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Grassmann Scalar Fields and Asymptotic Freedom

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Abstract We extend previous results about scalar fields whose Fourier components are even elements of a Grassmann algebra with given index of nilpotency. Their main interest in Particle Physics is related to the possibility that they describe fermionic composites analogous to the Cooper pairs of superconductivity.
We evaluate the free propagators for arbitrary index of nilpotency and we investigate a ϕ^4 model to one loop. Due to the nature of the integral over even Grassmann fields such a model exists for repulsive as well as attractive selfinteraction. In the first case the β-function is equal to that of the ordinary theory, while in the second one the model is asymptotically free. The bare mass has a peculiar dependence on the cutoff, being quadratically decreasing/ increasing for attractive/repulsive selfinteraction.

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1. Introduction and summary

Recently, in the study of a model where the gauge bosons are constructed in terms of the defining elements of a Grassmann algebra, nonlinear changes of variables in Berezin integrals have been considered [1]. In the preliminary investigation of such changes of variables the following program was outlined

i) to construct a perturbative scheme in terms of fermionic composites by introducing such composites as integration variables in the Berezin integral which defines the partition function of the constituents [2]

ii) to use such a formalism to construct models in terms of even Grassmann fields (even polynomials of the defining elements) assumed as independent variables.

The present paper is mainly devoted to the second part of the program, restricted to the case of scalar fields defined in terms of Fourier components of given index of nilpotency. For a model of this kind to be interesting in Particle Physics, such fields should have a particle interpretation. Now, while the meaning of even Grassmann fields is clear when they are introduced by a change of variables in the partition function of a fermionic system, it is no longer so when they are considered as independent variables from the outset. It would appear most natural to continue to interpret them as fermionic composites analogous to the Cooper pairs of superconductivity, which are represented in the partition function by an even field with Fourier components of index 2. But relativistic theories are severely constrained and we do not know when a given action of even fields can be obtained as the effective action of a fermionic system, even if such an effective action need not be entirely consistent: Lack of unitarity at some energy, for instance, can be the signal of the compositeness and of the importance of the constituents at that energy.

Surprisingly enough, there appears to be also the (highly specultaive) possibility that even fields can be associated to new particles (with unusual properties). We will in fact see that we have a consistent particle interpretation for the free theory, but we do not know if consistent interactions can be constructed.

The index of nilpotency of a variable ϕ is the smallest integer n^* such that

$$\phi^n = 0, \quad for\ n > n^*. \tag{1}$$

So in this paper we consider fields of given index of nilpotency in momentum space

$$(\tilde{\phi}(p))^n = 0, \quad for\ n > n^*,\ \ \forall\, p. \tag{2}$$

A following paper [3] will be devoted to the study of fields of given index of nilpotency in configuration space

$$(\phi(x))^n = 0, \quad for\ n > n^*,\ \ \forall\, x. \tag{3}$$

The present paper is organized in the following way. We will start by giving in Sec. 2 the rule of integration over even Grassmann variables and discussing some of its properties. We will report the full formalism, although its application in the present paper is confined to the simplest cases. This because we want to put our results into the perspective of a possible relation to fermionic composites, and therefore we want to show its generality. Moreover we need the full formalism to clarify, in Sec.3, the origin of the difference between the propagators of fields nilpotent in configuration and momentum space. Finally, we need it to explain the limitations in the interactions of fields nilpotent in momentum space and the difficulty in proving the reflection positivity of such interactions.

Sec. 3 deals with the free propagators of scalar fields. In ref. [1] it was already shown that it is possible to define the free action of a scalar field of index 2 in momentum space (this choice did not have any other reason that simplicity, 2 being the smallest index with the variables adopted there) in such a way that its propagator be equal to that of an ordinary scalar. Here we extend this result to arbitrary index of nilpotency. We will argue, however, that the thermodynamics of even Grassmann fields must be different from that of ordinary fields, the former ones obeying an exclusion principle. In spite of this, it seems from our derivation that it is consistent to associate these fields with new particles (with unusual properties) although we do not know if consistent interactions can be constructed (see Section 4). In the negative case the formalism would be interesting in Particle Physics only if these even fields could be related to fermionic composites.

In ref. [1] we also investigated the propagator of a complex scalar field of index 1 in configuration space. This propagator was related to the selfavoiding random walk. Here we clarify the origin of this difference with respect to fields of given index in momentum space. The extension to higher values of the index for fields nilpotent in configuration space can be found in the following paper.

In Sec. 4 we will consider the interactions of our fields. Since gauge transformations change the index of the Fourier components, in the framework of gauge symmetries a scalar field with given index in momentum space can only be introduced as the (gauge-invariant) polar radius of a Higgs field. We are thus led to selfinteractions, whose study was already started in ref.[1] with the ϕ^4 model. We will complete this investigation, whose main motivation is the hope to avoid the triviality of the model with ordinary fields, a feature whose relevance to the Higgs sector of the electroweak Standard Model has been discussed by many authors [4]. Triviality with ordinary fields could be avoided if it were possible to define the model for attractive selfcoupling, in which case it is perturbatively asymptotically free [5]. But there seems to be a rather general consensus that it is not possible to circumvent the obstruction of the euclidean action unbounded from below [6]. Now with even Grassmann fields the model can be defined also for attractive selfcoupling, at least in the presence of a suitable cutoff, and in its first investigation it was found to be asymptotically free in perturbation theory. In the presence of selfattraction, however, one expects a nontrivial vacuum. The natural way to investigate its structure would seem to use the so called Hubbard-Stratonovich transformation [7], but in the last Section we will show that this application is precluded by the lack of a small expansion parameter.

The cancellation of disconnected terms, which is more involved than in the ordinary case was not proven in ref. (1). In the present paper we show that it does occur, and we investigate the behaviour of the bare mass with the cutoff, which turns out to be rather surprising, being quadratically increasing/decreasing for repulsive/attractive selfcoupling. We perform the analysis for fields of index 1 and 2 finding the same behaviour.

We will not say anything about the physical interpretation of the model. The investigation of a possible relation to fermionic composites is in progress. The alternative that the ϕ-field be a fundamental one, on the other hand, depends on whether the model satisfies or not reflection positivity. The discussion of such a property following Osterwalder and Schrader [8] appears difficult to us for a reason explained below (while with fields nilpotent in configuration space it is similar to that of ordinary fields).

If a particle interpretation is possible, the model might find an application, admittedly rather artificial, in the Higgs sector of the Standard Model of electroweak interactions. Since, as we will see, a gauge symmetry cannot be introduced for a ϕ- field with given index in momentum space, as already said such a field could only be related to the gauge-invariant polar radius in a polar parametrization of the Higgs field. For the same reason, if the model is to be considered an effective theory, the "fundamental " fermionic constituents of the ϕ-field should not partecipate into the gauge transformations of the Electroweak Model and should therefore be new particles.

Summarizing, even fields with given index in momentum space are easy to treat perturbatively, but the possibility of constructing models is very limited, and the assessment of the physical interpretation is still lacking. The physical meaning of fields with given index in configuration space is instead much more transparent [3].

2. Nonlinear change of variables in a Berezin integral

Let us consider the composites

$$\phi_I = \sum_{i_1,i_2=1}^{2N} c_I^{i_1 i_2} \lambda_{i_1} \lambda_{i_2}, \tag{4}$$

where the λ-fields are the generating elements of a Grassmann algebra. The index i is associated to intrinsic as well as position or momentum degrees of freedom and for notational convenience we have put

$$\lambda_{N+i} = \lambda_i^*, \quad i = 1, \ldots N, \tag{5}$$

if the total number of degrees of freedom is N.

Our aim is to introduce the ϕ-fields as integration variables in the composite correlation functions defined by the Berezin integral

$$< \phi_{I_1} ... \phi_{I_n} >= \frac{1}{Z_\lambda} \int [d\lambda] dU \phi_{I_1} ... \phi_{I_n} e^{-S(\lambda,U)}, \tag{6}$$

where

$$[d\lambda] = \prod_{i=1}^{2N} d\lambda_i, \quad Z_\lambda = \int [d\lambda] dU e^{-S(\lambda,U)}, \tag{7}$$

dU is the measure over any additional variables U on which the euclidean action $S(\lambda, U)$ might depend. Therefore we must define an integral over the ϕ's in such a way that for an arbitrary function f

$$\int [d\phi] f(\phi) = \int [d\lambda] f[\phi(\lambda)]. \tag{8}$$

Eqs.(4) cannot obviously be inverted, namely the λ's cannot be expressed in terms of the ϕ's. In which sense they can be considered a change of variables will be specified below.

We recall that there is only one function (modulo a numerical factor) of the constituent fields which has a nonvanishing Berezin integral. This function is the product of all the λ-fields

$$\Lambda = \lambda_1 ... \lambda_{2N}, \tag{9}$$

and its Berezin integral is

$$\int [d\lambda] \Lambda = 1. \tag{10}$$

To find out the integration rule over the ϕ's, we must therefore determine all their functions which, when expressed in terms of the constituent fields, are proportional to Λ with nonzero coefficient. We call them relevant. Only when relevant functions can actually be constructed by means of the given composites, can the latter be introduced as new variables of integration.

The most general function of nilpotent variables is a polynomial. Hence it is sufficient to determine all the *relevant monomials*, which are the monomials of maximum degree. We call them *fundamental* if they are products of powers of the ϕ's with coefficient 1

$$\Theta_m = \phi_1^{m_1}\phi_2^{m_2}... = w_m\Lambda. \tag{11}$$

These monomials are characterized by the vector index m with components m_I restricted according to

$$\sum_I m_I = N \tag{12}$$

and by the weight w_m.

The fundamental monomials with the quantum numbers of the pion and the nucleon have been determined in the last of refs. [1].

The integral over the ϕ's can be expressed in terms of the weights. In fact any function of the ϕ's can be written in the form

$$f(\phi) = \sum_m f_m\Theta_m + irrelevant\ terms. \tag{13}$$

If we think of f as expressed, via the definition of the ϕ's, in terms of the λ's, its Berezin integral is

$$\int [d\lambda] f(\phi(\lambda)) = \sum_m f_m w_m. \tag{14}$$

This result can be directly obtained if we integrate over the new variables ϕ provided we give as rule of integration

$$\int [d\phi]\Theta_m = w_m \tag{15}$$

for all the fundamental monomials, all other integrals being zero. Note that, although in general different expansions

$$f(\phi) = \sum_m f_m\Theta_m + irr.\ terms = \sum_m f'_m\Theta_m + irr.\ terms \tag{16}$$

may exist, the above equality implies

$$\sum_m f_m w_m = \sum_m f'_m w_m, \tag{17}$$

since both the l.h.s. and the r.h.s. are equal to the coefficient of Λ in the expansion of $f(\phi)$ in terms of the generating elements, so that the value of the integral does not depend on the particular expresssion for f.

It should now be clear in which sense relations like Eq.(4) can be considered a change of variables. They cannot obviously be inverted, and therefore an action defined in terms of the λ's cannot in general be expressed through the ϕ's. But in any nonvanishing Berezin integral the integrand must be proportional to Λ, which we can always replace by relevant functions of the composite variables.

This integration rule has some characteristic features

1) The number of degrees of freedom need not be balanced. A discussion of this point can be found in the last of Refs. [1] and it will not be reported here.

2) The integral of a fundamental monomial is unaffected by an arbitrary shift $\phi \to \phi+\alpha$. The integral of an arbitrary function, however, does not enjoy such a property, because its expansion in fundamental monomials does change as a conseqence of the shift. Unlike the Berezin integral, therefore, the integral over even elements of a Grassmann algebra is not translation invariant. It enjoys, however, another property which is sufficient to derive an equation of motion for even Grassmann fields, as it will be shown in the subsequent paper .

3) The integral over even Grassmann variables is not invariant under a change from even to even variables which alters the index of nilpotency. This includes Fourier transforms and more generally unitary transformations. For this reason in the construction of models with even Grassmann fields as fundamental fields, for most symmetries we must restrict ourselves to fields nilpotent in configuration space. For instance local gauge transformations of such a scalar field

$$\phi(x) \to e^{i\theta(x)}\phi(x) \tag{18}$$

do not alter its index of nilpotency (but do change that of its Fourier transform). It is perhaps whorth while mentioning that the restriction to a given index of nilpotency in configuration space is also compatible with a model where the gauge fields are even elements of a Grassmann algebra (first of Refs. [1]).

Among the various symmetries, Lorentz invariance can be implemented for scalar fields with given index in configuration or momentum space. The first case is obvious, since $\phi(x)$ is Lorentz invariant, while for the second one we observe that under a Lorentz transformation (in the continuum, of course) the Fourier components do not change the index of nilpotency

$$\tilde{\phi}(p) \rightarrow e^{-id\Lambda p}\tilde{\phi}(\Lambda p). \tag{19}$$

4) It is easy to evaluate the integral of the exponential of a quadratic form of fields of index 1

$$\int [d\phi^* d\phi] e^{\sum_{I,J} \phi_I^* M_{IJ} \phi_J} = perM, \tag{20}$$

where $perM$ is the permanent of the matrix M. But the permanent of a matrix cannot be evaluated by diagonalization, which reflects the noninvariance of the integral under unitary transformations. As a consequence the evaluation of the propagator of even Grassmann fields cannot be performed in the standard way.

3. The free correlation functions of a scalar field nilpotent in momentum space

In this Section we show that it is possible to define the partition function of even Grassmann fields with given index in momentum space in such a way that their free correlation functions *at zero particle density* are equal to those of an ordinary, local scalar field.. We will not study the correlation functions *at finite particle density*, but we expect them, for a reason explained below, to be different from the corresponding ones of an ordinary scalar field, so that different should be the thermodynamics.

We will first discuss the case of a real field, and at the end we will report the result for a complex one.

It is convenient to use the parametrization

$$\tilde{\phi}(p) = \frac{1}{\omega(p)} A(p), \quad A^*(p) = A(-p), \tag{21}$$

where

$$\omega(p) = (p^2 + m^2)^{\frac{1}{2}}. \tag{22}$$

We start by the case where the $A(p)$'s have index 1

$$\int dA(p) A^m(p) = \delta(m-1), \ \ \forall p. \tag{23}$$

We assume the free action to be *minus* the standard one

$$S_0(\phi) = -\frac{1}{2} \int d^4x\, \phi(x)[-\Box + m^2]\phi(x), \tag{24}$$

and we define the correlation functions according to

$$< \tilde{\phi}(p_1)...\tilde{\phi}(p_{2n}) >_0 = \frac{1}{Z_0} \int [dA] \frac{A(p_1)}{\omega(p_1)} ... \frac{A(p_{2n})}{\omega(p_{2n})} e^{-S_0} \tag{25}$$

where

$$[dA] = \prod_{p \neq 0} dA(p), \quad Z_0 = \int [dA]\, e^{-S_0}. \tag{26}$$

This definition is analogous to that appropriate to ordinary scalars in terms of holomorphic variables. The reason why $A(0)$ is not integrated over is that it does not appear in S_0. In fact by inserting in Eq. (24) the Fourier transform of ϕ

$$\phi(x) = \frac{1}{L^2} \sum_p e^{ipx} \tilde{\phi(p)}, \ \ L = edge\ of\ the\ quantization\ volume, \tag{27}$$

we find

$$S_0 = -\frac{1}{2} \sum_p A(p) A(-p). \tag{28}$$

Since $A^2(0) = 0$, S_0 does not depend on $A(0)$. Moreover for $p \neq 0$ it is a function of the variables of index 1

$$B(p) = A(p) A(-p), \quad p \neq 0, \tag{29}$$

in terms of which

$$S_0 = -\frac{1}{2} \sum_{p \neq 0} B(p). \tag{30}$$

Since $B(p) = B(-p)$, it is convenient to define "star" sums, products, and integrals, where the momentum p ranges in the domain

$$\begin{aligned} \mathcal{P}^+ &: p_4 > 0 \\ p_4 &= 0,\ p_3 > 0 \\ p_4 &= p_3 = 0,\ p_2 > 0 \\ p_4 &= p_3 = p_2 = 0,\ p_1 > 0. \end{aligned} \tag{31}$$

We then have

$$\int [dA] = \int^* [dB] \tag{32}$$

$$S_0 = -\sum_p^* B(p), \tag{33}$$

$$e^{-S_0} = \prod_p^* [1 + B(p)]. \tag{34}$$

The fundamental monomials are the $B(p)$'s whose weight is 1. An alternative possibility is to use the variables $A(p), A^*(p)$ for $p \in \mathcal{P}^+$.

In order to evaluate the correlation functions we observe that they vanish unless the product of the A's can be arranged into a product of the B's, namely unless their arguments fall into pairs of opposite momenta. We can then use relations of the type

$$\begin{aligned} Z_0 &= \int [dA] e^{-S_0} = \int^* [dB] e^{-S_0} = 1, \\ &\int [dA] A(p_1) A(p_2) e^{-S_0} = \delta(p_1 + p_2) \int^* [dB] B(p_1^*)\, e^{-S_0} = \delta(p_1 + p_2), \end{aligned} \tag{35}$$

where

$$p^* = p,\ if\ p \in \mathcal{P}^+,\ p^* = -p,\ otherwise. \tag{36}$$

In such a way we get for the propagator the expression valid for ordinary scalars

$$< \tilde{\phi}(p_1) \tilde{\phi}(p_2) >_0 = \delta(p_1 + p_2) \frac{1}{\omega^2(p_1)}. \tag{37}$$

This result can easily be generalized to the $2n$-point correlation functions. We get a nonvanishing contribution only when all the momenta fall into *different* pairs of opposite momenta (in such a case we will say that the momenta are simply paired, while we will say that they are multiply paired with multiplicity m if a given momentum and its opposite occur m times)

$$\begin{aligned} < \tilde{\phi}(p_1)...\tilde{\phi}(p_{2n}) >_0 &= \frac{1}{Z_0} \sum_{all\ pairings} \int^* [dB] \frac{B(p^*_{i_1})}{\omega^2(p_{i_1})} \delta(p_{i_1} + p_{i_2}) ... \frac{B(p^*_{i_{2n-1}})}{\omega^2(p_{i_{2n-1}})} \delta(p_{i_{2n-1}} + p_{i_{2n}}) e^{-S_0} \\ &= \prod_{h<k} \theta(p_{2h}, p_{2k}) \sum_{all\ pairings} < \tilde{\phi}(p_{i_1}) \tilde{\phi}(p_{i_2}) >_0 ... < \tilde{\phi}(p_{i_{2n-1}}) \tilde{\phi}(p_{i_{2n}}) >_0, \\ & p_{i_{2h}} \neq \pm p_{i_{2k}},\ h \neq k. \end{aligned} \quad (38)$$

The θ-function appearing in the above equation is defined by

$$\theta(p, q) = 0, for\ \ p = \pm q, \ \ \theta(p, q) = 1\ otherwise. \quad (39)$$

Let us now consider these correlation functions in the thermodynamic limit. *At zero particle density*, the momenta excluded by the θ-functions belong to a subspace of vanishing measure, and the $2n$-point correlation functions are equal to those of a local free field.

The thermodynamics, however, is not expected to be the same as that of ordinary fields, because if we take the thermodynamic limit at *fixed nonvanishing density*, the restrictions of the θ-functions will act as an exclusion principle.

The above result can be easily generalized to an index of nilpotency $n^* > 1$. Confining ourselves for the sake of brevity to n^* even we normalize the integral over the A's and the B's according to

$$\begin{aligned} \int dA(p) A^m(p) &= n^*! \delta(m - n^*), \ \forall p, \\ \int dB(p) B^m(p) &= (n^*!)^2 \delta(m - n^*), \ \ p \neq 0 \ ; \int dB(0) B^m(0) = n^*! \delta(m - \frac{1}{2} n^*). \end{aligned} \quad (40)$$

Notice that

$$\int [dA] = \int dA(0) \int^{*} [dB], \quad for\ n^* > 1. \tag{41}$$

It is then easy to see that in order to get a 2-point function equal to that of a local free field, we must take as free action

$$S_0^{(n^*)}(\phi) = -\frac{1}{2} n^* S_0(\phi) = -\frac{1}{2} n^* B(0) - n^* \sum_p^{*} B(p). \tag{42}$$

In fact the exponential of this action has the expansion

$$e^{-S_0^{(n^*)}} = \sum_{s_0=1}^{\frac{n^*}{2}} \frac{1}{s_0!} [\frac{1}{2} n^* B(0)]^{s_0} \prod_p^{*} \sum_{s_p=1}^{n^*} \frac{1}{s_p!} [n^* B(p)]^{s_p}, \tag{43}$$

and repeating the steps which led to Eq.(38) we find the same result when the momenta are simply paired. But now some pairs of momenta can coincide. If the pairing has multiplicity m, with $m < n^*$, we must insert in the r.h.s. a factor $(n^*)^{-m}(n^*)!/(n^* - m)!$, while if $m > n^*$ the r.h.s. vanishes. The conclusion about the thermodynamic limit is the same as for index 1.

Let us finally report the result relative to a complex scalar. The definition of the Fourier transform remains of course valid, but without the reality condition, so that $A(p)$ and $A^*(p)$ are independent variables. We then find that to have a propagator equal to that of an ordinary scalar, for index n^* we must choose the action

$$S_0^{(n^*)}(\phi^*, \phi) = -n^* \int dx^4 \phi^*(x)[-\Box + m^2]\phi(x). \tag{44}$$

We conclude by two remarks. First, nothing in what we have done forces the interpretation of the ϕ-field as a fermionic composite, so that it seems theoretically consistent the existence of a new particle which behaves as an ordinary scalar at zero density, but has a different thermodynamics. But only if we can construct a consistent interaction, can such a field become the candidate for a new particle.

Second, we can clarify why the propagators of fields nilpotent in momentum and configuration space are so different from one another. In the first case there is only one fundamental monomial

$$\Theta = \prod_p [\tilde{\phi}(p)]^{n^*}. \quad (45)$$

If we perform a change of integration variables passing to the field in configuration space $\phi(x)$, we have infinitely many fundamental monomials if the field is defined in continuum space. If the field is defined on a lattice with N^4 lattice sites the fundamental monomials are

$$\Theta_m = \prod_x [\phi(x)]^{m_x}, \quad m = \{m_x\}, \quad \sum_x m_x = N^4 n^*. \quad (46)$$

This explains why it is easy to evaluate the propagator of fields of given index in momentum space, while we have no analytic expression for fields of given index in configuration space, whose propagator can only be related to the selfavoiding random walk.

4. The ϕ^4 theory

As already said we are unable to construct gauge interactions for fields with given index in momentum space, apart from the possibility mentioned in the introduction. We are then left with Yukawa and selfinteractions. In this Section we consider a ϕ^4 theory where the ϕ-field, assumed as an independent field, has index 1 or 2 in momentum space. The interaction

$$S_I = \frac{1}{4!} g \int d^4x \phi^4(x) = \frac{1}{4!} g \frac{1}{L^4} \sum_{q_1 q_2 q_3 q_4} \delta(q_1 + q_2 + q_3 + q_4) \prod_{i=1}^{4} \frac{A(q_i)}{\omega(q_i)} \quad (47)$$

is the standard function of the Fourier transform of the ϕ-field and the correlation functions

$$< \tilde{\phi}(p_1) ... \tilde{\phi}(p_{2n}) >= \frac{1}{Z} \int [dA] \tilde{\phi}(p_1) ... \tilde{\phi}(p_{2n}) e^{-(S_0+S_I)} \quad (48)$$

have the standard definition in terms of the partition function

$$Z = \int [dA] e^{-(S_0+S_I)}. \quad (49)$$

It should by now be clear why it is difficult to discuss reflection positivity for such a model. This property is best proven when the integration variables are the fields in configuration space, and if we perform the necessary Fourier transforms in the above equations, we are unable to evaluate the integrals, because of the uncontrollable number of fundamental monomials in configuration space.

Let us emphasize that the model is well defined irrespective of the sign of g, but when the selfinteraction is attractive we expect a nontrivial vacuum. In this Section we will confine ourselves, however, to the perturbative expansion with respect to the perturbative vacuum for both signs of g.

A qualification is in order for index of nilpotency 1. As already said S_0 does not depend on $A(0)$. But the terms involving this variable in S_I cannot contribute to any correlation function because they contain an odd number of momenta which therefore cannot be paired. As a consequence for index 1 the variable $A(0)$ is suppressed everywhere.

Let us introduce the renormalized action, with the counterterms necessary for a one-loop calculation, in the usual way

$$S_r = S_{0r} + S_{Ir}, \tag{50}$$

where

$$\begin{aligned} S_{0r} &= -n^* \int dx^4 \frac{1}{2}\phi_r(x)[-\Box + m_r^2]_\Lambda \phi_r(x), \\ S_{Ir} &= \int dx^4 \{-\frac{1}{2} n^* \delta m^2 \phi_r^2(x) + \frac{1}{4!} g_r Z_g \phi_r^4(x)\}, \end{aligned} \tag{51}$$

with

$$\phi = Z^{\frac{1}{2}}\phi_r, \quad g = g_r \frac{Z_g}{Z^2}, \quad m^2 = (m_r^2 + \delta m^2)/Z. \tag{52}$$

The suffix Λ means that the wave operator is regularized. For definiteness one can have in mind a lattice regularization or a cut off procedure implemented by the replacement

$$\omega(p) \to \omega(p) exp(\frac{p^2}{\Lambda^2}). \tag{53}$$

We will evaluate the 2- and 4-point functions at one loop. For the 2-point function we have

$$< \tilde{\phi}(p_1)\tilde{\phi}(p_2) >_1 = \mathcal{P}_1(p_1,p_2) - < \tilde{\phi}(p_1)\tilde{\phi}(p_2) >_0 \mathcal{D}_1, \tag{54}$$

where

$$\begin{aligned} \mathcal{D}_1 &= \frac{1}{Z_0}\int [dA](-S_{Ir})e^{-S_{0r}} \\ \mathcal{P}_1(p_1,p_2) &= \frac{1}{Z_0}\int [dA]\tilde{\phi}(p_1)\tilde{\phi}(p_2)(-S_{Ir})e^{-S_{0r}}. \end{aligned} \tag{55}$$

Similarly for the 4-point function

$$< \tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\tilde{\phi}(p_4) >_1 = \mathcal{A}_1(p_1,p_2,p_3,p_4) - \mathcal{A}_0(p_1,p_2,p_3,p_4)\mathcal{D}_1 \tag{56}$$

where

$$\begin{aligned} \mathcal{A}_0(p_1,p_2,p_3,p_4) &= \frac{1}{Z_0}\int [dA]\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\tilde{\phi}(p_4)(-S_{Ir})e^{-S_{0r}} \\ \mathcal{A}_1(p_1,p_2,p_3,p_4) &= \frac{1}{Z_0}\int [dA]\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\tilde{\phi}(p_4)\frac{1}{2}S_{Ir}^2 e^{-S_{0r}}. \end{aligned} \tag{57}$$

In the evaluation of the integrals, in order to single out from the beginning the pairings which occur in S_{Ir}, we write the latter in the form, valid for index 1 and 2

$$S_{Ir} = \sum_{i=1}^{9} T_i \tag{58}$$

where

$$\begin{aligned} T_1 &= \frac{1}{4!}g_r\frac{1}{L^4}6\tilde{\phi}^2(0)\sum_{q\neq 0}\tilde{\phi}(q)\tilde{\phi}(-q) \\ T_2 &= \frac{1}{4!}g_r\frac{1}{L^4}12\tilde{\phi}(0)\sum_{q\neq 0}\tilde{\phi}^2(q)\tilde{\phi}(-2q) \\ T_3 &= \frac{1}{4!}g_r\frac{1}{L^4}4\tilde{\phi}(0)\sum_{q_1,q_2,q_3\neq 0}\prod_{i<j=1}^{3}\theta(q_i,q_j)\delta(q_1+q_2+q_3)\prod_{h=1}^{3}\tilde{\phi}(q_h) \end{aligned}$$

$$
\begin{aligned}
T_4 &= \frac{1}{4!}g_r\frac{1}{L^4}3\sum_{q\neq 0}\tilde\phi(q)\tilde\phi(-q) \\
T_5 &= \frac{1}{4!}g_r\frac{1}{L^4}3\sum_{q_1,q_2\neq 0}\theta(q_1,q_2)\tilde\phi(q_1)\tilde\phi(-q_1)\tilde\phi(q_2)\tilde\phi(-q_2) \\
T_6 &= \frac{1}{4!}g_r\frac{1}{L^4}6\sum_{q_1,q_2,q_3\neq 0}\prod_{i<j=1}^{3}\theta(q_i,q_j)\delta(2q_1+q_2+q_3)\tilde\phi^2(q_1)\tilde\phi(q_2)\tilde\phi(q_3) \\
T_7 &= \frac{1}{4!}g_r\frac{1}{L^4}\sum_{q_1,q_2,q_3,q_4\neq 0}\prod_{i<j=1}^{4}\theta(q_i,q_j)\delta(q_1+q_2+q_3+q_4)\prod_{h=1}^{4}\tilde\phi(q_h) \\
T_8 &= -\frac{1}{2}n^*\delta m^2\tilde\phi^2(0) \\
T_9 &= -\frac{1}{2}n^*\delta m^2\sum_{q\neq 0}\tilde\phi(q)\tilde\phi(-q).
\end{aligned} \tag{59}
$$

To simplify the calculation we exclude the value $p_i = 0$ from the arguments of the correlation functions, and we restrict the arguments of the 4-point functions according to

$$
p_i \neq \pm p_j, \quad i \neq j. \tag{60}
$$

The contributions to $\mathcal{P}_1$ and $\mathcal{D}_1$ come only from the terms T_1, T_4, T_5, T_8, T_9

$$
\begin{aligned}
\mathcal{P}_1(p_1,p_2) &= <\tilde\phi(p_1)\tilde\phi(p_2)>_0 \frac{1}{Z_0}\int^*[dB]e^{-S_0}B(p_1^*)\{-\frac{1}{4!}g_r\frac{1}{L^4}[6\sum_q^*\frac{B^2(q)}{\omega^4(q)} \\
&+12\sum_{q_1q_2}^*\theta(q_1,q_2)\frac{B(q_1)B(q_2)}{\omega^2(q_1)\omega^2(q_2)}+12\frac{B(0)}{\omega^2(0)}\sum_q^*\frac{B(q)}{\omega^2(q)}] \\
&+\frac{1}{2}n^*\delta m^2[\frac{B(0)}{\omega^2(0)}+2\sum_q^*\frac{B(q)}{\omega^2(q)}]\},
\end{aligned} \tag{61}
$$

$$
\begin{aligned}
\mathcal{D}_1 &= \frac{1}{Z_0}\int^*[dB]e^{-S_0}\{-\frac{1}{4}g_r\frac{1}{L^4}[\sum_q^*\frac{B^2(q)}{\omega^4(q)}+2\sum_{q_1q_2}^*\theta(q_1,q_2)\frac{B(q_1)B(q_2)}{\omega^2(q_1)\omega^2(q_2)} \\
&+2\frac{B(0)}{\omega^2(0)}\sum_q^*\frac{B(q)}{\omega^2(q)}]+\frac{1}{2}n^*\delta m^2[\frac{B(0)}{\omega^2(0)}+2\sum_q^*\frac{B(q)}{\omega^2(q)}]\}.
\end{aligned} \tag{62}
$$

Because of the contribution of double pairings (the factors $B^2(p)$) the evaluation of $\mathcal{P}_1$ and $\mathcal{D}_1$ must be done separately for index 1 and 2.

The evaluation of $\mathcal{A}_0$ is straightforward and holds for both values of the index

$$
\mathcal{A}_0(p_1,p_2,p_3,p_4) = -g_r\frac{1}{L^4}\delta(p_1+p_2+p_3+p_4)\prod_{i=1}^{4}\frac{1}{\omega^2(p_i)}, \quad n^* = 1,2. \tag{63}
$$

Finally $\mathcal{A}_1$ gets non connected contributions from the products $T_1T_7, T_4T_7, T_5T_7, T_7T_8, T_7T_9$

$$\begin{aligned}\mathcal{A}_{1,nonconn}(p_1,p_2,p_3,p_4) &= \mathcal{A}_0(p_1,p_2,p_3,p_4)\int^*[dB]e^{-S_0}\prod_{l=1}^4 B(p_l)\\ &\{-g_r\frac{1}{L^4}[\frac{1}{2}\frac{1}{\omega^2(0)}B(0)\sum_q^* \frac{B(q)}{\omega^2(q)}+\frac{1}{2}\sum_{q_1q_2}^*\theta(q_1,q_2)\frac{B(q_1)B(q_2)}{\omega^2(q_1)\omega^2(q_2)}\\ &+\frac{1}{4}\sum_q^*\frac{B(q)}{\omega^4(q)}]+\frac{1}{2}n^*\delta m^2[\frac{B(0)^2}{\omega^2(0)}+2\sum_q^*\frac{B(q)}{\omega(q)^2}]\}.\end{aligned} \tag{64}$$

It must also be evaluated separately for the different values of the index.

The connected term comes entirely from the product T_7T_7

$$\begin{aligned}\mathcal{A}_{1,con}(p_1,p_2,p_3,p_4) &= \frac{1}{2}[\frac{1}{4!}g_r\frac{1}{L^4}]^2\sum_{q_1\ldots q_4\neq 0}\sum_{k_1\ldots k_4\neq 0}\delta(\sum_{m=1}^4 q_m)\delta(\sum_{n=1}^4 k_n)\\ &\prod_{i<j=1}^4\theta(q_i,q_j)\theta(k_i,k_j)\int[dA]\prod_{h=1}^4\tilde\phi(p_h)\tilde\phi(q_h)\tilde\phi(k_h)e^{-S_0}.\end{aligned} \tag{65}$$

Due to the restrictions imposed by the θ- and δ-functions and those on the arguments of the correlation functions, the above integral does not vanish only when 2 of the p_i are paired to 2 of the q_i, the remaining p_i to 2 of the k_i and the remaining q_i and k_i among themselves. There are 12^2 ways to do that. In general such pairings, for index 2, have multiplicity 2. Due to the aforementioned restrictions, however, only single pairings contribute so that the evaluation of the integral is the same for index 1 and 2

$$\begin{aligned}\mathcal{A}_{1,con}(p_1,p_2,p_3,p_4) &= -\mathcal{A}_0(p_1,p_2,p_3,p_4)\frac{3}{2}g_r\frac{1}{L^4}\sum_{k_1k_2\neq 0}\theta(k_1,k_2)\prod_{i=1}^2\prod_{j=1}^4\theta(k_i,p_j)\delta(p_1+p_2+k_1+k_2)\\ &\frac{1}{\omega^2(k_1)\omega^2(k_2)}\int[dA]\prod_{l=1}^4 B(p_l)B(k_1)B(k_2)e^{-S_0},\quad n^*=1,2.\end{aligned} \tag{66}$$

We must now transform the integral over the A's into an integral over the B's, and for this purpose we must rewrite the sum over k_1, k_2 as a sum over momenta belonging to $\mathcal{P}^+$. After that the evaluation is straightforward

$$\mathcal{A}_{1,con}(p_1,p_2,p_3,p_4) = -\mathcal{A}_0(p_1,p_2,p_3,p_4)\frac{3}{2}g_r\frac{1}{L^4}\sum_{k_1k_2}^{*}[\delta(p_1+p_2+k_1+k_2)+\delta(p_1+p_2-k_1-k_2)$$
$$+\delta(p_1+p_2+k_1-k_2)+\delta(p_1+p_2-k_1+k_2)]\frac{1}{\omega^2(k_1)\omega^2(k_2)}. \qquad (67)$$

Only the last 2 δ-functions contribute to the divergent part of this term

$$\mathcal{A}_{1,div}(p_1,p_2,p_3,p_4) = -\mathcal{A}_0(p_1,p_2,p_3,p_4)\frac{3}{2}g_r\frac{1}{L^4}\sum_k\frac{1}{\omega^4(k)} \qquad (68)$$

which is equal to that of the ordinary theory, so that also the counterterm is the same

$$\delta g_r = \frac{3}{2}g_r^2\frac{1}{L^4}\sum_k\frac{1}{\omega^4(k)}. \qquad (69)$$

It follows that the β-function is that of the ordinary theory

$$\beta = \frac{3}{16\pi^2}g_r^2, \qquad (70)$$

and the model is asymptotically free for attractive selfcoupling.

Let us now pass to the evaluation of the quantities which depend on the index of nilpotency.

For index of nilpotency 1 the terms containing $B(0)$ and $B^2(p)$ vanish. The surviving ones give

$$\mathcal{P}_1(p_1,p_2) = <\tilde\phi(p_1)\tilde\phi(p_2)>_0\{-\frac{1}{2}g_r\frac{1}{L^4}\sum_{q_1q_2}^{*}\frac{1}{\omega^2(q_1)\omega^2(q_2)}\theta(q_1,p_1)\theta(q_2,p_1)\theta(q_1,q_2)$$
$$+\delta m^2\sum_q^{*}\theta(q,p_1)\frac{1}{\omega^2(q)}\},$$
$$\mathcal{D}_1 = -\frac{1}{2}g_r\frac{1}{L^4}\sum_{q_1q_2}^{*}\theta(q_1,q_2)\frac{1}{\omega^2(q_1)\omega^2(q_2)}+\delta m^2\sum_q^{*}\frac{1}{\omega^2(q)}, \qquad (71)$$

so that the 2-point function to one loop results

$$<\tilde\phi(p_1)\tilde\phi(p_2)>_1=<\tilde\phi(p_1)\tilde\phi(p_2)>_0\frac{1}{\omega^2(p_1)}\{g_r\frac{1}{L^4}\sum_q^{*}\theta(q,p_1)\frac{1}{\omega^2(q)}-\delta m^2\}. \qquad (72)$$

Because of the restriction in the sum over q we see that the mass counterterm cannot be strictly local, but it is local modulo a term which vanishes in the thermodynamic limit

$$\delta m^2 = \frac{1}{2} g_r \frac{1}{L^4} \sum_q \frac{1}{\omega^2(q)} + \mathcal{O}(\frac{1}{(Lm)^4}). \tag{73}$$

Notice that δm^2 is increasing/decreasing for positive/negative g_r and its absolute value is equal to that of the ordinary theory.

The above result can also be obtained by an appropriate decomposition of any diagram vanishing because of nilpotency into the sum of a connected plus a disconnected graph with opposite values. The value of the disconnected graph must be such that the cancellation of disconnected graphs should be complete in the thermodynamic limit. This requires a positive/negative sign for a connected graph with an even/odd number of crossings respectively, as shown for fields with given index in configuration space [1].

Let us now pass to index 2. In this case we must retain the variable $A(0)$ and we have, neglecting terms $\mathcal{O}(1/(Lm)^4)$.

$$\begin{aligned}
\mathcal{P}_1(p_1, p_2) &= <\tilde{\phi}(p_1)\tilde{\phi}(p_2)>_0 \{-\frac{1}{4!} g_r \frac{1}{L^4}[3 \sum_q^* \theta(q, p_1) \frac{1}{\omega^2(q)}[\frac{1}{\omega^2(q)} + \frac{4}{\omega^2(p_1)} + \frac{4}{\omega^2(0)}] \\
&\quad +12 \sum_{q_1, q_2}^* \theta(q_1, q_2)\theta(q_1, p_1)\theta(q_2, p_1) \frac{1}{\omega^2(q_1)\omega^2(q_2)}] \\
&\quad +\delta m^2[\frac{1}{\omega^2(0)} - \frac{1}{\omega^2(p_1)} + 2 \sum_q^* \frac{1}{\omega^2(q)}]\}, \\
\mathcal{D}_1 &= -\frac{1}{4!} g_r \frac{1}{L^4}\{3 \sum_q^* \frac{1}{\omega^2(q)}[\frac{1}{\omega^2(q)} + \frac{4}{\omega^2(0)}] + 12 \sum_{q_1, q_2}^* \theta(q_1, q_2) \frac{1}{\omega^2(q_1)\omega^2(q_2)}\} \\
&\quad +\delta m^2[\frac{1}{\omega^2(0)} + 2 \sum_q^* \frac{1}{\omega^2(q)}].
\end{aligned} \tag{74}$$

Therefore the 2-point function is

$$<\tilde{\phi}(p_1)\tilde{\phi}(p_2)>_1 = <\tilde{\phi}(p_1)\tilde{\phi}(p_2)>_0 \frac{1}{\omega^2(p_1)}\{\frac{1}{2} g_r \frac{1}{L^4} \sum_q^* \frac{1}{\omega^2(q)} - \delta m^2\}, \tag{75}$$

and the cancellation of disconnected terms requires

$$\delta m^2 = \frac{1}{4} g_r \frac{1}{L^4} \sum_q \frac{1}{\omega^2(q)}, \tag{76}$$

whose absolute value is half that of the ordinary theory.

Finally $\mathcal{A}_{1,noncon}$ can be evaluated in a similar way, and it can be shown to vanish after use of the appropriate value of δm^2.

5. The Hubbard-Stratonovich transformation

When the selfcoupling is attractive we can introduce a truly bosonic field σ by means of the so called Hubbard-Stratonovich transformation

$$exp\{\frac{1}{4!}|g|.\int dx^4 \phi^4(x)\} \propto \int [d\sigma] exp\{-\int dx^4 [\frac{1}{2}\sigma^2(x) + \frac{\sqrt{|g|}}{2\sqrt{3}}\sigma(x)\phi^2(x)]\}. \tag{77}$$

We can then rewrite the partition function of a field with Fourier components of index n^* in the form

$$Z = \int [d\sigma][d\phi] exp\{-\int dx^4 [-\frac{1}{2} n^* \phi(x)[-\Box + m^2]\phi(x) + \frac{1}{2}\sigma^2(x) + \frac{\sqrt{|g|}}{2\sqrt{3}}\sigma(x)\phi^2(x)]\} \tag{78}$$

It is perhaps worth while noticing that the integral over ϕ would not exist if this field were truly bosonic. Now one can proceed to a loop expansion to evaluate this integral in order to obtain an effective action for the field σ, whose vacuum properties could be studied in the standard way. Unfortunately exponentiating the result of the loop expansion one gets in the action a logarithm whose expansion gives all the terms of order 1 irrespective of n^*. To have a small expansion parameter one should replace n^* by $(n^*)^2$ in quadratic action of the field ϕ. This would produce terms of order $(n^*)^{-s}$ for the s-th term of the expansion of the logarithm. There being at present no justification for such a choice we have not pursued the calculation.

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