to change coordinates during this analysis. Define

$$\boldsymbol{\eta}_i = \mathbf{y}_i - \mathbf{x}_i \quad \text{and} \quad \boldsymbol{\xi}_i = \mathbf{y}_i + \mathbf{x}_i\,.$$

Note that

$$A[\mathbb{AA}_n^m] = \mathbb{R}[\eta_{ij},\ \xi_{ij}]_{1\le i\le m,\ 1\le j\le n}.$$

Proposition 2 *The ring R_n^m is characterized differentially as the subring of all elements $f \in A[\mathbb{AA}_n^m]$ such that $\frac{\partial f}{\partial \xi_{ij}} \in (\eta_{ik})$ for all i,j,k, and such that $\frac{\partial f}{\partial \eta_{ij}} \in (\eta_{ik})$ for all i,j,k with $j \neq k$.*

Proof With our new coordinates, W_i can be defined by the equation $\prod_{k=1}^n \eta_{ik} = 0$. Let $W_{ik} \subseteq \mathbb{R}^n \times \mathbb{R}^n$ denote the hyperplane defined by the equation $\eta_{ik} = 0$, and note that $W_i = \bigcup_{k=1}^n W_{ik}$. Then R_n^m is the subring of $A[\mathbb{AA}_n^m]$ consisting of all functions whose restriction to W_i is independent of $\xi_{ij} - \eta_{ij}$ and $\xi_{ij} + \eta_{ij}$ for all i,j. Equivalently, R_n^m is the subring of $A[\mathbb{AA}_n^m]$ consisting of all functions whose restriction to W_{ik} is independent of ξ_{ij} and η_{ij} for all i,j,k.

Suppose $f \in \mathbb{R}[\xi_{ij}, \eta_{ij}]$. For any i,k, we can write

$$f(\boldsymbol{\xi},\boldsymbol{\eta}) = q(\boldsymbol{\xi},\boldsymbol{\eta})\eta_{ik} + r_{ik}(\boldsymbol{\xi},\boldsymbol{\eta}),$$

where η_{ik} does not appear in $r_{ik}(\boldsymbol{\xi},\boldsymbol{\eta})$. Since $f|_{W_{ik}} = r|_{W_{ik}}$, it follows that $f \in R_n^m$ if and only if for all i,j,k, $\frac{\partial r_{ik}}{\partial \xi_{ij}} = \frac{\partial r_{ik}}{\partial \eta_{ij}} = 0$. This in turn is equivalent to the condition that $\frac{\partial f}{\partial \xi_{ij}} \in (\eta_{ik})$ for all i,j,k and $\frac{\partial f}{\partial \eta_{ij}} \in (\eta_{ik})$ for all i,j,k with $j \neq k$.

Proposition 3 *The following set of monomials forms an $\mathbb{R}$-basis for R_n^m:*

$$\left\{ \prod_{i,j} \eta_{ij}^{a_{ij}} \xi_{ij}^{b_{ij}} \;\middle|\; (\forall i) \left(\left(\exists j \text{ such that } a_{ij} > 0 \text{ or } b_{ij} > 0 \right) \Longrightarrow (\forall k, a_{ik} > 0) \right) \right\}.$$

Proof The differential conditions given in proposition 2 which are necessary and sufficient for a polynomial $f \in A[\mathbb{AA}_n^m]$ to belong to R_n^m are true for a polynomial f if and only if they are true for each of the monomial summands of f. Hence, we know that there exists a basis of monomials for R_n^m. The monomials which satisfy the conditions of proposition 2 are exactly those listed in the statement of this proposition.

Having completed these initial calculations, it remains to calculate $A_{fin}\left[\widetilde{Sp}(\mathbb{AA}_n^1)\right]$. Recall that the Hilbert series $HS_\Omega(t)$ of a non-negatively-graded vector space $\Omega = \bigoplus_{i=0}^{\infty} \Omega_i$ is defined to be the sum

$$HS_\Omega(t) = \sum_{i=0}^{\infty} \dim(\Omega_i)\, t^i.$$

Theorem 2 *$A_{fin}\left[\widetilde{Sp}(\mathbb{AA}_n^1)\right]$ is freely generated as an $\mathbb{R}$-algebra by the infinite symmetric polynomials*

$$p_{\mathbf{a},\mathbf{b}} = \sum_{i=1}^{\infty} \eta_{i1}^{a_1} \eta_{i2}^{a_2} \cdots \eta_{in}^{a_n} \xi_{i1}^{b_1} \xi_{i2}^{b_2} \cdots \xi_{in}^{b_n}$$

where $a_j \geq 1$ for all j.

Proof Let $\mathcal{A}$ denote the $\mathbb{R}$-algebra generated by the $p_{\mathbf{a},\mathbf{b}}$ such that $a_j \geq 1$ for all j. By Lemma 3, $\mathcal{A}$ is freely generated by these $p_{\mathbf{a},\mathbf{b}}$. Since each of these $p_{\mathbf{a},\mathbf{b}}$ is in $A_{fin}\left[\widetilde{Sp}(\mathbb{AA}_n^1)\right]$, we have that $\mathcal{A}$ is a freely generated sub-algebra of $A_{fin}\left[\widetilde{Sp}(\mathbb{AA}_n^1)\right]$. It remains to show that these two algebras are in fact equal.

To prove equality, we need only show that $HS_{\mathcal{A}}(t) = HS_{A_{fin}\left[\widetilde{Sp}(\mathbb{AA}_n^1)\right]}(t)$. This equality follows from the results of Lemma 4 and Lemma 5 below.

Lemma 4

$$HS_{\mathcal{A}}(t) = \prod_{d=0}^{\infty} \left(1 - t^{n+d}\right)^{-\binom{d+2n-1}{2n-1}}.$$

Proof Since we require that $a_j \geq 1$ for all generators $p_{\mathbf{a},\mathbf{b}}$ of $\mathcal{A}$, it follows that $\mathcal{A}$ has 0 generators in degrees 1 through $n-1$. In degrees $d \geq n$, $\mathcal{A}$ has $\binom{d+n-1}{d-n} = \binom{d+n-1}{2n-1}$ generators (this multinomial coefficient represents the number of ways to put $d-n$ balls into $2n$ buckets).

Lemma 5

$$HS_{A_{fin}\left[\widetilde{Sp}(\mathbb{AA}_n^1)\right]}(t) = \prod_{d=0}^{\infty} \left(1 - t^{n+d}\right)^{-\binom{d+2n-1}{2n-1}}.$$

Proof For ease of notation, let $f(t) = HS_{A_{fin}\left[\widetilde{Sp}(\mathbb{AA}^1_n)\right]}(t)$. Additionally, we define $g(t) = \prod_{d=0}^{\infty}\left(1-t^{n+d}\right)^{-\binom{d+2n-1}{2n-1}}$. We must show that $f(t) = g(t)$.

Let $<_\ell$ denote the lexicographic order on $\mathbb{N}^n_+ \times \mathbb{N}^n$. Let $\prec$ denote the following linear order on $\mathbb{N}^n_+ \times \mathbb{N}^n$: for $(\mathbf{a},\mathbf{b}),(\mathbf{a}',\mathbf{b}') \in \mathbb{N}^n_+ \times \mathbb{N}^n$, $(\mathbf{a}',\mathbf{b}') \prec (\mathbf{a},\mathbf{b})$ if either

$$\sum(\mathbf{a}'+\mathbf{b}') < \sum(\mathbf{a}+\mathbf{b}),$$

or

$$\sum(\mathbf{a}'+\mathbf{b}') = \sum(\mathbf{a}+\mathbf{b}) \text{ and } (\mathbf{a}',\mathbf{b}') <_\ell (\mathbf{a},\mathbf{b}).$$

Let $(\mathbf{a},\mathbf{b}) \in \mathbb{N}^n_+ \times \mathbb{N}^n$. Define $f_{\mathbf{a},\mathbf{b}}(t)$ to be the Hilbert series of the free subalgebra of $A_{fin}\left[\widetilde{Sp}(\mathbb{AA}^1_n)\right]$ generated by the symmetrizations of the monomials

$$\prod_{i=1}^{l}\prod_{j=1}^{n}\eta_{ij}^{a_{ij}}\xi_{ij}^{b_{ij}} \text{ such that } (\mathbf{a},\mathbf{b}) \geq (\mathbf{a}_1,\mathbf{b}_1) \geq (\mathbf{a}_2,\mathbf{b}_2) \geq \cdots \geq (\mathbf{a}_l,\mathbf{b}_l),$$

where $(\mathbf{a}_i,\mathbf{b}_i) \in \mathbb{N}^n_+ \times \mathbb{N}^n$. For $(\mathbf{a},\mathbf{b}) \in \mathbb{N}^n_+ \times \mathbb{N}^n$, let $(\mathbf{a}',\mathbf{b}') \in \mathbb{N}^n_+ \times \mathbb{N}^n$ be the immediate predecessor to $(\mathbf{a},\mathbf{b})$ under the $\preceq$ ordering (if it exists). Then we have

$$f_{\mathbf{1},\mathbf{0}}(t) = (1-t^n)^{-1}$$

and

$$f_{\mathbf{a},\mathbf{b}}(t) = (1-t^{\sum(\mathbf{a}+\mathbf{b})})^{-1} f_{\mathbf{a}',\mathbf{b}'}(t).$$

We now analyze the function $g(t)$. For

$$D_{\mathbf{a},\mathbf{b}}(t) = \prod_{0\leq d<\sum(\mathbf{a}+\mathbf{b})-n}\left(1-t^{d+n}\right)^{-\binom{d+2n-1}{2n-1}},$$

$$A_{\mathbf{a},\mathbf{b},i}(t) = \prod_{0\leq\alpha<a_i-1}\left(1-t^{\sum(\mathbf{a}+\mathbf{b})}\right)^{-\binom{n+\sum(\mathbf{a}+\mathbf{b})-\left(\sum_{j<i}a_j\right)-\alpha-2}{2n-i-1}},$$

and

$$B_{\mathbf{a},\mathbf{b},i}(t) = \prod_{0\leq\beta<b_i}\left(1-t^{\sum(\mathbf{a}+\mathbf{b})}\right)^{-\binom{n+\sum(\mathbf{b})-i-\left(\sum_{j<i}b_j\right)-\beta-1}{n-i-1}},$$

set

$$g_{\mathbf{a},\mathbf{b}}(t) = \left(1-t^{\sum(\mathbf{a}+\mathbf{b})}\right)^{-1} D_{\mathbf{a},\mathbf{b}}(t)\left(\prod_{i=1}^{n}A_{\mathbf{a},\mathbf{b},i}(t)\right)\left(\prod_{i=1}^{n-1}B_{\mathbf{a},\mathbf{b},i}(t)\right).$$

Note that

$$g_{\mathbf{1},\mathbf{0}}(t) = (1-t^n)^{-1}.$$

Lemma 6 below shows that

$$g_{\mathbf{a},\mathbf{b}}(t) = \left(1-t^{\sum(\mathbf{a}+\mathbf{b})}\right)^{-1} g_{\mathbf{a}',\mathbf{b}'}(t).$$

Since $f_{\mathbf{1},\mathbf{0}} = g_{\mathbf{1},\mathbf{0}}$ and the $f_{\mathbf{a},\mathbf{b}}$ and $g_{\mathbf{a},\mathbf{b}}$ satisfy the same recurrence relation, it follows that $f_{\mathbf{a},\mathbf{b}} = g_{\mathbf{a},\mathbf{b}}$ for each $\mathbf{a},\mathbf{b}$, and thus

$$f(t) = \lim_{(\mathbf{a},\mathbf{b})\to\infty} f_{\mathbf{a},\mathbf{b}}(t) = \lim_{(\mathbf{a},\mathbf{b})\to\infty} g_{\mathbf{a},\mathbf{b}}(t) = g(t).$$

Lemma 6 *In the notation given in Lemma 5, the functions $g_{\mathbf{a},\mathbf{b}}$ satisfy*

$$g_{\mathbf{a},\mathbf{b}}(t) = \left(1 - t^{\sum(\mathbf{a}+\mathbf{b})}\right)^{-1} g_{\mathbf{a}',\mathbf{b}'}(t).$$

Proof Due to the complicated definition of the total order $\prec$ used above, this lemma must be proven separately for the following five cases:

1. $\mathbf{a}' = \mathbf{a}$, ${b'}_i = b_i$ for $1 \leq i \leq n-2$, ${b'}_{n-1} + 1 = b_{n-1}$, and ${b'}_{n-1} - 1 = b_{n-1}$.
2. $\mathbf{a}' = \mathbf{a}$, and there exists an index $i_0 < n-1$ such that ${b'}_i = b_i$ for $i < i_0$, ${b'}_{i_0} + 1 = b_{i_0}$, ${b'}_{i_0+1} - 1 = b_n$, ${b'}_n = b_{i_0+1} = 0$, and ${b'}_i = b_i = 0$ for $i_0 + 2 \leq i \leq n-1$.
3. ${a'}_i = a_i$ for $i < n$, ${a'}_n + 1 = a_n$, ${b'}_1 - 1 = b_n$, ${b'}_n = b_1 = 0$, and ${b'}_i = b_i = 0$ for $2 \leq i \leq n-1$.
4. there exists an index $i_0 \leq n-1$ such that ${a'}_i = a_i$ for $i < i_0$, ${a'}_{i_0} + 1 = a_{i_0}$, ${a'}_{i_0+1} - 2 = b_n$, $a_{i_0+1} = 1$, ${a'}_i = a_i = 1$ for $i \geq i_0 + 2$, $\mathbf{b}' = 0$, and $b_i = 0$ for all $i < n$.
5. ${a'}_1 = b_n$, $a_1 = 1$, ${a'}_i = a_i = 1$ for all $i > 1$, ${b'}_i = b_i = 0$ for $i < n$, and ${b'}_n = 0$.

For case (1), we have:

$$(\mathbf{a}', \mathbf{b}') = \left(\begin{pmatrix} a_1 \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} - 1 \\ b_n + 1 \end{pmatrix}\right) \text{ and } (\mathbf{a}, \mathbf{b}) = \left(\begin{pmatrix} a_1 \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix}\right).$$

Note that $B_{\mathbf{a},\mathbf{b},n-1}(t) = \left(1 - t^{\sum(\mathbf{a}+\mathbf{b})}\right)^{-1} B_{\mathbf{a}',\mathbf{b}',n-1}(t)$. As all other factors of $g_{\mathbf{a},\mathbf{b}}$ and $g_{\mathbf{a}',\mathbf{b}'}$ are identical, the lemma holds.

The proofs of cases (2) through (5) are very similar. Of these, for ease, we only prove case (2). For this case, we have:

$$(\mathbf{a}', \mathbf{b}') = \left(\begin{pmatrix} a_1 \\ \vdots \\ a_{i-1} \\ a_i \\ a_{i+1} \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_{i-1} - 1 \\ b_n + 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right) \text{ and } (\mathbf{a}, \mathbf{b}) = \left(\begin{pmatrix} a_1 \\ \vdots \\ a_{i-1} \\ a_i \\ a_{i+1} \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_{i-1} \\ 0 \\ 0 \\ \vdots \\ b_n \end{pmatrix}\right).$$

It suffices to show that

$$B_{\mathbf{a},\mathbf{b},i-1}(t) = B_{\mathbf{a}',\mathbf{b}',i-1}(t) B_{\mathbf{a}',\mathbf{b}',i}(t) \left(1 - t^{\sum(\mathbf{a}+\mathbf{b})}\right)^{-1},$$

as all other factors of $g_{\mathbf{a},\mathbf{b}}$ and $g_{\mathbf{a}',\mathbf{b}'}$ are identical. Both sides of the equation above are merely products of powers of $1 - t^{\sum(\mathbf{a}+\mathbf{b})}$, so we need only show that the exponents of $1 - t^{\sum(\mathbf{a}+\mathbf{b})}$ are the same on both sides of the equality. That is, we must show that

$$\begin{aligned} &\binom{n + \sum(\mathbf{b}) - (i-1) - \left(\sum_{j<i-1} b_j\right) - (b_{i-1} - 1) - 1}{n - (i-1) - 1} \\ &\qquad = 1 + \sum_{\beta=0}^{b_n} \binom{n + \sum(\mathbf{b}) - i - \left(\sum_{j<i} b_j\right) + 1 - \beta - 1}{n - i - 1}. \end{aligned}$$

This simplifies to

$$\binom{n+\sum(\mathbf{b})-i-\sum_{j<i}b_j+1}{n-i} = 1+\sum_{\beta=0}^{b_n}\binom{n+\sum(\mathbf{b})-i-(\sum_{j<i}b_j)-\beta}{n-i-1}.$$

Because $b_j = 0$ for $i+1 \leq j < n$, we can further simplify to

$$\begin{aligned}\binom{(n-i)+(b_n+1)}{(n-i)} &= 1+\sum_{\beta=0}^{b_n}\binom{(n-i)+(b_n-\beta)}{(n-i)-1} \\ &= \sum_{\beta=0}^{b_n+1}\binom{(n-i)+(b_n-\beta)}{(n-i)-1} \\ &= \sum_{\beta'=0}^{b_n+1}\binom{(n-i)+\beta'-1}{(n-i)-1}.\end{aligned}$$

This equality holds due to the combinatorial identity

$$\binom{x+k}{x} = \sum_{k'=0}^{k}\binom{x+k'-1}{x-1},$$

which is easily proven by induction on k using Pascal's rule.

4 Extending and calculating the invariants

Having determined the ring of graded-finite global sections of the "cubical" persistence modules $\mathcal{R}(k,n)$, we are now faced with two problems. First, it is not clear how (or if) these functions might extend to the class of arbitrary persistence modules $\mathcal{M}(k,n)$. Second, although we have determined the ring of graded-finite global sections of $\mathcal{R}(k,n)$, it is not clear how one might calculate them for any given persistence module. This section addresses both of these issues.

4.1 Defining the invariants

For an integral multidimensional persistence module M over a field k and $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$, the rank invariant $\rho_{\mathbf{u},\mathbf{v}}(M)$ is defined by

$$\rho_{\mathbf{u},\mathbf{v}}(M) = \begin{cases}\dim_k\left(\mathrm{Im}\left(M_{\lfloor\mathbf{u}\rfloor} \to M_{\lceil\mathbf{v}\rceil}\right)\right) & \mathbf{u}\leq\mathbf{v} \\ 0 & \text{otherwise}\end{cases}.$$

Remark 5 For a real (rather than integral) treatment of multidimensional persistence, one would remove the floor and ceiling symbols in the definition of $\rho_{\mathbf{u},\mathbf{v}}$. An algorithm to calculate the rank invariant is given in [6].

For $\mathbf{a} \in \mathbb{N}_+^n$, let $I = \{i \mid a_i = 1\}$, and let $J = \{i \mid a_i > 1\}$. Let $(\mathbf{z}, \mathbf{z}') \in \mathbb{R}^{n+|J|}$ denote a variable constructed so that $\mathbf{z}, \mathbf{z}' \in \mathbb{R}^n$ denote variables such that for all $i \in I$, the ith coordinate of $\mathbf{z}$ is the same as the ith coordinate of $\mathbf{z}'$.

As an example of the above construction, if we have $n = 3$ and $\mathbf{a} = (1,2,1)$, then $I = \{1,3\}$, $J = \{2\}$, and $(\mathbf{z}, \mathbf{z}') \in \mathbb{R}^4$ where $\mathbf{z} = (z_1, z_2, z_3)$ and $\mathbf{z}' = (z_1, z_2', z_3)$.

For any persistence module M and $(\mathbf{a}, \mathbf{b}) \in \mathbb{N}_+^n \times \mathbb{N}^n$, define

$$F_{\mathbf{a},\mathbf{b}}(M) = \int_{\mathbb{R}^{n+|J|}} (\mathbf{z}' - \mathbf{z})^{\mathbf{a}_J - 2} (\mathbf{z})^{\mathbf{b}_I} (\mathbf{z} + \mathbf{z}')^{\mathbf{b}_J} (\rho_{\mathbf{z},\mathbf{z}'}(M)) \, d(\mathbf{z}, \mathbf{z}').$$

4.2 Proving equivalence of invariants

In this section, we will prove that on $\mathcal{R}(k,n)$, the invariants $\{F_{\mathbf{a},\mathbf{b}}\}$ defined in Section 4.1 are equivalent to the invariants $\{p_{\mathbf{a},\mathbf{b}}\}$ defined in Theorem 2 (equivalent in the sense that they encode the same information about an element $M \in \mathcal{R}(k,n)$). This shows that the invariants $\{F_{\mathbf{a},\mathbf{b}}\}$ extend to all of $\mathcal{M}(k,n)$ the invariants $\{p_{\mathbf{a},\mathbf{b}}\}$ defined on $\mathcal{R}(k,n)$.

More concretely, we will prove that on $\mathcal{R}(k,n)$,

$$\text{Span}\left(\{F_{\mathbf{a},\mathbf{b}}\}_{(\mathbf{a},\mathbf{b}) \in \mathbb{N}_+^n \times \mathbb{N}^n}\right) = \text{Span}\left(\{p_{\mathbf{a},\mathbf{b}}\}_{(\mathbf{a},\mathbf{b}) \in \mathbb{N}_+^n \times \mathbb{N}^n}\right).$$

We prove this fact in three iterations - first for $\mathcal{R}^1(k,1)$, then for $\mathcal{R}^1(k,n)$, and finally for $\mathcal{R}(k,n)$.

In proving equivalence, it will be necessary to define a partial order $\preceq$ on $\mathbb{N}_+^n \times \mathbb{N}^n$ (which is different than the linear order $\leq$ defined in the proof of Lemma 5). For $(\mathbf{a},\mathbf{b}), (\mathbf{a}',\mathbf{b}') \in \mathbb{N}_+^n \times \mathbb{N}^n$, we say that $(\mathbf{a},\mathbf{b}) \preceq (\mathbf{a}',\mathbf{b}')$ if and only if

$$\sum(\mathbf{a} + \mathbf{b}) = \sum(\mathbf{a}' + \mathbf{b}') \text{ and } \mathbf{a} \leq \mathbf{a}' \text{ (coordinate-wise)}.$$

We extend this partial order to monomials. We say that $\mathbf{x}^{\mathbf{a}} \mathbf{y}^{\mathbf{b}} \preceq \mathbf{x}^{\mathbf{a}'} \mathbf{y}^{\mathbf{b}'}$ if and only if $(\mathbf{a},\mathbf{b}) \preceq (\mathbf{a}',\mathbf{b}')$, and we consider monomials $\mathbf{x}^{\mathbf{a}'} \mathbf{y}^{\mathbf{b}'}$ with $\mathbf{x}^{\mathbf{a}'} \mathbf{y}^{\mathbf{b}'} \succeq \mathbf{x}^{\mathbf{a}} \mathbf{y}^{\mathbf{b}}$ as "higher order terms" (it will be clear from context what the monomial represented here by $\mathbf{x}^{\mathbf{a}} \mathbf{y}^{\mathbf{b}}$ is).

Lemma 7 *On $\mathcal{R}^1(k,1)$,*

$$\text{Span}\left(\{F_{a,b}\}_{(a,b) \in \mathbb{N}_+ \times \mathbb{N}}\right) = \text{Span}\left(\{p_{a,b}\}_{(a,b) \in \mathbb{N}_+ \times \mathbb{N}}\right).$$

Proof It suffices to prove that for all k, the following holds (on $\mathcal{R}^1(k,1)$):

$$\text{Span}\left(\{F_{a,b}\}_{a+b=k}\right) = \text{Span}\left(\{p_{a,b}\}_{a+b=k}\right).$$

We proceed by showing that $F_{a,b}$ can be written as a (finite) linear combination of $p_{a,b}$ and higher order terms. It follows from this computation that

$$\text{Span}\left(\{F_{a,b}\}_{a+b=k}\right) \subseteq \text{Span}\left(\{p_{a,b}\}_{a+b=k}\right).$$

Since there are only finitely many pairs $(a',b') \in \mathbb{N}_+ \times \mathbb{N}$ with $a'+b' = a+b = k$, it follows from back substitution that

$$\text{Span}\left(\{F_{a,b}\}_{a+b=k}\right) \supseteq \text{Span}\left(\{p_{a,b}\}_{a+b=k}\right),$$

effectively proving the lemma.

We return now to the matter of calculating $F_{a,b}(M)$ for $M \in \mathcal{R}^1(k,1)$. Assume $M = [x,y] \in \mathcal{R}^1(k,1)$. Let $(a,b) \in \mathbb{N}_+ \times \mathbb{N}$. We first consider the case $a = 1$. In this case, we have that $\rho_{z,z}(M)$ is simply the characteristic function $\mathbb{1}_{[x,y]}$, and so:

$$\begin{aligned}
F_{1,b}(M) &= \int_{\mathbb{R}^1} z^b \left(\rho_{z,z}(M)\right)\, dz \\
&= \int_x^y z^b\, dz \\
&= \frac{1}{b+1}\left(y^{b+1} - x^{b+1}\right) \\
&= \left(\frac{1}{b+1}\right)\left(\left(\frac{(y+x)+(y-x)}{2}\right)^{b+1} - \left(\frac{(y+x)-(y-x)}{2}\right)^{b+1}\right) \\
&= \left(\frac{1}{(b+1)2^{b+1}}\right)\left(\sum_{i=0}^{b+1}\left(1-(-1)^i\right)\binom{b+1}{i}(y-x)^i(y+x)^{b+1-i}\right) \\
&= \left(\frac{1}{2^b}\right) p_{1,b}(M) + \text{higher order terms}\,.
\end{aligned}$$

Next, we calculate $F_{a,b}(M)$ for $M = [x,y] \in \mathcal{R}^1(k,1)$ and $(a,b) \in \mathbb{N}_+ \times \mathbb{N}$ with $a \geq 2$. Note that in this case, $\rho_{z,z'}(M)$ is the characteristic function of the solid triangular region

$$T_{x,y} = \left\{(z,z') \in \mathbb{R}^2 \;\middle|\; (z,z') \in [x,y] \times [x,y] \text{ and } z' \geq z\right\}.$$

In the calculation of $F_{a,b}$, we will change variables by defining $\alpha = z' - z$ and $\beta = z' + z$. With these coordinates,

$$T_{x,y} = \left\{(\alpha,\beta) \in \mathbb{R}^2 \;\middle|\; 2x + \alpha \leq \beta \leq 2y - \alpha \text{ and } 0 \leq \alpha \leq y - x\right\}.$$

We calculate:

$$\begin{aligned}
F_{a,b}(M) &= \int_{\mathbb{R}^2} (z'-z)^{a-2}(z'+z)^b \left(\rho_{z,z'}(M)\right)\, d(z,z') \\
&= \frac{1}{2}\int_0^{y-x}\int_{2x+\alpha}^{2y-\alpha} \alpha^{a-2}\beta^b \, d\beta\, d\alpha \\
&= \frac{1}{2(b+1)}\int_0^{y-x} \alpha^{a-2}\left((2y-\alpha)^{b+1}-(2x+\alpha)^{b+1}\right) d\alpha \\
&= \frac{1}{2(b+1)}\int_0^{y-x} \alpha^{a-2}(2y-\alpha)^{b+1} - \alpha^{a-2}(2x+\alpha)^{b+1}\, d\alpha \\
&= \frac{1}{2(b+1)}\left[\sum_{i=0}^{b+1}\left(\frac{(a-2)!\,(b+1)!}{(a-1+i)!\,(b+1-i)!}\right)\alpha^{a-1+i}(2y-\alpha)^{b+1-i}\right. \\
&\qquad \left. -(-1)^i\left(\frac{(a-2)!\,(b+1)!}{(a-1+i)!\,(b+1-i)!}\right)\alpha^{a-1+i}(2x+\alpha)^{b+1-i}\right]_{\alpha=0}^{y-x} \\
&= \sum_{i=0}^{b+1}\left(\frac{1-(-1)^i}{2}\right)\left(\frac{(a-2)!\,b!}{(a-1+i)!\,(b+1-i)!}\right)(y-x)^{a-1+i}(y+x)^{b+1-i} \\
&= \left(\frac{1}{a(a-1)}\right)p_{a,b}(M) + \text{higher order terms}\,.
\end{aligned}$$

Lemma 8 *On $\mathcal{R}^1(k,n)$,*

$$\text{Span}\left(\{F_{\mathbf{a},\mathbf{b}}\}_{(\mathbf{a},\mathbf{b})\in\mathbb{N}_+^n\times\mathbb{N}^n}\right) = \text{Span}\left(\{p_{\mathbf{a},\mathbf{b}}\}_{(\mathbf{a},\mathbf{b})\in\mathbb{N}_+^n\times\mathbb{N}^n}\right).$$

Proof The proof of this lemma will be similar to that of Lemma 7. That is, we prove that for all k, the following holds (on $\mathcal{R}^1(k,n)$):

$$\text{Span}\left(\{F_{\mathbf{a},\mathbf{b}}\}_{\mathbf{a}+\mathbf{b}=k}\right) = \text{Span}\left(\{p_{\mathbf{a},\mathbf{b}}\}_{\mathbf{a}+\mathbf{b}=k}\right)$$

by showing that $F_{\mathbf{a},\mathbf{b}}$ can be written as a (finite) linear combination of $p_{\mathbf{a},\mathbf{b}}$ and higher order terms.

Let $M \in \mathcal{R}^1(k,n)$. Write M as a product $M = \prod_{i=1}^n [x_i, y_i]$. Note that M is determined by vertices $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ with $\mathbf{x} \leq \mathbf{y}$. Let $(\mathbf{a},\mathbf{b}) \in \mathbb{N}_+^n \times \mathbb{N}^n$. As discussed previously, we let $I = \{i \mid a_i = 1\}$ and $J = \{i \mid a_i > 1\}$. We can write the rank function $\rho_{\mathbf{z},\mathbf{z}'}(M)$ as a convenient product:

$$\rho_{\mathbf{z},\mathbf{z}'}(M) = \left(\prod_{i\in I}\mathbb{1}_{[x_i,y_i]}\right)\left(\prod_{i\in J}\mathbb{1}_{T_{x_i,y_i}}\right),$$

where the region T_{x_i,y_i} is as defined in the proof of Lemma 7. This allows us to reduce the calculation of $F_{\mathbf{a},\mathbf{b}}(M)$ to the calculations performed in Lemma 7:

$$\begin{aligned}
F_{\mathbf{a},\mathbf{b}}(M) &= \int_{\mathbb{R}^{n+|J|}} (\mathbf{z}'-\mathbf{z})^{\mathbf{a}_J-2}\,(\mathbf{z})^{\mathbf{b}_I}\,(\mathbf{z}+\mathbf{z}')^{\mathbf{b}_J}\,(\rho_{\mathbf{z},\mathbf{z}'}(M))\;d(\mathbf{z},\mathbf{z}') \\
&= \left(\prod_{i\in I}\int_{\mathbb{R}^1} z_i^{b_i}\left(\mathbb{1}_{[x_i,y_i]}(z_i)\right)\,dz_i\right) \\
&\qquad \left(\prod_{i\in J}\int_{\mathbb{R}^2}(z_i'-z_i)^{a_i-2}(z_i'+z_i)^{b_i}\left(\mathbb{1}_{T_{x_i,y_i}}(z_i,z_i')\right)\,d(z_i,z_i')\right) \\
&= \left(\prod_{i\in I}\left(\frac{1}{2^{b_i}}\right)(y_i-x_i)(y_i+x_i)^{b_i}+\text{higher order terms}\right) \\
&\qquad \left(\prod_{i\in J}\left(\frac{1}{a_i(a_i-1)}\right)(y_i-x_i)^{a_i}(y_i+x_i)^{b_i}+\text{higher order terms}\right) \\
&= \left(\prod_{i\in I}\frac{1}{2^{b_i}}\right)\left(\prod_{i\in J}\frac{1}{a_i(a_i-1)}\right)p_{\mathbf{a},\mathbf{b}}(M)+\text{higher order terms}\,.
\end{aligned}$$

The previous two lemmas allow us to prove the main theorem of this section:

Theorem 3 *On $\mathcal{R}(k,n)$,*

$$\text{Span}\left(\{F_{\mathbf{a},\mathbf{b}}\}_{(\mathbf{a},\mathbf{b})\in\mathbb{N}_+^n\times\mathbb{N}^n}\right)=\text{Span}\left(\{p_{\mathbf{a},\mathbf{b}}\}_{(\mathbf{a},\mathbf{b})\in\mathbb{N}_+^n\times\mathbb{N}^n}\right).$$

Proof The proof of this lemma will be similar to that of Lemma 8. That is, we prove that on $\mathcal{R}(k,n)$,

$$\text{Span}\left(\{F_{\mathbf{a},\mathbf{b}}\}_{\mathbf{a}+\mathbf{b}=k}\right)=\text{Span}\left(\{p_{\mathbf{a},\mathbf{b}}\}_{\mathbf{a}+\mathbf{b}=k}\right)$$

for all k by showing that $F_{\mathbf{a},\mathbf{b}}$ can be written as a (finite) linear combination of $p_{\mathbf{a},\mathbf{b}}$ and higher order terms.

Each element M of $\mathcal{R}(k,n)$ can be viewed as the finite direct sum

$$M=\bigoplus_k M_k,$$

where $M_k\in\mathcal{R}^1(k,n)$. Hence, for $(\mathbf{a},\mathbf{b})\in\mathbb{N}_+^n\times\mathbb{N}^n$, we have that

$$\rho_{\mathbf{z},\mathbf{z}'}(M)=\sum_k\rho_{\mathbf{z},\mathbf{z}'}(M_k).$$

This allows us to calculate $F_{\mathbf{a},\mathbf{b}}(M)$ from the calculation of $F_{\mathbf{a},\mathbf{b}}(M_k)$ from Lemma 8:

$$\begin{aligned}
F_{\mathbf{a},\mathbf{b}}(M) &= \int_{\mathbb{R}^{n+|J|}} (\mathbf{z}'-\mathbf{z})^{\mathbf{a}_J-2}\,(\mathbf{z})^{\mathbf{b}_I}\,(\mathbf{z}+\mathbf{z}')^{\mathbf{b}_J}\,(\rho_{\mathbf{z},\mathbf{z}'}(M))\;d(\mathbf{z},\mathbf{z}') \\
&= \int_{\mathbb{R}^{n+|J|}} (\mathbf{z}'-\mathbf{z})^{\mathbf{a}_J-2}\,(\mathbf{z})^{\mathbf{b}_I}\,(\mathbf{z}+\mathbf{z}')^{\mathbf{b}_J}\left(\sum_k \rho_{\mathbf{z},\mathbf{z}'}(M_k)\right) d(\mathbf{z},\mathbf{z}') \\
&= \sum_k \int_{\mathbb{R}^{n+|J|}} (\mathbf{z}'-\mathbf{z})^{\mathbf{a}_J-2}\,(\mathbf{z})^{\mathbf{b}_I}\,(\mathbf{z}+\mathbf{z}')^{\mathbf{b}_J}\,(\rho_{\mathbf{z},\mathbf{z}'}(M_k))\;d(\mathbf{z},\mathbf{z}') \\
&= \sum_k \left(\left(\prod_{i\in I}\frac{1}{2^{b_i}}\right)\left(\prod_{i\in J}\frac{1}{a_i(a_i-1)}\right)p_{\mathbf{a},\mathbf{b}}(M_k) + \text{higher order terms}\right) \\
&= \left(\prod_{i\in I}\frac{1}{2^{b_i}}\right)\left(\prod_{i\in J}\frac{1}{a_i(a_i-1)}\right)p_{\mathbf{a},\mathbf{b}}(M) + \text{higher order terms}\,.
\end{aligned}$$

5 Group completions and geometric insights

Recall that we have defined spaces $Sp^\infty(\mathcal{J}) = \varinjlim Sp^m(\mathcal{J})$ and $\widetilde{Sp}(\mathbb{R}^n\times\mathbb{R}^n)$ by

$$\widetilde{Sp}(\mathbb{R}^n\times\mathbb{R}^n) = \frac{\coprod_m Sp^m(\mathbb{R}^n\times\mathbb{R}^n)}{\simeq}.$$

Both $Sp^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ and $\widetilde{Sp}(\mathbb{R}^n\times\mathbb{R}^n)$ are commutative monoids generated by $Sp^1(\mathbb{R}^n\times\mathbb{R}^n)$ with monoid operation given by the natural maps

$$+ : Sp^m(\mathbb{R}^n\times\mathbb{R}^n)\otimes Sp^{m'}(\mathbb{R}^n\times\mathbb{R}^n)\to Sp^{m+m'}(\mathbb{R}^n\times\mathbb{R}^n).$$

This section is dedicated to the definition and properties of the group completions $K\left(Sp^\infty(\mathbb{R}^n\times\mathbb{R}^n)\right)$ and $K\left(\widetilde{Sp}(\mathbb{R}^n\times\mathbb{R}^n)\right)$ of these monoids. We show that there is a bijection between elements of $K\left(Sp^\infty(\mathbb{R}^n\times\mathbb{R}^n)\right)$ and (suitably defined) generalized rank invariants. This bijection provides an alternative method of finding $p_{\mathbf{a},\mathbf{b}}(M)$. We also show that $A_{fin}\left[\widetilde{Sp}(\mathbb{AA}^1_n)\right]$ separates the elements of $K\left(\widetilde{Sp}(\mathbb{R}^n\times\mathbb{R}^n)\right)$.

5.1 $K\left(Sp^\infty(\mathbb{R}^n\times\mathbb{R}^n)\right)$ and $K\left(\widetilde{Sp}(\mathbb{R}^n\times\mathbb{R}^n)\right)$.

For a (cancellative) monoid M, we define the group completion $K(M)$ of M by

$$K(M) = \frac{M\times M}{\simeq},$$

where for $(s_1,t_1),(s_2,t_2)\in M\times M$, we define an equivalence $(s_1,t_1)\simeq(s_2,t_2)$ if and only if $s_1+t_2 = s_1+t_1$. It is implied that the element $(s,t)\in K(M)$ should be thought of as "$s-t$".

We have discussed in Lemma 2 and Remark 3 that $Sp^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ can be identified with a certain subset of the $\mathbb{R}$-points of $Sp^\infty(\mathbb{AA}^1_n)$. We now discuss the extension of this identification to $K(Sp^\infty(\mathbb{R}^n \times \mathbb{R}^n))$.

For any $X \in Sp^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, we may associate an $\mathbb{R}$-point φ^*_X of $Sp^\infty(\mathbb{AA}^1_n)$. The underlying homomorphism $\varphi_X : A_{fin}[Sp^\infty(\mathbb{AA}^1_n)] \to \mathbb{R}$ has the potential of producing geometrically meaningful summaries of X, especially in the case where X has arisen (in the sense of Corollary 1) from an element of $\mathcal{R}(k,n)$. Indeed, if we fix some $f_i \in A[Sp^\infty(\mathbb{AA}^1_n)]$, the numbers $\varphi_X(f_i)$ might provide insightful information about X (we explore this idea further in Section 6). We now extend this methodology to the case $X \in K(Sp^\infty(\mathbb{R}^n \times \mathbb{R}^n))$ and $X \in K(\widetilde{Sp}(\mathbb{R}^n \times \mathbb{R}^n))$.

For $(X_1, X_2) \in Sp^\infty(\mathbb{R}^n \times \mathbb{R}^n) \times Sp^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, we define a ring homomorphism

$$\varphi_{(X_1,X_2)} : A_{fin}[Sp^\infty(\mathbb{AA}^1_n)] \to \mathbb{R}$$

on the generators $p_{\mathbf{a},\mathbf{b}}$ of $A_{fin}[Sp^\infty(\mathbb{AA}^1_n)]$ (defined in Lemma 3) by

$$\varphi_{(X_1,X_2)}(p_{\mathbf{a},\mathbf{b}}) = \varphi_{X_1}(p_{\mathbf{a},\mathbf{b}}) - \varphi_{X_2}(p_{\mathbf{a},\mathbf{b}}).$$

Proposition 4 *The map*

$$Sp^\infty(\mathbb{R}^n \times \mathbb{R}^n) \times Sp^\infty(\mathbb{R}^n \times \mathbb{R}^n) \longrightarrow \left(A_{fin}[Sp^\infty(\mathbb{AA}^1_n)] \to \mathbb{R}\right)$$

defined by

$$(X_1, X_2) \mapsto \varphi_{(X_1,X_2)}$$

factors through $K(Sp^\infty(\mathbb{R}^n \times \mathbb{R}^n))$, and can thus be viewed as a map

$$K(Sp^\infty(\mathbb{R}^n \times \mathbb{R}^n)) \longrightarrow \left(A_{fin}[Sp^\infty(\mathbb{AA}^1_n)] \to \mathbb{R}\right).$$

Proof It suffices to observe that for $X_1, X_2, Y \in Sp^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, the following equality holds (by definition):

$$\varphi_{(X_1+Y,\ X_2+Y)} = \varphi_{(X_1,X_2)}.$$

By combining Theorem 2 and Proposition 4, we obtain:

Proposition 5 *The map*

$$Sp^\infty(\mathbb{R}^n \times \mathbb{R}^n) \longrightarrow \left(A_{fin}[Sp^\infty(\mathbb{AA}^1_n)] \to \mathbb{R}\right)$$

defined by

$$X \mapsto \varphi_X$$

descends to a map

$$\widetilde{Sp}(\mathbb{R}^n \times \mathbb{R}^n) \longrightarrow \left(A_{fin}[\widetilde{Sp}(\mathbb{AA}^1_n)] \to \mathbb{R}\right).$$

Moreover, the map

$$\widetilde{Sp}(\mathbb{R}^n \times \mathbb{R}^n) \times \widetilde{Sp}(\mathbb{R}^n \times \mathbb{R}^n) \longrightarrow \left(A_{fin}[\widetilde{Sp}(\mathbb{AA}^1_n)] \to \mathbb{R}\right)$$

defined by

$$(X_1, X_2) \mapsto \varphi_{(X_1,X_2)}$$

factors through $K(\widetilde{Sp}(\mathbb{R}^n \times \mathbb{R}^n))$, and can thus be viewed as a map

$$K(\widetilde{Sp}(\mathbb{R}^n \times \mathbb{R}^n)) \longrightarrow \left(A_{fin}[\widetilde{Sp}(\mathbb{AA}^1_n)] \to \mathbb{R}\right).$$

5.2 Injectivity results

Having defined maps φ_X for X in various spaces (e.g., $Sp^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $K(\widetilde{Sp}(\mathbb{R}^n \times \mathbb{R}^n))$), we now show that $\varphi_X \neq \varphi_Y$ given $X \neq Y$. These lemmata generalize a result of [2].

Lemma 9 *Let $X, Y \in Sp^m(\mathbb{R}^n \times \mathbb{R}^n)$ such that $X \neq Y$. Then $\varphi_X \neq \varphi_Y$ (as maps $A[Sp^m(\mathbb{AA}^1_n)] \to \mathbb{R}$).*

Proof Let $Z = \{(\mathbf{z}_i, \mathbf{w}_i)\}_{i=1}^m$. Let $f \in A[Sp^m(\mathbb{AA}^1_n)]$ be defined by

$$f(\mathbf{x}, \mathbf{y}) = \prod_{\sigma \in S_m} \left(\sum_{i=1}^m \sum_{j=1}^n \left(x_{\sigma(i),j} - z_{i,j}\right)^2 + \left(y_{\sigma(i),j} - w_{i,j}\right)^2 \right).$$

Then for any $W \in Sp^m(\mathbb{R}^n \times \mathbb{R}^n)$, $\varphi_W(f) = 0$ if and only if $W = Z$.

Remark 6 We have presented a concrete proof to Lemma 9. However, this lemma also follows from the Nullstellensatz and the fact that for any real variety V, the affine coordinate ring $A[V]$ separates the (real) points of V.

Lemma 10 *Let $X, Y \in Sp^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that $X \neq Y$. Then $\varphi_X \neq \varphi_Y$ (as maps $A_{fin}[Sp^\infty(\mathbb{AA}^1_n)] \to \mathbb{R}$).*

Proof We may represent X and Y by elements $X_{fin}, Y_{fin} \in Sp^m(\mathbb{R}^n \times \mathbb{R}^n)$. By Lemma 9, there exists $f_{fin} \in A[Sp^m(\mathbb{AA}^1_n)]$ such that $\varphi_{X_{fin}}(f_{fin}) \neq \varphi_{Y_{fin}}(f_{fin})$. By Lemma 3, we may write f_{fin} as a finite sum $f_{fin} = \sum_i p_{\mathbf{a}_i, \mathbf{b}_i, m}$. Define an element $f \in A_{fin}[Sp^\infty(\mathbb{AA}^1_n)]$ by $f = \sum_i p_{\mathbf{a}_i, \mathbf{b}_i}$. Then

$$\varphi_X(f) = \varphi_{X_{fin}}(f_{fin}) \neq \varphi_{Y_{fin}}(f_{fin}) = \varphi_Y(f).$$

Lemma 11 *Let $X, Y \in K(Sp^\infty(\mathbb{R}^n \times \mathbb{R}^n))$ such that $X \neq Y$. Then $\varphi_X \neq \varphi_Y$ (as maps $A_{fin}[Sp^\infty(\mathbb{AA}^1_n)] \to \mathbb{R}$).*

Proof Choose coset representatives for X and Y in $Sp^\infty(\mathbb{R}^n \times \mathbb{R}^n) \times Sp^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and write $X = (X_+, X_-)$ and $Y = (Y_+, Y_-)$. Since $X_+ + Y_- \neq Y_+ + X_-$, there exists by Lemma 10 some $f' \in A_{fin}[Sp^\infty(\mathbb{AA}^1_n)]$ such that $\varphi_{X_+ + Y_-}(f') \neq \varphi_{Y_+ + X_-}(f')$.

Since $A_{fin}[Sp^\infty(\mathbb{AA}^1_n)]$ is generated as an algebra by the power sums $p_{\mathbf{a},\mathbf{b}}$, there exists a power sum $f = p_{\mathbf{a}_0, \mathbf{b}_0} \in A_{fin}[Sp^\infty(\mathbb{AA}^1_n)]$ such that $\varphi_{X_+ + Y_-}(f) \neq \varphi_{Y_+ + X_-}(f)$. Because f is a power sum,

$$\varphi_{X_+ + Y_-}(f) = \varphi_{X_+}(f) + \varphi_{Y_-}(f) \qquad \text{and} \qquad \varphi_{Y_+ + X_-}(f) = \varphi_{Y_+}(f) + \varphi_{X_-}(f).$$

Combining these equalities and inequalities shows that

$$\varphi_{X_+}(f) - \varphi_{X_-}(f) \neq \varphi_{Y_+}(f) - \varphi_{Y_-}(f),$$

and hence

$$\begin{aligned} \varphi_X(f) &= \varphi_{(X_+, X_-)}(f) \\ &= \varphi_{X_+}(f) - \varphi_{X_-}(f) \\ &\neq \varphi_{Y_+}(f) - \varphi_{Y_-}(f) \\ &= \varphi_{(Y_+, Y_-)}(f) \\ &= \varphi_Y(f). \end{aligned}$$

Lemma 12 *Let $X, Y \in \widetilde{Sp}(\mathbb{R}^n \times \mathbb{R}^n)$ such that $X \neq Y$. Then $\varphi_X \neq \varphi_Y$ (as maps $A_{fin}[\widetilde{Sp}(\mathbb{AA}^1_n)] \to \mathbb{R}$).*

Proof By the definition of $\widetilde{Sp}(\mathbb{R}^n \times \mathbb{R}^n)$, we can represent any $Z \in \widetilde{Sp}(\mathbb{R}^n \times \mathbb{R}^n)$ by some $Z_{fin} \in Sp^{m'}(\mathbb{R}^n \times \mathbb{R}^n)$ for some m'. Choose m minimal and elements $X_{fin}, Y_{fin} \in Sp^m(\mathbb{R}^n \times \mathbb{R}^n)$ so that we can represent X by X_{fin} and Y by Y_{fin}. By Proposition 1, it now suffices to produce some $f_{fin} \in R^m_n \cap A[Sp^m(\mathbb{AA}^1_n)]$ such that $\varphi_{X_{fin}}(f_{fin}) \neq \varphi_{Y_{fin}}(f_{fin})$. Indeed, if $f \in A_{fin}[\widetilde{Sp}(\mathbb{AA}^1_n)]$ restricts to f_{fin}, then

$$\varphi_X(f) = \varphi_{X_{fin}}(f_{fin}) \neq \varphi_{Y_{fin}}(f_{fin}) = \varphi_Y(f).$$

Let

$$g_{fin} = \prod_{i=1}^{m} \prod_{j=1}^{n} (y_{ij} - x_{ij}).$$

Note that $g_{fin} \in A[Sp^m(\mathbb{AA}^1_n)]$. By the minimality of m, it is not the case that $\varphi_{X_{fin}}(g_{fin}) = \varphi_{Y_{fin}}(g_{fin}) = 0$.

If $\varphi_{X_{fin}}(g_{fin}) \neq \varphi_{Y_{fin}}(g_{fin})$, then we may let $f_{fin} = g_{fin}$. On the other hand, if $\varphi_{X_{fin}}(g_{fin}) = \varphi_{Y_{fin}}(g_{fin}) \neq 0$, then by Lemma 9, there exists $h_{fin} \in A[Sp^m(\mathbb{AA}^1_n)]$ such that $\varphi_{X_{fin}}(h_{fin}) \neq \varphi_{Y_{fin}}(h_{fin})$. In this case, let $f_{fin} = g_{fin}h_{fin}$, and observe that

$$\begin{aligned}
\varphi_{X_{fin}}(f_{fin}) &= \varphi_{X_{fin}}(g_{fin}h_{fin}) \\
&= \varphi_{X_{fin}}(g_{fin})\varphi_{X_{fin}}(h_{fin}) \\
&= \varphi_{Y_{fin}}(g_{fin})\varphi_{X_{fin}}(h_{fin}) \\
&\neq \varphi_{Y_{fin}}(g_{fin})\varphi_{Y_{fin}}(h_{fin}) \\
&= \varphi_{Y_{fin}}(g_{fin}h_{fin}) \\
&= \varphi_{Y_{fin}}(f_{fin}).
\end{aligned}$$

Regardless of which definition we choose for f_{fin}, we have that $f_{fin} \in R^m_n$ by Proposition 2 and that $f_{fin} \in A[Sp^m(\mathbb{AA}^1_n)]$ because its constituent factors are elements of $A[Sp^m(\mathbb{AA}^1_n)]$.

Lemma 13 *Let $X, Y \in K(\widetilde{Sp}(\mathbb{R}^n \times \mathbb{R}^n))$ such that $X \neq Y$. Then $\varphi_X \neq \varphi_Y$ (as maps $A_{fin}[\widetilde{Sp}(\mathbb{AA}^1_n)] \to \mathbb{R}$).*

Proof The proof of Lemma 13 from Lemma 12 is identical to the proof of Lemma 11 from Lemma 10.

5.3 Generalized rank invariants.

The definition of the invariants $F_{\mathbf{a},\mathbf{b}}$ of Section 4.1 relies quite heavily on the rank invariants $\rho_{\mathbf{u},\mathbf{v}} : \mathcal{M}(k,n) \to \mathbb{N}$. Having fixed $M \in \mathcal{M}(k,n)$, we may view the rank invariant $\rho_{\mathbf{u},\mathbf{v}}(M)$ as a function of its subscripts

$$\rho_{-,-}(M) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{N}.$$

Note that $\rho_{\mathbf{u},\mathbf{v}}(M) = 0$ unless $\mathbf{u} \leq \mathbf{v}$. Moreover, because we require elements of $\mathcal{M}(k,n)$ to be finite, there exist $\mathbf{u}_0, \mathbf{v}_0 \in \mathbb{Z}^n$ such that for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, we have that $\rho_{\mathbf{u},\mathbf{v}}(M) = 0$ unless $\mathbf{u} \geq \mathbf{u}_0$ and $\mathbf{v} \leq \mathbf{v}_0$. Finally, even though we may evaluate $\rho_{\mathbf{u},\mathbf{v}}(M)$ for arbitrary $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, the values $\rho_{\mathbf{u},\mathbf{v}}(M)$ are fully determined for $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^n$ (since, by definition, M is an integral persistence module).

These observations inspire the following definition:

Definition 5 A generalized rank invariant is a function $\rho_{-,-} : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ such that:

1. $\rho_{\mathbf{u},\mathbf{v}} = 0$ unless $\mathbf{u} \leq \mathbf{v}$.
2. There exist $\mathbf{u}_0, \mathbf{v}_0 \in \mathbb{Z}^n$ such that $\rho_{\mathbf{u},\mathbf{v}} = 0$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^n$ except if $\mathbf{u} \geq \mathbf{u}_0$ and $\mathbf{v} \leq \mathbf{v}_0$.

For the remainder of Section 5.3, we define

$$J(n) = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}^n \times \mathbb{Z}^n \mid \mathbf{x} \leq \mathbf{y}\}.$$

For $(\mathbf{x}, \mathbf{y}) \in J(n)$, we can define a generalized rank invariant $\rho_{-,-}((\mathbf{x}, \mathbf{y}))$ by

$$\rho_{\mathbf{u},\mathbf{v}}((\mathbf{x}, \mathbf{y})) = \begin{cases} 1 & \mathbf{x} \leq \mathbf{u} \leq \mathbf{v} \leq \mathbf{y} \\ 0 & \text{otherwise} \end{cases}.$$

We may symmetrize the above definition: for $X = \sum_{i=1}^m (\mathbf{x}_i, \mathbf{y}_i) \in Sp^\infty(J(n))$, define a generalized rank invariant $\rho_{-,-}(X)$ by

$$\rho_{\mathbf{u},\mathbf{v}}(X) = \sum_{i=1}^m \rho_{\mathbf{u},\mathbf{v}}((\mathbf{x}_i, \mathbf{y}_i)).$$

Finally, for $X \in K(Sp^\infty(J(n)))$ represented by $(X_+, X_-) \in Sp^\infty(J(n)) \times Sp^\infty(J(n))$, define a generalized rank invariant $\rho_{-,-}(X)$ by

$$\rho_{\mathbf{u},\mathbf{v}}(X) = \rho_{\mathbf{u},\mathbf{v}}(X_+) - \rho_{\mathbf{u},\mathbf{v}}(X_-).$$

Lemma 14 *The map from $K(Sp^\infty(J(n)))$ to the set of generalized rank invariants defined by $X \mapsto \rho_{-,-}(X)$ is injective.*

Proof Fix a total order $\preceq$ on $J(n)$ such that $(\mathbf{x}, \mathbf{y}) \preceq (\mathbf{z}, \mathbf{w})$ if $\mathbf{x} \leq \mathbf{z}$ or if $\mathbf{x} = \mathbf{z}$ and $\mathbf{y} \geq \mathbf{w}$.

Let $X \in K(Sp^\infty(J(n)))$ such that $\rho_{\mathbf{u},\mathbf{v}}(X) = 0$ for all $(\mathbf{u}, \mathbf{v}) \in J(n)$. If $X \neq 0$, then we may write $X = \sum_{i=1}^m c_i(\mathbf{x}_i, \mathbf{y}_i)$, where $(\mathbf{x}_i, \mathbf{y}_i) \prec (\mathbf{x}_j, \mathbf{y}_j)$ for $i < j$, $c_i \in \mathbb{Z}^*$, and $m \geq 1$. Then by the minimality of $(\mathbf{x}_1, \mathbf{y}_1)$ under the order $\preceq$, we have that $\rho_{\mathbf{x}_1,\mathbf{y}_1}(X) = c_1 \neq 0$, which is a contradiction.

Lemma 15 *The map from $K(Sp^\infty(J(n)))$ to the set of generalized rank invariants defined by $X \mapsto \rho_{-,-}(X)$ is surjective.*

Proof As in the proof of Lemma 14, we choose a total order $\preceq$ on $J(n)$ such that $(\mathbf{x},\mathbf{y}) \preceq (\mathbf{z},\mathbf{w})$ if $\mathbf{x} \leq \mathbf{z}$ or if $\mathbf{x} = \mathbf{z}$ and $\mathbf{y} \geq \mathbf{w}$. Let $\rho_{-,-}$ be a generalized rank invariant. We now describe an algorithm that produces an $X \in K(Sp^{\infty}(J(n)))$ such that $\rho_{-,-} = \rho_{-,-}(X)$.

If $\rho_{\mathbf{u},\mathbf{v}} = 0$ for all $(\mathbf{u},\mathbf{v}) \in \mathbb{Z}^n \times \mathbb{Z}^n$, let $X = 0$.

Otherwise, by conditions (1) and (2) of Definition 5, there exist $\mathbf{u}_0, \mathbf{v}_0 \in \mathbb{Z}^n$ such that $\rho_{\mathbf{u},\mathbf{v}} = 0$ for all $\mathbf{u},\mathbf{v} \in \mathbb{Z}^n$ except if $\mathbf{u}_0 \leq \mathbf{u} \leq \mathbf{v} \leq \mathbf{v}_0$. Note that the set all such $(\mathbf{u},\mathbf{v}) \in \mathbb{Z}^n \times \mathbb{Z}^n$ such that $\mathbf{u}_0 \leq \mathbf{u} \leq \mathbf{v} \leq \mathbf{v}_0$ is finite. Choose $(\mathbf{x},\mathbf{y})$, minimal under the total order $\preceq$, such that $\rho_{\mathbf{x},\mathbf{y}} \neq 0$. We complete the proof of Lemma 15 by induction on

$$\left|\{(\mathbf{u},\mathbf{v}) \in \mathbb{Z}^n \times \mathbb{Z}^n \mid (\mathbf{x},\mathbf{y}) \preceq (\mathbf{u},\mathbf{v}) \text{ and } \mathbf{u}_0 \leq \mathbf{u} \leq \mathbf{v} \leq \mathbf{v}_0\}\right|.$$

Define a generalized rank invariant $\rho'_{-,-}$ by

$$\rho'_{-,-} = \rho_{-,-} - \rho_{\mathbf{x},\mathbf{y}} \cdot \rho_{-,-}((\mathbf{x},\mathbf{y})).$$

Then $\rho'_{\mathbf{u},\mathbf{v}} = 0$ for all $(\mathbf{u},\mathbf{v}) \preceq (\mathbf{x},\mathbf{y})$. Moreover, $\rho_{\mathbf{u},\mathbf{v}} = 0$ for all $\mathbf{u},\mathbf{v} \in \mathbb{Z}^n$ except if $\mathbf{u}_0 \leq \mathbf{u} \leq \mathbf{v} \leq \mathbf{v}_0$. By induction, there exists $X' \in K(Sp^{\infty}(J(n)))$ such that $\rho'_{-,-} = \rho_{-,-}(X')$. Let $X = X' + \rho_{\mathbf{x},\mathbf{y}} \cdot (\mathbf{x},\mathbf{y})$. Then $\rho_{-,-} = \rho_{-,-}(X)$.

Theorem 4 *The map from $K(Sp^{\infty}(J(n)))$ to the set of generalized rank invariants defined by $X \mapsto \rho_{-,-}(X)$ is a bijection.*

Proof Injectivity follows Lemma 14. Surjectivity follows from Lemma 15.

We may now use Theorem 4 to assist in the computation of $p_{\mathbf{a},\mathbf{b}}(M)$ for arbitrary $M \in \mathcal{M}(k,n)$. Recall that by Theorem 3, $p_{\mathbf{a},\mathbf{b}}(M)$ is a linear combination of the invariants $F_{\mathbf{a}',\mathbf{b}'}(M)$ (introduced in Section 4.1), which are defined solely in terms of the rank invariants $\rho_{\mathbf{u},\mathbf{v}}(M)$. That is, for $M \in \mathcal{M}(k,n)$ and $X \in K(Sp^{\infty}(J(n)))$, we will have $p_{\mathbf{a},\mathbf{b}}(M) = p_{\mathbf{a},\mathbf{b}}(X)$ for all $\mathbf{a},\mathbf{b}$ provided that $\rho_{\mathbf{u},\mathbf{v}}(M) = \rho_{\mathbf{u},\mathbf{v}}(X)$ for all $\mathbf{u},\mathbf{v}$.

Thus, to compute $p_{\mathbf{a},\mathbf{b}}(M)$, we first use the rank invariants $\rho_{-,-}(M)$ and the algorithm presented in the proof of Lemma 15 to find some $X \in K(Sp^{\infty}(J(n)))$ such that $\rho_{-,-}(M) = \rho_{-,-}(X)$. We may now calculate $p_{\mathbf{a},\mathbf{b}}(M)$ using the equality $p_{\mathbf{a},\mathbf{b}}(M) = p_{\mathbf{a},\mathbf{b}}(X)$.

Remark 7 The results of this section also hold when working in the real (rather than integral) formulation of multidimensional persistence. However, in the real case, we would need to replace $\mathbb{Z}^n$ with $\mathbb{R}^n$ throughout this subsection and add a finiteness condition to the definition of a generalized rank invariant that ensures the finite termination of the algorithm in Lemma 15.

6 Recovering the point cloud

Section 3 examines the K-finite sections of $\mathcal{R}(k,n)$, and Section 4 provides a concrete method for calculating them. In this section, we provide a way to recover information about $M \in \mathcal{R}(k,n)$ (including M itself) from the invariants $p_{\mathbf{a},\mathbf{b}}(M)$ (where, as before, $(\mathbf{a},\mathbf{b}) \in \mathbb{N}_+^n \times \mathbb{N}^n$). In fact, these techniques provide a theoretical method of

recovering any $X \in \widetilde{Sp}(J(n))$ from the values $p_{\mathbf{a},\mathbf{b}}(X)$. In practice, the numbers encountered in the limit formulae given in this section can be too large and unwieldy to allow for the recovery of $X \in \widetilde{Sp}(J(n))$ from the values $p_{\mathbf{a},\mathbf{b}}(X)$. Nonetheless, these results justify our previous assertion that the values $p_{\mathbf{a},\mathbf{b}}(M)$ can provide useful geometric information about the multidimensional persistence module M.

First, we prove two technical lemmas:

Lemma 16 *Let $a_1, \ldots, a_n \in \mathbb{R}$. Assume that $|a_1| < 1$. Then*

$$\lim_{k\to\infty}\left(\prod_{j=1}^{n} a_j^{k^{n+1-j}}\right) = 0.$$

Proof We prove this lemma by induction on n. The case $n = 1$ is trivial. If the lemma is true when $n = m - 1$, it is also true for the case $n = m$, since

$$\lim_{k\to\infty}\left(\prod_{j=1}^{m-1} a_j^{k^{(m-1)+1-j}}\right) = 0$$

implies

$$\lim_{k\to\infty}\left(\prod_{j=1}^{m} a_j^{k^{m+1-j}}\right) = \lim_{k\to\infty}\left(a_m\left(\prod_{j=1}^{m-1} a_j^{k^{(m-1)+1-j}}\right)\right)^k = 0.$$

Lemma 17 *Let $w_i \in \mathbb{R}_{>0}$ and $\mathbf{z}_i \in \mathbb{R}^n_{>0}$ for $1 \leq i \leq m$. Further assume that the $\mathbf{z}_i$ are in decreasing lexicographic order. Then*

$$\lim_{k\to\infty}\left(\sum_{i=1}^{n}\left(w_i\prod_{j=1}^{n}\left(\frac{z_{i,j}}{z_{1,j}}\right)^{k^{n+1-j}}\right)\right)^{1/k} = 1.$$

Proof It suffices to show that there exist uniform (positive) lower and upper bounds on the quantity

$$Q_k = \sum_{i=1}^{n}\left(w_i\prod_{j=1}^{n}\left(\frac{z_{i,j}}{z_{1,j}}\right)^{k^{n+1-j}}\right).$$

Note that, as a sum of positive terms, Q_k is greater than its first term w_1. It remains to show that the Q_k are uniformly bounded from above.

Since Q_k is the finite sum of terms of the form

$$T_{k,i} = w_i\prod_{j=1}^{n}\left(\frac{z_{i,j}}{z_{1,j}}\right)^{k^{n+1-j}},$$

it suffices to bound the $T_{k,i}$ uniformly from above. Because the $\mathbf{z}_i$ are decreasingly lexicographically ordered, if $T_{k,i}$ is not identically equal to w_i, then there must be

some least j, denoted j_i, such that $z_{i,j} < z_{1,j}$. Note that $0 < \frac{z_{i,j_i}}{z_{1,j_i}} < 1$. Moreover, for all $j < j_i$, we must have $z_{i,j} = z_{1,j}$, and so

$$T_{k,i} = w_i \prod_{j=i_j}^{n} \left(\frac{z_{i,j}}{z_{1,j}}\right)^{k^{n+1-j}} = w_i \prod_{j=1}^{n+1-i_j} \left(\frac{z_{i,j+i_j-1}}{z_{1,j+i_j-1}}\right)^{k^{n+2-i_j-j}}.$$

This last term now falls under the auspices of Lemma 16; thus, $\lim_{k\to\infty} T_{k,i} = 0$ for all i.

The next theorem provides a means of recovering information about the cubical summands of some $M \in \mathcal{R}(k,n)$ from the values $p_{\mathbf{a},\mathbf{b}}(M)$.

Theorem 5 *Let $\mathcal{A}$ be an algebra of nonnegative real functions on a set $\mathcal{X}$. Suppose $f_1,\ldots,f_n \in \mathcal{A}$ and $x_1,\ldots,x_m \in \mathcal{X}$. Further assume that the x_i are ordered so that the vectors $\{[f_j(x_i)]_{j\geq 1}\}_i \in \mathbb{R}^n$ are arranged in decreasing lexicographic order. Furthermore, assume that we only have access to the values*

$$\left\{\sum_{i=1}^{m} f(x_i)\right\}_{f\in\mathcal{A}}.$$

Then we can recover the set $\{f_j(x_i)\}$ inductively via the formula

$$f_j(x_i) = \lim_{k\to\infty} \left(\frac{\sum_{i=1}^{m} \left(\prod_{j'\leq j} \left(f_{j'}(x_i)\right)^{k^{j+1-j'}}\right) - \sum_{i'<i} \left(\prod_{j'\leq j} \left(f_{j'}(x_i)\right)^{k^{j+1-j'}}\right)}{\prod_{j'<j} \left(f_{j'}(x_i)\right)^{k^{j+1-j'}}} \right)^{1/k}.$$

Letting $\mathcal{A}$ equal the algebra generated by the $p_{\mathbf{a},\mathbf{b}}$ for $(\mathbf{a},\mathbf{b}) \in \mathbb{N}^n_{>0} \times \mathbb{N}^n$ and letting $\mathcal{X}$ equal $\mathcal{R}^1(k,n)$ will allow us to recover the values of various functions on the individual cubes that constitute an element M of $\mathcal{R}(k,n)$. For example, letting $f_1 = p_{\mathbf{1},\mathbf{0}}$ will allow us to determine in decreasing order the volumes of the cubes which constitute M.

Proof Theorem 5 follows from Lemma 17 by setting $w_i = 1$ and $z_{i,j} = f_j(x_i)$. The quantity $\sum_{i=1}^{m} \left(\prod_{j'\leq j} \left(f_{j'}(x_i)\right)^{k^{j+1-j'}}\right)$ is available to us by hypothesis, and all other values $f_{j'}(x_i)$ of the limit above can be determined by induction.

Theorem 5 allows us to recover the values of any function in the ring of graded-finite global sections of $\widetilde{Sp}(\mathbb{R}^n \times \mathbb{R}^n)$. However, the individual coordinate functions η_i and ξ_i are not, unfortunately, elements of this ring. The following variant of Theorem 5 provides a solution to this quandary.

Theorem 6 *Let $M \in \mathcal{R}(k,n)$, and assume that*

$$M = \bigoplus_{i=1}^{m} M_i,$$

for $M_i \in \mathcal{R}^1(k,n)$. Assume that the M_i are ordered so that the vectors

$$\begin{bmatrix} p_{\mathbf{1},\mathbf{0}}(M_i) \\ \eta_j(M_i) \\ \xi_j(M_i) \end{bmatrix} \in \mathbb{R}^{2n+1}$$

are arranged in decreasing lexicographic order. Let

$$f_j(M_i) = (p_{\mathbf{1},\mathbf{0}}(M_i))^{k^{j+1}} \left(\prod_{j' \leq j} \left(\eta_{j'}(M_i) \right)^{k^{j+1-j'}} \right)$$

$$g_j(M_i) = (p_{\mathbf{1},\mathbf{0}}(M_i))^{k^{n+j+1}} \left(\prod_{j'=1}^{n} \left(\eta_{j'}(M_i) \right)^{k^{n+j+1-j'}} \right) \left(\prod_{j' \leq j} \left(\xi_{j'}(M_i) \right)^{k^{j+1-j'}} \right).$$

Then we can recover the values $\eta_j(M_i)$ and $\xi_j(M_i)$ inductively from the values that $A\left[\widetilde{Sp}(\mathbb{R}^n \times \mathbb{R}^n)\right]$ takes on M via the following:

$$\eta_j(M_i) = \lim_{k \to \infty} \left(\frac{\sum_{i'=1}^{m} f_j(M_{i'}) - \sum_{i'<i} f_j(M_{i'})}{(p_{\mathbf{1},\mathbf{0}}(M_i))^{k^{j+1}} \prod_{j'<j} \left(\eta_{j'}(M_i) \right)^{k^{j+1-j'}}} \right)^{1/k}$$

$$\xi_j(M_i) = \lim_{k \to \infty} \left(\frac{\sum_{i'=1}^{m} g_j(M_{i'}) - \sum_{i'<i} g_j(M_{i'})}{(p_{\mathbf{1},\mathbf{0}}(M_i))^{k^{n+j+1}} \left(\prod_{j'=1}^{n} \left(\eta_{j'}(M_i) \right)^{k^{n+j+1-j'}} \right) \prod_{j'<j} \left(\xi_{j'}(M_i) \right)^{k^{j+1-j'}}} \right)^{1/k}.$$

Proof We can determine the necessary values of $p_{\mathbf{1},\mathbf{0}}(M_i)$ using Theorem 5. The values $\sum_{i'=1}^{m} f_j(M_{i'})$ and $\sum_{i'=1}^{m} g_j(M_{i'})$ are in the ring of algebraic functions on $\mathcal{R}(k,n)$. All other values of $\eta_j(M_i)$ and $\xi_j(M_i)$ in the limits shown above can be determined by induction.

The evaluation of the limits follows from Lemma 17 with $w_i = 1$, $z_{i,1} = p_{\mathbf{1},\mathbf{0}}(x_i)$, $z_{i,j+1} = \eta_j(x_i)$ for $1 \leq j \leq n$, and $z_{i,j+1} = \xi_j(x_i)$ for $n+1 \leq j \leq 2n$.

Corollary 2 *A one dimensional persistence module M is completely recoverable from the values $\{p_{a,b}(M)\}_{(a,b) \in \mathbb{N}_{>0} \times \mathbb{N}}$ (equivalently, from the values $\{F_{a,b}(M)\}_{(a,b) \in \mathbb{N}_{>0} \times \mathbb{N}}$).*

Proof This follows from the fact that $\mathcal{M}(k,1) = \mathcal{R}(k,1)$.

The previous two theorems merely give an idea of how one may use Lemma 17 to recover information about an element $M \in \mathcal{R}(k,n)$. One may, of course, use Lemma 17 with other functions defined on $\mathcal{R}(k,n)$.

6.1 The biggest box

The previous results discussed in this section assume that we are working with the class of persistent cubes $\mathcal{R}(k,n)$ rather than the more general class of multidimensional persistence modules $\mathcal{M}(k,n)$. We now discuss how our previous results might be extended to $\mathcal{M}(k,n)$. As mentioned previously, Theorem 4 allows us to construct some $X \in K(Sp^{\infty}(J(n)))$ such that $\rho_{-,-}(M) = \rho_{-,-}(X)$. We may use methods similar to those in Theorem 5 and Theorem 6 to extract information about the most prominent summand(s) of X from the values $p_{\mathbf{a},\mathbf{b}}(X)$. Like Theorem 6, Theorem 7 is not very attractive from the perspective of computational feasibility. However, the theorem supports our opinion that the values $p_{\mathbf{a},\mathbf{b}}(M)$ provide insightful geometric information about $M \in \mathcal{M}(k,n)$.

We begin by presenting a variant of Lemma 17 which does not make any assumptions concerning non-negativity.

Lemma 18 *Let $w_i = \pm 1$ and $\mathbf{z}_i \in \mathbb{R}^n_{>0}$ for $1 \leq i \leq m$. Assume that the values $\mathbf{z}_i$ are in decreasing lexicographic order and that $w_1 = 1$. Let i_0 denote the least i such that $w_i = -1$. Assume that for all $i \geq i_0$ we have that $z_{i,1} < z_{1,1}$. Then*

$$\lim_{k\to\infty}\left(\sum_{i=1}^{n}\left(w_i\prod_{j=1}^{n}\left(\frac{z_{i,j}}{z_{1,j}}\right)^{k^{n+1-j}}\right)\right)^{1/k} = 1.$$

Proof This proof will mirror that of Lemma 17. It suffices to show that there exists some k_0 such that there exist uniform (positive) lower and upper bounds on the quantity

$$Q_k = \sum_{i=1}^{n}\left(w_i\prod_{j=1}^{n}\left(\frac{z_{i,j}}{z_{1,j}}\right)^{k^{n+1-j}}\right)$$

for $k \geq k_0$. By Lemma 16,

$$\lim_{k\to\infty}\sum_{i=i_0}^{n}\left(w_i\prod_{j=1}^{n}\left(\frac{z_{i,j}}{z_{1,j}}\right)^{k^{n+1-j}}\right) = 0.$$

Since the first summand of Q_k is 1 and the first $i_0 - 1$ summands of Q_k are positive, it follows that there exists k_0 such that the Q_k are uniformly bounded below by some positive quantity for $k \geq k_0$.

To show that the Q_k are uniformly bounded from above, we first note that

$$Q_k = \sum_{i=1}^{n}\left(w_i\prod_{j=1}^{n}\left(\frac{z_{i,j}}{z_{1,j}}\right)^{k^{n+1-j}}\right) \leq \sum_{i=1}^{n}\left(\prod_{j=1}^{n}\left(\frac{z_{i,j}}{z_{1,j}}\right)^{k^{n+1-j}}\right).$$

This latter quantity is uniformly bounded above by the proof of Lemma 17.

The following theorem provides a recipe for recovering the biggest box in a box decomposition of some $M \in \mathcal{M}(k,n)$. In fact, it shows that, if we provide a box decomposition $X \in K\left(\widetilde{Sp}(J(n))\right)$ of M, then we can use the values $p_{\mathbf{a},\mathbf{b}}(M)$ to recover all positive summands of X bigger than the largest negative summand of X.

Theorem 7 *Let $M \in \mathcal{M}(k,n)$. Let $X \in K\left(\widetilde{Sp}(J(n))\right)$ have the same rank invariants as M (i.e., $\rho_{-,-}(M) = \rho_{-,-}(X)$), and assume that*

$$X = \bigoplus_{i=1}^{m} c_i X_i,$$

where $c_i = \pm 1$, m is minimal, and $X_i \in \mathcal{R}^1(k,n)$. Assume that the X_i are ordered so that the vectors

$$\begin{bmatrix} |p_{\mathbf{1},\mathbf{0}}(X_i)| \\ |\eta_j(X_i)| \\ |\xi_j(X_i)| \end{bmatrix} \in \mathbb{R}^{2n+1}$$

are arranged in decreasing lexicographic order. Let

$$f_j(X_i) = (p_{\mathbf{1},\mathbf{0}}(X_i))^{k^{j+1}} \left(\prod_{j' \leq j} \left(\eta_{j'}(X_i)\right)^{k^{j+1-j'}} \right)$$

$$g_j(X_i) = (p_{\mathbf{1},\mathbf{0}}(X_i))^{k^{n+j+1}} \left(\prod_{j'=1}^{n} \left(\eta_{j'}(X_i)\right)^{k^{n+j+1-j'}} \right) \left(\prod_{j' \leq j} \left(\xi_{j'}(X_i)\right)^{k^{j+1-j'}} \right).$$

Let i_0 denote the least index such that there exists some i_{neg} with $p_{\mathbf{1},\mathbf{0}}(X_{i_{neg}}) < 0$ and $|p_{\mathbf{1},\mathbf{0}}(X_{i_0})| = \left|p_{\mathbf{1},\mathbf{0}}(X_{i_{neg}})\right|$.

Then we can recover the values $\eta_j(X_i)$ and $\xi_j(X_i)$ for $i < i_0$ inductively from the values that $A\left[\widetilde{Sp}(\mathbb{R}^n \times \mathbb{R}^n)\right]$ takes on X via the following:

$$\eta_j(X_i) = \lim_{k\to\infty} \left(\frac{\sum_{i'=1}^{m} f_j(X_{i'}) - \sum_{i'<i} f_j(X_{i'})}{(p_{\mathbf{1},\mathbf{0}}(X_i))^{k^{j+1}} \prod_{j'<j} \left(\eta_{j'}(X_i)\right)^{k^{j+1-j'}}} \right)^{1/k}$$

$$\xi_j(X_i) = \lim_{k\to\infty} \left(\frac{\sum_{i'=1}^{m} g_j(X_{i'}) - \sum_{i'<i} g_j(X_{i'})}{(p_{\mathbf{1},\mathbf{0}}(X_i))^{k^{n+j+1}} \left(\prod_{j'=1}^{n} \left(\eta_{j'}(X_i)\right)^{k^{n+j+1-j'}} \right) \prod_{j'<j} \left(\xi_{j'}(X_i)\right)^{k^{j+1-j'}}} \right)^{1/k}.$$

Proof The proof of Theorem 7 from Lemma 18 is identical to the proof of Theorem 6 from Lemma 17.

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