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Cost Allocation in Distribution Planning

Stefan Engevall

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Abstract

This thesis concerns cost allocation problems in distribution planning. The cost allocation problems we study are illustrated using the distribution planning situation at the Logistics department of Norsk Hydro Olje AB. The planning situation is modeled as a Traveling Salesman Problem and a Vehicle Routing Problem with an inhomogeneous fleet. The cost allocation problems are the problems of how to divide the transportation costs among the customers served in each problem. The cost allocation problems are formulated as cooperative games, in characteristic function form, where the customers are defined to be the players. The games contain five and 21 players respectively. Game theoretical solution concepts such as the core, the nucleolus, the Shapley value and the τ-value are discussed. From the empirical results we can, among other things, conclude that the core of the Traveling Salesman Game is large, and that the core of the Vehicle Routing Game is empty. In the accounting of Norsk Hydro the cost per m^3 can be found for each tour. We conclude that for a certain definition of the characteristic function, a cost allocation according to this principle will not be included in the core of the Traveling Salesman Game. The models and methods presented in this thesis can be applied to transportation problems similar to that of Norsk Hydro, independent of the type of products that are delivered. 96 refs, 11 figs, 26 tabs

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Contents

Preface

About half a year after I enrolled as a PhD-student at the Division of Optimization at Linköping Institute of Technology, I happened to talk to a friend of mine, who just recently had begun working for Norsk Hydro.

I was involved in a project concerning cost allocation in transportation problems in general. When I described examples of what kind of problems we considered, I used the following example: 'Suppose that a distribution system involves some vehicles, many customers and an enormous amount of products. The initial problem is that of deciding which vehicle should carry what products to which customers, so that total cost is minimized. We study the next the problem; how to allocate the total cost to the products or to the customers involved.' To me the problem was thrilling, since it was easy to see the relations to reality and its importance in practice. It also had an obvious economic interpretation which I, having a M.Sc. in Industrial Engineering and Management, appreciated.

If the problem was thrilling in theory it is easy to understand how I felt, when my friend at Norsk Hydro said after my description, 'That is exactly a part of my responsibilities!'

The project has been going on for about two years. My hopes are that the results of this thesis can be of use in the analysis of real life cost allocation problems. In particular I hope that the results can be of use for Norsk Hydro.

Acknowledgments

First of all, I want to thank my supervisors Maud Göthe-Lundgren and Peter Värbrand. The process has not included supervising, but rather cooperation, with large portions of support and inspiring academic discussions. I also appreciate our non-academic discussions.

I want to thank Johan Söderberg and Hans Stålnacke at Norsk Hydro for the time they have devoted to our project, and the resources that they have supplied. I also want to thank Bernt Nordqvist for showing me the activities at the depot in Göteborg.

My colleagues at the Division of Optimization provide an inspiring and friendly atmosphere. Especially I would like to thank my former room-mate Magnus, who is a good listener and always in a happy mood. Torbjörn Larsson has given me many ideas to improve my research. Jan Lundgren has helped me improve the thesis (especially the abstract and the introduction chapter) both academically and linguistically. Michael Patriksson and Martin Joborn provided me with LaTeXfiles, and Olof Damberg helped me all the times when I managed to crash the computer.

I would also like to thank Joachim Samuelsson, my teacher in Linear and Non-linear Programming, who inspired me so much that I eventually enrolled as a PhD student in optimization.

Pamela Vang corrected the English in an early draft of the thesis. I hope that I have not added too many errors since then.

Thanks to my friends, I may enjoy my spare time and find inspiration for other things than academic activities.

My family, Kerstin, Sigfrid, Katarina and Thomas and his family are always at my side.

Finally, I wish I could express my gratitude towards Anneli, for providing me with all the love, support and encouragement I can ever ask for. And she never got tired of proofreading the many versions of the thesis.

Linköping, November 1996

Stefan Engevall

Abstract

This thesis concerns cost allocation problems in distribution planning. The cost allocation problems we study are illustrated using the distribution planning situation at the Logistics department of Norsk Hydro Olje AB. The planning situation is modeled as a Traveling Salesman Problem and a Vehicle Routing Problem with an inhomogeneous fleet. The cost allocation problems are the problems of how to divide the transportation costs among the customers served in each problem. The cost allocation problems are formulated as cooperative games, in characteristic function form, where the customers are defined to be the players. The games contain five and 21 players respectively. Game theoretical solution concepts such as the core, the nucleolus, the Shapley value and the τ-value are discussed. From the empirical results we can, among other things, conclude that the core of the Traveling Salesman Game is large, and that the core of the Vehicle Routing Game is empty. In the accounting of Norsk Hydro the cost per m^3 can be found for each tour. We conclude that for a certain definition of the characteristic function, a cost allocation according to this principle will not be included in the core of the Traveling Salesman Game. The models and methods presented in this thesis can be applied to transportation problems similar to that of Norsk Hydro, independent of the type of products that are delivered.

Chapter 1

Introduction

Whenever it is necessary or desirable to divide a common cost between several users or items, a *cost allocation method* is needed. Moriarity (1981) gives a number of reasons for why an allocation of common costs can be necessary or desirable. It may be required for external reporting on inventory values. It may serve as an aid for cost control. A cost allocation may provide a guidance for product pricing. It may encourage cooperation and discourage wasteful consumption. In a decision making situation a cost allocation may signal optimal capacity adjustments and the relative profitability of products. An example related to the problems treated in this thesis, is when several users decides to cooperate in order to realize economies of scale, and therefore use a common service. The cost of providing this service is then to be allocated among the users.

In practice the cost allocation could be done in several ways, e.g., using a rule of thumb, activity based costing or cooperative game theory. In this licenciate thesis a cost allocation problem found in a distribution planning situation is analyzed, and we discuss cost allocation methods based on concepts from cooperative game theory.

A *cooperative game situation* can be described as a situation including several *players*, who can choose to either participate or not participate in a cooperation. The motivation for the players to cooperate could be that they gain more (or loose less) by cooperating. The *cooperative game* is the question of how to allocate the total gains (or savings) among the participating players. A *cooperative cost game* is a game in which the players can reduce their total cost by cooperating, and where the total cost is to be allocated to the participating players.

The cost allocation problems we study in this thesis are illustrated using the distribution planning situation at the Logistics department at Norsk Hydro Olje AB[1]. The Logistics department at Norsk Hydro handles the supply and distribution of gas and gas-oil (i.e., heating oil and diesel oil) to customers in Sweden. The distribution is carried out from 10 depots located in southern Sweden. The transportation of the products is done by carrier companies independent of Norsk Hydro. The amounts paid to the carrier companies are

[1] In this thesis we shall refer to Norsk Hydro Olje AB simply as Norsk Hydro, except for when there is a risk of misunderstanding.

computed using a tariff, and it is these costs that are to be allocated. . The tariff makes it possible to compute what the cost of any tour would be, which is important when using cooperative game theory.

The primary reason for why the transportation costs of Norsk Hydro are to be allocated is because there are two different cost centers (Gas and Gas-oil) within Norsk Hydro. If only one product type is transported by a vehicle, the cost can be allocated to the right cost center immediately. However, sometimes a vehicle deliver both products, in which case the allocation is more difficult. The primary reason for doing the cost allocation at Norsk Hydro is thus for book-keeping purposes.

From an accounting perspective it is also interesting to be able to estimate the cost of a specific customer or a customer group. If the costs are allocated to specific customers or customer groups, the costs for the customers or customer groups can be aggregated into costs for the cost centers Gas and Gas-oil respectively. This is one approach that can be used to solve the problem of allocating the cost to the two cost centers.

The game we study is disaggregated into a Traveling Salesman Game and into a Vehicle Routing Game, where the customers are defined to be the players. The Traveling Salesman Game we study is the problem of how to allocate the cost of one tour, to the customers that were served by the tour. The Vehicle Routing Game we study is the problem of how to allocate the cost of all the tours from one depot during one day, to the customers served that day. We study only the allocation of costs to gas customers.

1.1 Purpose and contributions

The purpose of this thesis is to investigate a cost allocation problem in a specific distribution planning situation, using the perspective of cooperative game theory.

The main contributions of the thesis are:

- The *application of cooperative game theory* on a real-life cost allocation problem. We discuss considerations that have to be taken when modeling and solving the real-life problem of Norsk Hydro. We also compare and discuss the results of various solution concepts in cooperative game theory applied on a practical problem.
- The solution concepts we use, give raise to *large scale optimization problems*. We present models and methods that can be used to solve these large problems.
- We study the Vehicle Routing Game with an *inhomogeneous* fleet. The nucleolus (Schmeidler, 1969) is computed for this game. This is done using models and methods adopted from those that can be used to solve the Vehicle Routing Game with a homogeneous fleet (Göthe-Lundgren et. al.,1996).

The modeling of the games and the solution methods does not depend explicitly on the products gas and gas-oil. Thus the models and methods presented in the thesis can be applied in transportation problems similar to that of Norsk Hydro, independent of the type of products that are delivered.

1.2 Method of the Thesis

Cost allocation using cooperative game theory

Dror (1990) defines a cost allocation problem in a cooperative game setting as the problem of finding the best cost allocation method. A cost allocation method is a function defined for all cost games such that the total cost of the game is divided among the participants in the game. Using the definition of Dror (1990), we do not aim to *solve* the cost allocation problem of Norsk Hydro, since we do not aim to suggest *the best* method. However we *study* the cost allocation problem and suggest and evaluate different cost allocation methods (i.e., functions that divides the costs among the participants), that fulfill some fairness requirements.

Profitability assessment

If a number of participants, or players, choose to cooperate in a game situation where a common cost arise, they have to decide on a cost allocation method. A certain cost allocation method can be conceived as more or less fair to the players, but if the apprehension of what is fair differs, it might be impossible to agree on a cost allocation method. The subjective apprehension of fairness is important. A player may have difficulties in expressing why a certain cost allocation method is conceived as more or less fair. In cooperative game theory, several values (or concepts) and corresponding methods have been suggested, that fulfill various fairness conditions. Some of the values are uniquely defined, if a set of fairness conditions are to be fulfilled at the same time. As pointed out in Jensen (1977), in order to find a cost allocation method that a number of players can agree upon, it might be easier to discuss these fairness conditions, rather than to discuss specific cost allocation methods.

Furthermore, one of the drawbacks of many cost allocation methods based on some rule of thumb, is that they fail to capture the strategic gains customers could make by cooperating. Individuals or individual firms normally have the possibility of making strategic decisions to cooperate. Strategic possibilities play a significant role in cooperative game theory.

The games we formulate are suggestions for methods of allocating the transportation costs to the customers, that capture some strategic possibilities. The games we present can be seen as simulations of plausible games, which describe a quantitative measure of fairness. The main objective of Norsk Hydro to do the cost allocation, is to get an instrument for book keeping, that allocates the costs of the transportation in a fair manner.

If the aim is to compute the short term profitability of a specific customer or customer group in order to decide whether or not to serve him or them, a cost allocation may not be advisable. The users share common costs, and excluding a customer might give the effect that a part of this customer's cost is only transferred to the customers still served. Thus by excluding a customer, the total cost of the system does not necessarily decrease by the same amount as the cost that was allocated to the excluded customer.

A traditional approach could be used in a decision-making situation which involves short term profitability assessment. The customer or customer group is served if the marginal revenue (marginal income minus marginal cost) is positive. If the marginal revenue is positive, the company will gain more (or lose less) than if that customer or customer group is not served.

A game theoretic approach to analyze the short term profitability of a certain customer or customer group would be to make a revenue allocation. This would require more data (total income and total cost of each customer and customer group), which would increase the uncertainty, since data always contains more or less uncertainty. Another problem when attempting to make a revenue allocation, is that in a company it is generally easier to assess cost data, than revenue data. In our case it would also mean that we would have to analyze the *total* cost of each customer, not only the cost of the transportation of the products. All the customers benefit in some sense from all the activities of Norsk Hydro, including e.g., marketing & sales, computer services, accounting, financing and the activities at the depot. The gas stations are tied to Norsk Hydro by long term contracts, making the cost structure even more complicated. The costs for the oil customers might be a little easier to assess, since some of the costs can be traced to a certain instance of delivery. The principles that are presented when doing the cost allocation, could be used in a similar way when doing a revenue allocation.

There are theoretical arguments *against* the allocation of common costs. Thomas (1969, 1974) and Stigler (1966) among others, even argue that if a cost that is common to several users is to be divided between the users, this cost allocation is arbitrary by nature, and therefore no strategic decisions should be made using these data. If the allocation was not arbitrary, the cost could be traced to a user, and thus it would not be common. Callen (1978) argues that if certain cost allocation axioms are accepted, cost allocations that are not arbitrary can be found.

However, when making long term profitability assessments, some kind of allocation of common costs could be advisable. If a firm is to survive in the long run, it is of course not enough if the marginal costs are covered by marginal income. In the long run the fixed costs must also be covered. Furthermore, in spite of the theoretical arguments against the allocation of common costs, it is frequently done in practice. It is an advantage if there is some way of estimating what the major reasons for the common costs are. The estimation of the reasons for the common costs can be done by applying activity based costing (see e.g., Lumsden, 1995 or Brimson, 1991), where the idea is to analyze which activities are the *cost drivers* for the common costs. In the perspective of activity based costing, the game that we formulate could be seen as a way of analyzing what the cost drivers are in the types of distribution problems we study.

Criticism of the game theoretic approach

One criticism of the game theoretic approach for the problem we are studying, is that there may not be a real game situation at all, since we are studying tours and transportations that have already been realized. Even if the games were modeled before the tours were carried out, the real game situation in the case of Norsk Hydro, would still not be as in our models. The customers do not have the alternative to leave the game and solve the transportation problem themselves, which we assume. The gas customers are tied to Norsk Hydro by long term contracts, and their alternative would be to break this contract and form a new agreement with another gas company. For the gas-oil customers, the alternative would simply be to get their deliveries from another oil company. Furthermore, these decisions clearly depend on more factors than just the cost of the deliveries of the products[2]. The customers do not even get a specification of the cost of the transport; this cost is included in the cost of the product. Theoretically, a game closer to the real game could be modeled e.g., by estimating the true alternative cost for each customer and each set of customers. However, it would be very difficult and very time consuming to estimate these costs.

Data collection

The data used in the problems studied has been collected from Norsk Hydro's records of deliveries. Alveson & Sköldberg (1991) argue that data *always* has been rendered, in one way or another. The data we collected has been modified to some extent (e.g., volumes have been rounded off, the location of customers has been estimated). Thus we say that we study the problem using data based on reality. When analyzing all the data, we deliberately searched for problems that we assumed would give interesting results. This resembles the way *case studies* (see e.g., Merriam, 1994) are performed. The possibility of generalizing from a case study may be limited. Even if the results obtained in this thesis may not be generalized, the methods, principles and concepts presented can be used in many other situations that resembles the situation at Norsk Hydro.

The material collected for the description of the company has been collected from various written sources received from Norsk Hydro, and from interviews and discussions with representatives of Norsk Hydro throughout the project.

1.3 Outline of the Thesis

The thesis is arranged in the following way:

In Chapter 2 we present the Norsk Hydro case. The distribution planning situation and the cost structure of the transportations is described. The tariff is formulated mathematically for further use.

[2]The cost for the delivery of gas from depot to customer is around 80 SEK per m^3 on the average, while the product price for the gas customers are around 2200 SEK per m^3 (excluding tax).

Chapter 3 begins with a brief introduction to game theory. Solution concepts from cooperative game theory are presented. Chapter 3 also includes a literature survey that covers cost allocation using cooperative game theory.

The disaggregation of Norsk Hydros problem into a Traveling Salesman Game and a Vehicle Routing Game is done in Chapter 4. Important presumptions we have made in order to model the situation of Norsk Hydro are discussed. Chapter 4 also includes the relevant data for the games that we will study in Chapters 5 and 6.

Chapter 5 discusses the Traveling Salesman Game. The mathematical models and methods used to compute a solution in the core, and the nucleolus are presented. The Shapley value and the τ-value are also computed and presented.

In Chapter 6 the Vehicle Routing Game is treated. Methods to find a solution in the core, and to find the nucleolus are presented.

Conclusions that can be drawn from the games studied are discussed in Chapter 7. The most important numerical results from Chapter 5 and Chapter 6 are repeated.

Finally, in Chapter 8 some questions for further research are discussed.

Chapter 2

Case description of Norsk Hydro Olje AB

In Chapter 2, we describe the distribution planning situation of Norsk Hydro Olje AB. The discussion include aspects concerning both the product groups Gas and Gas-oil, but the focus is on aspects related to the product group Gas.

Recently Norsk Hydro bought the company Uno-X, adding roughly 350 gas stations to Norsk Hydro, and increasing the turnover for the product group Gas by around 50%. Chapter 2 describes the planning situation as it was before the take over of Uno-X.

The primary reason for why Norsk Hydro is interested in studying its cost allocation problem further, is that they feel that the current cost allocation principles (which are presented in Chapter 2.2.4) are not fair, especially when allocating the cost of tours where both gas and gas-oil are delivered (so called *shared deliveries*). They believe intuitively that the cost center Gas sponsors the cost center Gas-oil, simply because true costs do not increase linearly with demanded volume, and gas customers generally have a higher demand than gas-oil customers. The cost allocation made between the two cost centers Gas and Gas-oil is made for book-keeping reasons.

During the initial discussions about the project, other problems which are interesting for further study were revealed. These include the problem of how to allocate the cost of a tour or a set of tours to the customers, and how to allocate the cost among different customer groups, in order to be able to make a better profitability estimate than what is done today.

To solve the problem of how to allocate the costs between the cost centers Gas and Gas-oil using all the data from one year, one week or even one day would be too big a problem to begin with. The amount of data needed, not only require too much time and too many resources in the collecting phase, it would also be very difficult to analyze, model and solve. The problem necessarily calls for a disaggregation. Once the disaggregated levels are modeled, solved and analyzed, an aggregation can be done, in order to arrive at a solution of the original problem. The disaggregation that is done depends on the categorization of the problem. This is treated in Chapter 4.1.

2.1 The Distribution Planning Situation

Company description

Norsk Hydro Olje AB, a subsidiary of Norsk Hydro Sverige AB, is a sales and marketing company for gas-oil and gas in Sweden. Norsk Hydro Sverige AB is a subsidiary of Norsk Hydro a.s., which in turn is owned by the Norwegian state (51 %), and by other international stockholders (49 %). Norsk Hydro Olje AB has around 180 employees, but including the gas stations there are more than 1000 people involved in the business. The net turnover in 1995 was around 3.7 billion SEK.

Norsk Hydro Olje AB started its operations in Sweden in 1972. Before the take over of Uno-X there were around 220 Norsk Hydro gas stations (not including the ICA gas pumps; see below). Some of the gas stations are owned by Norsk Hydro (mostly automatic pump stations). However the majority of the gas stations are franchising companies (i.e., the stations are owned and run by private persons, but their business is regulated by contracts made with Norsk Hydro) or stations operated in close cooperation with Volvo retailers, IKEA department stores and ICA (a Swedish food store chain) stores. The heating oil (gas-oil) is sold both to smaller and larger house-owners. A system of truck stops (HDS) where diesel (gas-oil) is sold, is also being developed. Some of the gas stations also carry diesel.

The logistics department

The Logistics department at Norsk Hydro is responsible for the transportation of different qualities of gas and gas-oil, to the customers of Norsk Hydro. The head office in Stockholm has 5 employees, working with both supply and distribution. Their responsibilities include the management of the department, e.g., developing the overall structure of the Logistic department, making contracts and having contact with the carrier companies, technical development and support to the dispatchers. The daily operations of the logistics department are carried out at 10 depots in southern Sweden[1] (Gävle, Stockholm, Norrköping, Kalmar, Karlshamn, Malmö, Göteborg, Klädesholmen, Lysekil and Karlstad). At each depot there are order receivers who book the orders from the customers in the region. At each depot, the actual planning of the tours, i.e., assignment of the demand of a customer to a specific tour and a specific truck, is done by a dispatcher. The dispatcher has too keep the expenses within the limits of a transport budget, and she[2] is also responsible for a preliminary follow-up of the transportation costs. The main goal of the planning is to ensure deliveries to all customers at a minimal cost. In practice, there might be many considerations to be taken into account, that make the cost larger than

[1] In northern Sweden all logistics services, including planning and transportation, were bought from ODAB, Olje Distributions Aktie Bolaget. However, this contract has been terminated.

[2] The dispatcher may of course also be a man. I shall for simplicity refer to the dispatcher as she, rather than to use she/he.

a theoretical minimum. Some of these considerations are discussed when the planning situation is described below.

The products

Two qualities of gas are delivered to the gas stations: 95 octane (regular) and 98 octane (premium). Both qualities are only sold unleaded. A third quality, 96 octane (also unleaded), is sold at some gas stations. This quality is a mixture of regular and premium, in the gas pump. There are three types of gas-oil: MK1 (diesel oil used as fuel for vehicles), F10 and F32 (mostly used as heating oil). All three qualities can also be delivered in two different taxation classes. The low-taxed products are colored with a very strong pigment, added when the product is loaded onto the truck.

For quality reasons, it is of course very important not to mix the different qualities. Furthermore since the pigment used in the low-taxed products is so strong, only a small amount of a low taxed product in another quality would ruin the load completely. The dispatcher may take this into consideration when planning a tour, by trying not to load too many different products or qualities on the same truck, even if it is technically possible. The dispatcher in Göteborg generally tries to plan as few shared deliveries as possible.

The transportations

The frequency of gas deliveries to customers ranges normally from about once a week to about three times a week. The dispatcher at the Göteborg (Gothenburg) depot is responsible for the distribution planning to a total of 63 gas customer, and a large number of gas-oil customers. In most cases the stock of gas at the gas stations is owned by the gas stations, but in some cases it is owned by Norsk Hydro. However, it is always the responsibility of the gas stations to place the orders so that they do not run out of stock.

The actual transportations are mostly carried out by independent carrier companies. Norsk Hydro have six trucks of their own, but the majority of transportation of gas and gas-oil are bought from these companies. There are long term contracts between Norsk Hydro and the carrier companies. The contracts contain regulations on e.g., what truck type and what equipment is to be used and how the truck should be painted, the availability of the truck, how to account for a tour, how the payment is computed. The contracts run over a period of one year, but they are renewed as long as neither side gives notice of termination of the contract.

Since the availability of the truck is specified, and since the truck has to be painted with the logotype and colors of Norsk Hydro, the carrier companies can not carry any other products than those of Norsk Hydro. The dispatcher may take this dependency on Norsk Hydro into consideration, and try to divide the tours evenly among the different carrier companies. In so doing, the dispatcher may in fact not choose the truck that would be the optimal one with respect to the cost, in each case. Apart from an aspect of fairness, dividing the tours among the carrier companies is done in order to allow for deliveries to be carried out as far as possible during daytime hours. However, in extreme situations (for example before a price raise), the trucks can be run both at nights and weekends, if the customers are willing to receive deliveries during these hours. The possibility of using the trucks in shifts allows for flexibility in capacity.

The policy of Norsk Hydro is that a customer should have her[3] demand satisfied within three days of the receipt of the order. This is regulated in internal contracts between the Logistics department and the Marketing department. In practice the customers may get even quicker service, since the dispatcher tries to serve customers at shorter notice, if necessary and possible. At the Göteborg depot, the gas customers have a verbal promise from the dispatcher to be served within 36 hours of the reception of the order if it is necessary, even though this is not guaranteed. The customers (especially gas customers) do not generally want deliveries any earlier than necessary. One reason is that they might include the future demands of *their* customers, when placing the order to Norsk Hydro. Thus there would not be room in the tanks for an earlier delivery. Another reason is that if the customers receive the delivery earlier than necessary they also have to pay for the delivery earlier, and hence increase the inventory costs. For those stations where the inventory is owned by Norsk Hydro, an early delivery would not affect the inventory cost, and an early delivery would be possible, if there is enough room in the tanks.

If Norsk Hydro could deliver earlier, or if they could deliver the products later than within three days, there would be many more possible tours. The problem would then be more like an inventory-routing problem (see e.g., Dror & Ball, 1987). This would most certainly lower the optimal cost of the transportation. From case to case, Norsk Hydro may ask the customer if it is possible to come later than within the promised three days, if this would improve the possibility of constructing good tours. The customer may accept, but she is always guaranteed delivery within three days, if she wants. The planning situation gets more complicated the more days that are included, and without a good planning instrument the cost could even increase in practice. Furthermore, the service to the customers would decrease.

The dispatcher has access to a commercial software package, RouteLogix[4], that solves a Traveling Salesman Problem, using the Swedish road database. RouteLogix is not used operationally to construct the tours, neither is it used to guide the driver. Customers assigned to a tour are served in an order chosen by the driver unless there are special considerations to be taken, such as time windows[5] or the urgency to serve a specific customer. RouteLogix is sometimes used by the dispatcher when planning the tours, to see if a tour seems 'reasonable'. The dispatcher sometimes also uses RouteLogix to check how the reported length of a tour corresponds to an optimal tour. If the difference is too large and there is no specific reason for this, the carrier company may not be paid according to the reported length, which is normally the case. The truck driver may not choose to drive the shortest distance for several reasons other than the ones mentioned

[3]The customer may of course also be a man. I shall for simplicity refer to the customer as she, rather than to use she/he.

[4]In Sweden RouteLogix is sold by Distribution Planning Software Scandinavia AB, Askim, Sweden, which is a subsidiary of Distribution Planning Software Ltd, Birmingham.

[5]A restriction on the time during which the customer is to be served.

above. This is discussed in Chapter 2.2.3.

There are also many other restrictions that must be taken into consideration in designing a tour, e.g., laws that regulate which roads can or can not be used for large trucks, laws that, for safety reasons, regulate which roads to use when driving through a city (e.g., forbidding certain tunnels).

In order to utilize the capacity of the trucks as far as possible, the dispatcher may overplan[6] the trucks. It can happen that some of the customers do not have room in their tanks for the whole volume they ordered, and all customers may then have their actual demand fulfilled. It can also happen that some customer does not get as much delivered as she demanded. The first case is more common in the deliveries of gas-oil (smaller house owners do not know their exact demand). The second case is more common in the deliveries of gas. Since the demand of the gas customers usually is quite large, the gas customers normally does not mind if a small portion of their demand is not delivered, at least if the relationship with Norsk Hydro (in particular with the dispatcher) is good. The dispatcher can also take into consideration that a customer did not receive its full demand when planning the next delivery, e.g., by giving priority to her the next time.

The tariff used by Norsk Hydro includes 14 different types of trucks. The gross capacities for each truck types are presented in Table 2.1 below.

Truck type	20	21	22	23	30	31	32	33	34	40	46	64	66	68
Capacity (m^3)	10	19	30	35	15	25	35	44	48	10	35	35	44	48

Table 2.1 Gross capacities for the truck types in the tariff.

The first digit in the truck type 20-34 is the number of axles of the truck, and the second digit the number of axles of the trailer (where 0 means no trailer). The truck types 40-68 correspond to trucks that are used in shifts. The numbers are divided by two, to get the corresponding system as with the trucks 20-33 (e.g., truck type 40 means that a truck of type 20 is used in shifts).

The net capacity of a specific truck may differ from the gross capacity of that truck type. The strongest limitation for the capacity of the truck is the weight, not the volume, of the truck. The net capacity of a truck depends on the supplementary equipment of the specific truck, since this affects the weight. A certain truck type may also have a different capacity for gas and for gas-oil respectively. Not only do the two products have different weights, but a truck carrying gas-oil needs special pumping equipment, adding to the weight. This is not needed on a truck exclusively designed to carry gas, since gas can be tapped without special pump facilities.

[6] A demand higher than the capacity of the truck is assigned to the truck.

2.2 Cost Structure of the Distribution

2.2.1 Principles of the Tariff

The costs of Norsk Hydro is based on a tariff negotiated with the carrier companies. The tariff is negotiated once a year, but it may be revised every quarter of the year, especially if the fuel price changes a lot. The tariff contains a number of different components, which are taken into consideration when Norsk Hydro compute what they have to pay to the carrier company. These components are the following:

- A *Base-time cost*, which is a fixed start-up cost for each tour. This cost is supposed to cover waiting time, e.g., the time it takes at the depot to collect the necessary documents for a tour.
- A fixed *City driving supplement* is paid when the tour starts (i.e., the depot is) in either Göteborg or Stockholm. The City driving supplement is supposed to compensate for a lower (average) speed when passing through these big cities.
- A *load/unload cost* that covers the time it takes to fill the truck at the depot and to empty the truck in the customers tanks. For the delivery of gas, the cost is proportional to delivered volume. For the delivery of gas-oil, there are three different intervals ($< 3m^3, 3m^3 - 5m^3, > 5m^3$) of delivered volume to a specific customer. Within each interval, and for each order (i.e., customer), the cost is proportional to the volume delivered.
- A cost which is paid for each customer visited. This cost covers the time it takes to stop at a customer and to connect the first tank. The term used at Norsk Hydro for this cost is *A-stop*, and therefore A-stop is used here as well.
- A cost paid for each supplementary tank filled at a customer. This cost is to compensate for the time it takes to shift a hose from one tank to another, as well as for shifting between different compartments in the truck. The term that is used is *B-stop*.
- A *mileage allowance* that is proportional to reported length of the tour. This is supposed to cover e.g., the drivers wages, the fuel consumption, the maintenance and the depreciation of the truck.

All the above types of costs (except the B-stop cost), are also dependent on the truck-type, since a larger truck is assumed to have a higher cost of depreciation, maintenance, fuel consumption etc.

The costs of transportation during the period 18-29 September 1995, using the truck OMB575 (license plate number), delivering only gas from Göteborg, were divided in the categories mentioned above. This division is presented in Table 2.2 below.

Cost type	Cost (SEK)	% of total cost
Base-time	5 152.9	8.2
City driving supplement	1 871.7	3.0
A-stop	2 320.5	3.7
B-stop	1 820.0	2.9
Load/Unload	11 739.6	18.7
Mileage allowance	39 791.3	63.6

Table 2.2. Cost types of truck OMB575 during the period 18-29 September 1995.

The majority of the trucks in the fleet of Norsk Hydro are owned by independent carrier companies, but six trucks used for the transportations are owned by Norsk Hydro. The construction of the tariff used by Norsk Hydro is very simple, and therefor task of computing the cost for a (real or imaginary) tour is elementary. The cost for the carrier company has components such as the wages for the driver, the depreciation of the truck, maintenance, fuel consumption etc., which would be difficult to compute exactly, since much more data for each tour would be required. The time spent driving, the time spent at each customer, the fuel consumption for the specific tour, the true rate of depreciation of the truck etc. would have to be known. This data would probably also be different for each carrier company, and in turn different for each truck and driver within the carrier company. To compute the cost of a tour that has not taken place, which is needed in our formulations, would be impossible. An imaginary tariff could be constructed using the cost structure of each carrier company. However, this would mean that a cost allocation in some sense already would have been made. This allocation is also made today, but only from the perspectives of the carrier companies. From the perspective of Norsk Hydro no allocation of costs has been done. This licenciate thesis studies the cost allocation problem of Norsk Hydro, hence only costs according to the tariff are discussed, and we do not investigate further the allocations that have been made in constructing the tariff.

The existence of the tariff is very important for us, since it allows us to compute exactly what the cost of any possible tour would be, given the demand and location of each customer.

2.2.2 Categorization of Costs

The different costs may be expressed mathematically (to be used in later chapters), and they can be divided into categories. A mathematical formulation of the tariff is the following:

Define:

B_k = Base-time cost for truck type k.
$\tau = \begin{cases} 1 & \text{if the depot is situated in a big city (i.e., Göteborg or Stockholm)} \\ 0 & \text{otherwise} \end{cases}$.
t_k = City driving supplement for truck type k.
a_k = A-stop cost for truck type k.
β = B-stop cost.
b_i = Number of B-stops at customer i.
d_k = Load/unload cost per m^3 for truck type k.
D_i = Delivered (demanded) volume (m^3) at customer i.
V_k = Capacity (m^3) for truck type k.
σ_k = Cost per km (Mileage cost)[7]for truck type k.
l_r = Distance of arc r (representing a road), in km.
S = A set of customers, i.e., a *coalition.*
R^S = The set of arcs (representing roads) in the optimal tour that covers the customers in S.

Then the optimal cost for serving coalition S, $c(S)$, using truck type k (i.e., $\sum_{i\in S} D_i \leq V_k$) transporting gas, can be expressed as:

$$c(S) = B_k + \tau t_k + |S|a_k + \beta\sum_{i\in S} b_i + d_k\sum_{i\in S} D_i + \sigma_k \sum_{r\in R^S} l_r$$

This expression has to be modified slightly to also hold for gas-oil, since the Load/unload cost for gas-oil is not linear in delivered volume.

If a truck type is chosen, $c(S)$ can be divided into three categories:

- A customer specific cost that can be traced to the demand of a certain customer.
- A fixed cost that depends only on whether the depot is situated in a big city or not.
- A common transportation cost.

Suppose that truck type $\hat{k}$ has been chosen. Then the division is:

Fixed cost,	c_0	=	$B_{\hat{k}} + \tau t_{\hat{k}}$
Customer specific cost,	c_i	=	$a_{\hat{k}} + \beta b_i + d_{\hat{k}} D_i, \quad i \in S$
Common transportation cost,	C_S	=	$\sigma_{\hat{k}} \sum_{r\in R^S} l_r$

This division into three categories can not be done unless the truck type is chosen. The truck type that could be used for a tour depends on the total demand of the tour, which in

[7]Mileage cost means cost per distance unit, while mileage allowance means the cost for the traveled distance, i.e., (mileage cost)*(distance).

turn depends on the sum of the demands of each customer on the tour. Thus the demand of a single customer can affect *all* other costs (except B-stop) of the tour, including the customer specific cost of all other customers, since it can affect the type of truck that can be used. For a tour that has been made, the truck type used is known, and it is meaningful to divide the cost into the three categories. However, for a tour that has not been made, a truck type has not been chosen, and it is not straightforward to make the division into categories.

For each tour that is carried out, there is practically always a difference between optimal cost, and the actual cost for Norsk Hydro, i.e., what Norsk Hydro pays the carrier company. One important and easily discovered reason, is that there is a difference between the optimal length, and the actual length of a tour which is used when computing the cost according to the tariff. We call the difference between optimal and actual cost the *cost remainder*. It is discussed in more detail in the next chapter.

2.2.3 The Cost Remainder

The cost remainder, or simply the *remainder* can be interpreted as the difference in cost between the optimal solution to the planning situation, and the solution that was applied. The main reason for the remainder is that the actual length of a tour, or a set of tours, is larger than what the optimal length is.

It is only meaningful to consider the remainder for a tour or a set of tours that have taken place, since the remainder is the difference between the actual and the optimal solutions. For tours that have not taken place, it is difficult to tell what the actual solution would be. The reasons for the remainder are slightly different in the Traveling Salesman case and the Vehicle Routing case.

The Traveling Salesman Case

Define:

$c(S)$	=	The cost for an optimal tour covering the customers in S.
$c_\gamma(S)$	=	The actual cost for a tour covering the customers in S.
γ_S	=	The remainder, i.e., the difference between the actual cost and the optimal cost of a tour covering the customers in S.

Then the remainder can be expressed as:

$$\gamma_S = c_\gamma(S) - c(S)$$

In the Traveling Salesman case it is only meaningful to compute the remainder for the *grand coalition* N (i.e., the coalition that consists of all the customers served on the tour), $c_\gamma(N)$, as this is the only tour that has been made.

There are a number of reasons why there is a remainder. Some of them are discussed below.

The driver may not choose to drive the shortest path (which is always less costly for Norsk Hydro). She[8] may for example choose to drive the quickest path. A smaller road may be shorter, but it can be faster e.g., to drive to, on and from a highway. There may be a temporary road block on the shortest path, something that is not included in the road database used to compute optimal distances. It may be forbidden for the truck to use certain roads. It may also be technically impossible to use a certain road (e.g., if it is too narrow). The driver may also have other incentives for not choosing the shortest path. Even though she is not supposed to, the driver may also report a distance that is longer than what actually was driven, e.g., including the distance between her house and the depot.

The driver may want to deliver all of the same quality first, in a certain area. It can allow the driver to do fewer connections and disconnections of the hoses, and this may also help her save time in emptying the hoses between the deliveries, which can cause the length of the tour to increase.

The customers can have requirements on the time when they can receive the products, i.e., there exists time window restrictions. In this case it can be an optimal decision to drive a detour to serve the customers in an order that does not minimize the length of the tour.

A customer may call in an urgent order very late, and Norsk Hydro may decide to serve the customer. This can lead to rescheduling, or redirecting of trucks that have already left the depot, which increase total length of the tour.

If a truck has made all the deliveries scheduled for a tour and there still are products left in the truck, the dispatcher may redirect the truck to deliver the rest of the products to a customer initially not scheduled on the tour. Compared with the inclusion of the extra customer initially, this tour may be more expensive.

All the above reasons are connected to a non-optimal length of a tour. There are also other possible reasons for the remainder. The dispatcher may choose a truck that is not of an optimal size, i.e., a truck that is larger than the smallest possible truck. However this is not very frequent, since the dispatcher often over plans the trucks.

Some further reasons for the remainder (as we compute it), come from errors in the data we use in the models. Examples of possible errors are errors in the software RouteLogix or in the road database tied to it, errors in placing the customers geographically in the database, and possible round-offs that we make. The effect of these errors could be that what we compute as optimal in our models, may not be optimal in reality. These components of the remainder can be positive as well as negative. However, the errors due to our modeling are assumed to be fairly small.

The Vehicle Routing Case

In addition to the reasons in the Traveling Salesman case above, at least one more reason

[8]The dispatcher may of course also be a man. I shall for simplicity refer to the driver as she, rather than to use she/he.

for the remainder can be found in the Vehicle Routing case. This reason is that the dispatcher may not assign customers to tours in an optimal way, i.e., she does not use the optimal routes in the solution to the Vehicle Routing Problem.

The remainder in the grand coalition can then be divided into two components. One component appears because each of the actual routes is not driven optimally. The other component appears because the actual routes are not part of the optimal Vehicle Routing solution. As in the Traveling Salesman case, it is only meaningful to consider the remainder for the grand coalition. Therefore, we only define the remainder for N.

Define:

γ_N	$=$	The total remainder in the Vehicle Routing case.
γ_S	$=$	The remainder, due to the difference between the actual cost and the optimal cost of a tour covering the customers in S.
γ^T	$=$	The remainder due to the difference in cost between the actual routes and the optimal Traveling Salesman solutions to these routes.
γ^V	$=$	The remainder due to the difference in cost between an optimal Vehicle Routing solution and the sum of optimal Traveling Salesman solutions. for the coalitions that correspond to the actual routes.
S^A	$=$	The set coalitions that correspond to the actual routes.
S^O	$=$	The set coalitions that correspond to the routes in the optimal Vehicle Routing solution.

Then the total actual cost, $c_\gamma(N)$, in the Vehicle Routing case can be expressed as:

$$c_\gamma(N) = c(N) + \gamma_N$$

Since the total actual cost is the sum of the costs of the actual routes, we can also write:

$$\begin{aligned} c_\gamma(N) &= \sum_{S \in S^A} c_\gamma(S) = \sum_{S \in S^A} c(S) + \sum_{S \in S^A} \gamma_S = \\ &= \sum_{S \in S^A} c(S) + \gamma^T = \sum_{S \in S^O} c(S) + \gamma^V + \gamma^T = c(N) + \gamma^V + \gamma^T \end{aligned}$$

i.e.,

$$\begin{aligned} \gamma^V &= \sum_{S \in S^A} c(S) - \sum_{S \in S^O} c(S) \\ \gamma^T &= \sum_{S \in S^A} \gamma_S \\ \gamma_N &= \gamma^V + \gamma^T \end{aligned}$$

2.2.4 Current Cost Allocation Principles

The only allocation of transportation costs that is carried out regularly at Norsk Hydro today, is the allocation of costs between the two cost centers Gas and Gas-oil. When the truck carries only one of these product groups, there is actually no need for a cost allocation of this tour, since the whole cost for a tour can be directly traced to a product group. In the case of shared deliveries, the customer specific costs, c_i, can be directly traced to a product group since it can be traced to a product[9]. The other costs, i.e., Fixed cost (c_0) and Mileage allowance (C_S) are simply divided equally between the cost centers Gas and Gas-oil. This is done without any consideration to the demand or location of the customers, i.e., no consideration is taken as to how much gas and gas-oil respectively that are delivered in one tour. The conclusion is that if a larger volume of one product is delivered, this product carries less of the common cost, per m^3.

The other costs of the Logistics department, such as the cost of the depots and the dispatcher, and the cost of the head-office, are allocated to the two cost centers Gas and Gas-oil, according to a fixed percentage.

In the accounting of the tours, a cost per m^3 of each product group is computed for each tour. This implies a cost allocation principle, where the costs are allocated in proportion to demand. This principle means that the geographical location of a customer, is not taken into consideration at all. In this case only the demand of a customer is significant. In the following the principle of allocating costs in proportion to demand is referred to as 'the principle implied by Norsk Hydro'.

Today, no regular allocation of the transportation cost to the customers is made by Norsk Hydro. However, a profitability evaluation is done each month for each station. In this evaluation transportation cost is one of the components taken into consideration. It is based on a rough template.

An example of how cost allocation is done in a situation similar to that of Norsk Hydro, can be found in the method used by another Swedish company[10] in their distribution.

They allocate the cost of the tour to the customers receiving deliveries, in the following way:

The fixed cost (c_0) is split evenly among the customers on the tour. The customer specific costs (c_i) are directly allocated to the customers. The common transportation cost (C_S) is computed using the following principle:

[9]If a customer gets a delivery of gas and gas-oil at the same time (which is hardly ever the case), the customer could be split into two imaginary customers, and then the customer specific costs could be traced to a product group.

[10]The company name is not mentioned for integrity reasons.

Define:

N = The set of customers on the tour.
D_i = Delivered volume to customer i.
σ_i = Mileage allowance if only customer i is served.
C_S = Common transportation cost.

One component of C_S, λ_i, is computed as:

$$\lambda_i = \sigma_i \frac{D_i}{\sum_{i \in N} D_i}$$

The second component of C_S, μ, is what is not covered by λ_i, divided evenly among the customers:

$$\mu = \frac{C_S - \sum_{i \in N} \lambda_i}{|N|}$$

Hence the cost y_i allocated to customer i is:

$$y_i = \frac{c_0}{|N|} + c_i + \lambda_i + \mu$$

The method takes into consideration the Delivered volume and the Mileage allowance for each customer. Therefore we will refer to it as the DM-method in later chapters.

Chapter 3

Cost Allocation and Cooperative Game Theory

In Chapter 3.1 a brief introduction to game theory is presented. Chapter 3.2 presents basic concepts in cooperative game theory. In Chapter 3.3 some solutions concepts are described, and Chapter 3.4 is a literature survey over cost allocation using cooperative game theory.

3.1 Introduction to Game Theory

The history of Game theory dates back to the Babylonian Talmud. Cases are described in which the estate of a deceased man are to be divided among his three widows. The widows have claims of 100, 200 and 300 units respectively on the estate. In Talmud, three cases are described, where the estate consists of 100, 200 and 300 units. In each case the prescription for the division is in Talmud: $(33\frac{1}{3}, 33\frac{1}{3}, 33\frac{1}{3})$, (50, 75, 75) and (50, 100, 150) respectively. It turns out that for a certain definition of the characteristic function, the solution is the nucleolus (Schmeidler, 1969) in each case. The problem, characterized as a bankruptcy game, is discussed in more detail in Aumann & Maschler (1985).

Modern game theory is considered to begin with von Neuman & Morgenstern's publication 'Theory of Games and Economic Behavior' (von Neuman & Morgenstern, 1944), even though the study of the theory of games was started in 1928 by von Neuman (1928). Nash published a number of important papers on game theory between in the beginning of 1950's (Nash, 1950a, 1950b, 1951 and 1953).

For an outline of the history of game theory we refer to Walker (1995).

Problems in game theory can be classified in a number of ways. The most obvious classification is into cooperative and non-cooperative games. In cooperative games it is possible for the players to form binding contracts, while this is not possible in non-cooperative games. An example of a cooperative game is when a number of players decide to work together in order to save costs. The cost allocation game is how to divide the cost (or the savings) among the players. Examples of non-cooperative games are chess or war situations.

A game can contain two players, n players or infinitely many players. An example of a two-person game is chess and an example of a n-person game is when a number of players cooperate in order to save costs. Examples of games with infinitely many players are in market situations (where each player is too small to affect the market), and in games where a cost is to be allocated to a continuous amount of products.

Games can also be divided into transferable utility (TU) and non transferable utility (NTU) games. TU-games often involves monetary, or physical, units that can be transfered between the players. An example of a TU-game is again a game where a number of players work together to save costs, where the cost (or the savings) can be transfered among the players. An example of a NTU-game is again the game of chess, where the 'utility' (the victory or loss) can not be transfered.

In this thesis we study n-person cooperative TU-games.

3.2 Basic Concepts

A *cooperative n-person game* is a pair $(N;v)$ where $N = \{1, 2, \ldots, n\}$ is the set of players and v is a real valued function, called the *characteristic function*, on $S \subseteq N$ with $v(\emptyset) = 0$. Each subset $S \subseteq N$ is called a *coalition*. We denote by N the *grand coalition*. The *cardinality* of a coalition, $|S|$, is equal to to number of players in S.

The characteristic function usually refers to a payoff that the players in S can receive from cooperating. In this thesis we study cooperative *cost games* $(N;c)$, and we denote the characteristic function $c(S)$ instead of $v(S)$. The characteristic function in a cost game refers to the cost that arises when a coalition chooses to cooperate. A cost game can be transfered to a game of payoffs (a *cost savings game*), if the payoff to a coalition is seen as the savings that a coalition can achieve from cooperating. Since we study cost games, the concepts that are defined will be defined for cost games. Sometimes the prefix 'anti' is put in front of the concepts in cost games (e.g., anti-core, anti-nucleolus). We will not do this.

An *outcome* (or a *pre-imputation*) y is a vector such that player i is allocated the cost y_i, and $\sum_{i \in N} y_i = c(N)$[1]. An *imputation* is an outcome that fulfills the requirement $y_i \leq c(\{i\})$ for all i[2], i.e., individual rationality (see Chapter 3.3).

If the characteristic function c is *monotone*, i.e., $c(S) \leq c(T)$ for $S \subset T \subset N$, then the game $(N;c)$ is *monotone*. The game $(N;c)$ is *proper* if the characteristic function is

[1]For simplicity, we will write $y(S) = \sum_{i \in S} y_i$.

[2]For simplicity, we will write $c(i) = c(\{i\})$.

subadditive, i.e., $c(S) + c(T) \geq c(S \bigcup T)$ for all $S, T \subset N, S \bigcap T = \emptyset$. In a game with a subadditive characteristic function, it is always profitable (or at least not unprofitable) to form larger coalitions. The weakest form of subadditivity is if the characteristic function is *additive*, i.e., $c(S) + c(T) = c(S \bigcup T)$ for all $S, T \subset N, S \bigcap T = \emptyset$. A game with an additive characteristic function is called an *inessential* game. All other games are called *essential*.

An interesting class of games is the class of *convex* games. A cost game is convex if its cost function is *concave* (or *submodular*), i.e., $c(S \bigcup T) + c(S \bigcap T) \leq c(S) + c(T)$ for all $S, T \subseteq N$. This can also be expressed as $c(S \bigcup \{i\}) - c(S) \geq c(T \bigcup \{i\}) - c(T)$ for all $i \in N$ and $S \subseteq T \subseteq N \backslash \{i\}$. This implies a sort of 'snowballing' effect, i.e., increasing cost savings with the size of the coalition.

A subclass of convex games, *semiconvex games*, was introduced by Driessen & Tijs (1985).

Two games are *strategically equivalent* if they both have the same set of players N, and the characteristic function of one game, $c(S)$, can be related to the characteristic function of the other game, $\hat{c}(S)$, as: $\hat{c}(S) = kc(S) + \sum_{i \in S} c_i$, for $S \subseteq N$, where k is a positive number and $c_i, i \in N$, are arbitrary real numbers.

The *excess* of a nonempty coalition S with respect to a (cost allocation) vector y is $e(S, y) = c(S) - y(S)$.

The *marginal cost* of a player, m_i is the marginal cost of that player in the grand coalition, i.e., $m_i = c(N) - c(N \backslash \{i\})$. For a monotone game, $m_i \geq 0$ for all i.

Nilssen (1987) has an excellent survey (unfortunately in Norwegian) of basic concepts and solution concepts. He discusses the concepts in the context of cost allocation problems concerning cooperation in the Norwegian gas-oil industry.

3.3 Solution Concepts

In order to characterize different solution concepts a number of properties can be defined. A solution concept may fulfill some of these properties. Depending on what properties a decision maker regards as necessary for a fair cost allocation, different solution concepts can be chosen. It is also necessary to evaluate the possibility of computing the different solution concepts. If the characteristic function is difficult to compute, it may not be possible to use a solution concept that requires the characteristic function value for all coalitions to be known in advance. Some important properties for solution concepts are the following:

Efficiency (or *group rationality*[3] or *Pareto optimality*) states that the sum of the allocated cost should be equal to the total cost of the game, i.e., $y(N) = c(N)$.

[3]Sometimes in the literature group rationality refers to the condition $y(S) \leq c(S), S \subset N$.

Individual rationality states that no player i should be allocated a cost that is higher than the cost of serving only player i, i.e., $y_i \leq c(i)$ for all i. We will refer to $c(i)$ as the *stand-alone cost* of player i.

The *kick-back* criterion requires that no player gets a negative cost allocation, i.e., $y_i \geq 0$.

The *incremental cost* condition states that no player should be charged less than the marginal cost of including this player, i.e., $y_i \geq m_i$.

The *dummy player property* states that if player i contributes nothing (in the reduction of total cost) to any coalition, i.e., $c(S) = c(S\backslash\{i\}) + c(i)$ for all $S \subseteq N, S \ni i$, then the cost allocated to i, y_i, is equal to $c(i)$.

The *anonymity* (or *neutrality* or *symmetry*) condition, states that the order in which the players are numbered should not affect the cost allocation.

Regarding the *monotonicity* condition of a solution concept there is some confusion[4] in the literature. Monotonicity is sometimes used to describe *Monotonicity in the aggregate*, which implies that if the overall cost increases, no player should be allocated a lower cost, i.e., if $c^1(S) = c^2(S), S \subset N$ and $c^1(N) \geq c^2(N)$, then $y_i^1 \geq y_i^2$ for all i. We will use monotonicity for this condition. Sometimes monotonicity is used to describe *coalitional monotonicity*, which implies that if the cost increases for a particular coalition, T, and stays the same for all other coalitions, $S \neq T$, then no member of T can get a lower cost allocated than before the increase, i.e., if $c(T) \geq \bar{c}(T)$ and $c(S) = \bar{c}(S), S \neq T$, means that $y_i \geq \bar{y}_i, i \in T$. We will use coalitional monotonicity to describe this property.

Additivity requires that if the cost matrix $C = \{c_{ij}\}$ is divided into two independent cost matrices, $C^1 = \{c_{ij}^1\}$ and $C^2 = \{c_{ij}^2\}$, where $c_{ij} = c_{ij}^1 + c_{ij}^2$ for all i, j, then $y_i = y_i^1 + y_i^2$ for all i.

Young (1985) refers to the property *covariance* (or *relative invariance under S-equivalence* in e.g., Driessen, 1988) if two games that are strategically equivalent, yield corresponding results in the cost allocation. This means that if y is the solution to game $(N; c)$ and $\hat{y}$ is the solution to game $(N; \hat{c})$, where: $\hat{c}(S) = kc(S) + \sum_{i \in S} c_i, S \subseteq N$, then $\hat{y}_i = ky_i + c_i$.

Hartman & Dror (1996) discusses another set of criteria that a good cost allocation method should fulfill, in the context of cost allocation for an inventory model. They suggest three necessary criteria: *Stability* (a core solution), *justifiability* (consistency of benefits with costs) and *polynomial computability.* They show that the Louderback value (Louderback, 1976) fulfill these three criteria for a specific example. In the example the Shapley value, the nucleolus, and four more methods fail to fulfill all three criteria.

A cost allocation concept that is obvious and very easy to compute, is the *egalitarian* method, which divides the cost $c(N)$ equally among the players in N. It is efficient, kickback-free, additive, anonymous and monotonic. However, it is not individually ra-

[4] To add to this confusion monotonicity as describes as a property of cost allocation methods, is not the same as the property that a game (or a characteristic function) can be monotone, as described in Chapter 3.2.

tional, and it does not possess the dummy player property. Furthermore it is clearly unattractive since it does not relate the cost allocation to the marginal cost of a player.

The concepts we present further are the core, the nucleolus, the Shapley value and the τ-value. There are also a number of other solution concepts that we present but not discuss in detail.

Core

The core is probably the most intuitive solution concept conceived as fair. It is defined as those imputations that fulfill:

$$\sum_{i \in S} y_i \leq c(S), \quad S \subset N \quad [3.1]$$
$$\sum_{i \in N} y_i = c(N) \quad [3.2]$$

Constraints [3.1] say that no player or coalition, should together be allocated a cost when forming the grand coalition, that is higher than if the individual or coalition would act alone. Constraint [3.2] is the efficiency axiom.

It is possible that the constraints [3.1] and [3.2] define an empty set, i.e., that the core is empty. One strong condition that guarantees non-emptiness of the core is that the game is convex (for a proof, see e.g., Shapley, 1971 or Driessen, 1988). However, the core may be non-empty even if the game is not convex.

The core was presented in Gillies (1959), but according to Shubik (1982) Shapley developed the core as a solution concept. However, the idea behind the core is older than that. The contract curve of Edgeworth (1881) and Böhm-Bawerks (1889) solution of horse bargaining can be seen as precursors of the core. According to Straffin & Heaney (1981), Ransmeier (1942) discusses preliminary criterion of a satisfactory allocation, as being the conditions of the core. The idea of the core is also mentioned in von Neuman & Morgenstern (1944).

The two main theoretical drawbacks of the core, are that in general the core will not consist of a unique solution, and that the core can be empty. The core can be seen as a description of candidate allocations, rather than a concept that can be used to find a particular allocation.

The kick-back criterion and the incremental cost condition are fulfilled in any solution in a non-empty core. Since we do not in general get a unique solution using the core concept, it is not meaningful to discuss the neutrality condition or the monotonicity condition.

It is neither meaningful to discuss the monotonicity condition. Two games with the same cost matrix (i.e., $C^1 \equiv C^2 \equiv C$) could generate two different solutions in the core, which would not fulfill the monotonicity condition. If one $y_i^1 < y_i^2$, then necessarily at least one $y_j^1 > y_j^2, j \neq i$, since $y(N) = c(N)$. On the other hand, the monotonicity condition can be fulfilled when looking at two different cost matrixes C^1 and C^2, again depending on which solutions in the core that are chosen.

The additivity condition is fulfilled, in the sense that if two games with the cost matrices C^1 and C^2 are formulated, and if y_i^1 and y_i^2 are in the core of their respective game, then the solution $y_i = y_i^1 + y_i^2$ will be one of the solutions in the core of the game with cost matrix $C = C^1 + C^2$. However, the converse is not necessarily true.

Peleg (1992) presents some further properties and proves that a number of these properties uniquely defines the core.

The *strong ϵ-core* (sometimes just called the ϵ-core) are those solutions y, that fulfill the requirements:

$$\begin{array}{lcll} \sum_{i \in S} y_i & \leq & c(S) & +\epsilon, \quad S \subset N \\ \sum_{i \in N} y_i & = & c(N) & \end{array}$$

The *weak ϵ-core* are those solutions y, that fulfills the requirements:

$$\begin{array}{lcll} \sum_{i \in S} y_i & \leq & c(S) & +|S|\epsilon, \quad S \subset N \\ \sum_{i \in N} y_i & = & c(N) & \end{array}$$

The core is the same as the strong or weak 0-core, i.e., when ϵ=0. Note that a solution in an ϵ-core does not necessarily fulfill the individual rationality conditions. The weak and the strong ϵ-core were introduced by Shapley & Shubik (1963, 1966).

If ϵ is large enough, the strong and the weak ϵ-core are always non-empty (for a proof see e.g., Kannai, 1992). The minimal ϵ-value that produces a non-empty ϵ-core could be seen as either a measure of the 'distance' from a non-empty core, or as an indication of the size of the core. The ϵ-value can also be viewed as a value that takes into account the cost of forming a coalition (such as communication costs).

The minimal ϵ-value that makes the strong ϵ-core non-empty is computed as a sub-procedure when computing the nucleolus (see below). The solutions in the strong ϵ-core for the minimal ϵ-value is called the *least ϵ-core.* For further discussions on approximate cores, see e.g., Kannai (1992).

Nucleolus

The concept of the *nucleolus* was introduced by Schmeidler (1969). The nucleolus intuitively lies in the center of the core, since it minimizes maximal discontent (or maximizes minimal content, or gain) for the coalitions.

To define the nucleolus we need the following:

In a game $(N;c)$, define for each imputation y an *excess vector* $\theta(y)$ of dimension $2^{|N|}-2$. Let the excess vector contain the excesses, $e(S,y)$, of each $S \subset N, S \neq \emptyset$ with respect to y, in a non-decreasing order. This implies that if $i < j$, $\theta_i(y) \leq \theta_j(y)$ for all $1 \leq i < j \leq n$. If there exists a positive integer q, such that $\theta_i(y) = \theta_i(\bar{y})$ whenever $i < q$ and $\theta_i(y) > \theta_i(\bar{y})$ for $i = q$, we say that $\theta(y)$ is *lexicographically greater* than $\theta(\bar{y})$, and denote this by $\theta(y) >_L \theta(\bar{y})$. With $\theta(y) \geq_L \theta(\bar{y})$ we will mean that either $\theta(y) >_L \theta(\bar{y})$ or $\theta(y) = \theta(\bar{y})$.

The nucleolus is defined as those imputations y that have the lexicographically greatest associated excess vector, i.e.,

$$\theta(y) \geq_L \theta(\bar{y}), \text{ for all } \bar{y} \in \{y | y(N) = c(N), y_i \leq c(i)\}.$$

Schmeidler (1969) showed that the nucleolus always exists, and that it is a unique point. He also showed that the nucleolus is always included in the kernel (see e.g., Davis & Maschler 1965). For convex games, Maschler et. al. (1972) proved that the kernel coincides with the nucleolus. Schmeidler (1969) also showed that the nucleolus is also included in every nonempty ϵ-core (i.e., in any nonempty core). Furthermore he showed that the nucleolus is a continuous function of the characteristic function.

The fact that the nucleolus always exists and defines a unique point that (in some sense) is in the center of any non-empty core, makes it an appealing cost allocation concept. The nucleolus is efficient, individually rational, anonymous and possesses the dummy player property. Axiomatizations of the nucleolus are presented in Potters (1990) and Snijders (1995).

The main theoretical drawbacks of the nucleolus are that it is not additive and it is not monotonic. In fact, Young (1985) shows that for $|N| \geq 5$ there exists no coalitionally monotonic core allocation method. Another drawback is that the nucleolus does not take into consideration possible differences in importance of certain coalitions. All coalitions are equally important in the computation of the nucleolus.

The *pre-nucleolus* is defined in a similar way to the nucleolus. The pre-nucleolus is defined as those pre-imputations y that have the lexicographically greatest associated excess vector, i.e.,

$$\theta(y) \geq_L \theta(\bar{y}), \text{ for all } \bar{y} \in \{y | y(N) = c(N)\}$$

The pre-nucleolus is also a unique point, and except for the individual rationality constraint, the pre-nucleolus fulfills all the properties of the nucleolus. An axiomatization of the pre-nucleolus is presented in Maschler (1992). If the pre-nucleolus is an imputation, i.e., fulfills the individual rationality conditions, the pre-nucleolus coincides with the nucleolus.

Grotte (1970) introduced the *normalized nucleolus* (also called the *weak nucleolus* e.g., in Young et. al., 1982 or the *per capita nucleolus* e.g., in Young, 1985). It is defined in the same manner as the nucleolus except that the excesses $e(S,y)$ are divided by the cardinality (size) of the coalition, i.e., $e^N(S,y) = \frac{e(S,y)}{|S|}$. The effect of the normalized nucleolus is that the large coalitions gain in importance, since the excesses are reduced more for the large coalitions than for small coalitions. Therefore the large coalitions can get a lower cost allocated to them in the normalized nucleolus compared to what they get in the nucleolus. It is pointed out in Young et. al. (1982) that the normalized nucleolus may not fulfill the dummy-player property. However, it is monotonic, although not coalitionally monotonic.

The normalized nucleolus is subjected to some axiomatic criticism in e.g., Shapley (1981) and Young (1985). The idea behind the criticism is that when formulating a game, all the relevant information of the game $(N;c)$ should be included in the set of players N, and in the value of the characteristic function $c(S)$. This means that no other information, like the number of players in the coalition, should need to be taken into account. It is necessary that all information of the game is included in N and c, if the properties (or axioms) presented earlier in this chapter are to hold, for several of the solution concepts.

Young et. al. (1982) also suggests the *proportional nucleolus.* It is defined in the same manner as the nucleolus except that the excesses $e(S,y)$ are divided by the value of the characteristic function of the coalition, i.e., $e^N(S,y) = \frac{e(S,y)}{|c(S)|}$.

Shapley Value

The rationale behind the Shapley value (Shapley, 1953) is that the marginal cost of each player when successively forming the grand coalition is reflected. Each way of forming the grand coalition is considered to be equally probable. Given a set N (of size n), there are $n!$ permutations, i.e., different ways to order the members of N (or to form the coalition N).

Suppose that the grand coalition is formed by successively adding players in the order $p_1, \ldots, p_s, \ldots, p_n$. There are $(s-1)!(n-s)!$ ways of adding players (i.e., forming the grand coalition), such that player $i = p_s$. Furthermore let S (of size s) be the coalition that correspond to the players in the order $p_1, \ldots, p_s$. Then the marginal cost of player i in coalition S, is $c(S) - c(S\backslash\{i\})$

The Shapley value for player i is computed as the sum over all the coalitions S, of the marginal cost of player i in the coalition S, multiplied with the probability that the grand coalition is formed that way, i.e.,

$$\phi_i = \sum_{S \subseteq N | i \in S} \frac{(|S|-1)!(|N|-|S|)!}{|N|!}(c(S) - c(S\backslash\{i\}))$$

The Shapley value is a unique solution to a game. It is the only value that satisfies the three properties additivity, symmetry and the dummy-players property (for a proof, see e.g. Driessen, 1988). Furthermore, the Shapley value is efficient and fulfills the anonymity conditions. It is also covariant.

The main theoretical drawback of the Shapley value is that even if the core is non-empty, the Shapley value may not be included in the core, e.g., it does not necessarily fulfill the individual rationality conditions. However, the Shapley value is in the core of every convex game. If certain multiplicity of the extreme points of the core is taken into consideration, the Shapley value is in the center of gravity of the core of a convex game (see e.g., Shapley, 1971 or Driessen, 1988). Another drawback of the Shapley value is that in many cases, the assumption that all possibilities to successively form the coalition are equally probable, may not hold.

A concept related to the Shapley Value, is the *Shapley-Shubik power index* (see e.g., Roth, 1988). It is the Shapley Value adopted to *simple n-person games* (e.g., voting games). A simple game is a game $(N;v)$[5] such that: $v(S) \in \{0,1\}, S \subset N$, $v(N) = 1$ and $v(S) \leq v(T), S \subset T \subset N$.

τ-value

Tijs (1981) introduced the τ-value. Tijs & Driessen (1986) discuss the *cost gap allocation method.* This is the equivalence to the τ-value, for cost games. We will call this method the τ-value, also for cost games. Tijs & Driessen (1986) define a *(cost) gap function* $g(S)$ of a game $(N;c)$, as:

$$g(S) = \begin{cases} c(S) - \sum\limits_{i \in S} m_i & \text{for } S \neq \emptyset \\ 0 & \text{for } S = \emptyset \end{cases} .$$

The rationale of the value is that if a player i is allocated a cost $y_i > m_i + g(S)$ for some coalition S where $i \in S$, then the player i will form the coalition S. In this coalition all players $j \in S, j \neq i$ can get a cost of m_j and player i can absorb the cost $m_i + g(S)$, which is less than the cost allocated to player i.

The non-separable cost m_i, can be seen as a *lower bound* for the cost of player i (since this is the marginal contribution of player i to the grand coalition). Equivalently $m_i + w_i$ can be seen as an *upper bound* for the cost of player i (what i receives if all other players are allocated their marginal cost, in the best coalition S, from the view of i). The τ-value is the unique efficient allocation on the line between the vectors m and w.

The τ-value is based on *separable* (or marginal) and *non-separable* costs. In the first step of such a method, the separable cost of each player $i \in N$, is allocated to player i. In the second step, the remaining, non-separable cost, $g(N) = c(N) - \sum\limits_{i \in N} m_i$, is allocated according to some weight vector $w = (w_1, w_2, \ldots, w_n)$, where $w_i \geq 0, i \in N$. The allocation is $y_i = m_i + \frac{w_i}{\sum\limits_{i \in N} w_i} g(N)$ The weight vector in the τ-value is computed as $w_i = \min\limits_{S | i \in S} g(S)$.

[5] These types of games are most frequently games where the characteristic function refers to payoffs. Therefore we let the characteristic function be v (instead of c as in the cost games).

The τ-value is based on the assumption that for subsets $S \subset N$ one has $c(S) \geq \sum_{i \in S} m_i$, and $\sum_{i \in N} m_i \geq g(N)$. This does not hold for all games, which means that the τ-value can not be applied to all games.

For games with the properties $c(S) \geq \sum_{i \in S} m_i$, and $\sum_{i \in N} m_i \geq g(N)$, the τ-value is efficient, individually rational, possesses the dummy-player and the anonymity property and it is covariant. It is also a continuous function of the characteristic function.

A drawback of the τ-value is that it does not necessarily lie in the core. Driessen & Tijs (1985) present conditions which guarantee that the τ-value lies in the core. In particular they show that for games with a constant gap function, i.e., games such that $g(S) = g(N) \geq 0, S \subseteq N \backslash \emptyset$, the nucleolus, the Shapley value and the τ-value coincide.

The τ-value is discussed in detail in e.g., Driessen (1988). An axiomatization that uniquely defines the τ-value (using properties not described in this thesis) is presented in Tijs (1987).

Other solution concepts

There are a number of methods other than the τ-value, that are based on separable and non-separable costs. Three methods of this type are described by Tijs & Driessen (1986). These methods are efficient and anonymous, but they do not necessarily fulfill the individual rationality condition (i.e., the solutions are not necessarily in the core). The methods are the following: In the *equal charge method*, where the weight vector is $w_i = 1, i \in N$, the non-separable cost is divided equally among the players. In the *alternative cost avoided method*, where $w_i = c(i) - m_i, i \in N$, i.e., the non-separable cost is divided in proportion to the savings that are made for each player by joining the grand coalition instead of acting alone. A method related to the alternative cost avoided method is the *separable cost remaining benefits method*, where $w_i = \min\{c(i), b_i\} - m_i, i \in N$, where b_i is the estimated benefit to player i if only the purposes of player i are served. The idea is that a project would not be undertaken, if the benefit was less than the cost (i.e., if $b_i < m_i$).

Gately (1974) proposed a solution method that uses a cost savings game. He defines each player's *propensity to disrupt*. As pointed out in Straffin & Heaney (1981) the alternative cost avoided method is the same as minimizing the maximal propensity to disrupt.

A (pre-)imputation $\bar{y}$ is said to be *dominated* by a (pre-)imputation vector y if there is a non-empty $S \subseteq N$ such that $y(S) \geq c(S)$ and $y_i \leq \bar{y}_i, i \in S$. von Neuman & Morgenstern (1944) suggested the solution concept of *stable sets* (or *von Neuman-Morgenstern solutions*), as the set, V, of (pre-)imputations that are not dominated by any other vector in V. Every stable set contains the core. Shapley (1971) proves that for convex games the stable set is unique and coincides with the core.

Other solution concepts which we will not discuss further are the *bargaining set* (Aumann & Maschler, 1964 and Peleg 1963), and the *kernel* (Davis & Maschler, 1965).

3.4 Literature Survey

This literature survey covers cost allocation problems, using cooperative game theory. The focus is on cost allocation in games that are closely related to our problems. Much attention is paid to tree games, since these games have been studied extensively in the literature. Tree Games are also related to Traveling Salesman Games. Since this thesis covers the Traveling Salesman Game and the Vehicle Routing Game, the literature covering these problems is studied in detail.

There are also other games that are related in the sense that they produce interesting combinatorial problems. These are for example the Bin-packing Game (Dror, 1990), and the Knapsack Game (Dror, 1990). Other game situations producing combinatorial games, that have been studied are e.g., the Network Design Game (see e.g., Kubo & Kasugai, 1992) and the Assignment Game (see e.g., Shapley & Shubik, 1972). However, these games are not discussed in this thesis.

The *Bin Packing Problem* (BPP) can be described as: Given a finite set of items, $U = \{u_1, u_2, \ldots, u_n\}$, each of a rational size between zero and one, find a partition of U with the smallest number of disjoint subsets $U_1, U_2, \ldots$, such that the total size of each subset does not exceed one. If the cost c_j of each bin j is arranged in an increasing order, the cost will be minimized if the number of bins is minimized.

The BPP Game is the problem of how to divide the total cost between the different items in U. A practical situation that is a BPP is the following: Suppose several companies wish to transport products overseas, using standard containers. The total cost is minimized if the number of containers are minimized. Since the companies may share a container, the total cost has to be allocated among the companies.

The *0-1 Knapsack Problem* can be described as: Suppose that the cost of transportation of a carrier is fixed to C, and that there is a finite set $U = \{u_1, u_2, \ldots, u_n\}$ of potential items to load in the carrier. Each item has an associated volume $v_i > 0$, and a revenue r_i. The total size of the carrier is V. Let $y_i = 1$ indicate that item u_i is loaded, and let $y_i = 0$ indicate that item u_i is not loaded. The 0-1 Knapsack problem is the problem of which items to load, so that $\sum_{i=1}^{n} v_i y_i \leq V_i$ and that $\sum_{i=1}^{n} r_i y_i$ is maximized.

A difference between the 0-1 Knapsack Problem and the BPP, is that in the 0-1 Knapsack Problem only the most profitable items are packed, while in the BPP all items have to be packed. The *Knapsack Game*, is how to assign the shipping cost C to each of the items loaded.

Linear Production Games were studied by Owen (1975). The article of Owen (1975) is interesting since he uses mathematical programming to draw conclusions about the core of the game. Linear Production Games can be described in the following way: Suppose that a set of players, N, is given, where each player is given a vector $b_i, i \in N$, of resources. The resources can be used to produce goods which can be sold at a given market price. The Linear Production game is the problem of how to allocate the total earnings among the players that contributed with the resources. The characteristic function of this game is a Linear Programming (LP) problem, thus it is a special case of LP-Games (see e.g., Tijs, 1992). Owen (1975) uses duality theory in LP to obtain equilibrium prices for the resources, which are used to prove that the core of the Linear Production Game is non-empty.

A number of references in the intersection of mathematical programming and cooperative game theory are mentioned in Tijs (1992).

3.4.1 Tree Games

A *tree* (see Figure 3.1) is a connected graph that contains m nodes and $m - 1$ edges. A *spanning tree* (see Figure 3.2) is a tree that spans all the nodes N in a graph.

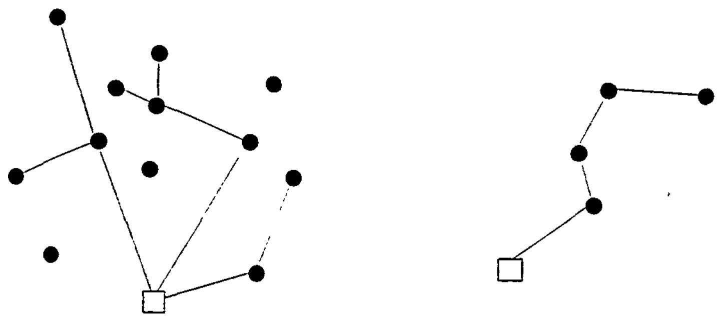


Figure 3.1. Two examples of trees.

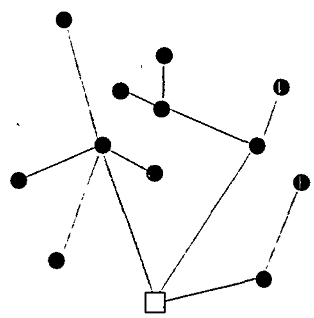


Figure 3.2. A spanning tree.

A *minimum cost spanning tree* (MCST) is a spanning tree where the sum of the edge costs is minimal among all spanning trees.

Given a graph $G = (V, E), V = \{0, 1, \ldots, n\}$ where 0 is the common supplier (or the *root*) and the set $N = \{1, 2, \ldots, n\}$ is the users' set, and E are the edges each with a cost c_{ij} between node i and j. Given the cost data for a graph, presented in Table 3.1, the MCST to the graph of this cost data would be the the tree in Figure 3.3.

Cust.	**0**						
1	5	**1**					
2	8	5	**2**				
3	11	9	4	**3**			
4	6	6	3	6	**4**		
5	9	11	8	7	5	**5**	
6	9	13	12	13	9	6	**6**
7	4	10	11	13	8	9	5

Table 3.1. Cost data ($C = \{c_{ij}\}, c_{ij} = c_{ji}$) for a graph.

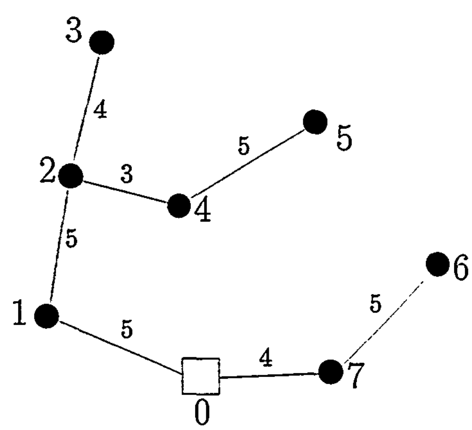


Figure 3.3. A MCST to the cost data in Table 3.1.

A tree structure can be found in many practical situations. A need for cost allocation in a tree would arise when sharing the cost for constructing e.g., a cable-TV network, a telephone network, or a water pipe system (where there is no doubling of connections for e.g., security reasons).

Tree Games, i.e., the problem of how to divide the cost arising in various sorts of trees, is the problem of how to divide the cost between the players in the tree (situated at the nodes of the tree). Tree Games were first introduced by Claus & Kleitman (1973). They discuss a number of plausible cost allocation methods for a MCST. They do not use the notion of the core, but the requirements that they give for an acceptable cost allocation method implies the core.

A MCST-Game is a cooperative game (N;c) where the characteristic function $c(S)$ is a MCST problem.

Bird (1976) gives an erroneous proof (see Granot & Huberman, 1981) that the MCST-Game has a non-empty core. He defines the *irreducible core* of a MCST-Game. The irreducible core is shown to be *stable under union of additional players*, i.e, for solutions y in the irreducible core, if a set of players are added to the MCST-Game $(N;c)$ to form the game $(\bar{N},c)$, there exists solutions $\bar{y}_i \leq y_i, i \in N$. The core is not stable under union of other players. Bird (1976) also gives an example of a Tree Game where multiple sources are allowed, that has an empty core.

Granot & Huberman (1981) showed that a solution in the core of a MCST-Game, referred to as the L-solution, can be read from an associated MCST graph. Thus, the core of a MCST-Game is never empty. Consider a complete graph $G = (V, E)$. In this graph let $\Gamma_N = (V_N, E_N)$ be a MCST. This tree induces a partial order $>$ on $0 \bigcup N$. Write $i > j$ if node j is on the (unique) path connecting node i and node 0 in Γ_N. Under the order $>$ each node $i \in N$ has one immediate predecessor $p(i)$ and a (possibly empty) set of immediate followers $F(i)$. Granot & Huberman (1981) showed that the vector $L(c) \equiv (y_1, y_2, \cdots, y_n)$ (the L-solution) is in the core of the MCST-Game $(N; c)$, where $y_i = c_{ip(i)}$. When the MCST solution is not unique, L(c) may not be unique either. The L-solution to the problem in Figure 3.3 would be $L(c) = (5,5,4,3,5,5,4)$.

A weakness of this solution is that it discriminates users close to the depot. All the followers of a node i benefit from the connection that node i has to the subtree connected to the depot, but in the L-solution node i pays the whole cost of this connection. In this way a node i with $F(i) = \emptyset$ (i.e., each leaf), pays its minimum allocation. Even though the L-solution is a core solution, it does not seem to be the most fair, since it is an extreme solution.

Granot & Huberman (1981) also discuss the core, the nucleolus and the Shapley value of a MCST-Game (N;c) where the MCST, Γ_N, has more than one edge, $p > 1$, incident to the common supplier 0. They show how this game can be divided into p MCST-Games and how the core and the nucleolus easily can be constructed from the solutions to these games. They also show that the simple method can not be applied to compute the Shapley value.

Granot & Huberman (1984) present two efficient procedures for generating cost allocation vectors in the core of a MCST-Game, using so called *weak demand operations* and *strong demand operations.* They also show how the nucleolus can be computed efficiently using strong demand operations. Finally they show that in a MCST-Game, the nucleolus is the unique point of the intersection of the kernel and the core.

Even though MCST-Games always posses an non-empty core, they differ in general from convex games (see e.g., Granot & Huberman, 1982). However, there exists convex MCST-Games (e.g., the games presented in Bird, 1976, Megiddo, 1978a, and Littlechild, 1974). Granot & Huberman, (1982) introduce the class of *permutationally convex* games, and shows that permutationally convex games posses a non-empty core. They also show that both convex games and MCST-Games are permutationally convex games.

Granot et. al. (1996) studies the *Tree Enterprise* and its game. In particular they study the *Standard Tree Enterprise.* This is a tree, where there are non-negative costs on the arcs, one or more players are located in each vertex, the root is not occupied, and only one arc leaves the root. They show that the core of the Standard Tree Enterprise Game is non-empty, and that the kernel consists of a unique point in the core (and therefore coincides with the nucleolus). They describe a procedure to find the nucleolus/kernel, that is based on eliminations of arcs and condensations of players of the tree. For certain trees (e.g., chains) the procedure to find the nucleolus involves $O(n)$ operations.

Megiddo (1978a) has presented a procedure to find the nucleolus of a tree game in $O(n^3)$ operations, and Galil (1980) has reduced the number of operations needed for Megiddo's procedure to $O(n \log n)$.

There are also other Tree Games that have been studied. Megiddo (1978b) treats the Steiner Tree Game, which is a game where the members of $N \bigcup 0$ is just a subset of the nodes of G, i.e., the players are not limited to use only arcs linking two members of $N \bigcup 0$, but they may use some additional arcs. Megiddo (1978b) proves that the core of such a game may be empty. He also shows that if a MCST-Game $(N; c)$ on a graph G is modified such that the characteristic function $\bar{c}(S)$ in the game $(N; \bar{c})$ is a Steiner Tree on G, the core of $(N; \bar{c})$ is included in the core of $(N; c)$, and that the L-solution of Granot & Huberman (1981) is included in the core of $(N; \bar{c})$.

The 1-tree Game is studied by Göthe-Lundgren et. al. (1992). The 1-tree problem (see e.g., Held & Carp 1970, 1971) is the problem of connecting the n users to each other by a MCST, Π_N, and Π_N to the common supplier by the two least cost edges incident to the supplier. The 1-tree problem is interesting, since it is a relaxation to the Traveling Salesman Problem. Göthe-Lundgren et. al. (1992) show that the core of the 1-tree Game is non-empty. They also present a procedure to find the nucleolus, using a constraint generation approach, similar to the procedure of finding the nucleolus in the Traveling Salesman Game and the Vehicle Routing Game which is presented in Chapters 5 and 6. The subproblem in the constraint generation procedure is the Node Weighted Steiner Tree problem, which is studied in e.g., Engevall et. al. (1995) and Segev (1987).

Bjørndal (1995) also studies the computation of the nucleolus of tree and 1-tree Games.

3.4.2 Traveling Salesman Games

Hoffman & Wolfe (1985) introduce the Traveling Salesman Problem (TSP) as follows:

> *'If a salesman, starting from his home city, is to visit exactly once each city on a given list and then return home, it is plausible for him to select the order in which he visits the cities so that the total of the distances traveled in his tour is as small as possible.'*

A *Hamiltonian cycle* is a cycle that visits each node in a graph exactly once. The TSP can in graph-theory terminology be defined as the problem of finding the minimum weight

Hamilton cycle in a weighted complete graph. The TSP can also be defined mathematically as:

$$
\begin{array}{rlcll}
z = & \min \displaystyle\sum_{i\in N}\sum_{j\in N} c_{ij}x_{ij} & & & \\
& \displaystyle\sum_{i\in N} x_{ij} = & 1, & j \in N & [3.3] \\
& \displaystyle\sum_{j\in N} x_{ij} = & 1, & i \in N & [3.4] \\
& \displaystyle\sum_{i\in S}\sum_{j\in S} x_{ij} \leq & |S|-1, & \left\{\begin{array}{l} S \subset N \\ |S| \geq 2 \end{array}\right. & [3.5] \\
& x_{ij} \in & \{0,1\}, & i,j \in N & [3.6]
\end{array}
$$

where:

$$x_{ij} = \begin{cases} 1 \text{ if the arc between nodes } i \text{ and } j \text{ is used} \\ 0 \text{ otherwise} \end{cases}$$

Conditions [3.3] states that exactly one arc should be used entering node j. Conditions [3.4] states that exactly one arc should be used leaving node i. A *subtour* is a tour that covers only the nodes in some set $S \subset N$. A subtour covering the nodes S, has exactly $|S|$ arcs. In the TSP, no subtours may be included and the conditions [3.5] are the subtour-breaking inequalities.

The TSP-Game is the problem of how to divide the total cost $c(N) = z$ of the optimal solution to the TSP, among the players covered by the tour.

Dror (1990) shows that the core of a TSP-Game without a home city is empty. This is shown in the following example. For each player i we have $c(i) = 0$. The individual rationality constraints then require that $y_i \leq c(i) = 0$. Furthermore, assume that $c(N) > 0$. Then we require that $\sum_{i\in N} y_i = c(N)$, which clearly is a contradiction. Thus no outcome y is in the core.

Dror (1990) shows in the following example that the characteristic function of a TSP-Game (N;c), without a home city, may not be subadditive: In the graph in Figure 3.4, let the distances of the arcs not shown be very large, and let $S = \{1,4\}$ and $T = \{2,3\}$. Then $c(S) = c(T) = 2$ and $c(S \bigcup T) = 22$. Clearly $c(S) + c(T) = 4 \not\geq c(S \bigcup T) = 22$. Thus the TSP-Game without a home city is not subadditive in general.

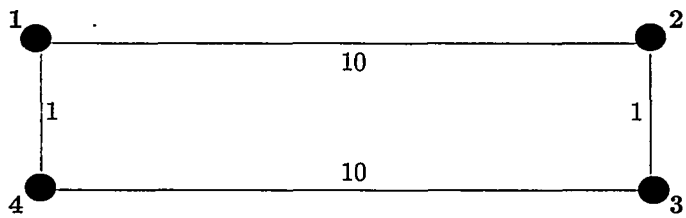


Figure 3.4. A TSP-Game without a home city, that is not subadditive. (Dror, 1990).

Dror (1990) also discusses the TSP-Game with a home city. If a home city is included, the TSP-Game is subadditive.

Tamir (1989) gives several examples of games where the core is empty. Two examples of graphs with an empty core for the corresponding TSP-Games are shown in Figure 3.5 (all arcs have unit length).

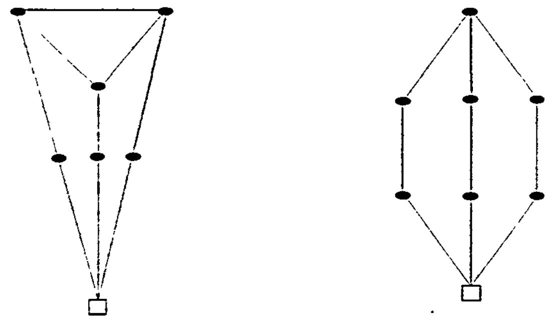


Figure 3.5. Examples of graphs for TSP-Games with an empty core.

Tamir (1989) also gives some sufficient conditions for the core to be nonempty in special cases of the TSP-Game. As a corollary he also shows that if the graph G of a TSP-Game with a symmetric cost matrix[6], and has at most 5 nodes ($|N| \leq 4$), the TSP-Game has a non-empty core.

Kuipers (1993) shows that the core is also non-empty for TSP-Games with a symmetric cost matrix that has 6 nodes ($|N| \leq 5$).

Potters et. al. (1992) describe the fixed route TSP-Game. In this TSP-Game the order in which the players are served, remains the same for all coalitions. They show that if the cost matrix fulfills the triangle inequality[7]. , and the fixed route is a minimal cost TSP-route, then the core is non-empty for the fixed route TSP-Game. Fishburn & Pollak (1983) gives three conditions that should be fulfilled for a cost allocation method. These conditions are modified slightly by Potters et. al. (1992), into efficiency, individual rationality, and that each player pays at least his marginal cost. They then describe the simple allocation method $y_i = \lambda c(i) + (1-\lambda)m_i$, where λ is chosen such that $\sum_{i \in N} y_i = c(N)$. This method gives a cost allocation satisfying the three conditions. However, the allocation does not necessarily lye in the core.

Potters et. al. (1992) give an example of a TSP-Game with a cost matrix that fulfills the triangle inequality, and that has an empty core. However they show that if any cost matrix $C = \{c_{ij}\}$ is modified in the way described below, the corresponding modified TSP-Game has a non-empty core. Define a matrix $L(a,b) = \{l_{ij}^{ab}\}$:

$$l_{ij}^{ab} = \begin{cases} a_i + b_j \text{ if } i,j \in N \bigcup 0 \text{ and } i \neq j. \\ 0 \text{ if } i = j. \end{cases}$$

The numbers a_i and b_i can be seen as an entry and exit tax that has to be paid when entering and leaving, respectively, city i. If $C' = C + L(a,b)$, Potters et. al. (1992)

[6]A symmetric cost matrix, C has $c_{ij} = c_{ji}$ for all $i,j \in N$.
[7]Triangle inequality states that for a cost (or distance) matrix $C = \{c_{ij}\}$, $c_{ik} \leq c_{ij} + c_{jk}$

show among other things that there exists a real number $\beta(C)$ such that the TSP-Game corresponding to the cost matrix C' has a non-empty core, if and only if $a_0 + b_0 \geq \beta(C)$. Potters et. al. (1992) finally show that TSP-Games in a graph have a non-empty core if $|N| \leq 3$, and if the cost matrix fulfills the triangle inequality (even if the matrix is not symmetric).

In a working paper, Engevall et. al. (1996) shows an example of an empty core in a (Euclidean TSP-Game). The Euclidean TSP-Game is the game where the graph is complete (i.e., there is an arc between each pair of players $i, j \in N \bigcup 0$) and the length of an arc is equal to the Euclidean distance between the players. The example they present is the following:

Suppose that a large number of customers are spread out on a circle, and along 3 radii, where the depot is located in the center of the circle. Suppose that the six coalitions, $S_1, \dots, S_6$ are identified, where S_1, S_2 and S_3 correspond to the customers along each of the radii, and S_4, S_5 and S_6 correspond to parts (in a natural fashion) of the circle (see Figure 3.6a).

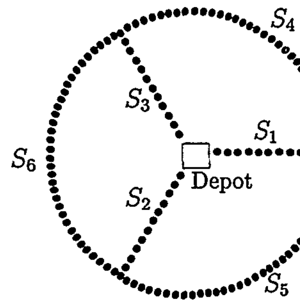


Figure 3.6a. Example of a TSP, where the corresponding Euclidean TSP-Game has an empty core. (Engevall et. al., 1996).

An optimal solution to the TSP in Figure 3.6a is e.g., serving the coalitions in the order $S_1, S_5, S_2, S_6, S_4, S_3$ (see Figure 3.6b). If the radius of the circle is r, the optimal objective function value is $(2\pi + 4 + \sqrt{3})r - \epsilon$, where ϵ is small if the customers are close to each other.

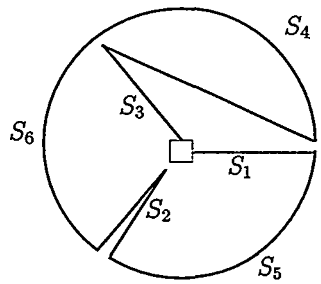
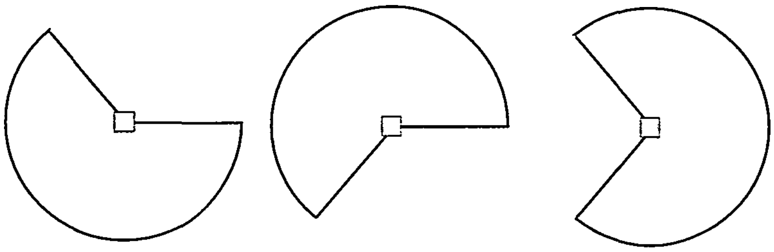


Figure 3.6b. An optimal solution to the TSP of Figure 3.6a.

The characteristic function value for the grand coalition, $c(N)$ in the corresponding TSP-Game is $(2\pi+4+\sqrt{3})r-\epsilon$. We also know that for a solution in the core, the inequalities [3.6a]-[3.6c] (see Figure 3.6c) must hold:

$$
\begin{array}{llllllllll}
+y_1 & & +y_3 & & +y_5 & +y_6 & \leq c(\{S_1,S_3,S_5,S_6\}) & = & (\frac{4\pi}{3}+2)r & [3.6a] \\
+y_1 & +y_2 & & +y_4 & & +y_6 & \leq c(\{S_1,S_2,S_4,S_6\}) & = & (\frac{4\pi}{3}+2)r & [3.6b] \\
 & +y_2 & +y_3 & +y_4 & +y_5 & & \leq c(\{S_2,S_3,S_4,S_5\}) & = & (\frac{4\pi}{3}+2)r & [3.6c]
\end{array}
$$



$\{S_1,S_3,S_5,S_6\}$ [3.6a] $\{S_1,S_2,S_4,S_6\}$ [3.6b] $\{S_2,S_3,S_4,S_5\})$ [3.6c]

Figure 3.6c. Optimal solutions to the TSPs for the coalitions corresponding to constraints [3.6a]-[3.6c].

The inequalities [3.6a] - [3.6c] imply that

$2y_1+2y_2+2y_3+2y_4+2y_5+2y_6 \leq (4\pi+6)r$ [3.7]

which is equivalent to

$y(N) \leq (2\pi+3)r$ [3.8]

However, the efficiency condition in the core formulation require that $y(N) = c(N) = (2\pi+4+\sqrt{3})r-\epsilon$, which can not hold (for ϵ small enough) at the same time as [3.6a]-[3.6c] hold. Thus, the TSP-Game corresponding to Figure 3.6a has an empty core.

3.4.3 Vehicle Routing Games

In the *Vehicle Routing Problem* (VRP) a set of customers, each with a specific demand, and a central depot from which the customers are supplied by a number of vehicles, are considered. Each vehicle has a specified capacity, and it is assumed that each vehicle is used in one route at the most. A route is a path that starts at and returns to the depot, and passes at least one customer on the way. Given the cost of transportation using each vehicle, the Vehicle Routing Problem is the problem of minimizing the total transportation cost, given that the capacity of each vehicle is not exceeded, and the demand of the customers is satisfied.

If the fleet is homogeneous, i.e., the capacities and costs of all trucks are equal, and if it is assumed that each customer is visited exactly once, the VRP is the *basic* VRP. The basic VRP[8] can be formulated mathematically in the following way:

Assume that for each subset of customers for which the total demand does not exceed the vehicle capacity, a minimal cost route, r, is known. Denote the cost of such a route q_r, and the set of all such cost routes R. Let:

$$a_{ir} = \begin{cases} 1 \text{ if customer } i \text{ is covered by route } r \\ 0 \text{ otherwise} \end{cases}$$

$$x_r = \begin{cases} 1 \text{ if route } r \text{ is chosen} \\ 0 \text{ otherwise} \end{cases}$$

Then the VRP can be formulated as a set partitioning problem (SPP):

$$\begin{aligned} z = \min \quad & \sum_{r \in R} q_r x_r \\ \text{s.t.} \quad & \sum_{r \in R} a_{ir} x_r = 1, \quad i \in N \\ & x_r \geq 0, \quad r \in R \\ & x_r \text{ integer}, \quad r \in R \end{aligned}$$

The minimal cost, q_r, of each route r is the solution to a TSP (see Chapter 3.4.2.) Since the number of columns is large, this formulation of the VRP is most useful if column generation (see e.g., Balinski & Quandt, 1964) is applied.

The VRP has many natural applications in distribution situations. The basic VRP and many of its extensions have been studied extensively in the literature (see e.g., Golden & Assad, 1988).

The *Vehicle Routing Game* is a game (N;c) where the total cost of the VRP is to be divided among the players (i.e., the customers). The VRP-Game is studied in Göthe-Lundgren et. al. (1996), where the corresponding VRP is the basic VRP.

Göthe-Lundgren et. al. (1996) prove that the number of inequalities that defines the core in the VRP-Game can be reduced significantly by only consider coalitions that can be served by one vehicle. They give an example of a VRP-Game with an empty core:

The location of three customers and a depot is shown in Figure 3.7. The costs of the arcs are also given in the figure. Each customer has a demand of one unit, and the capacity of each vehicle is two units (suppose a supply of at least two trucks). The characteristic function values of the game $(N;c)$ where $N = \{1,2,3\}$, are:

$$\begin{aligned} c(S) &= 2, \quad |S| = 1 \\ c(S) &= 3.7, \quad |S| = 2 \\ c(N) &= 5.7 \end{aligned}$$

The value of the grand coalition correspond to e.g., the routes Depot-1-2-Depot and Depot-3-Depot. Due to the symmetrical roles of the three customers, the nucleolus is

[8]For simplicity we will refer to the basic VRP simply as VRP

$y = (1.9, 1.9, 1.9)$. For any coalition $|S| = 2$ we have $y(S) = 3.8 \not\leq c(S)$, i.e., the nucleolus does not fulfill the core inequalities. Since the nucleolus belongs to any non-empty core, we can conclude that the core is empty for this game.

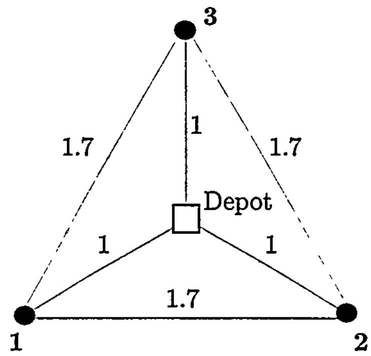


Figure 3.7. Example of a VRP-Game with an empty core. (Göthe-Lundgren et. al., 1996).

Göthe-Lundgren et. al. (1996) prove that the core of the VRP-Game is empty if and only if there is an integrality gap between the optimal solution to the SPP formulation and the optimal solution to the LP-relaxation of the SPP formulation. Finally they show how the nucleolus can be computed using a constraint generation approach.

The models and methods presented in Göthe-Lundgren et. al. (1996) will be discussed in detail in Chapters 5 and 6.

3.4.4 Applications

Solution concepts from Game Theory have been applied to several real-life problems. Some of them will be described in this chapter.

Aircraft landing fees

Littlechild & Thompson (1977) present a case where aircraft landing fees are computed using cooperative game theory. Using computations based on data from the airport of Birmingham in 1968-69, they investigate:

- If the airport of Birmingham was built at the right size, i.e., if the size of the airport is such that maximal net present value of benefits is maximized.
- If the movement charges applied were fair, i.e., if:
  - A smaller aircraft pays less than a larger (fairness criterion I).
  - The charge of a larger aircraft does not exceed the charge of a smaller aircraft by more than the difference in cost for the two aircraft (fairness criterion II).

- If the charges were efficient[9], i.e., if:
  - The charges give the airport authority incentive to build an airport of optimal size.
  - There is no cross subsidization between different sizes of aircraft.
  - The pricing scheme gives all aircraft that can use the airport an incentive to do so.
- How the charges compare to various rules of thumb derived from game theory. They study the Shapley value and the nucleolus.

Littlechild & Thompson (1977) conclude from their analysis that the chosen runway size of the airport is justified, provided that the smaller runway sizes are justified. They can not conclude weather it would be optimal to build a larger runway, since data corresponding to a larger runway than the actual one, can not be collected.

They also conclude that the landing charges not quite (but almost) fulfill neither the efficiency condition, nor any of the conditions on fair charges.

In the game theoretical context they define each movement (take-off and landing) as one player, each with a requirement on the runway depending on the size of the aircraft. Littlechild & Thompson (1977) compute both the benefit and the cost of each movement. Thus they are able to compute the nucleolus for two different games, and conclude that the nucleolus in the game concerning benefits is close to fairness criterion I, and the nucleolus in the cost game is close to fairness criterion II. However, the actual charges are far from the nucleolus computed in each of the games.

Littlechild & Thompson (1977) can not compute the Shapley value for the benefit game exactly, but they compute an approximative value. If the cost game is studied, the Shapley value can be computed easily, through a simplification of the formula for computing it. They conclude that the approximative Shapley value for the benefit game is closer to the actual charges than the nucleolus, and that the Shapley value for the cost game is even closer to the actual charges. They also say that airport economists actually have suggested allocation rules for allocating construction costs, that coincides with the Shapley value for the cost game.

Water resource planning (Tennessee Valley Authority)

Straffin & Heaney (1981) analyze the Tennessee Valley Authority (TVA) from a game theoretical perspective (see also e.g., Parker, 1943). The TVA was formed in the 1930's and was authorized to undertake large scale, multiple purpose water resource development projects in the Tennessee River basin.

[9]This efficiency conditions is not to be mixed up with the efficiency requirement in a cooperative game, described in Chapter 3.3.

The major purposes to be served were improved navigation, flood control and provision of electric power, with subsidiary benefits to irrigation, national defense and the production of fertilizer. It was decided that the cost were to be allocated to the different purposes involved.

TVA engineers and consultants examined a number of cost allocation methods. Some of these methods are closely related to concepts in cooperative game theory. These concepts are discussed further in Straffin & Heaney (1981).

Straffin & Heaney (1981) also present the computations of a specific game using methods suggested in the TVA, and concepts from cooperative game theory (e.g., the nucleolus and the Shapley value). According to Parker (1943), the final cost allocation of the TVA was not based on any one mathematical formula, but rather fixed by judgment.

Water resource development

Young et. al. (1982) discuss different methods for allocating costs of water supply projects among users. They discuss both simple proportional methods, as well as solution concepts from cooperative game theory (e.g., the Shapley value and variants of the core).

Young et. al. (1982) exemplify the different methods in a case study. An association called Sydvatten Company was formed by some municipalities in Skåne in southern Sweden. The association started to design a project for obtaining water from a lake from outside the region, via a tunnel. A total of 18 municipalities (not only those forming Sydvatten Company) could participate in the project. 18 players can form 2^{18} -1 = 262 143 different coalitions. To compute or to estimate the characteristic function of each coalition would be impossible. Therefore the 18 municipalities were divided into six independent units. When computing the characteristic function, considerations were taken that not all coalitions would actually form in practice. For some coalitions $S = S_1 \bigcup S_2$ the characteristic function was simplified to $c(S) = c(S_1) + c(S_2)$.

Young et. al. (1982) show that in the case studied, the proportional allocation principles suggested does not meet the individual rationality constraints. The separable cost remaining benefits method does not fulfill the constraints of the core, and neither does the Shapley value. Using an example they show that if the cost of the project turn out to be higher than was planned, the nucleolus will allocate a *lower* cost to one of the participants. The Shapley value and the normalized nucleolus allocates any increase in cost equally among the players. These properties of the nucleolus, the Shapley value and the normalized nucleolus are considered as undesirable properties by Young et. al. (1982). The remaining method they studied is the proportional nucleolus, which is included in any non-empty core, and allocates an increase in proportion to the cost savings of participating in the game. The proportional nucleolus is still fairly complicated to compute (as are the nucleolus, the normalized nucleolus and the Shapley value).

Straffin & Heaney (1981) conclude that the proportional nucleolus is a strong candidate for a cost allocation method in water resources development, even though they state that there is no one best method. In reality, the cost was simply allocated in proportion to population.

Investment in electric power

Gately (1974) studies the problem of distribution of gains from regional cooperation in planning investment in electric power. He uses the case of four states of the Southern Electricity Region of India. Gately uses a number of intuitive methods, as well as concepts from cooperative game theory (i.e., the core, the kernel and the Shapley value). Gately (1974) also introduces the concept of a players propensity to disrupt.

Analyzing the cost allocation using the different methods, Gately (1974) concludes that among the methods used, only the Shapley value and the value that equalizes each player's propensity to disrupt are candidates for mutually acceptable methods.

Telephone billing rates

Billera et. al. (1978) consider the problem of determining telephone billing rates at Cornell University. The problem is divided into two parts. The first part regards the problem of which services (out of four possible) to buy, in order to provide long-distance calling. The second part regards the problem of how to charge the users for the calls.

The costs consist to a large extent on components that are fixed, i.e., do not vary with the usage of the lines. Some of the services that can be bought, allow for a maximum usage to a fixed cost. The cost for the other services is mainly proportional to usage time or usage time and distance. The usage, specified to a maximum in some services, is accumulated during the month. This means that in the beginning of the month the marginal cost of a call is 0, while at the end of the month it may be greater than 0, if supplementary services have to be bought. However, it is not reasonable to charge little to calls in the beginning of the month, and more at the end of the month.

The client in the study (Cornell University) required that the rates must exactly cover the expenses. They must also be 'fair' or 'symmetric' in the sense that two calls made to the same destination during the same period of the day, must be charged the same, regardless of e.g., the purpose of the calls, or the office that placed the call.

The game is modeled as a non-atomic game (i.e., an infinite, or large, number of players), where the players represent the individual calls. Billera et. al. (1978) use the solution method of the Aumann-Shapley value (Aumann & Shapley, 1974) in order to suggest the telephone billing rates. The rates computed by Billera et. al. (1978) was published by Cornell University, and used for billing purposes.

Chapter 4

Modeling of the Norsk Hydro Case

In this chapter we present the categorization of the distribution planning situation of Norsk Hydro, and discuss the disaggregation of the situation that is made. We also discuss important presumptions that we have made in the modeling, including the problem of allocating the remainder. Finally we present the data collected in the cases that we have studied, i.e., in the Traveling Salesman Game and the Vehicle Routing Game.

4.1 Problem Categorization

The transportation problem of Norsk Hydro can be categorized as a Probabilistic Vehicle Routing Problem (PVRP) (see e.g., Jaillet & Odoni, 1988). A PVRP can be described as a Vehicle Routing Problem (VRP), where only a subset of the customers have to be visited, at a given instance of the problem. Since Norsk Hydro do not know exactly which day they will receive orders from their customers, their distribution planning problem is a PVRP. If it was Norsk Hydros responsibility to assure that the customers did not run out of stock, the problem could be seen as an Inventory Routing Problem (see e.g., Dror & Ball, 1987).

If Norsk Hydros cost allocation problem is to be studied, at least two cases can be identified, depending on the aim of the cost allocation.

If the aim is to allocate the actual costs of tours that have taken place, the problem can be seen as a game in a Multiperiod Vehicle Routing Problem (MVRP). It can be described as the problem of dispatching vehicles to satisfy multiple demands for services that evolve in a real-time fashion. The VRP is no longer stochastic, since the demand and the location of the customers are known.

Since one of our aims of the study is to acquire a deeper understanding of the characteristics of the cost allocation problem, it is necessary to further disaggregate the problem. The MVRP is a difficult problem to solve. A disaggregated problem allows us to study what details that are important to take into consideration e.g., when solving the more complicated MVRP-Game.

The MVRP may be simplified and disaggregated into a number of single period VRPs, e.g., a number of VRPs that each cover the customers served during one day. The (single period) VRP can in turn be simplified by studying a number of Traveling Salesman Problems (TSP), e.g., the tours that were carried out.

The study of the less complex TSP-Game and VRP-Game, can facilitate the understanding of the aspects needed to be considered when studying a more complicated game. It can also be interesting to compare the cost allocations for various solution concepts in the TSP-Game, with the results in the VRP-Game. These comparisons may indicate which solution concepts that can be used in a less complicated game to approximate the solutions of a more complicated game. This would be useful in the development of an operational tool for the cost allocation problem.

If the aim is to roughly estimate the long-term profitability of a customer or customer group, the stochasticity of the PVRP should not be excluded. The cost allocated to a customer or customer group, would risk to depend too much on the situation in the case studied. If a day when the planning situation was more difficult than usual (in the sense that it leads to higher costs than usual), was included in the game, this could lead to an disadvantageous estimation of the costs of the customers that happened to be served that day. The conclusion is that the game chosen for the study either has to be kept stochastic in some way, or it has to be large enough for any unusual situations to be of minor importance. We do not further discuss these possible approaches.

In the TSP-Game we study one tour that has been carried out. In the VRP-Game we study the VRP that corresponds to the tours carried out during one day, delivering one product from one depot.

4.2 Presumptions

In order to model the problem, a number of presumptions are necessary. Some have been mentioned implicitly in previous chapters, some are new to this chapter.

The most important presumption, is that we assume that the goal of Norsk Hydro, or more precisely of the dispatcher, is to minimize total transportation cost. As mentioned in Chapter 2.1, the dispatcher may take into account other objectives than the minimization of costs. If these other objectives that she may have, not are based on rational decisions, they are difficult or impossible to model. In our models we need to compute the cost of tours that never took place. The best assumption we can make about these tours is that they would be designed optimally (according to our model), i.e., at a minimal cost.

This leads us to the formulation of the characteristic function, which will be defined as the solution to an optimization problem that corresponds to the planning situation in each case (i.e., a TSP and a VRP respectively).

Another important presumption, is that we assume that the goal of the customers (i.e., gas stations) is to minimize the transportation cost allocated to them. This means that a customer i is assumed to prefer a cost allocation vector y^1 to a vector y^2, if and only if $y_i^1 < y_i^2$. In practice it could occur that the solution y^2 is preferred to y^1 for some non-monetary reason, e.g., because applying solution y^1 would mean that customer i is served at an too early hour, according to her preferences.

At the Göteborg depot, only truck types 30, 32, 33 and 34 are available normally. We only consider these to be in the fleet of the TSP-Game, even if occasionally trucks of other sizes are borrowed from a neighboring depot. Furthermore, in the VRP-Game we only consider the types 33 and 34 to be in the fleet. These are the only truck sizes that would actually be used for gas transport. Our models and solution methods will still be applicable if more truck types were included in each of the games.

The maximum capacities of the trucks in the TSP are assumed to be the gross capacities, presented in Table 2.1 in Chapter 2.1, except for truck type 33. Since this is the truck used in the TSP, and it carried 45.1 m^3, we assume the capacity of truck type 33 to be this volume. The maximum capacity of the trucks in the VRP, has been decided by evaluating old data about the three trucks that were actually used the day of the VRP that we study. The capacity of the truck OMB575 (type 33) was set to 46.0 m^3 and the capacity of the trucks KFK382 and BSS420 (type 34) was set to 53.4 m^3. In both the TSP and the VRP, we assume the supplies of each truck type to be large.

When we study the problem of gas deliveries, we do not take into account that two different qualities of gas actually are delivered (regular and premium). We simply study *gas* as one product. This simplification does not have a large effect. Of course the qualities can not be mixed in the different compartments of the truck. However, as the greatest limitation of a truck is the weight and not the volume, it is assumed that the truck can take any combination of the two qualities as long as the *sum* of the weights (and volumes) does not exceed the capacity limit of the truck.

All deliveries are given in m^3 and are rounded off to one decimal. The deliveries to each customer are presented in Table 4.5 in Chapter 4.5. The costs based on quantity are computed after the round offs have been made. We suppose that the demand of a customer is equal to the amount delivered to her. In reality, true demand may be more as well as less than what was delivered.

We assume that the customers actually served in the TSP-case and the VRP-case respectively, were the only customers that had to be served. It is possible that one of the customers served, did not have to be served that day. It is also possible that a customer that should have been served was *not* served, for some reason.

We have placed the customers geographically as precisely as possible, given the data that we had. The placement in the data base of RouteLogix has been made according to the area code in the address of each customer, except for a few cases where we have tried to be even more detailed. The distances in the distance matrix are computed using

RouteLogix. The distance matrix is assumed to be symmetrical[1]. This is not always the case in reality as it may not be possible to use the same roads, e.g., because a road used (in one direction) is a one-way road. We have only computed the distance between each pair of customers once, and simply assumed the distance matrix (and thus the cost matrices) to be symmetric. The error due to this should be very small.

When we compute the costs for the arcs between all customers, the costs (in SEK) are rounded off to one decimal. As will be seen in Chapter 5.1.2 this leads to a different theoretical conclusion about the TSP-Game that we study. However it should not be of any significane to the computational results or to the conclusions. All costs and figures presented in the thesis are also rounded off to one decimal even if more decimals are used in the computations.

We only consider the possibility of delivery from one depot. In practice it is possible that customers located close to an area covered from another depot, are served from this other depot, if the dispatchers of the two areas form an agreement for this. We do not consider the possibility of *split delivery*, i.e., to get the total demand of a customer satisfied from two deliveries.

We do not allocate the cost remainder. A discussion of how this could be done in a practical situation, and a motivation to why we choose to not allocate it, is discussed in the next chapter.

4.3 Allocation of the Cost Remainder

As in Chapters 2.2.3 and 3.2, define:

γ_N	=	The total remainder in the Vehicle Routing case.
γ_S	=	The remainder, due to the difference between the actual cost and the optimal cost of a tour covering the customers in S.
γ^T	=	The remainder due to the difference in cost between the actual routes and the optimal Traveling Salesman solutions to these routes.
γ^V	=	The remainder due to the difference in cost between an optimal Vehicle Routing solution and the sum of optimal Traveling Salesman solutions. for the coalitions that correspond to the actual routes.
S^O	=	The set coalitions that correspond to the routes in the optimal Vehicle Routing solution.
$c(S)$	=	The cost for an optimal tour covering the customers in S.
$c_\gamma(S)$	=	The actual cost for a tour covering the customers in S.
$y(S)$	=	The sum of the costs allocated to the customers in $S \subseteq N$.

If the actual total cost $c_\gamma(N)$ is to be divided, there are of course methods for doing this in practice.

[1]The distance going from customer i to customer j is equal to the distance going in the opposite direction.

In Chapter 5.1 and 6.1, we define the characteristic function for the games we study. The characteristic function is defined as c(S). If γ_S could be estimated for each coalition, $S \subset N$, this could lead us to another characteristic function $c_\gamma(S)$, which would define another game. This would be a possible way of handling the remainder, using the same principles as those we suggest.

The remainder, γ_S, could be estimated to be in the same proportion to $c(S)$, as γ_N is to $c(N)$. A game with such a remainder used to compute $c_\gamma(S)$ would be strategically equivalent to the game with characteristic function $c(S)$. The result of the computations would be the same as allocating the remainder γ_N in the same proportion as $c(N)$ is allocated.

Another way to allocate $c_\gamma(N)$ is to let the grand coalition pay the total cost $c_\gamma(N)$, but still compute $c(S)$ as the optimal cost for each coalition $S \subset N$. Since we will define the characteristic function to be the *optimal* solution to a TSP and a VRP respecitvely, this would turn the game that we define into another game. In the VRP-case, if the total cost, $c_\gamma(N)$, was to be allocated, while computing $c(S)$ as optimal solutions to a VRP covering the customers in S, the core would be empty as soon as $\gamma_N > 0$. This can be shown in the following way: If $\gamma_N > 0$ in the VRP-case, and the total cost $c_\gamma(N)$ is to be allocated this means that a solution y in the core must fulfill:

$$y(N) = c_\gamma(N) = c(N) + \gamma_N = c(N) + \gamma^V + \gamma^T > c(N) = \sum_{S \in S^O} c(S)$$

However for any core solution, we also have the requirements $y(S) \leq c(S), S \subset N$, and thus:

$$y(N) = \sum_{S \in S^O} y(S) \leq \sum_{S \in S^O} c(S) < y(N)$$

which is a contradiction, and proves that the core is empty for this game. Solution concepts such as the nucleolus and the Shapley value could still be computed, since they do not require the core to be nonempty.

If more data was collected about each tour, it could be possible to reduce the remainder. If the reason for a part of the remainder could be traced to a certain customer, that part of the remainder could be interpreted as a *customer specific remainder* of that customer. However, it would still be difficult to compute what that customer specific remainder would be in any coalition other than the actual coalition. For example, suppose that a time-window restriction says that customer i has to be served before 8.00 a.m. If this restriction is included in the mathematical model, there would be a problem of how to take this restriction into consideration when computing the costs for serving coalitions that does not include i, $S|\imath \notin S$.

Another possible way of dividing the total cost, $c_\gamma(N)$, in practice would be to simply divide γ_N equally among the players in N.

It is difficult to argue that the customers have no reason to cover any cost that is non-optimal and that therefore the remainder should be paid by (i.e., allocated to) Norsk

Hydro. The objective is to allocate the costs of Norsk Hydro. If part of the cost is returned to Norsk Hydro, the problem is not solved. Furthermore, in the accounts this would mean that not all of the costs are allocated, which might be undesirable.

Since we can not find any theoretical motivations for allocating γ_N in one way or another, we will not allocate the remainder. We shall only consider the optimal cost $c(S), S \subseteq N$ when we do our computations. In Chapters 4.4 and 4.5 the size of the remainder is presented for the problems studied.

4.4 Data in the Traveling Salesman Game

Since we already had quite detailed data for the deliveries of gas and gas-oil, during the period September 18, 1995 to September 29, 1995, we chose to study a tour during this time period.

We wanted to study a Traveling Salesman Game that was large enough to show interesting properties in the modeling phase as well as in the solution phase. Yet, the problem should be small enough to allow a study of depth from different perspectives. On most gas tours, between one and five customers are served, while on the gas-oil tours most frequently between five and 20 customers are served. We thought that computationally it would be possible to handle around five customers and that this number would be large enough to show some interesting properties. We also searched for a tour that described a loop, since a tour back and forth along the same road may not have a structure typical to a TSP.

To be able to extend the problem into a cost allocation problem in a VRP-Game, the chosen tour should be on a day when there were several tours from the same depot, but still not too many to be impossible to handle computationally. To make the problem less complicated, there should be no shared deliveries from that depot during that day.

The choice fell on studying gas deliveries from the Göteborg depot, the second tour made by truck OMB575 (license plate) on September 20, 1995. This tour satisfies all the above criteria. The tour involves five customers in a loop. The same day there were 8 gas tours from Göteborg to a total of 21 gas customers and no shared deliveries.

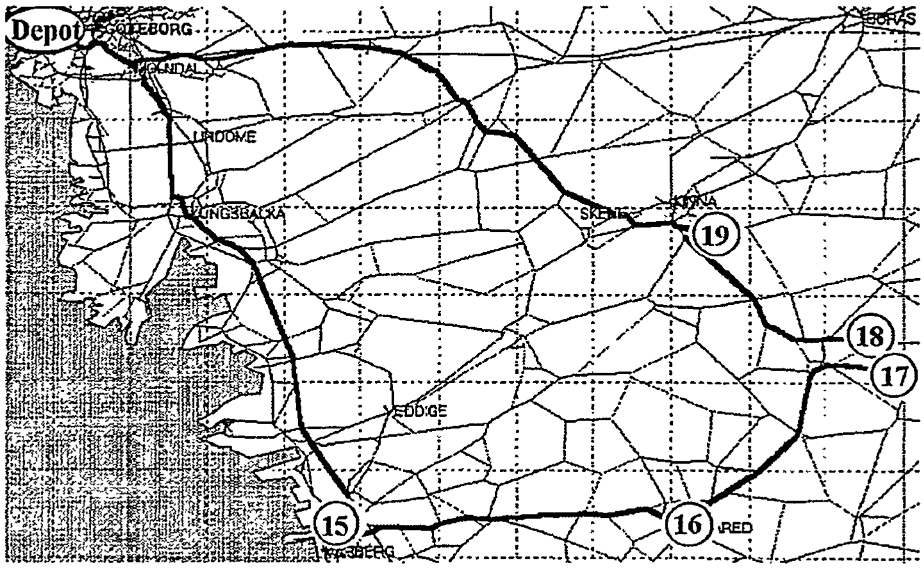


Figure 4.1. Map of the studied TSP-tour.

The detailed data for the tour and the customers of the tour are the following:

Number of customers, $\|N\| =$	5
Truck type:	33
A-stops:	5
B-stops:	3
Actual length:	250 km
Optimal length:	239.7 km
Delivered volume:	45.1 m^3

The customer specific data are as follows:

Customer i^2	Customer location	Demanded volume (m^3)	Customer specific cost (Truck type 33) (SEK)
15	Varberg	27.6	399.3
16	Ullared	3.0	79.6
17	Överlida	8.5	162.1
18	Mjöbäck	3.0	79.6
19	Öxabäck	3.0	79.6

Table 4.1. Customer specific data of the studied TSP-tour.

In the tour we have been studying the costs types are:

Cost type		Cost (SEK)	% of total cost
Base-time	(B_{33})	198.2	5.6
City driving supplement	(τt_{33})	72.0	2.0
A-stop	$(\lvert S \rvert a_{33})$	227.5	6.5
B-stop	$(\beta \sum_{i \in N} b_i)$	60.0	1.7
Load/Unload	$(d_{33} \sum_{i \in N} D_i)$	512.8	14.5
Mileage allowance	$(\sigma_{33} \sum_{r \in R^N} l_r)$	2 355.1	66.8
Remainder	(γ_N)	101.2	2.9

Table 4.2. Cost types in the studied TSP-tour.

Using the categories discussed in Chapter 2.2.2, the costs are:

Cost type		Cost (SEK)	% of total cost
Fixed cost	(c_0)	270.2	7.7
Customer specific costs	$(\sum_{i \in N} c_i)$	800.3	22.7
Common cost	(C_S)	2 355.1	66.8
Remainder	(γ)	101.2	2.9

Table 4.3. Cost types in the studied TSP-tour.

The distances between the customers can be found as a sub-matrix of the distance matrix in appendix 1.

4.5 Data in the Vehicle Routing Game

One of the criteria we used to select a TSP-tour, was that there should be several tours delivering gas, and no shared deliveries, from that depot on the chosen day. This was to allow us to compare the results of a TSP-Game with the results of a VRP-Game. The chosen day, the customers were located as below.

[2]Since the five customers are numbered 15-19 in the VRP-problem, they are numbered the same way here.

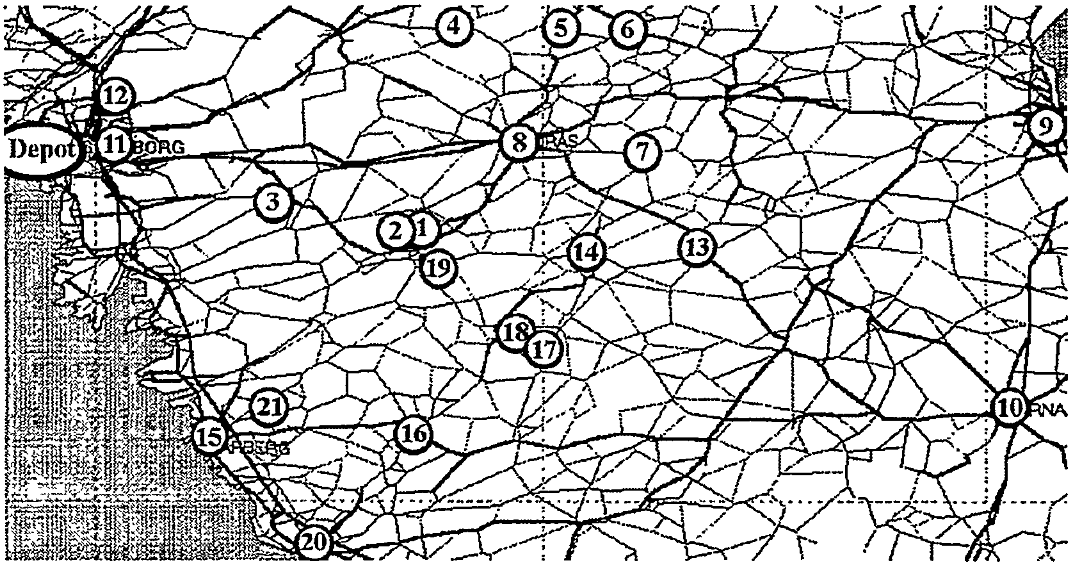


Figure 4.2. Map of the customers in the studied VRP.

The 8 tours that day has the following characteristics:

Tour no.[3]	Truck type	A-stops	B-stops	Actual length (km)	Optimal length (km)	Demanded volume
BSS 1	34	3	1	149	137.6	49.4
BSS 2	34	5	5	252	234.9	52.1
BSS 3	34	1	2	340	304.6	50.1
KFK 1	34	1	3	380	340.0	50.0
KFK 2	34	2	5	60	46.5	50.0
OMB 1	33	2	2	260	222.3	45.0
OMB 2	33	5	3	250	239.7	45.1
OMB 3	33	2	5	220	218.9	38.0

Table 4.4a. Characteristics of the actual tours in the studied VRP.

[3] The letters are the letters in the license plate of the truck, and the number is a counter, e.g., KFK 1 means the first tour of the truck with license plate KFK382.

Tour no.[3]	Fixed cost (SEK)	Customer specific costs (SEK)	Mileage allowance (SEK)	Remainder γ^T (SEK)
BSS 1	274.6	684.2	3 426.5	403.3
BSS 2	274.6	770.5	468.6	136.3
BSS 3	274.6	729.8	1 386.7	114.8
KFK 1	274.6	933.5	2 367.3	171.4
KFK 2	274.6	665.4	2 069.8	356.3
OMB 1	270.2	642.7	2 184.1	370.7
OMB 2	270.2	800.3	2 355.1	101.2
OMB 3	270.2	623.1	2 150.7	11.0

Table 4.4b. Characteristics of the actual tours in the studied VRP.

The customer specific data are as follows:

Tour no.	Customer i	Customer location	Demanded volume m^3	Customer specific cost (Actually used truck) (SEK)
BSS 1	1	Kinna	12.0	185.0
	2	Kinna	14.0	208.1
	3	Hällingsjö	23.4	336.7
BSS 2	4	Ljurhalla	11.5	346.3
	5	Ljung	1.5	63.6
	6	Ljung	4.1	199.2
	7	Dannike	12.5	113.6
	8	Borås	22.5	210.7
BSS 3	9	Jönköping	50.1	665.4
KFK 1	10	Värnamo	50.0	684.2
KFK 2	11	Björkekärr	25.7	383.3
	12	Angered	24.3	387.2
OMB 1	13	Tranemo	19.0	281.5
	14	Svenljunga	26.0	361.1
OMB 2	15	Varberg	27.6	399.3
	16	Ullared	3.0	79.6
	17	Överlida	8.5	162.1
	18	Mjöbäck	3.0	79.6
	19	Öxabäck	3.0	79.6
OMB 3	20	Falkenberg	26.0	401.1
	21	Varberg	12.0	222.0

Table 4.5. Customer specific data in the studied VRP.

The aggregation of the costs in the VRP-Game give the following proportions of the costs:

Cost type		Cost (SEK)	% of total cost
Remainder,	γ^T	1 665.0	6.1
Remainder,	γ^V	257.8	1.0
Remainder,	$\gamma = \gamma^T + \gamma^V$	1 922.8	7.1
Fixed cost,	c_0	2 183.6	8.1
Customer specific costs,	$\sum_{i \in N} c_i$	5 849.5	21.6
Common cost,	C_s	17 408.8	64.2

Table 4.6. Cost categories in the studied VRP.

For this VRP, the distance matrix is presented in appendix 1.

Chapter 5

Cost allocation in a Traveling Salesman Game

In this chapter, we present the mathematical models and methods, and the computational results of several different solution concepts, in a cost allocation game of a Traveling Salesman Problem (TSP). The TSP-Game is the problem of how to divide the cost for a TSP-tour, among the customers that were served on the tour.

5.1 The Core

5.1.1 Mathematical Models and Methods

Recalling the mathematical definition of the core as being all imputations $y = \{y_i\}$, where y_i is the cost allocated to customer i, that fulfills:

$$\begin{aligned} \sum_{i \in S} y_i &\leq c(S), \quad S \subset N \quad [5.1] \\ \sum_{i \in N} y_i &= c(N) \qquad\qquad [5.2] \end{aligned}$$

where $c(S), S \subseteq N$, is the characteristic function of the game (N;c).

The triangle inequality

The distance matrix in the studied TSP fulfills the triangle inequality. If l_{ij} is the distance between customer i and j, the triangle inequality states that for customers i, j and m:

$l_{im} \leq l_{ij} + l_{jm}$,

i.e., the distance to go straight between two customers is never longer compared to (going between the two customers and) passing another customer between these two. In theory, a TSP-tour passes each customer exactly once, but in practice there is no reason to exclude the possibility of passing a certain customer several times, if it would be less costly. If this was the case, an artificial arc with the distance $l_{ij} + l_{jm}$ could be constructed between customers i and m. A distance matrix with artificial arcs constructed in this way, corresponds to a complete network (i.e., arcs between all pair of nodes) that fulfills the triangle inequality. Since the distance matrix we have used contains the shortest path between customers i and j, for all $i, j \in (N \bigcup 0)$, the distance matrix fulfills the triangle inequality. If the distance (or cost) matrix satisfies the triangle inequalities, the shortest (cheapest) tour is a TSP-tour.

The fixed cost, c_0, and the customer specific costs, c_i, can be combined with the distance matrix to construct a cost matrix as followings: Let $\sigma_{\hat{k}}$ be the mileage cost using truck type $\hat{k}$. The transportation cost of the (real or artificial) arc between customers i and j is then $\sigma_{\hat{k}} l_{ij}$. If c_0 is considered to be the customer specific cost of the depot, the customer specific cost of customer $i \in (N \bigcup 0), c_i$, is divided by 2 and added to all arcs connecting to node i. Since every TSP-tour leaves the depot, visits each customer exactly once and returns to the depot, the costs $c_i, i \in (N \bigcup 0)$, will be included in the cost of the tour.

For example, the cost $c_{ij}^{\hat{k}}$ of the arc between node i and node j using truck type $\hat{k}$ when the fixed and the customer specific costs are included, would be:

$$c_{ij}^{\hat{k}} = \sigma_{\hat{k}} l_{ij} + \tfrac{c_i}{2} + \tfrac{c_j}{2},$$

The cost $c_{ii}^{\hat{k}}$ is defined to be 0 for all $i \in (N \bigcup 0)$.

If the cost $c_{ij}^{\hat{k}}$ is considered, and $c_i \geq 0, i \in (N \bigcup 0)$ we have:

$$c_{im}^{\hat{k}} = \sigma_{\hat{k}} l_{im} + \tfrac{c_i}{2} + \tfrac{c_m}{2} \leq \sigma_{\hat{k}} l_{ij} + \sigma_{\hat{k}} l_{jm} + \tfrac{c_i}{2} + \tfrac{c_m}{2} \leq$$

$$\leq \sigma_{\hat{k}} l_{ij} + \tfrac{c_i}{2} + \tfrac{c_j}{2} + \sigma_{\hat{k}} l_{jm} + \tfrac{c_j}{2} + \tfrac{c_m}{2} = c_{ij}^{\hat{k}} + c_{jm}^{\hat{k}}$$

This proves that the triangle inequality is fulfilled also for the cost matrix $\{c_{ij}^{\hat{k}}\}$. The triangle inequality also implies that the TSP-Game is subadditive.

Defining the characteristic function

A game $(N; c)$ is defined by the players $N = \{1, \ldots, n\}$ and the characteristic function $c(S), S \subseteq N$. Therefore, it is very important to define the characteristic function and the players precisely. In the TSP cost allocation problem of Norsk Hydro, the customers served in the TSP are defined to be the players. There are several different ways to define the characteristic function $c(S)$, for a coalition S. For example:

1. (a) $c(S)$ could be defined assuming that the customers in S are served by the truck that was actually used in the tour (serving the customers in N), driving the shortest TSP tour.
   (b) $c(S)$ could be defined as in 1a, but assuming that the smallest possible truck is used when serving the customers in S, instead of the truck that was actually used.
2. (a) Another way to define the characteristic function is by letting $c(S)$ be the cost of the shortest tour covering the customers in S, served in the same order, with the same truck, as in the optimal tour for the grand coalition. This would be a fixed route TSP-Game, (see e.g., Fishburn & Pollak ,1983 or Potters et. al., 1992).
   (b) $c(S)$ could also be defined as in 2a, using the smallest possible truck instead of the actual one.

In this chapter refer to these games as $(N; c^{(1a)})$, $(N; c^{(1b)})$, $(N; c^{(2a)})$ and $(N; c^{(2b)})$ when the characteristic function is defined as in 1a, 1b, 2a and 2b respectively. The game $(N; c^{(1a)})$ $[(N; c^{(2a)})]$ can be seen as a special case of $(N; c^{(1b)})$ $[(N; c^{(2b)})]$, where there is only one truck type in the fleet. In all the games above $c(\emptyset) = 0$.

One of the reasons for studying the games $(N; c^{(1b)})$ or $(N; c^{(2b)})$ is that the demand of a customer (and of a coalition) is to some degree reflected in the formulation of the core. It is always less costly to use the smallest possible truck when serving a given coalition S. In the games $(N; c^{(1a)})$ and $(N; c^{(2a)})$ the demand is not at all taken into consideration when computing $c(S)$, thus neither in the formulation of the core. Since all customers in N actually were served by the truck, of course any coalition $S \subset N$ may also be served by that truck, without exceeding the capacity. The only factor that can affect the cost of a coalition, is the geographical placement of the customers in the coalition.

One reason for *not* studying the games $(N; c^{(1b)})$ and $(N; c^{(2b)})$ is that any other truck than the larger types, would never really be used by Norsk Hydro in the transportation of gas.

We present the results for the TSP-Games $(N; c^{(1a)})$ and $(N; c^{(1b)})$. We do not consider the games $(N; c^{(2a)})$ and $(N; c^{(2b)})$ explicitly, since in this particular case it turns out that they are equivalent to the games $(N; c^{(1a)})$ and $(N; c^{(1b)})$ respectively.

Define

$$
\begin{aligned}
K &= \text{The set of available truck types.} \\
V_k &= \text{Capacity } (m^3) \text{ of truck type } k. \\
D_i &= \text{Demand of customer } i. \\
x_{ij} &= \begin{cases} 1 \text{ if the arc between customers } i \text{ and } j \text{ is used} \\ 0 \text{ otherwise} \end{cases} \\
c_{ij}^k &= \text{The cost of the arc between customers } i \text{ and } j, \text{ using truck type } k.
\end{aligned}
$$

Choose the truck type $\hat{k}$. In the game $(N; c^{(1a)})$ the truck type is given in advance. In the game $(N; c^{(1b)})$ the truck type can be chosen as:

$$\hat{k} = \arg\min_{k \in K} \{V_k | V_k \geq \sum_{i \in S} D_i\}$$

Then the characteristic function $c(S)$ for the TSP-Game $(N; c)$ is formulated as a TSP in the following way:

$$\begin{array}{rlrcll} c(S) = & \min & \sum_{i \in S} \sum_{j \in S} c_{ij}^{\hat{k}} x_{ij} & & & \\ & \text{s.t.} & \sum_{i \in S} x_{ij} & = & 1, & j \in S \\ & & \sum_{j \in S} x_{ij} & = & 1, & i \in S \\ & & \sum_{i \in S_s} \sum_{j \in S_s} x_{ij} & \leq & |S_s| - 1, & \begin{cases} S_s \subset S \\ |S_s| \geq 2 \end{cases} \\ & & x_{ij} & \in & \{0,1\} & i,j \in S \end{array}$$

The tariff is constructed in such a way, that it is always less costly to choose the smallest possible truck to serve a given coalition S in a TSP-tour. However, it is possible that it is less costly to serve a coalition using two smaller trucks, instead of one larger, even if the demand of all customers would fit on the larger truck. The savings e.g., in mileage allowance can in extreme cases be more than the extra base-time cost, if using two smaller trucks instead of one larger one. Consider the following example:

In Figure 5.1 below suppose the following: The demand of customer A is 2 and the demand of customer B is 2. There are two types of vehicles available. Type (i) has a capacity of 5 and the cost is 5 per distance unit. Type (ii) has a capacity of 2 and the cost is 3 per distance unit. For each vehicle used, there is also a fixed cost of 5 (independent of vehicle type).

In this example, serving the two customers using vehicle type (i) would cost 110, while serving them using two vehicles of type (ii) would cost 65+17=82. Thus it would be cheaper to use two vehicles.

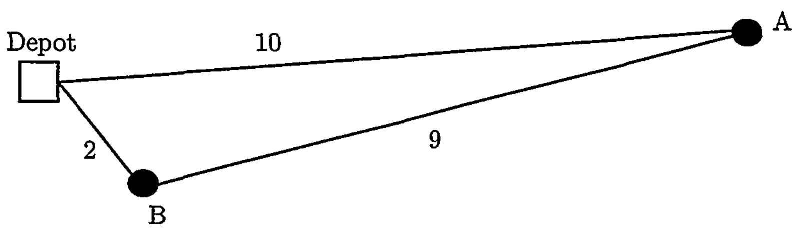


Figure 5.1. Transportation example where two vehicles are less costly than one.

The situation can only occur when adding one more customer means that a capacity limit is passed and a larger truck has to be chosen. Since the game $(N; c^{(1a)})$ requires that only one truck is used in the computations, the situation above can not occur in this game, and it can always be solved as a TSP-Game.

However, if the construction of the tariff and the location of the customers are such that there is a risk that the situation above could occur, the TSP-Game $(N; c^{(1b)})$ should instead be formulated as a VRP-Game with inhomogeneous fleet. If it was cheaper to use two trucks instead of one, this would be the optimal solution to the VRP. The VRP-Game with inhomogeneous fleet is studied in Chapter 6.

Finding a solution in the core

One of the problems when formulating the core, is that the number of inequalities in the formulation of the core (i.e., inequalities of type [5.1]) becomes very large when the number of customers, $|N|$, grows large. There are $2^{|N|} - 1$ number of constraints (including the grand coalition but excluding the empty set), and each constraint requires that a TSP is solved. It is a well-known fact that TSP belongs to the class of NP-hard problems. Therefore, the problem of evaluating the characteristic function for all the core defining inequalities is computationally difficult, as soon as N is of a nontrivial size.

If the aim is to find a particular solution in the core by adding a linear objective function to the core defining inequalities (e.g., to find the nucleolus or an extreme point), only a few of all the inequalities will be active. If the cost allocation problem has $|N|$ customers, the dual to the cost allocation problem has a basic solution with $|N|$ basic variables. This is equivalent to that at most $|N|$ constraints are enough in the core formulation. The problem is that there is no way of knowing in advance which these constraints are. In this setting a constraint generation approach can be applied. This approach is described in Chapter 6.1.1. When applying a constraint generation approach to the TSP-Game, the subproblem $\overline{\text{IVRP}}_D^{sub}$ in Chapter 6.1.1. will be replaced by a Profitable Tour Problem (PTP).

The PTP has been studied e.g., in Volgenant & Jonker (1987) and Dell'Amico et. al. (1995). Given a set of customers N and a vector $y \in R^{|N|}$ of profits, the PTP is to find a TSP-tour that covers one or more customers, and maximizes the sum of the profits made at the customers covered by the tour minus the arc costs of the tour. If the value of the objective function to the PTP is positive, the profits made are greater than the cost of the tour. In the cost allocation problem, where the profit correspond to a proposed allocation y, this is equivalent to a coalition S that is allocated a cost that is larger than the cost of serving them, i.e., $y(S) > c(S)$. This corresponds to a violated constraint in the formulation of the core, and should be included. A new allocation y is computed with the new constraint included, and the PTP is solved again. This procedure continues until the value of the objective function to the PTP is nonnegative, which means that $y(S) \leq c(S)$, for all $S \subset N$, and the solution y is a solution in the core, or until the core is proven to be empty. If there is no feasible solution y to the master problem after a constraint is added, the core is empty and the procedure can be terminated.

Volgenant & Jonker (1987) show that the PTP can be transformed into an Asymmetric TSP. They suggest that the Asymmetric TSP in turn is transformed into the (symmetric) TSP and thereafter solved. Dell'Amico et. al. (1995) present methods for solving the

Asymmetric TSP. Heuristics for solving the PTP are presented e.g., in Bienstock et. al. (1993).

Theoretically it is possible to generate all the extreme points (corresponding to feasible basic solutions) of the polyhedron described by the constraints of type [5.1] and type [5.2], (i.e., the polyhedron of the core). It is a way of getting an indication of the size of the core, even if it is only another way of representing the core. However, it is computationally very complicated to find all the extreme points. As mentioned, in the TSP-Game in general it is also complicated to find all the inequalities of type [5.1]. We take a simplified approach, evaluating the minimum and the maximum of what each customer would pay if the objective was to allocate as little and as much as possible respectively, for a single customer. This generates up to $2|N|$ extreme points of the core. It is of course only interesting to generate the extreme points, if the core is non-empty. Any convex combination of the generated extreme points in each game is also a solution in the core of each game respectively.

To evaluate the minimum amount that customer i has to pay, we solve a linear program with the constraints [5.1] and [5.2] and the objective function min y_i. The solution to this problem is an extreme point in the core (if it is non-empty). This is done repeatedly for each player. The same principle is used to evaluate the maximum amount that each player is willing to pay, where the objective function is max y_i. This is done both for the game $(N; c^{(1a)})$ and the game $(N; c^{(1b)})$.

5.1.2 Computational Results

In the Norsk Hydro case it is always optimal to serve a set of customers using one truck. The tariff is such that the costs of the smallest truck (type 30) is around 80% of the cost of the truck used for the grand coalition (type 33). Since the customers are almost at equal distance from the depot, if the grand coalition was divided in two smaller coalitions two trucks would together have to travel almost twice the distance in total. Even if this is a rough estimation, it is clear that using 20% less costly trucks, never would cover the cost of an extra truck.

In the TSP-Game we chose to study, there are only five customers giving rise to $2^5 - 1 = 31$ constraints (including the grand coalition, but excluding the empty coalition). With only five customers, it is computationally possible to solve the TSP using complete enumeration. However this involves $O(|N|!)$ operations, and becomes impossible as soon as N becomes large. A code that efficiently solves a TSP is necessary. The procedure we use to solve the TSP is the the same that we use for subproblem $\overline{\text{IVRP}}_D^{sub}$, described in Chapter 6.1.1.

In the TSP-Games $(N; c^{(1a)})$ and $(N; c^{(1b)})$, the core is defined by the constraints in Table 5.1:

Constraint ($\sum_{i \in S} y_i$)						$c(S)$ in game:	
						$(N;c^{(1a)})$	$(N;c^{(1b)})$
y_{15}					$\leq$	2 172.8	2 059.1
	y_{16}				$\leq$	2 338.4	1 902.4
		y_{17}			$\leq$	2 307.0	1 879.4
			y_{18}		$\leq$	2 177.2	1 769.8
				y_{19}	$\leq$	1 713.6	1 388.2
y_{15}	$+y_{16}$				$\leq$	2 791.8	2 645.2
y_{15}		$+y_{17}$			$\leq$	3 104.2	3 104.2
y_{15}			$+y_{18}$		$\leq$	3 004.0	2 845.9
y_{15}				$+y_{19}$	$\leq$	2 719.1	2 576.4
	y_{16}	$+y_{17}$			$\leq$	2 765.8	2 256.1
	y_{16}		$+y_{18}$		$\leq$	2 663.6	2 169.2
	y_{16}			$+y_{19}$	$\leq$	2 569.4	2 091.6
		y_{17}	$+y_{18}$		$\leq$	2 396.4	1 952.1
		y_{17}		$+y_{19}$	$\leq$	2 424.9	1 975.5
			y_{18}	$+y_{19}$	$\leq$	2 295.2	1 866.0
y_{15}	$+y_{16}$	$+y_{17}$			$\leq$	3 219.2	3 219.2
y_{15}	$+y_{16}$		$+y_{18}$		$\leq$	3 117.0	2 953.4
y_{15}	$+y_{16}$			$+y_{19}$	$\leq$	3 022.8	2 864.2
y_{15}		$+y_{17}$	$+y_{18}$		$\leq$	3 193.6	3 193.6
y_{15}		$+y_{17}$		$+y_{19}$	$\leq$	3 222.1	3 222.1
y_{15}			$+y_{18}$	$+y_{19}$	$\leq$	3 122.0	2 958.0
	y_{16}	$+y_{17}$	$+y_{18}$		$\leq$	2 855.2	2 328.8
	y_{16}	$+y_{17}$		$+y_{19}$	$\leq$	2 883.7	2 352.3
	y_{16}		$+y_{18}$	$+y_{19}$	$\leq$	2 781.6	2 265.4
		y_{17}	$+y_{18}$	$+y_{19}$	$\leq$	2 514.3	2 048.3
y_{15}	$+y_{16}$	$+y_{17}$	$+y_{18}$		$\leq$	3 308.6	3 308.6
y_{15}	$+y_{16}$	$+y_{17}$		$+y_{19}$	$\leq$	3 337.1	3 337.1
y_{15}	$+y_{16}$		$+y_{18}$	$+y_{19}$	$\leq$	3 235.0	3 235.0
y_{15}		$+y_{17}$	$+y_{18}$	$+y_{19}$	$\leq$	3 311.6	3 311.6
	y_{16}	$+y_{17}$	$+y_{18}$	$+y_{19}$	$\leq$	2 973.2	2 815.2
y_{15}	$+y_{16}$	$+y_{17}$	$+y_{18}$	$+y_{19}$	$=$	3 426.6	3 426.6

Table 5.1. Constraints for the TSP-Games[1].

In the TSP that we studied, we discovered that the games $(N;c^{(1a)})$ and $(N;c^{(1b)})$ (and $(N;c^{(2a)})$ and $(N;c^{(2b)})$ respectively) yield the same results[2] . The studied tour is constructed in such a way that serving a coalition, $S \subset N$, in an optimal order, is to serve the customers in the same order as they are served in the grand coalition N. This property is not general for TSP. Consider the following example: In the network in Figure 5.2, suppose that the length of the arcs not shown are large, and that one vehicle is enough

[1]For more details of each customer we refer to Chapter 4.4.

[2]This does not hold for one coalition, where the cost is lowered 0.1 SEK by reversing the order in which two customers are served, due to the fact that the elements of the cost matrix $\{c_{ij}^k\}$ have been rounded off to one decimal. If more exact numbers are used in the cost matrix, the statement is true.

to cover all four customers. The optimal order in which to serve all four customers is Depot-A-B-C-D-Depot. The optimal order in which to serve the subset of customers A, B and C is Depot-A-C-B-Depot. Thus the order in which the customers A, B and C are served, are different in the smaller coalition from that in the larger one.

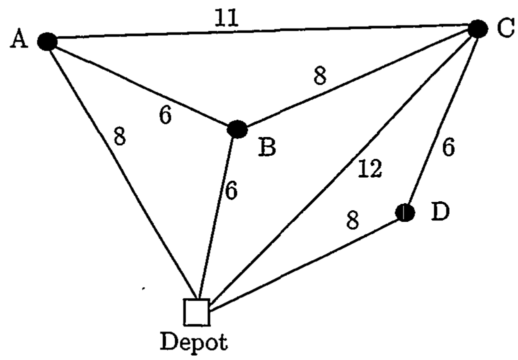


Figure 5.2. Example of a TSP where the order in which the customers are served changes between the large coalition and a smaller.

If the order in which the customers are served does not change for any coalition S, compared to the order in which the customer are served in the grand coalition N, the solution is same as if the game is formulated as a fixed route TSP-Game (see e.g., Fishburn & Pollak (1983) or Potters et. al. (1992)). Potters et. al. (1992) prove that the core of a fixed route TSP-Game is nonempty, if the solution to the TSP for the grand coalition is a minimal cost tour, and the cost matrix fulfills the triangle inequality. Thus we know[3] that the core of this particular TSP-Game will be non-empty.

In Tables 5.2-5.5 below we present the result of the computations. In Tables 5.2a, 5.2b and 5.3 we present the results for the game $(N; c^{(1a)})$, and in Tables 5.4a, 5.4b and 5.5 we present the results for the game $(N; c^{(1b)})$.

The computation of the minimum and maximum amount that can be allocated to each customer in a core solution, generates in this problem up to 10 of the at most $\binom{30}{4} = 27\,405$ extreme points[4] of the core.

[3]If a more exact cost matrix is used.

[4]Reducing the number of variables using the equality constraint, there are 30 conditions and 4 variables.

Customer i	Allocation using the objective function:[5] min y_{15}	min y_{16}	min y_{17}	min y_{18}	min y_{19}
15	453.4	2 172.8	2 172.8	2 172.8	2 172.8
16	2 338.4	115.0	619.0	619.0	619.0
17	427.3	931.3	191.6	427.3	427.3
18	89.5	89.5	212.2	89.5	89.5
19	118.0	118.0	231.0	118.0	118.0

Table 5.2a. A selection of extreme points in the core for the game $(N; c^{(1a)})$.

Customer i	Allocation using the objective function:[5] max y_{15}	max y_{16}	max y_{17}	max y_{18}	max y_{19}
15	2172.8	453.4	797.3	824.8	1 005.5
16	619.0	2 338.4	115.0	115.0	303.7
17	427.3	427.3	2 306.8	191.6	314.3
18	89.5	89.5	89.5	2 177.2	89.5
19	118.0	118.0	118.0	118.0	1 713.6

Table 5.2b. A selection of extreme points in the core for the game $(N; c^{(1a)})$.

It should be noted that when the objective function is $\min y_i$ ($\max y_i$), several extreme points where y_i attains its minimum (maximum) are possible, and that no particular method is used to choose between these extreme points. The most interesting results in Tables 5.2a and 5.2b are the minimum and maximum each customer is willing to pay (i.e., the diagonals of the tables). This is transferred to Table 5.3 below, that also presents the stand-alone cost for customer i ($c(i)$), the marginal cost (m_i), the cost allocation according to the principles implied by Norsk Hydro[6], and the cost allocation according to the DM method[7].

Customer i	$\min y_i$[8]	$\max y_i$[9]	$c(i)$	m_i	Norsk Hydro	DM
15	453.4	2 172.8	2 172.8	453.4	2 097.0	1 521.0
16	115.0	2 338.4	2 338.4	115.0	227.9	413.6
17	191.6	2 306.8	2 307.0	191.6	645.8	717.2
18	89.5	2 177.2	2 177.2	89.5	227.9	402.9
19	118.0	1 713.6	1 713.6	118.0	227.9	370.0

Table 5.3. Computational results for the game $(N; c^{(1a)})$.

[5]The objective function is used in a linear program with the core constraints, i.e., constraints [5.1] and [5.2].
[6]In proportion to demand.
[7]See Chapter 2.2.4.
[8]The minimum allocation can only be attained for one customer at a time.
[9]The maximum allocation can only be attained for one customer at a time.

Customer i	Allocation using the objective function:[10] min y_{15}	min y_{16}	min y_{17}	min y_{18}	min y_{19}
15	611.4	2 059.1	2 059.1	2 059.1	2 059.1
16	1 316.1	115.0	523.5	586.1	586.1
17	549.8	1 045.0	191.6	573.9	573.9
18	462.9	89.5	370.8	89.5	89.5
19	486.4	118.0	281.6	118.0	118.0

Table 5.4a. A selection of extreme points in the core for the game $(N; c^{(1b)})$.

Customer i	Allocation using the objective function:[10] max y_{15}	max y_{16}	max y_{17}	max y_{18}	max y_{19}
15	2 059.1	878.8	1 263.3	1 263.3	1 188.2
16	586.1	1 766.4	115.0	115.0	287.8
17	573.9	366.9	1840.8	355.2	472.9
18	89.5	195.5	89.5	1 575.1	89.5
19	118.0	219.0	118.0	118.0	1 388.2

Table 5.4b. A selection of extreme points in the core for the game $(N; c^{(1b)})$.

Customer i	$\min y_i$[11]	$\max y_i$[12]	$c(i)$	m_i	Norsk Hydro	DM
15	611.4	2 059.1	2 059.1	611.4	2 097.0	1 521.0
16	115.0	1 766.4	1 902.4	115.0	227.9	413.6
17	191.6	1 840.8	1 879.4	191.6	645.8	717.2
18	89.5	1 575.1	1 769.8	89.5	227.9	402.9
19	118.0	1 388.2	1 388.2	118.0	227.9	370.0

Table 5.5. Computational results for the game $(N; c^{(1b)})$.

Observe that the allocations according to the principle of Norsk Hydro, and the DM-method, are independent of the game, i.e., they are the same in Tables 5.3 and 5.5.

In Tables 5.2 and in Tables 5.4 above, we can see that the core is very large. As expected, it seems to be smaller in the game $(N; c^{(1b)})$ than in $(N; c^{(1a)})$.

Tables 5.3 and 5.5 also give an indication of the size of the core. The maximum of what each customer is willing to pay is close to her stand-alone cost ($c(i)$), and the minimum is her marginal cost (m_i). An interpretation of the result is that cooperation is very profitable.

[10]The objective function is used in a linear program with the core constraints, i.e., constraints [5.1] and [5.2].

[11]The minimum allocation can only be attained for one customer at a time.

[12]The maximum allocation can only be attained for one customer at a time.

The core is too large to give an indication of how a fair allocation should be done. It can be assumed that almost any cost allocation method that attempts to divide the cost between all the customers, will result in a solution in the core. Therefore, it is interesting to note that the cost allocation principle implied by Norsk Hydro gives a solution that is *not* in the core of the game $(N; c^{(15)})$! According to the principle implied by Norsk Hydro, i.e., to allocate cost in proportion to demand, customer 15 has to pay 2097.0, but she is only willing to pay 2059.1. This is an interesting result for Norsk Hydro, that clearly demonstrates one of the weaknesses of a rule of thumb, as well as motivating further research and discussion.

Finding a unique cost allocation vector

Since the core is generally not a unique solution, and the size of the core is large in the two TSP-Games studied, further investigation is necessary to suggest a cost allocation.

There are (at least) two possible ways to continue. The first way is to use a solution concept that gives a unique solution. We compute the nucleolus (and versions of it), the Shapley value and the τ-value. This is done in Chapters 5.2 and 5.3.

The second way is to enlarge the problem, to better capture the influence of the size of the demand on the cost allocation. This could be done for example, by studying the VRP-Game involving more customers on more tours. When more customers are included, the customers with a low demand in particular have a greater chance to form coalitions, thereby reducing what they would have to pay in a cost allocation in the core. We study this in Chapter 6, where the VRP-Game that includes all the tours from the Göteborg depot on the same day, is considered.

5.2 The Nucleolus

In this chapter we actually consider the pre-nucleolus. Since it turns out that the pre-nucleolus of the studied TSP-Game fulfills the requirements of individual rationality, i.e. $y_i \leq c(i)$ for all $i \in N$, the pre-nucleolus for this game coincides with the nucleolus.

5.2.1 Mathematical Models and Methods

If all constraints for the core are known, the pre-nucleolus can be found by solving successive, linear programs in the following way, suggested by Kopelowitz (1967) (see also Dragan, 1981):

$$
\begin{array}{llllll}
(P_1) & \max & & w_1 & & \\
& \text{s.t.} & \sum_{i\in S} y_i & +w_1 & \leq & c(S), \quad S \subset N \\
& & \sum_{i\in N} y_i & & = & c(N)
\end{array}
$$

Let $\Pi^1(S)$ be the dual variable that belongs to constraint $\sum_{i\in S} y_i + w_1 \leq c(S)$. Let Π^{1*} be the optimal dual solution to P_1. If P_1 has a unique (primal) solution (y^*, w_1^*), then y^* is the pre-nucleolus of the game.

The p_1 constraints ($p_1 \geq 1$,) that are binding and have a corresponding strictly positive dual solution ($\Pi^{1*}(S) > 0$), are binding *and* active. These constraints define the p_1 first elements of the lexicographically smallest excess vector $\theta(y)$, with the excesses $c(S) - y(S)$ arranged in increasing order (see Chapter 3.3). If the solution (y^*, w_1^*) is not unique at least element $p_1 + 1$ has to be found by solving another linear program, P_2, where the binding and active constraints in the optimal solution to P_1 are fixed:

$$
\begin{array}{llllll}
(P_2) & \max & & w_2 & & \\
& \text{s.t.} & \sum_{i\in S} y_i & +w_2 & \leq & c(S), \quad S \in \{S \subset N | \Pi^{1*}(S) = 0\} \\
& & \sum_{i\in S} y_i & +w_1^* & = & c(S), \quad S \in \{S \subset N | \Pi^{1*}(S) > 0\} \\
& & \sum_{i\in N} y_i & & = & c(N)
\end{array}
$$

If P_2 has a unique solution (y^*, w_2^*), then y^* is the pre-nucleolus of the game. If P_2 does not have a unique solution another problem, P_3, has to be solved. The problem P_3 is constructed in a way similar to P_2.

This procedure can be generalized as follows:

Let Γ_q be the set of all the coalitions that have corresponding dual variables $\Pi^{q*}(S) > 0$ (i.e., the excess $w_q^* = c(S) - y(S)$):

$\Gamma_q = \{S \subset N | \Pi^{q*}(S) > 0\}$.

At iteration t, solve the following linear program:

$$
\begin{array}{llllll}
(P_t) & \max & & w_t & & \\
& \text{s.t.} & \sum_{i\in S} y_i & +w_t & \leq & c(S), \quad \begin{array}{l} S \subset N \\ S \notin \bigcup_{q=1}^{t-1} \Gamma_q \end{array} \\
& & \sum_{i\in S} y_i & +w_q^* & = & c(S), \quad S \in \Gamma_q, q = 1, \ldots, t-1 \\
& & \sum_{i\in N} y_i & & = & c(N)
\end{array}
$$

This process has to be continued until P_t has a unique solution (y^*, w_t^*), where y^* is the pre-nucleolus.

The procedure above can also be used to find the nucleolus. In this case the additional constraints $y_i \leq c(i), i \in N$ must be added, to each linear program P_t.

The problem P_1 contains $2^{|N|} - 1$ constraints and $|N| + 1$ variables, which can be difficult to handle computationally, if the number of players is large. As mentioned in Chapter 5.1.1, the characteristic function itself adds significantly to the complexity. In the TSP-Game, each evaluation of the characteristic function requires a TSP to be solved, which is NP-hard. Thus the method described above can not always be used, for computational reasons.

However, in the studied TSP-Game there are 31 constraints, each requiring a fairly small TSP to be solved, and 5 players (i.e., 6 variables) in problem P_1. Therefore we chose to apply the procedure above.

For a problem where the method described above becomes too complicated computationally, a constraint generation procedure could be applied, similar to the procedure mentioned in Chapter 5.1.1, and more thoroughly described in Chapters 6.1.1 and 6.2.1.

5.2.2 Computational Results

Pre-nucleolus for the game $(N; c^{(1a)})$

The pre-nucleolus of the game $(N; c^{(1a)})$ was found in one iteration, i.e., only problem P_1 had to be solved.

The pre-nucleolus of the game $(N; c^{(1a)})$ is presented in Table 5.6. The stand-alone cost of each customer i, and the cost allocation according to the principle implied by Norsk Hydro are also presented. Finally the pre-nucleolus minus the customer specific costs c_i, is presented. These values are equivalent to the pre-nucleolus in a game when only the fixed and common transportation costs are included in the characteristic function. Since c_i for all $i \in N$ are independent of which coalition customer i is included in, the game that does not include the customer specific costs, is a strategic equivalent game to the game $(N; c^{(1a)})$.

Customer i	Pre-nucleolus (Nucleolus)	$c(i)$	Norsk Hydro	Pre-nucleolus$-c_i$
15	945.2	2 172.8	2 097.0	545.9
16	606.8	2 338.4	227.9	527.2
17	683.4	2 307.0	645.8	521.3
18	581.3	2 177.2	227.9	501.7
19	609.8	1 713.6	227.9	530.2

Table 5.6. Computational results for the game $(N; c^{(1a)})$.

As can be seen in Table 5.6, the pre-nucleolus fulfills the individual rationality constraints (since it is less than $c(i)$ for all $i \in N$), and thus the pre-nucleolus coincides with the nucleolus.

The value of w_1^* (i.e., the excess in the least content coalitions) is 491.8. The binding constraints in the solution are all the $\{N \backslash i\}$ coalitions. An interpretation of the result is that cooperation is very profitable. The larger the coalition is, the more will be saved. Since the grand coalition correspond to the optimal solution, the larger coalitions will be the least satisfied (i.e., they do not gain as much by cooperating as the smaller coalitions do).

All the customers can be served by one truck, and therefore demand does not have any influence on the fixed cost and the transportation cost for a coalition. This could be seen in the strategically equivalent game, where only the fixed cost and the transportation cost is considered. This game would have a pre-nucleolus (and nucleolus) that coincide with the pre-nucleolus minus c_i, presented above. The customers get an almost equal allocation, which is to be suspected given the geographical placement of the customers. The customers 15 and 19 get a somewhat higher cost (even if they are closer to the depot than the others), since they are located on the side compared to the others. They do not have as many opportunities to form good coalitions as customers 16, 17 and 18, who are located more centrally.

It does not seem fair and reasonable that the demand of the customers should not have any significance on the allocation of the cost. The smaller customers do not have as many options to cooperate in the TSP-Game, as they would have in a real situation. If more customers than the customers from one tour are included in the problem, the smaller customers would have a wider range of opportunities. They could either choose to cooperate with many other small customers, or with a few larger customers. The larger customers would not have as many possibilities, as they only could cooperate with one or a few of the smaller customers. An extension that includes more customers is the VRP-Game studied in Chapter 6.

Pre-nucleolus for the game $(N; c^{(1b)})$

When the game $(N; c^{(1b)})$ is studied, the differences in demand between the customers are reflected to some extent. In Table 5.7 below, the pre-nucleolus, the stand-alone cost for customer i, and the allocation implied by Norsk Hydro are presented. The pre-nucleolus found in game $(N; c^{(1a)})$ is also presented. The game $(N; c^{(1b)})$ is not strategically equivalent to the game where the customer specific costs not are included, since these not are the same for each customer in all coalitions (as the customer specific costs varies with the size of the truck, i.e., with the demand of the coalition). Thus this value is not included for this game.

Customer i	Pre-nucleolus $(N; c^{(1b)})$	$c(i)$ $(N; c^{(1b)})$	Norsk Hydro	(Pre-)nucleolus $(N; c^{(1a)})$
15	1143.5	2 059.1	2 097.0	945.2
16	647.1	1 902.4	227.9	606.8
17	603.9	1 879.4	645.8	683.4
18	501.8	1 769.8	227.9	581.3
19	530.3	1 388.2	227.9	609.8

Table 5.7. Computational results for the game $(N; c^{(1b)})$.

As in the game $(N; c^{(1a)})$ the pre-nucleolus fulfills the individual rationality constraints, and therefore it coincides with the nucleolus.

In game $(N; c^{(1b)})$ two iterations were necessary. In the optimal solution to P_1, $w_1^* = 412.3$, corresponding to the excesses in the least content coalitions. The constraints corresponding to coalitions $\{17, 18, 19\}$, $\{15, 16, 17, 18\}$, $\{15, 16, 17, 19\}$ and $\{15, 16, 18, 19\}$ were included in Γ_1. In problem P_2 the pre-nucleolus was found. The value of $w_2^* = 532.1$. The two remaining $\{N \backslash i\}$-coalitions were the two new binding constraints.

As expected, the nucleolus in the game $(N; c^{(1b)})$ captures some of the difference in demand between the customers. The smaller customers gain somewhat in importance and get a lower cost allocated to them, except for customer 16. She is rather far away from 18 and 19, the other two small customers, and therefore it is more difficult for customer 16 to form good coalitions. The principle implied by Norsk Hydro does not at all correspond to the nucleolus.

It still seems like demand does not have as much influence on the allocations as would be desirable in an allocation that was to be accepted by all the customers. Therefore we study some more concepts related to cooperative game theory in the next chapter.

5.3 Other Solution Concepts

We have computed the normalized nucleolus (Grotte, 1970). In the game $(N; c^{(1a)})$, the normalized nucleolus coincides with the nucleolus, since the larger coalitions already correspond to the binding constraints. In the game $(N; c^{(1b)})$, there is a minor difference.

Another version of the nucleolus, that takes into consideration the demand of the customers, is the *demand nucleolus.* The demand nucleolus is defined in the same way as the nucleolus (see Chapter 3.3), but using the excess $e^D(S, y)$ instead of $e(S, y)$ where $e^D(S, y)$ is the excess $e(S, y)$ multiplied with the total demand of the coalition:

$$e^D(S, y) = e(S, y) \sum_{i \in S} D_i$$

The demand nucleolus suffers from the same axiomatic criticism as the normalized nucleolus (see Chapter 3.3). However, the effect that the demand nucleolus has on the cost allocation, might be desirable. This effect is that the importance of coalitions having a large demand is reduced, when computing the nucleolus. A larger portion of the total cost can be allocated to customers that participate in coalitions with a large demand. It is a way of including what might be felt as reasonable; that the customers with a higher demand should carry a larger share of the common costs.

We have also computed the Shapley value and the τ-value (see Chapter 3.3 for mathematical descriptions) for the games $(N; c^{(1a)})$ and $(N; c^{(1b)})$.

The values are presented in Table 5.8 and Table 5.9 below.

Customer i	Nucleolus (Normalized nucleolus)	Demand nucleolus	Shapley value	τ-value	Norsk Hydro
15	945.2	1 756.3	913.3	794.1	2 097.0
16	606.8	263.2	716.3	682.4	227.9
17	683.4	954.1	720.0	709.3	645.8
18	581.3	228.6	609.0	627.2	227.9
19	609.8	224.3	467.9	613.6	227.9

Table 5.8. A selection of values for the game $(N; c^{(1a)})$.

Customer i	Nucleolus	Normalized nucleolus	Demand nucleolus	Shapley value	τ-value	Norsk Hydro
15	945.2	1 102.3	1 788.3	1 158.9	898.4	2 097.0
16	606.8	605.9	234.1	625.9	689.0	227.9
17	683.4	631.4	999.9	681.5	697.6	645.8
18	581.3	529.3	201.5	539.3	628.0	227.9
19	609.8	557.8	202.7	421.0	513.5	227.9

Table 5.9. A selection of values for the game $(N; c^{(1b)})$.

The discussion about the results is the same for both games. The results show that the normalized nucleolus gives almost the same result as the nucleolus. The demand nucleolus seems to capture the influence of the demand of the customers fairly well, since the result is close to the results of Norsk Hydro. The procedure of computing the demand nucleolus terminated when all the constraints corresponding to the singletons were binding. It also seems like the truck size is of minor importance when computing the demand nucleolus. The Shapley value and the τ-value does not seem to capture the importance of the demand in a better way than the nucleolus, for this particular game.

The most interesting values from a theoretical and practical viewpoint, is in our opinion the nucleolus, the Shapley value, and the τ-value. The demand nucleolus is interesting since it seems to be the best approximation of the allocation principle implied by Norsk Hydro, for this particular game . The maximum that each player would be willing to pay in any solution, and the allocation implied by Norsk Hydro are interesting from a practical viewpoint. The values mentioned above are presented in Table 5.10, for the game $(N; c^{(1a)})$, which is the most interesting game for Norsk Hydro, as this game correspond to the trucks that would be used in the transportation:

Customer i	Nucleolus	Shapley value	τ-value	Demand Nucleolus	max y_i	Norsk Hydro
15	945.2	913.3	794.1	1 756.3	2 172.8	2 097.0
16	606.8	716.3	682.4	263.2	2 338.4	227.9
17	683.4	720.0	709.3	954.2	2 306.8	645.8
18	581.3	609.0	627.2	228.6	2 177.2	227.9
19	609.8	467.9	613.6	224.3	1 713.6	227.9

Table 5.10. A selection of values for the games $(N; c^{(1a)})$.

The smaller customers should be able to reduce their cost significantly, by having the possibility to form coalitions with many other customers. These possibilities should have a larger effect, than the possibilities of reducing the cost by choosing a smaller truck (as in game $(N; c^{(1b)})$). To be able to compare what the difference is between the allocation of the cost for one tour, and the allocation of costs in a larger game with more opportunities to form coalitions, we study the VRP-Game. This is done in Chapter 6.

Chapter 6

Cost allocation in a Vehicle Routing Game

In this chapter we study cost allocation in a vehicle routing problem where the fleet is inhomogeneous. We call this problem the Inhomogeneous Vehicle Routing Problem (IVRP), and the corresponding game the IVRP-Game. We show how the methods presented in Göthe-Lundgren et. al. (1996), studying a cost allocation game in the Vehicle Routing Problem where all the vehicles have identical capacity (which we denote by VRP), can be adopted for the IVRP. The IVRP-Game is the problem of how to divide the cost for a IVRP among the customers that are served on the routes.

6.1 The Core

6.1.1 Mathematical Models and Methods

The VRP-Game

We start with a brief description of the VRP-Game. The mathematical definitions of the core and the nucleolus for the VRP-Game are the same as in the TSP-Game (or any cooperative game). The difference is in the characteristic function $c(S), S \subseteq N$, which in the VRP-Game is the optimal solution to a VRP.

We introduce the following definitions for use in this chapter:

$\bar{R}$ = The set of all possible routes (i.e., routes where the demand of the customers covered by the route, does not exceed the capacity limit for a vehicle).
q_r = The cost of route $r \in \bar{R}$.
$a_{ir} = \begin{cases} 1 \text{ if customer } i \text{ is covered by route } r \\ 0 \text{ otherwise} \end{cases}$
$x_r = \begin{cases} 1 \text{ if route } r \text{ is chosen} \\ 0 \text{ otherwise} \end{cases}$

The characteristic function $c(S)$ for the VRP-Game can be defined as:

$$
\begin{aligned}
c(S) = \min \quad & \sum_{r \in \bar{R}} q_r x_r \\
\text{s.t.} \quad & \sum_{r \in \bar{R}} a_{ir} x_r = 1, && i \in S \\
& x_r \geq 0, && r \in \bar{R} \\
& x_r \text{ integer}, && r \in \bar{R}
\end{aligned}
$$

The IVRP-Game

The formulation of the IVRP, similar to the formulation of the VRP above, is the following:

Denote a coalition S such that $\sum_{i \in S} D_i \leq V_k$ a *feasible* coalition with respect to truck type k. Denote a coalition that is not a feasible coalition with respect to truck type k an *infeasible* coalition with respect to truck type k. Such a coalition is infeasible in the sense that it can not be served by one single truck (of type k). An IVRP has to be solved for such a coalition.

In the IVRP-Game, we define the customers served in the IVRP as the players. Furthermore, define:

D_i = The demand of customer i.
K = The set of truck types in the fleet.
V_k = The capacity of truck type k, $k \in K$.
k_{max} = The truck type with the largest capacity, i.e., $V_{k_{max}} \geq V_k, k \in K$.
R = The set of all feasible coalitions $\{S_1, \ldots, S_m\}$ with respect to the largest truck type, i.e., $R = \{S \subset N | \sum_{i \in S} D_i \leq V_{k_{max}}\}$.
$\bar{c}(S)$ = The least costly route that serves the customers in coalition $S \in R$, using the cheapest truck[12].
$a_{ir} = \begin{cases} 1 \text{ if customer } i \in S_r \\ 0 \text{ otherwise} \end{cases}$
$x_r = \begin{cases} 1 \text{ if the route covering the customers in } S_r \text{ is chosen} \\ 0 \text{ otherwise} \end{cases}$

[2]For a given tour, a smaller truck is always less costly than a larger.
[2]S_r and $\bar{c}(S_r)$ will be used for coalitions $\in R$, when it is necessary to refer to a specific route, e.g.,

We define the characteristic function, $c(S)$ in the IVRP-Game as the optimal solution to an IVRP, involving the customers in S. Mathematically, $c(S), S \subseteq N$, is expressed as:

$$
\begin{array}{llrcll}
\text{(IVRP)} & c(S) = & \min & \sum_{r|S_r \in R} \tilde{c}(S_r)x_r & & \\
& & \text{s.t.} & \sum_{r|S_r \in R} a_{ir}x_r & = 1, & i \in S \\
& & & x_r & \geq 0, & r|S_r \in R \\
& & & x_r & \text{integer}, & r|S_r \in R
\end{array}
$$

In the game above, define $c(\emptyset) = 0$.

The function $\tilde{c}(S)$ for each $S \in R$ can be computed by solving a TSP using the cheapest possible truck. This formulation of IVRP contains as many columns as the VRP has, if the VRP is formulated for the largest truck type k_{max}. The number of columns can be reduced, since it is possible that some of the columns in IVRP never would be chosen in an optimal solution.

Consider the following example: Suppose coalition $S_j \in R$ is feasible with respect to truck type $k_j \neq k_{max}$, and that $S_l \in R$ is feasible with respect to truck type $k_l \neq k_{max}$. Suppose also that $S_m = \{S_j \bigcup S_l\} \in R$ is feasible with respect to truck type k_{max}. If $\tilde{c}(S_m) > \tilde{c}(S_j) + \tilde{c}(S_l)$, then the customers in S_j and S_l will not form coalition S_m, and x_m will never be chosen in an optimal solution to the IVRP.

For those coalitions $S \in R$ that can be served at a lower cost using more than one truck we have that $\tilde{c}(S) > c(S)$. For those coalitions $S \in R$ that are served at an optimal cost using one truck, we have equality, i.e., $\tilde{c}(S) = c(S)$. Thus $\tilde{c}(S)$ is an upper bound to $c(S)$.

It is possible to reduce the number of columns, by eliminating the columns that correspond to coalitions S where $\tilde{c}(S) > c(S)$. However this would mean that an IVRP (instead of a TSP) has to be solved for each coalition $S \in R$, and from a computational aspect, it would probably not be worth the effort of eliminating columns.

The core defining inequalities

If all the constraints of the IVRP-Game would be explicitly formulated it would be necessary to solve $2^{|N|} - 1$ problems of type IVRP. This would be too complicated computationally, for any nontrivial size of N. However, it is possible to reduce the number of core defining inequalities significantly. The proof below is adapted from Göthe-Lundgren et. al. (1996):

Consider any infeasible coalition $\hat{S}$ ($\hat{S} \neq N$) with respect to the largest truck type, and a corresponding optimal route configuration with routes $\{r_1, \ldots, r_m\}$. Denote by $\{S_1, \ldots, S_m\}$ the corresponding feasible (with respect to any truck type) disjoint coali-

when relating S_r to x_r.

tions. Since $\sum_{j=1}^{m}\sum_{i\in S_j} y_i = \sum_{i\in\hat{S}} y_i$ and $\sum_{j=1}^{m} c(S_j) = c(\hat{S})$ it follows that the core defining inequalities $\sum_{i\in S_j} y_i \le c(S_j), j=1,\ldots,m$ imply $\sum_{i\in\hat{S}} y_i = \sum_{j=1}^{m}\sum_{i\in S_j} y_i \le \sum_{j=1}^{m} c(S_j) = c(\hat{S})$.

This proves that the constraints $\sum_{i\in\hat{S}} y_i \le c(\hat{S})$ are redundant in the definition of the core. Thus, only the feasible coalitions (with respect to at least one truck type) are of interest, and it is only necessary to evaluate the characteristic function $c(S)$ for feasible coalitions $S \in R$.

Thus the core of the IVRP-Game can be expressed using these constraints:

$$(\overline{\text{IVRP}}\text{core}) \quad \begin{array}{lcll} \sum_{i\in S} y_i & \le & c(S) \quad S\in R & [6.1] \\ \sum_{i\in N} y_i & = & c(N) & [6.2] \end{array}$$

Observe that one IVRP still has to be solved, in order to find the value of $c(N)$ in constraint [6.2].

It is also interesting to observe that in any core solution to the IVRP, the customers on each optimal route, have to carry the full cost of the route. There can be no subsidizing between routes. The following proof of this is adopted from Göthe-Lundgren et. al. (1996):

Let the route configuration in the optimal solution to the IVRP of the grand coalition be $\{r_1^N,\ldots,r_m^N\}$, and the corresponding optimal routes $\{S_1^N,\ldots,S_m^N\}$. Since the coalitions $S_j^N, j=1,\ldots,m$ are feasible (at least with respect to the largest truck type), we know that for every solution in the core we must have:

$$\sum_{i\in S_j^N} y_i \le c(S_j^N), \quad j=1,\ldots,m$$

We also know that:

$$\sum_{j=1}^{m}\sum_{i\in S_j^N} y_i = \sum_{i\in N} y_i = c(N) = \sum_{j=1}^{m} c(S_j^N)$$

This is satisfied if and only if

$$\sum_{i\in S_j^N} y_i = c(S_j^N), \quad j=1,\ldots,m$$

This does not mean that the IVRP-Game can be disaggregated into several TSP-Games, each including one route in the optimal route configuration of the IVRP. The formulation of the core includes constraints corresponding to coalitions that would not be formed in the optimal IVRP solution. Thus the constraints in the definition of the core in each TSP-Game are only a subset of all the constraints for the core of the IVRP-Game. However, any core solution to the IVRP-Game, are solutions in the core of each TSP-

Game (corresponding to the routes in the optimal IVRP-solution) respectively.

Investigation of the emptiness of the core

Even if the number of core defining inequalities can be reduced significantly by including only those constraints that correspond to feasible coalitions, as described above, the number of core defining inequalities may still be too large to generate computationally.

If the aim is to find a particular solution in the core by adding a linear objective function to the core defining inequalities, as in the TSP-Game, only a few of all the inequalities will be active. If the game has $|N|$ players, the dual to the core formulation has a basic solution with $|N|$ basic variables. In this setting a constraint generation approach (e.g., Gilmore & Gomory, 1961) can be applied. Constraint generation to find solutions to cooperative games has successfully been applied by e.g., Hallefjord et. al. (1995), and Göthe-Lundgren et. al. (1996).

As in the VRP-Game, the core of the IVRP-Game can be empty or non-empty, depending on the data of the problem. In order to investigate whether the core is non-empty in a large game *where* $c(N)$ *is not known*, the following procedure (including constraint generation) may be applied (adopted from Göthe-Lundgren et. al. 1996).

Formulate the IVRP for the grand coalition in the following way:

$$
\begin{array}{llrcll}
\text{(IVRP)} & z = & \min & \sum_{r|S_r\in R} \tilde{c}(S_r)x_r & & \\
& & \text{s.t.} & \sum_{r|S_r\in R} a_{ir}x_r & = & 1, \quad i \in N \\
& & & x_r & \geq & 0, \quad r|S_r \in R \\
& & & x_r & & \text{integer}, \quad r|S_r \in R
\end{array}
$$

The LP-relaxation to the IVRP is denoted $\overline{\text{IVRP}}$, where $\overline{z}$ is the optimal objective function to $\overline{\text{IVRP}}$, i.e.,

$$
\begin{array}{llrcll}
(\overline{\text{IVRP}}) & \overline{z} = & \min & \sum_{r|S_r\in R} \tilde{c}(S_r)x_r & & \\
& & \text{s.t.} & \sum_{r|S_r\in R} a_{ir}x_r & = & 1, \quad i \in N \\
& & & x_r & \geq & 0, \quad r|S_r \in R
\end{array}
$$

Then the dual to $\overline{\text{IVRP}}$, which is denoted $\overline{\text{IVRP}}_D$, can be formulated as:

$$
\begin{array}{llrcll}
(\overline{\text{IVRP}}_D) & u = & \max & \sum_{i\in N} y_i & & \\
& & \text{s.t.} & \sum_{i\in S} y_i & \leq & \tilde{c}(S), \quad S \in R \quad [6.3]
\end{array}
$$

In any imputation, the efficiency condition requires that the optimal cost in the IVRP solution is allocated among all the players,

$$\sum_{i \in N} y_i = c(N) = z$$

We also know from $\overline{\text{IVRP}}_D$ and $\overline{\text{IVRP}}$ that

$$u = \bar{z} \leq z = \sum_{i \in N} y_i = c(N),$$

and therefore that $\sum_{i \in N} y_i = u = c(N)$ exactly when $z = \bar{z}$.

Thus, if $\overline{\text{IVRP}}$ is solved in order to find (a bound for) the optimal solution to the IVRP, we can conclude that if there is an integrality gap between z and $\bar{z}$, i.e., $\bar{z} < z$, the core is empty. Otherwise it is non-empty. Furthermore we know that there will be an integrality gap, if all the optimal solutions to the $\overline{\text{IVRP}}$ are non-integer.

Computing a solution in the core

To investigate whether this particular IVRP-Game has a non-empty core or not, we solve the $\overline{\text{IVRP}}$ by solving $\overline{\text{IVRP}}_D$.

If at least one optimal solution to $\overline{\text{IVRP}}$ is integer, i.e., $\bar{z} = z$, we have found the right hand side of the efficiency constraint [6.2] in the core formulation, i.e., $y(N) = c(N) = \bar{z}$. At the same time we have proven that the core is non-empty.

If all optimal solutions to $\overline{\text{IVRP}}$ are non-integer, we can conclude that the core is empty. If $\overline{\text{IVRP}}$ has multiple optimal solutions, it is necessary to investigate weather at least one is integer. The constraints generated in the procedure below, can be transformed into columns in $\overline{\text{IVRP}}$. A branch-and-bound procedure can the be applied on $\overline{\text{IVRP}}$, using column generation, to investigate weather there are any optimal solutions to $\overline{\text{IVRP}}$ that are integer. If it is proven that the core is empty, the column generation and branch-and-bound procedure may also be used to solve IVRP for the grand coalition, since it is necessary to solve IVRP if the value $c(N)$ is needed, for example to compute the nucleolus.

We apply the following procedure to solve $\overline{\text{IVRP}}_D$, to find out weather the core is empty or not:

0. Begin with a number of constraints (that prevents the problem from being unbounded), corresponding to a set of coalitions $\Omega \subset R$.

1. Solve the problem $\overline{\text{IVRP}}_D^R$ (which is a relaxation to the problem $\overline{\text{IVRP}}_D$ in the sense that not all $S \in R$ are included):

$$(\overline{\text{IVRP}}_D^R) \quad w = \max \sum_{i \in N} y_i$$
$$\text{s.t.} \quad \sum_{i \in S} y_i \leq \tilde{c}(S), \quad S \in \Omega \quad [6.4]$$

Identify the solution y^*.

2. For each truck type $k \in K$:
   Find a coalition[3] $S^k \in \{S | S \text{ is a feasible coalition with respect to truck type } k\} \backslash \Omega$ that most disagrees with the proposed cost allocation y^*.
   This can be found by solving $\overline{\text{IVRP}}_D^{sub}$ described below.
   Let the objective function value to $\overline{\text{IVRP}}_D^{sub}$ be z_k.

3. Choose a most unsatisfied coalition $S^{\hat{k}}$ among all truck types, i.e., choose $\hat{k} \in \arg\max_{k \in K} z_k$

4. If $z_{\hat{k}} \leq 0$, the $\overline{\text{IVRP}}_D$ is solved, and the procedure can terminate.
   If $z_{\hat{k}} > 0$ then let $\Omega = \Omega \bigcup S^{\hat{k}}$, and return to step 1.

If the solution to $\overline{\text{IVRP}}$ that is found is integer, we have a solution y in the core. Otherwise we have to investigate weather there is at least one optimal solution to $\overline{\text{IVRP}}$ that is integer, for example by using a branch-and-bound procedure.

The subproblem $\overline{\text{IVRP}}_D^{sub}$

The procedure we use to solve $\overline{\text{IVRP}}_D^{sub}$ has been adopted from Laporte & Martello (1990). They use the procedure to solve the Selective Traveling Salesman Problem (STSP). Given a graph with costs on the arcs and profits at the nodes, the STSP consists of selecting a simple circuit of maximal profit, whose route cost does not exceed a pre-specified bound.

Göthe-Lundgren et. al. (1996) have adopted the procedure to solve $\overline{\text{IVRP}}_D^{sub}$
Given a graph with costs on the arcs and profits at the nodes (corresponding to a suggested cost allocation y), select a simple circuit, including the depot, that maximizes the profits at the nodes visited minus the costs of the arcs used, such that the vehicle capacity is not exceeded. The procedure is based on implicit enumeration in a branch-and-bound procedure. For a more detailed description of the procedure, see Göthe-Lundgren et. al. (1996)[4].

The sub-procedure $\overline{\text{IVRP}}_D^{sub}$ only handles one vehicle capacity. Therefore, step 2 in the procedure has to be repeated for each available truck type $k \in K$. From a computational aspect this is not very efficient. The problem $\overline{\text{IVRP}}_D^R$ is a linear program, and is easily solved. Furthermore, it does not have to be solved from the beginning in each iteration, since only one constraint is added. It is more efficient to re-optimize in each iteration.

However, the procedure $\overline{\text{IVRP}}_D^{sub}$ is computationally difficult. The procedure to solve $\overline{\text{IVRP}}_D$ would be more efficient if $\overline{\text{IVRP}}_D^{sub}$ is adopted in such a way that for each coalition

[3]The problem of regenerating constraints corresponding to coalitions $S \in \Omega$ is easy to solve, by forbidding these coalitions in the implicit enumeration.
[4]The procedure can also be used to solve a TSP. If sufficiently high profits are assigned to the nodes, *all* nodes will be included in the solution, to a minimal arc cost, i.e., the TSP is solved.

considered, only the least costly (i.e, smallest possible) truck type is considered. This is possible, since a coalition feasible with respect to a truck type $\bar{k} \neq k_{max}$ that is unsatisfied when evaluating truck k_{max}, is even more unsatisfied when truck type $\bar{k}$ is evaluated. The procedure to solve $\overline{\text{IVRP}}_D$ would also be more efficient if *all* coalitions that disagrees with the proposed coalitions were generated in step 2-4. A simple heuristic to find unsatisfied coalitions could also be implemented in step 2, and only use $\overline{\text{IVRP}}_D^{sub}$ to verify optimality.

Since our aim has not been to implement an efficient code we solve $\overline{\text{IVRP}}_D^{sub}$ more times than necessary.

The subproblem $\overline{\text{IVRP}}_D^{sub}$ may generate constraints corresponding to coalitions S such that $\tilde{c}(S) > c(S)$. These constraints will be redundant in the optimal solution to $\overline{\text{IVRP}}_D^R$ (and in any solution to $\overline{\text{IVRP}}_D$). This is shown by the following example:

Suppose that $S_m = S_j \bigcup S_l$, where $S_m, S_j, S_l \in R$. Suppose also that the division is made in such a way that the least costly way to serve S_m is by serving S_j and S_l separately. This implies that $\tilde{c}(S_m) > \tilde{c}(S_j) + \tilde{c}(S_l) = c(S_j) + c(S_l)$, and that $c(S_m) = c(S_j) + c(S_l)$.

If all constraints in the formulation of the core of IVRP would be formulated, we would have the following three constraints among them:

$$\begin{aligned}
\sum_{i \in S_j} y_i &\leq c(S_j) && [6.5a]\\
\sum_{i \in S_l} y_i &\leq c(S_l) && [6.5b]\\
\sum_{i \in S_j \bigcup S_l} y_i &\leq c(S_m) && [6.5c]
\end{aligned}$$

Since $c(S_m) = c(S_j) + c(S_l)$, constraints [6.5a] and [6.5b] make constraint [6.5c] redundant.

Now when we consider the problem $\overline{\text{IVRP}}_D$ we have the following constraints corresponding to the coalitions S_m, S_j, S_l:

$$\begin{aligned}
\sum_{i \in S_j} y_i &\leq c(S_j) && [6.6a]\\
\sum_{i \in S_l} y_i &\leq c(S_l) && [6.6b]\\
\sum_{i \in S_m \bigcup S_l} y_i &\leq \tilde{c}(S_m) && [6.6c]
\end{aligned}$$

Since $c(S_j) + c(S_l) > \tilde{c}(S_m)$, constraints [6.6a] and [6.6b] make [6.6c] redundant.

Finally, if we consider the problem $\overline{\text{IVRP}}_D^R$ we could generate constraint [6.6c] before constraints [6.6a] and [6.6b]. However, as long as the constraint [6.6c] remains binding, at least one of the coalitions S_j or S_l will be unsatisfied with the allocation y, and the the corresponding constraint will be generated before the optimal solution to $\overline{\text{IVRP}}_D^R$ is found. Thus it is redundant in the optimal solution to $\overline{\text{IVRP}}_D^R$.

The constraints [6.4] in $\overline{\text{IVRP}}_D^R$ represent a relaxation of the core, that still gives the core.

6.1.2 Computational Results

In the IVRP-Game that we chose to study, there are $2^{21} - 1 = 1\ 048\ 575$ constraints in the definition of the core. Each of these constraints requires an IVRP to be solved. Our computations showed that the IVRP for the grand coalition N was difficult to solve. Hence it would be computationally impossible to generate all the constraints.

As shown in Chapter 6.1.1, the number of constraints can be reduced significantly by only considering the core defining inequalities that correspond to feasible coalitions, with respect to at least one truck type. It turned out that the number of constraints could be reduced to a total of 4976. These constraints are divided as presented in Table 6.1 below.

Size of the coalition, $\|S\|$	Number of constraints (feasible coalitions)
1	21
2	177
3	705
4	1 460
5	1 550
6	839
7	204
8	20

Table 6.1. Distribution of core defining inequalities in the studied IVRP-Game

Each of these constraints involves the computation of an optimal solution to a TSP. Even though the TSP is NP-hard, it is possible to solve these TSPs (since there are only eight customers in the largest problems), either using the subproblem $\overline{\text{IVRP}}_D^{sub}$ presented in Chapter 6.1.1, or even by using complete enumeration. Furthermore, a linear program with 22 variables[5] and 4976 constraints, is fairly easy to solve. Thus in the IVRP-Game we chose to study, it would be computationally possible to generate all core defining inequalities [6.3].

However, we want to evaluate a solution method that could be applied also for larger problems than the one we chose to study. We also need the solution to the IVRP for the coalition N, to be able to compute the nucleolus. Therefore we chose to investigate the emptiness of the core, by solving $\overline{\text{IVRP}}_D$.

In our computations we began the solution procedure of $\overline{\text{IVRP}}_D$ with the constraints corresponding to the coalitions $\Omega = \{S|S$ is a singleton $\} \cup \{S|S$ is a coalition in the solution applied by Norsk Hydro $\}$. From the beginning, Ω contained the 21 singletons and six coalitions corresponding to actual routes (Two of the actual routes were to singletons). The solution procedure terminated when a total of 77 constraints were included in the problem (i.e., 50 generated constraints). The optimal objective function value to $\overline{\text{IVRP}}_D$

[5]The number of variables in the first iteration in the computation of the nucleolus.

was 24 928.9. The distribution of the sizes of the coalitions, corresponding to the generated constraints, is presented in Table 6.2 below.

Size of the coalition, $\|S\|$	Number of constraints in the optimal solution to $\overline{\text{IVRP}}_D^R$	Total number of constraints (feasible coalitions)
1	21	21
2	10	177
3	6	705
4	18	1 460
5	16	1 550
6	6	839
7	0	204
8	0	20

Table 6.2. Distribution of core defining inequalities generated in the optimal solution to $\overline{\text{IVRP}}_D^R$.

The dual optimal solution, i.e., the optimal solution to $\overline{\text{IVRP}}$ is unique and *non-integer.* Thus we can conclude that the core of this particular IVRP-Game is empty.

We still want to find a cost allocation for the IVPR-Game, and therefore we compute the nucleolus and the pre-nucleolus. This is done in Chapter 6.2.

6.2 The Nucleolus

6.2.1 Mathematical Models and Methods

Before we can begin to compute the nucleolus and the pre-nucleolus, we need to know the optimal solution to the IVRP for the grand coalition.

This was done using a branch-and-bound procedure where in each node the linear problem $\overline{\text{IVRP}}$ is solved by using constraint generation (see e.g., Desrochers et. al., 1992).

Once the optimal objective function value to the IVRP-problem is known, the pre-nucleolus (and the nucleolus) can be computed through successive linear programs.

The procedure uses the same principles as in Göthe-Lundgren et. al. (1996), and as when solving $\overline{\text{IVRP}}_D$ in Chapter 6.1.1. It can be described as follows:

0. Begin with a number of constraints corresponding to a non-empty set of coalitions $\Omega \subseteq R$. For example, if the emptiness of the core has been investigated using constraint generation, these constraints may be included in Ω from the beginning. Let $\Gamma_0 = \emptyset$, and $t = 1$.

1. Solve master problem t:

$$
\begin{array}{llllll}
(P_M^t) & \max & w_t & & & \\
& \text{s.t.} & \sum_{i\in S} y_i & +w_t & \leq & \tilde{c}(S), & S\in\Omega \\
& & & & & & S\notin \bigcup_{\tau=0}^{t-1}\Gamma_\tau \\
& & \sum_{i\in S} y_i & +w_\tau^* & = & \tilde{c}(S), & S\in\Gamma_\tau, \tau=0,\dots,t-1 \\
& & \sum_{i\in N} y_i & & = & c(N) &
\end{array}
$$

   Identify the solution (y^*, w_t^*). If y^* is a unique solution to P_M^t, then y^* is the pre-nucleolus.
   Otherwise go to step 2.

2. For each truck type $k \in K$:
   Find a coalition $S^k \in \{S|S \text{ is a feasible coalition with respect to truck type } k\}\backslash\Omega$ that most disagrees with the proposed cost allocation y^*.
   This can be found by solving $\overline{\text{IVRP}}_D^{sub}$ described in Chapter 6.1.1.
   Let the objective function value in $\overline{\text{IVRP}}_D^{sub}$ be z_k.

3. Choose a most unsatisfied coalition $S^{\hat{k}}$ among all truck types i.e., choose $\hat{k} \in \arg\max_{k\in K} z_k$

4. We know that if the coalition $S^{\hat{k}}$ is unsatisfied with respect to any truck, $S^{\hat{k}}$ will be most unsatisfied with respect to the smallest possible truck that can serve them. Thus, we have the excess in $S^{\hat{k}}$:
   $e(S^{\hat{k}}, y) = \tilde{c}(S^{\hat{k}}) - \sum_{i\in S^{\hat{k}}} y_i = -z_{\hat{k}}$
   If $e(S^{\hat{k}}, y) < w_t^*$ this corresponds to a violated constraint in P_M^t, since we require: $w_t^* \leq e(S, y)$ for all $S \in R\backslash S \bigcup_{\tau=0}^{t-1}\Gamma_\tau$.
   Let $\Omega = \Omega \bigcup S_{\hat{k}}$, and return to step 1. If $e(S^{\hat{k}}, y) \geq w_t^*$, all the necessary constraints for P_M^t are included and step 5 should be performed.

5. $\Gamma_t = \{S \in R\backslash \bigcup_{\tau=0}^{t-1}\Gamma_\tau | \Pi^{t*}(S) > 0\}$. Let t=t+1, and go to step 1.

The efficiency of the procedure to find the pre-nucleolus (and the nucleolus) can be improved by modifying it, in the same way as the suggested modifications of the procedure to solve $\overline{\text{IVRP}}_D^{sub}$, which were described in Chapter 6.1.2.

The nucleolus can be found using the procedure above, if the constraints $y_i \leq c(i), i \in N$ are added to the problems P_M^t.

6.2.2 Computational results

We only considered two truck types in our computations. These were the two largest types, 33 and 34. In the tariff, the smaller truck type (33) has costs that are 97-98% of the cost of the larger type (34). To serve the customer that is closest to the depot would cost around 300 SEK, in fixed cost only. If a coalition, feasible with respect to truck type 34, would be served at a lower cost using two trucks of type 33, this would mean that the 300 SEK are saved on the 3 % lower cost of transportation. This means that the tour should be at least 1000 *km*. Clearly this is not likely, and we can conclude that in the studied IVRP, it will always be less costly to serve a feasible coalition using one truck. The cost matrices for the truck types 33 and 34 in the IVRP, can be found in Appendix 2.

In our computations we started to solve IVRP for the grand coalition, to obtain $c(N)$. The optimal objective function value of IVRP was 25 185.0 (compared to 24 928.9 for the $\overline{\text{IVRP}}$). The optimal route configuration is presented in table 6.3:

Customers	Cost of the route	Truck type used
$\{1,2,3\}$	2 390.8	34
$\{4,5,6,7,8\}$	3 573.4	34
$\{9\}$	4 009.8	34
$\{10\}$	4 385.4	34
$\{11,12\}$	1 512.7	34
$\{13,14\}$	3 096.9	33
$\{15,21\}$	2 519.4	33
$\{16,17,18,19,20\}$	3 696.6	33

Table 6.3. The optimal route configuration in the IVRP.

The only difference between the optimal IVRP-solution and the actual tours that were made, is that customers 15 and 20 are switched in the last two coalitions.

All constraints corresponding to route configurations generated in the branch-and-bound tree, could be included from the beginning in the set Ω, when computing the nucleolus. However, for practical reasons, we let Ω contain the 77 constraints generated when solving $\overline{\text{IVRP}}_D$.

When computing the pre-nucleolus the procedure went through 39 major iterations, thus generating Γ_t and w_t^* for $t = 1,\ldots,39$. The first value, w_1^*, was -33.4. This can be interpreted to measure the dissatisfaction of the most unsatisfied feasible coalition. The vector w^* was (-33.4, -25.8, -23.6, -3.3, -3.3, -0.9, 12.2, 14.6, 29.8, 42.3, 58.1, 59.1, 61.5, 61.5, 63.9, 68.3, 70.1, 75.2, 75.8, 81.0, 86.0, 90.0, 94.7, 96.3, 97.9, 98.4, 99.3, 100.8, 101.7, 102.5, 105.5, 107.1, 107.2, 109.5, 116.1, 116.9, 118.2, 118.2, 120.2)[6].

A total of 105 constraints were generated (i.e., another 28 from the solution to $\overline{\text{IVRP}}$). The distribution of the sizes of the coalitions corresponding to the generated constraints, is presented in Table 6.4 below.

[6]When two successive values are equal in the vector above, there is a difference in the decimals not shown.

Size of the coalition, $\lvert S \rvert$	Number of constraints generated to find the pre-nucleolus (New from $\overline{\text{IVRP}}$)	Total number of constraints (feasible coalitions)
1	21(0)	21
2	12(2)	177
3	13(7)	705
4	33(15)	1 460
5	20(4)	1 550
6	6(0)	839
7	0	204
8	0	20

Table 6.4. Distribution of coalitions corresponding to generated constrains in the computation of the pre-nucleolus.

When computing the nucleolus the procedure went through 41 major iterations. The first value, w_1^*, was -39.1. The vector w^* was (-39.1, -35.8, -32.6, -31.2, -5.7, -5.7, 0.0, 0.0, 6.0, 10.3, 19.8, 39.4, 45.9, 55.6, 62.4, 62.4, 62.6, 62.8, 67.6, 76.5, 77.8, 78.0, 79.0, 81.6, 87.0, 88.7, 92.8, 93.1, 97.1, 99.0, 105.3, 105.5, 105.5, 107.4, 108.4, 112.4, 114.5, 114.5, 117.0, 117.6)[7].

A total of 107 constraints were generated (i.e., another 30 from the $\overline{\text{IVRP}}$). The distribution of the sizes of the coalitions corresponding to the generated constraints, is presented in Table 6.5 below.

Size of the coalition, $\lvert S \rvert$	Number of constraints generated to find the pre-nucleolus (New from $\overline{\text{IVRP}}$)	Total number of constraints (feasible coalitions)
1	21(0)	21
2	12(2)	177
3	14(8)	705
4	35(17)	1 460
5	19(3)	1 550
6	6(0)	839
7	0	204
8	0	20

Table 6.5. Distribution of coalitions corresponding to generated constrains in the computation of the nucleolus.

The pre-nucleolus and the nucleolus are presented in Table 6.6 below. Two allocations according to the principles implied by Norsk Hydro are presented. NH_1 corresponds to

[7]When two successive values are equal in the vector above, there is a difference in the decimals not shown.

the principle that the cost of *each optimal route* is divided in proportion to the demand of the customers on the route. NH_2 corresponds to the principle that the *total cost of the* IVRP is divided in proportion to the demand of the customers of the IVRP. Finally, the stand-alone cost for each customer, $(c(i))$, is also presented.

Customer no.	Pre-nucleolus	Nucleolus	NH_1	NH_2	$c(i)$
1	625.6	625.3	580.8	795.9	1 760.8
2	725.7	732.0	677.6	928.6	1 763.8
3	1 072.8	1 072.5	1 132.5	1 552.1	1 401.4
4	866.3	864.8	788.8	762.8	1 918.6
5	182.0	201.5	102.9	99.5	2 138.6
6	510.5	494.0	281.2	271.9	2 351.2
7	813.2	812.4	857.3	829.1	2 242.4
8	1 234.8	1 239.7	1 543.2	1 492.4	2 022.4
9	4 043.2	4 009.8	4 009.8	3 323.1	4 009.8
10	4 386.3	4 385.4	4 385.4	3 316.4	4 385.4
11	719.8	722.6	777.5	1 704.6	950.4
12	826.3	829.1	735.2	1 611.8	1 037.2
13	1 542.7	1 544.9	1 307.6	1 260.2	2 723.0
14	1 587.6	1 587.8	1 789.3	1 724.5	2 443.0
15	1 828.5	1 834.4	1 755.9	1 830.7	2 172.8
16	301.3	306.5	254.9	199.0	2 338.4
17	654.8	647.4	722.3	563.8	2 307.0
18	185.9	192.5	254.9	199.0	2 177.2
19	185.0	188.3	254.9	199.0	1 713.6
20	2 168.3	2 169.9	2 209.5	1 724.5	2 709.0
21	724.3	724.0	763.5	795.9	2 032.6

Table 6.6. Computational results for the IVRP-Game.

In Table 6.6 it can be seen that the pre-nucleolus is close to the nucleolus. The largest differences are found for customers 5, 6 and 9. The individual rationality constraints are not fulfilled for customers 9 and 10 (corresponding to the two routes serving singletons, in the optimal IVRP-solution). The principle NH_2 does not fulfill the individual rationality constraint for customer 3. NH_1 seems to be the best approximation to the nucleolus. The differences between NH_1 and the nucleolus are largest for customers 5 and 6.

6.3 Other Solution Concepts

In order to compute, the normalized nucleolus, the demand nucleolus, the Shapley value and the τ-value it is required that *all* core constraints are known or generated (i.e., not only the constraints corresponding to feasible coalitions with respect to any truck). Since that would mean that more than 1 000 000 IVRPs would have to be solved, these values are impossible to compute in the studied IVRP.

It is neither possible to compute the maximum nor the minimum that each player could pay, since the core is empty.

The marginal cost of each customer, m_i could be computed, but this was not done since it would require 21 IVRPs with 20 customers to be solved. We found it computationally difficult even to find the solution to *one* IVRP (with 21 customers).

Thus the only solution concept we have studied, that is possible to compute in the IVRP-Game studied, is the nucleolus.

Chapter 7

Conclusions

An interesting comparison is that of the nucleolus of the IVRP-Game and solutions according to different solution concepts in each of the TSP-Games (corresponding to the tours that were actually carried out). The comparison is found in Table 7.1 below. It contains the nucleolus in the IVRP-Game ((IVRP-)nucleolus), the nucleolus[1], the Shapley value, the τ-value and the demand nucleolus[2] for each of the TSP-Games ((TSP-)nucleolus, (TSP-)Shapley value, (TSP-)τ-value and (TSP-)demand nucleolus, respectively). In the TSP-Games the game $(N; c^{(1a)})$ is considered in each case (See Chapter 5.1.1). The principle implied by Norsk Hydro (NH_1 in Chapter 6)[3] is also presented.

In Table 7.1, the solutions in the TSP-Games do not sum up to the same amount as the IVRP-nucleolus. The difference is due to the part γ^V of the cost remainder (which is discussed in Chapter 2.2.3). The size of γ^V is 257.8, compared to the value of the optimal IVRP-solution which is 25 185.0.

The double lines in Table 7.1 divides the customers into the corresponding actual tours.

[1]The nucleolus for each TSP-Game is computed in the same way as presented in Chapter 5.

[2]The demand nucleolus for each TSP-Game is computed in the same way as presented in Chapter 5.

[3]In Table 7.1 the actual tours are considered. In Chapter 6, Table 6.6, the routes in the optimal solution to the IVRP are considered, when computing NH_1.

Customer i	(IVRP-) nucleolus	(TSP-) nucleolus	(TSP-) Shapley value	(TSP-) τ-value	(TSP-) Demand nucleolus	NH_1
1	625.3	742.2	830.2	808.1	675.4	580.8
2	732.0	745.3	833.4	811.2	839.5	677.6
3	1 072.5	903.3	727.1	771.5	875.9	1 132.5
4	864.8	668.7	606.7	627.3	740.9	788.8
5	201.5	538.1	599.6	581.1	145.4	102.9
6	494.0	639.5	751.8	709.4	427.1	281.2
7	812.4	907.0	879.0	887.7	942.6	857.3
8	1 239.7	819.9	736.4	767.6	1317.4	1 543.2
9	4 009.8	4 009.8	4 009.8	4 009.8	4 009.8	4 009.8
10	4 385.4	4 385.4	4 385.4	4 385.4	4 385.4	4 385.4
11	722.6	712.2	712.2	712.2	719.5	777.5
12	829.1	800.6	800.6	800.6	793.2	735.2
13	1 544.9	1 688.5	1 688.5	1 688.5	1 528.3	1 307.6
14	1 587.8	1 408.5	1 408.5	1 408.5	1 568.6	1 789.3
15	1 834.4	945.2	794.1	887.4	1 756.3	2 097.0
16	306.5	606.8	682.4	676.3	263.2	227.9
17	647.4	683.4	709.3	725.6	954.1	645.8
18	192.5	581.3	627.2	616.5	228.6	227.9
19	188.3	609.8	613.6	520.2	224.3	227.9
20	2 169.9	1 860.2	1 860.2	1 860.2	2 172.4	2 082.7
21	724.0	1 183.8	1 183.8	1 183.8	871.5	961.2

Table 7.1. A collection of values for customers 1-21.

It can be seen in table 7.1 that the allocation principles implied by Norsk Hydro seems to be a very good approximation to the(IVRP-)nucleolus. The biggest differences are for customers 6, 8, 13, 14, 15 and 21. As explained in Chapter 2.2.4, the principle implied by Norsk Hydro only takes demand into consideration (and not the geographical location of the customers).

The (TSP-)demand nucleolus seems to be an even better approximation to the (IVRP-)nucleolus. It behaves considerably better than the principle implied by Norsk Hydro, for the customers 6, 8, 13, 14, 15 and somewhat better for customer 21. However, it is considerably worse for customers 3 and 17. Compared to the principle implied by Norsk Hydro, the (TSP-)demand nucleolus is computationally difficult. Furthermore, the (TSP-)demand nucleolus is not a better approximation of the (IVRP-)nucleolus in all cases (even considerably worse in some). Compared to the (IVRP-)nucleolus, the computations of the (TSP-)demand nucleolus is computationally much easier.

In the computations of the nucleolus, the possibilities of forming coalitions is important for a customer to reduce the costs that are allocated to her. The principle implied by Norsk Hydro only take demand into consideration. The (TSP-)demand nucleolus take both demand and the geographical location into consideration. Therefore it is not surprising that the allocation according to the (TSP-)demand nucleolus is somewhat better than the principle implied by Norsk Hydro. When computing the (TSP-)nucleolus only the geographical locations of the customers are considered. The (TSP-)nucleolus differs much more from the (IVRP-)nucleolus than what the principle implied by Norsk Hydro and the (TSP-)demand nucleolus do. A conclusion of this is that it seems as the size of the demand is more important than the geographical location, for the possibilities of a customer to form (good) coalitions.

The (TSP-)nucleolus, the (TSP-)Shapley-value and the (TSP-)τ-value are almost the same[4]. All these three concepts aim at finding a 'fair' allocation. If the customers of each game participate under the same conditions, all three values will divide the total cost equally (due to the anonymity property, see Chapter 3.3.3). Since in the studied TSP-Games, the customers of each game participate under *almost* the same conditions (their stand-alone cost is high, and cooperation in all coalitions is very profitable), the three methods divides the cost *almost* equally using the three values. The (IVRP-)nucleolus is quite different from the (TSP-)nucleolus. It would be interesting to see weather the Shapley value and the τ-value behave in a similar way. However, this is unfortunately not computationally possible.

[4]For the TSP-Games where $|N| \leq 2$ the nucleolus, the Shapley-value and the τ-value coincide. This property holds in general for games where $|N| \leq 2$.

Chapter 8

Further research

There are a number of possible ways to continue the research on questions related to the Norsk Hydro case.

Our motive to study the IVRP-Game instead of the TSP-Game, was that the customers should have more opportunities to form coalitions with other customers than just those that happened to be on the same tour. The extension of this would be to study what the results would be, if the customers also had a possibility to form coalitions with some of the customers that were served within delivery time, i.e., within three days. The planning problem (i.e., the characteristic function to be defined), could be seen as an IVRP-Game with time windows, or a multi-period IVRP-Game. If more days were included in the game, the game could also be used to estimate the long-term cost of the customers. The extensions of the types described would be interesting both from an theoretical and a practical aspect.

Another way to estimate the long-term cost of customers, would be to study a number of IVRPs, each corresponding to the deliveries made during one day. This would be necessary in order to prevent conclusions to be made about a customer or a customer group, based on data from unusual situations (e.g., unusual planning situations). In order to develop an efficient tool that solves many IVRP-Games, research would have to be done on how to make the solution procedure described in this thesis, more efficient.

It is also interesting to study weather the games concerning gas-oil deliveries, and shared deliveries have the same characteristics as the games concerning gas deliveries. The study of games concerning shared deliveries is needed if an allocation is to be done between the two cost centers Gas and Gas-oil. However, not only the shared delivery tours would have to be studied. A customer served on a shared delivery tour, has also the possibility to form coalitions with customers on a tour where only one product is delivered. It would be necessary to define what coalitions are feasible in such a game. If the aim is to allocate the costs to the two cost centers, another approach is to define the products as players in a game. This would turn the game in a non-atomic game (see e.g., Owen, 1995). A non-atomic game is a game with a continuum of players, instead of an $n-$person game (with n distinct players), which we have studied in this thesis.

We decided to not allocate the cost remainder. In a practical problem, such as the Norsk Hydro case, it would be interesting to treat the remainder, e.g., by including it in the mathematical modeling.

From a methodological aspect it would also be interesting to find procedures that succeeds in computing or approximating game theoretic solution concepts other than the nucleolus (e.g., the Shapley value or the τ-value), in large games such as the IVRP-Game.

Finally, it would be interesting to study cost allocation problems in other real-life problems, e.g., production planning and scheduling, financial planning and problems concerning investments in infra-structure (traffic planning, telecommunication investments etc).

Appendix 1

Distance matrix

The distances (in km) between the customers in the IVRP is presented below. It is assumed that $d_{ij} = d_{ji}$ for $i, j \in N$, where d_{ij} is the distance between customers i and j. Customer 0 is the depot.

Cust.	**0**										
1	66.6	**1**									
2	65.6	1.6	**2**								
3	40.7	29.7	28.7	**3**							
4	73.9	60.5	61.3	58.0	**4**						
5	91.9	58.3	59.1	68.2	19.6	**5**					
6	100.2	62.4	63.2	72.3	27.9	10.4	**6**				
7	89.8	44.5	45.3	58.2	47.1	42.2	39.0	**7**			
8	71.8	29.7	30.5	42.1	33.0	30.9	34.9	20.4	**8**		
9	152.3	107.3	108.1	122.6	97.0	81.8	72.2	69.2	81.9	**9**	
10	170.0	108.6	108.0	136.2	135.0	126.4	116.8	88.7	105.4	67.0	**10**
11	15.4	58.0	57.0	35.0	59.1	77.1	85.4	81.2	63.2	143.7	161.4
12	19.6	62.9	61.9	41.0	61.7	79.7	88.0	86.0	67.9	148.6	166.3
13	110.5	49.3	48.7	75.1	71.8	69.2	62.4	29.2	41.2	65.8	66.3
14	92.2	30.8	30.2	58.4	64.6	62.0	60.4	26.2	34.0	82.6	79.8
15	76.5	56.5	55.5	66.5	116.2	114.0	118.1	98.4	85.6	157.7	121.6
16	101.2	51.8	50.8	77.6	109.3	107.1	110.7	77.9	78.7	132.3	92.2
17	95.4	34.5	33.7	60.3	89.5	87.3	85.3	51.5	58.9	102.6	82.9
18	93.0	32.1	31.1	57.9	86.7	84.5	85.7	51.5	56.1	103.1	84.2
19	69.4	8.5	7.5	34.3	65.6	64.8	67.5	45.7	35.0	109.0	107.7
20	103.7	81.3	80.3	93.7	138.8	136.6	140.2	107.4	108.0	161.8	114.6
21	78.4	50.8	49.8	65.6	109.3	107.1	111.2	90.9	78.5	147.9	115.8

Cust.	11									
12	11.4	**12**								
13	101.9	106.8	**13**							
14	83.6	88.5	19.6	**14**						
15	73.4	82.0	91.6	75.5	**15**					
16	98.1	106.7	66.2	53.9	30.2	**16**				
17	87.0	91.7	36.5	26.9	59.4	32.8	**17**			
18	84.4	89.3	37.0	26.4	60.0	33.2	3.4	**18**		
19	60.7	65.7	44.6	26.1	54.6	47.2	29.9	27.5	**19**	
20	100.6	109.2	95.7	83.4	31.2	30.3	62.7	76.7	62.3	**20**
21	75.3	83.9	83.2	67.1	10.8	24.4	54.1	46.2	53.7	36.8

Appendix 2

Cost matrices

The cost (in SEK) matrices for truck type 33 and truck type 34 in the IVRP-Game are presented below. It is assumed that $c_{ij} = c_{ji}$ for $i, j \in N$, where c_{ij} is the (modified) cost of going between customers i and j. Customer 0 is the depot.

Cost matrix for truck type 33.

Cust.	0	1	2	3	4	5	6
1	880.4						
2	881.9	209.0					
3	700.7	548.6	550.1				
4	959.3	783.5	802.7	833.8			
5	1069.3	695.0	714.3	867.1	322.0		
6	1175.6	760.1	779.3	932.2	428.3	189.5	
7	1121.2	632.0	651.2	841.4	664.7	549.7	543.0
8	1011.2	553.4	572.7	750.1	593.0	505.5	569.6
9	1959.0	1472.8	1492.0	1697.9	1378.7	1162.5	1093.0
10	2142.3	1495.0	1500.4	1840.9	1761.5	1610.2	1540.6
11	475.2	849.7	851.2	698.5	867.6	977.6	1084.0
12	518.6	899.9	901.4	759.5	895.2	1005.2	1111.6
13	1361.5	716.1	721.6	1044.4	944.3	851.9	809.9
14	1221.5	574.1	579.6	920.1	913.4	821.0	830.0
15	1086.4	845.7	847.3	1018.8	1439.4	1351.0	1416.0
16	1169.2	639.7	641.3	968.0	1211.8	1123.3	1183.5
17	1153.5	511.0	514.5	839.3	1058.5	970.1	975.2
18	1088.6	446.2	447.7	774.5	989.8	901.3	937.9
19	856.8	214.3	215.8	542.6	782.5	707.7	759.1
20	1354.5	1090.3	1091.8	1286.9	1662.4	1573.9	1634.1
21	1016.3	701.0	702.6	921.3	1283.0	1194.5	1259.6

Cust.	7						
8	471.0	8					
9	1111.3	1302.9	9				
10	1312.3	1543.2	1322.8	10			
11	1090.5	980.5	1928.3	2111.6	11		
12	1139.7	1028.7	1978.5	2161.8	491.8	12	
13	531.5	716.2	1114.8	1129.2	1330.8	1381.0	13
14	541.8	685.3	1319.7	1301.6	1190.8	1241.0	513.9
15	1270.2	1211.3	2076.6	1731.4	1109.7	1196.2	1240.4
16	909.0	983.7	1667.2	1282.7	1192.5	1279.0	831.0
17	690.9	830.4	1416.7	1232.6	1124.7	1172.9	580.5
18	649.6	761.7	1380.3	1204.1	1057.9	1108.1	544.1
19	592.6	554.3	1438.3	1435.0	825.0	876.2	618.8
20	1359.6	1432.3	2117.8	1663.5	1377.8	1464.3	1281.6
21	1107.9	1052.9	1891.7	1585.7	1039.6	1126.2	1069.2

Cust.	14						
15	1122.0	15					
16	749.9	536.2	16				
17	525.9	864.3	443.1	17			
18	479.7	829.0	405.8	154.3	18		
19	476.8	775.9	543.4	414.6	349.8	19	
20	1200.5	706.8	538.1	897.7	993.9	852.5	20
21	950.8	416.7	390.5	723.6	604.7	678.4	673.1

Cost matrix for truck type 34.

Cust.	0						
1	900.9	1					
2	902.5	212.5	2				
3	715.8	560.1	561.6	3			
4	981.7	801.7	821.4	852.5	4		
5	1095.3	711.7	731.4	887.5	328.9	5	
6	1203.9	778.1	797.8	953.8	437.6	193.4	6
7	1147.7	646.2	665.9	860.3	679.6	562.5	555.2
8	1034.1	564.9	584.6	765.8	605.3	516.4	581.7
9	2004.9	1506.4	1526.2	1736.6	1409.9	1188.9	1117.2
10	2192.7	1529.0	1534.6	1883.1	1802.2	1647.8	1576.1
11	484.2	868.6	870.2	712.8	886.9	1000.5	1109.1
12	528.4	919.9	921.4	775.1	915.0	1028.6	1137.3
13	1393.9	732.2	737.8	1068.2	966.1	872.1	828.6
14	1249.9	586.2	591.8	940.3	934.0	840.0	848.9
15	1110.9	864.4	866.0	1041.2	1473.3	1383.3	1449.7
16	1197.7	654.9	656.5	990.9	1241.6	1151.6	1212.9
17	1181.0	522.3	525.9	858.3	1083.8	993.8	998.7
18	1115.0	456.3	457.9	792.3	1013.8	923.8	961.0
19	877.2	218.5	220.1	554.5	801.2	725.3	777.5
20	1385.8	1115.1	1116.7	1316.1	1701.8	1611.8	1673.2
21	1039.9	716.8	718.4	942.0	1313.6	1223.6	1290.0

Cust.	7						
8	480.1	8					
9	1135.5	1331.3	9				
10	1341.4	1577.5	1350.0	10			
11	1115.4	1001.8	1972.6	2160.4	11		
12	1165.7	1051.0	2023.9	2211.7	500.1	12	
13	542.6	731.3	1138.8	1153.2	1361.6	1412.8	13
14	552.8	699.2	1348.5	1329.7	1217.6	1268.9	523.9
15	1299.7	1238.5	2124.7	1770.3	1134.0	1222.6	1268.7
16	930.9	1006.8	1706.5	1311.8	1220.8	1309.4	850.6
17	706.6	849.0	1449.0	1259.8	1150.7	1200.0	593.0
18	664.8	779.0	1412.2	1231.1	1082.7	1134.0	556.3
19	606.4	566.4	1471.7	1468.0	843.9	896.2	632.9
20	1391.1	1465.0	2166.7	1700.5	1408.9	1497.5	1310.8
21	1133.9	1076.8	1935.7	1621.6	1063.0	1151.6	1093.9

Cust.	14						
15	1146.9	15					
16	767.1	547.5	16				
17	536.7	883.5	453.3	17			
18	489.9	847.8	415.5	157.0	18		
19	486.9	793.4	556.6	424.0	358.1	19	
20	1227.3	720.5	549.2	917.5	1016.8	871.7	20
21	972.1	424.0	398.8	739.9	618.5	694.1	686.8

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