American Nuclear Society
9th Annual Meeting
Salt Lake City, Utah
June 17-19, 1963

CONF-39-11

NSA 2U

THEORY OF CORRELATIONS AND FLUCTUATIONS IN NEUTRON DISTRIBUTIONS*

R. K. Osborn and S. Yip
The University of Michigan
Ann Arbor, Michigan

*Work supported in part by the National Science Foundation.

DISCLAIMER

Fluctuations and correlations in neutron distributions have been the subject of lively investigations for several years. In recent studies,[1-3] the theory was advanced by the introduction of an appropriate ensemble probability for a reactor, and the phenomenlogical derivation of an equation to describe it. However, because of the intuitive origin of this equation, it is difficult, if not impossible, to explore the limits of its validity or to obtain its generalization.

We have investigated this problem from the point of view of the quantum Liouville equation[4] for the reactor, i.e.,

$$\frac{\partial D}{\partial t} = \frac{i}{\hbar}[D,H], \tag{1}$$

where the density matrix, D, is identifiable, after appropriate specialization, as essentially the same as the ensemble probability mentioned above. The Hamiltonian for the reactor, H, is, of course, not known explicitly because of lack of knowledge of nuclear forces. Nevertheless, enough of its properties are inferrable to enable an illuminating derivation of an approximate equation describing D which incorporates the usual notions of particle streaming and binary interactions. Furthermore, particle densities and variances are readily definable in terms of averages of appropriate dynamical variables with respect to the density matrix, and equations describing them are deducible from the equation for the density matrix.

Since comparison between theory and experiment takes place at the level of particle densities and variances rather than at the level of the density matrix itself, it is the equations describing the particle densities that are of primary interest. For the sake of simplicity and concreteness, but with no intent to imply lack of generality, we consider here only first and second order densities for neutrons and alpha particles, the latter often being the particles detected in an experiment. We introduce the operators $\rho_1^{(n)}$, $\rho_1^{(\alpha)}$, $\rho_2^{(n)}$, $\rho_2^{(\alpha)}$ and $\rho_2^{(n\alpha)}$ such that their expected values, in the sense of

$$f_i^{(A)}\ (\underline{X},\underline{P},t) = \mathrm{Tr}\ \rho_i^{(A)}\ (\underline{X},\underline{P})\ D(t), \tag{2}$$

are respectively the singlet densities for neutrons and alphas, the doublet densities for neutrons and alphas, and the doublet cross-density for neutrons and alphas. All densities are defined in a coarse-grained phase space[5] so that they are all positive definite. The phase points $(\underline{X},\underline{P})$ form a discrete lattice in which each point is the center of a six-dimensional hypercell whose volume is h^3. Any particle in a given hypercell is assigned the coordinates of its center, hence the uncertainty principle is always satisfied.

The set of kinetic equations necessary to describe the various densities have been obtained. In the classical limit,* the equations are given by the following

*This implies that effects due to quantum statistics are ignored.

$$\left[\frac{\partial}{\partial t}+\underline{v}\cdot\underline{\nabla}+v\Sigma_t(\underline{v})\right] f_1^{(n)}\ (\underline{x},\underline{v},t) = \int d^3v'v'f_1^{(n)}\ (\underline{x},\underline{v}',t)G(\underline{v}'\rightarrow\underline{v}), \tag{3}$$

$$\left(\frac{\partial}{\partial t}+\underline{V}\cdot\underline{\nabla}\right) f_1^{(\alpha)}\ (\underline{x},\underline{V},t) = \int d^3v'v'\Sigma_D(\underline{v}'\rightarrow\underline{V})\ f_1^{(n)}(\underline{x},\underline{v}',t), \tag{4}$$

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+\underline{v}\cdot\underline{\nabla}+\underline{v}'\cdot\underline{\nabla}'+v\Sigma_t(\underline{v})+v'\Sigma_t(\underline{v}')\right] f_2^{(n)}\ (\underline{x}',\underline{v}',\underline{x},\underline{v},t)\\ &= \int d^3v_1v_1f_2^{(n)}\ (\underline{x}',\underline{v}_1,\underline{x},\underline{v},t)G(\underline{v}_1\rightarrow\underline{v}')\\ &+\int d^3v_1v_1f_2^{(n)}\ (\underline{x}',\underline{v}',\underline{x},\underline{v}_1,t)G(\underline{v}_1\rightarrow\underline{v})\\ &+\delta(\underline{x}-\underline{x}')\Lambda(\underline{v},\underline{v}',\underline{x},t),\end{aligned} \tag{5}$$

$$\begin{aligned}\Lambda(\underline{v},\underline{v}',\underline{x},t) &= \delta(\underline{v}-\underline{v}')\Big\{v\Sigma_t(\underline{v})f_1^{(n)}\ (\underline{x},\underline{v},t)\\ &+\int d^3v_1v_1f_1^{(n)}\ (\underline{x},\underline{v}_1,t)\Big[\Sigma_s\ (\underline{v}_1\rightarrow\underline{v})\\ &+\sum \alpha^2B_\alpha^j(\underline{v}_1,\underline{v})\Big]\Big\}\ -vf_1^{(n)}\ (\underline{x},\underline{v},t)G(\underline{v}\rightarrow\underline{v}')\\ &-v'f_1^{(n)}\ (\underline{x},\underline{v}',t)G(\underline{v}'\rightarrow\underline{v})\\ &+\int d^3v_1v_1\Sigma_f(\underline{v}_1)f_1^{(n)}(\underline{x},\underline{v}_1,t)\sum_{j\alpha\beta}\alpha\beta[B_{\alpha\beta}^j(\underline{v}_1|\underline{v},\underline{v}')\\ &-B_{\alpha\beta}^j(\underline{v}_1|\underline{v},\underline{v})],\end{aligned} \tag{5a}$$

$$G(\underline{v}\rightarrow\underline{v}') = v\Sigma_s(\underline{v}\rightarrow\underline{v}') + \Sigma_f(v)\sum_{j\alpha}\alpha B_\alpha^j(\underline{v},\underline{v}'), \tag{5b}$$

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\underline{V}\cdot\underline{\nabla}+\underline{V}'\cdot\underline{\nabla}'\right) f_2{}^{(\alpha)}(\underline{x}',\underline{V}',\underline{x},\underline{V},t)\\ &= \int d^3v\ v\Sigma_D(\underline{v}\rightarrow\underline{V})f_2{}^{(n\alpha)}(\underline{x}',\underline{V}',\underline{x},\underline{v},t)\\ &+\int d^3v\ v\Sigma_D(\underline{v}\rightarrow\underline{V}')f_2{}^{(n\alpha)}(\underline{x}'\underline{v},\underline{x},\underline{V},\underline{t}),\end{aligned} \tag{6}$$

$$\left[\frac{\partial}{\partial t}+\underline{V}\cdot\underline{\nabla}+\underline{v}'\cdot\underline{\nabla}'+v'\Sigma_t(\underline{v}')\right]f_2^{(n\alpha)}(\underline{x}',\underline{v}',\underline{x},\underline{V},t)$$
$$= \int d^3v_1 v_1 f_2^{(n\alpha)}(\underline{x}',\underline{v}_1,\underline{x},\underline{V},t)G(\underline{v}_1 \to \underline{v}')$$
$$+ \int d^3v_1 v_1 \Sigma_D(\underline{v}_1 \to \underline{V})f_2^{(n\alpha)}(\underline{x}',\underline{v}',\underline{x},\underline{v}_1,t) \tag{7}$$
$$- \ \delta(\underline{x}-\underline{x}')v'\Sigma_D(\underline{v}' \to \underline{V})f_1^{(n)}(\underline{x},\underline{v}',t).$$

In writing these expressions, we have introduced continuous configuration and momentum spaces for the densities. The cross-section $\Sigma_D(\underline{v} \to \underline{V})$ can be regarded as $\Sigma_D(\underline{v})\gamma^D(\underline{v} \to \underline{V})$, where $\Sigma_D(\underline{v})$ is the ordinary macroscopic cross-section for the absorption of a neutron with velocity $\underline{v}$ and $\gamma^D(\underline{v} \to \underline{V})d^3V$ is the conditional probability that given such an absorption an alpha particle with velocity $\underline{V}$ in d^3V will be produced. The quantity $B_\alpha^j(\underline{v},\underline{v}')d^3v'$ is the probability that a fission induced by a neutron at $\underline{v}$ produces j neutrons — α of them in d^3v' about v'. In Eq. (5a) $B_{\alpha\beta}^j(\underline{v}_1|\underline{v},\underline{v}')d^3v d^3v'$ represents the probability that given a fission reaction induced by a neutron with velocity $\underline{v}_1$ j neutrons will be produced, α of which have velocity in d^3v about $\underline{v}$ and β of which have velocity in d^3v' about v'.

The set of Eqs. (3) through (7) provides a partial basis for a systematic investigation of neutron fluctuations due to space and velocity correlations. They can also be used to study the approximations inherent in existing theories.[1-3] For the present discussion, we shall focus our attention upon the neutron singlet and doublet densities which are given by Eqs. (3), (5) and (5ab). It should be noted that neutron sources—which would be required to complete the description of subcritical systems—have not been included here. Furthermore, delayed neutrons have also been neglected, so that the above system of equations must necessarily be regarded as merely illustrative. These effects have not been ignored because of any difficulty in principle, but solely because their inclusion would add great bulk to the present discussion without shedding significant new light on the subject. But, of course, a working set must describe sources, delayed neutrons, and anything else pertinent to a given experimental situation.

In an effort to provide a little more insight into the implications of these equations and in order to compare some of them with descriptions provided by other investigators, we consider a crude reduction to their "diffusion theory" equivilants (keeping in mind that we consider only the neutron densities henceforth). As the reduction of the transport equations for the neutron singlet density, Eq. (3), is well-known,[6] we shall make no comment on it here but rather will confine our attention to Eq. (5) for the doublet density. As a first step, we integrate Eq. (5) over all $\underline{\Omega}$ and $\underline{\Omega}'$ ($\underline{\Omega}$ being the unit vector in the direction of $\underline{v}$, for example) to obtain

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + v\Sigma_t(v) + v'\Sigma_t(v')\right) \Phi(\underline{x}',E';\underline{x},E;t) \\
&\quad + v'\underline{\nabla}'\cdot\underline{Q}'(\underline{x}',E';\underline{x},E;t) + v\underline{\nabla}\cdot\underline{Q}(\underline{x}',E';\underline{x},E;t) \\
&= \int dE_1 v_1 \Phi(\underline{x}',E_1;\underline{x},E;t)\, G_0(E_1 \rightarrow E') \\
&\quad + \int dE_1 v_1 \Phi(\underline{x}',E';\underline{x},E_1;t)\, G_0(E_1 \rightarrow E) \\
&\quad + \delta(\underline{x}-\underline{x}')\Lambda_0(E,E',\underline{x},t) \ .
\end{aligned} \tag{8}$$

In Eq. (8) we have introduced the notation:

$$\Phi(\underline{x}',E';\underline{x},E;t) \equiv \int d\Omega d\Omega' f_2^{(n)}(\underline{x}',E',\underline{\Omega}';\underline{x},E,\underline{\Omega};t), \tag{9a}$$

$$\underline{Q}'(\underline{x}',E';\underline{x},E;t) \equiv \int d\Omega d\Omega' \underline{\Omega}' f_2^{(n)}(\underline{x}',E',\underline{\Omega}';\underline{x},E,\underline{\Omega};t), \tag{9b}$$

$$\underline{Q}(\underline{x}',E';\underline{x},E;t) \equiv \int d\Omega d\Omega' \underline{\Omega} f_2^{(n)}(\underline{x}',E',\underline{\Omega}';\underline{x},E,\underline{\Omega};t), \tag{9c}$$

$$G_0(E \to E') \equiv \int d\Omega' G(E,\underline{\Omega} \to E',\underline{\Omega}'), \tag{9d}$$

$$\Lambda_0(E,E',\underline{x},t) \equiv \int d\Omega d\Omega' \Lambda(E,\underline{\Omega};E',\underline{\Omega}';\underline{x},t). \tag{9e}$$

Next we multiply Eq. (5) by $\underline{\Omega}'$ and again integrate over all $\underline{\Omega}$ and $\underline{\Omega}'$. We get

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + v\Sigma_t(v) + v'\Sigma_t(v')\right) \underline{Q}'(\underline{x}',E';\underline{x},E;t) \\
&+ v'\int d\Omega d\Omega' \underline{\Omega}'\Omega_j' \frac{\partial f_2^{(n)}(\underline{x}',E',\underline{\Omega}';\underline{x},E,\Omega;t)}{\partial x_j'} \\
&+ v\int d\Omega d\Omega' \underline{\Omega}'\Omega_j \frac{\partial f_2^{(n)}(\underline{x}',E',\underline{\Omega}';\underline{x},E,\underline{\Omega};t)}{\partial x_j} \\
&= \int dE_1 v_1 G_0(E_1 \to E)\underline{Q}'(\underline{x}',E';\underline{x},E_1;t) \\
&+ \int dE_1 v_1 G_1(E_1 \to E')\underline{Q}'(\underline{x}',E_1;\underline{x},E;t) \\
&+ \delta(\underline{x}-\underline{x}')\int d\Omega d\Omega' \underline{\Omega}'\Lambda(E,\underline{\Omega};E',\underline{\Omega}';\underline{x},t) .
\end{aligned} \tag{10}$$

In this equation we have introduced the further notation,

$$G_1(E_1 \to E') \equiv \int d\Omega' (\underline{\Omega}' \cdot \underline{\Omega}_1) G(E_1,\underline{\Omega}_1 \to E',\underline{\Omega}') . \tag{11}$$

Another equation, similar to (10), is obtained by multiplying by $\underline{\Omega}$ and integrating over $\underline{\Omega}$ and $\underline{\Omega}'$. We shall not bother to display this equation explicitly, but will refer to it where necessary as Eq. (10a).

The implications of the coupled system of relations (8), (10), and (10a) have not begun to be explored to date. However, our purposes here will be at least partially served if we simply discard all embarrasing terms in (10) and (10a) without regard for justification until we obtain a sort of Fick's rule relating $\underline{Q}$ and $\underline{Q}'$ to Φ. Thus (all with reference to (10) and (10a)) we:

(a) Neglect all time derivatives;

(b) Set $\int d\Omega d\Omega' \underline{\Omega}' \Omega_j \frac{\partial f_2^{(n)}}{\partial x'_j} = \frac{1}{3} \underline{\nabla}' \Phi$;

(c) Set $\int d\Omega d\Omega' \underline{\Omega}' \Omega_j \frac{\partial f_2^{(n)}}{\partial x_j} = 0$; and

(d) Neglect all terms on the right-hand sides of these equations.

We then find that

$$\underline{Q}'(\underline{x}', E'; \underline{x}, E; t) = - \frac{v'}{3(v'\Sigma_t' + v\Sigma_t)} \underline{\nabla}' \Phi , \tag{12a}$$

and

$$\underline{Q}(\underline{x}', E'; \underline{x}, E; t) = - \frac{v}{3(v'\Sigma_t' + v\Sigma_t)} \underline{\nabla} \Phi . \tag{12b}$$

Inserting (12a and b) into Eq. (8), we obtain

$$\left(\frac{\partial}{\partial t} + v\Sigma_t + v'\Sigma_t'\right)\Phi - \frac{v'^2}{3(v'\Sigma_t'+v\Sigma_t)} \nabla'^2\Phi$$

$$- \frac{v^2}{3(v'\Sigma_t'+v\Sigma_t)} \nabla^2\Phi = \int dE_1 v_1 G_o(E_1 \to E)\Phi(E',E_1) \tag{13}$$

$$+\int dE_1 v_1 G_o(E_1 \to E')\Phi(E_1,E) + \delta(\underline{x}-\underline{x}')\Lambda_o \;.$$

From here on, we discontinue the explicit indication of the arguments of functions except where deemed necessary.

To proceed further, we consider only those systems in which we may treat the singlet density as space-independent, and then obtain the "one-speed" equations by integrating (13) over all E and E'. The first restriction guarantees that Λ_o will be space-independent, and hence that Φ will depend only on the space variables $\underline{x}-\underline{x}'$. It then follows that $\nabla^2 = \nabla'^2$ and also that Φ is symmetric under the interchange of E and E'. Introducing the space variable $\underline{X} = \underline{x}-\underline{x}'$ and the notation;

$$n_2(\underline{X},t) \equiv \int dEdE'\Phi \;, \tag{14a}$$

$$\bar{\bar{R}}n_2 \equiv \int dEdE' v\Sigma\Phi$$

$$= \int dEdE' v'\Sigma'\Phi \;, \tag{14b}$$

and

$$\bar{\bar{D}}n_2 \equiv \int dEdE' \frac{v'^2+v^2}{3(v'\Sigma_t'+v\Sigma_t)}\Phi \;, \tag{14c}$$

we find after integrating (13) over E and E';

$$\left(\frac{\partial}{\partial t} + 2\bar{\bar{R}}_c - 2\bar{\bar{R}}_f(\bar{j}-1) + D\nabla^2\right) n_2$$

$$= \delta(\underline{X})\,[\bar{R}_c n_1 + \overline{(j-1)^2}\,\bar{R}_f n_1] \; . \qquad (15)$$

The subscripts, c and f, designate averaged reaction rates for capture and fission respectively. The single-barred averages have been defined with respect to the energy dependence of the singlet density. The quantities $\bar{j}$ and $\overline{j^2}$ are the mean and squared-mean number of neutrons produced by fission respectively. Again we caution that neutron sources, other than prompt fission, have not been accounted for—and must be before (15) may be credited as even partially descriptive of a real system.

Lastly, we observe that a steady state solution of Eq. (15) implies that n_2 depends upon position according to

$$n_2(\underline{x},\underline{x}',t) = (\text{constant})\,\frac{e^{-|\underline{x}-\underline{x}'|/x_o}}{|\underline{x}-\underline{x}'|} \,, \qquad (16)$$

Thus, it is seen that in situations in which it may well be appropriate to approximate the singlet density as spatially uniform, it does not necessarily follow that the same approximation applies equally to the doublet density.

REFERENCES

1. P. Pluta, Thesis, The University of Michigan, 1962.

2. W. Matthes, Nukleonik, Band 4, Heft 5, 213 (1962).

3. G. I. Bell, Ann. Phys. 21, 243 (1963).

4. R. C. Tolman, The Principles of Statistical Mechanics, Oxford University Press, Oxford, 1948.

5. S. Ono, Prog. Theor. Phys. (Japan), 12, 113 (1954).

6. See, for example, R. V. Meghreblian and D. K. Holmes, Reactor Analysis, McGraw-Hill, New York, 1960.