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CERTAIN PROBLEMS CONCERNING WAVELETS AND WAVELET PACKETS

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ABSTRACT

Wavelet analysis is the outcome of the synthesis of ideas that have emerged in different branches of science and technology, mainly, in the last decade. The concept of wavelet packets, which are superpositions of wavelets, has been introduced a couple of years ago. They form bases which retain many properties of wavelets like orthogonality, smoothness and localization. The Walsh orthonormal system is a special case of wavelet packet. The wavelet packets provide at our disposal a library of orthonormal bases, each of which can be used to analyse a given signal of finite energy. The optimal choice is decided by the entropy criterion. In the present paper we discuss results concerning convergence, coefficients, and approximation of wavelet packets series in general and wavelets series in particular. Wavelet packet techniques for solutions of differential equations are also mentioned.

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1 Introduction

Wavelet analysis is the outcome of the synthesis of ideas that have emerged in different branches of Mathematics, Physics and Engineering. Since the days of Fourier, scientists and engineers, besides mathematicians themselves, have made vigorous efforts to represent square integrable functions (signals having finite energy) as linear combination of functions having some nice properties. Radamacher, Haar, Walsh, Franklin and Vilenkin have constructed non-trigonometric orthonormal systems in their endeavour to accomplish this goal. The Walsh function was extensively studied and applied by Electrical and Electronic Engineers during the seventies and eighties prior to the invention of wavelets in the mid eighties (see for example [89] and [92] and references therein). In 1981, Strömberg [98] constructed an orthonormal spline system on the real line which is now termed as the first example of wavelet constructed by a mathematician. However, without having the knowledge of this work, physicists like Grossman and geophysicsts like Morlet were developing a technique to study non-stationary signals which led to the development of the wavelet theory in the last decade (see refs.[48]-[50]). Meyer, Daubechies, Mallat (see references specially [30], [73], [74], [79], [80], [81]) have put this theory on firm foundation through the multiresolution analysis and establishing relationship between function spaces and wavelet coefficients. This scientific discipline of vital importance has been introduced in an excellent way by Meyer [80], where he has also explained the relationship between fractals (another exciting scientific discipline) and wavelets along with future avenues of researches specially in understanding the hierarchial organization and formation of distant galaxies. For further interaction of fractals and wavelets we refer to Arneodo *et al.* in [81] 1. Wavelet transform of fractals: from transition to chaos to fully developed turbulence. 2. Optical wavelet transform of fractal growth phenomena, pp.286–352; Holschneider [56], [60a], Hardin *et al.* [54], Wornell and Oppenheim in [36], pp.785–800, and Hazewinkel in [68b], wavelet understand fractals, 217–219. Most of the important results on theory and applications of wavelets can be found in either references [3], [8], [9], [10]–[18], [21]. [23], [25], [30], [31], [36], [37], [39], [40], [45], [46], [47], [56], [60a], [61], [64], [65], [67], [68], [71], [72], [77], [78], [79], [80], [81], [82], [88], [91], [93], [94], [97], [108], [109], [110], [111] and [115] or the references given therein. Since 1991 a generalization of wavelets, known as the wavelet packet has been studied by Wickerhauser, Meyer, Coifman and others (see Refs. [22], [23], [61], [72], [80], [85], [93], [94], [113], [114] and [116]. Wavelet packets are also called arborescent wavelets and they are particular linear combinations or superpositions of wavelets. Discrete wavelet packets have been thoroughly studied by Wickerhauser [113] who has also developed computer programmes and implemented them. The study of convergence, coefficients, and approximation of wavelet packet series is the main objective of this paper.

The discovery of wavelets (small waves) was a result of an attempt to search a function which will generate the space of square integrable functions over the real line. The ideal tool for studying stationary signals is Fourier transformation, that is, natural stationary signals decompose into the linear combination of waves (sines and cosines). In the same manner non-stationary signals decompose into linear combinations of wavelets. The study of non-stationary signals, where transient events appear that cannot be predicted necessitates techniques different from Foureir analysis. These techniques, which are specific to the non-stationarity to the signal include wavelets of the time frequency type and wavelets of the time scale time. Time frequency wavelets are suited specially to the analysis of quasi-stationary signals, while time scale wavelets are adopted to signals having a fractal structure. Time scale analysis in the case of image processing is called the Multiresolution analysis. This involves a vast range of scales for signal analysis. In the Fourier analysis there is no relation between Fourier coefficients and Fourier transforms while there is a close relationship between wavelet coefficients and wavelet transforms. The discrete wavelet transform is faster than the fast Fourier transform as it requires only $O(N)$ operations while the fast Fourier transform needs $O(N \log_2 N)$. Well known Daubechies orthogonal wavelets [p.43,80] are a special case of wavelet packets. Wavelet packets are organized naturally into collections, and each collection is an orthonormal basis for $L_2(R)$. It is a simple but very powerful extension of wavelets and multiresolution analysis. The wavelet packets allow more flexibility in adopting the basis to the frequency contents of a signal and it is easy to develop a fast wavelet packet transform. The power of the wavelet packet lies in the fact that we have much more freedom in deciding which basis function we use to represent the given function. The best basis selection criteria and applications to image processing can be found in [22], [112] and [113].

2 Basic definitions and notations

Haar function is defined as follows:

$$\begin{aligned} h(x) &= 1, \text{ if } 0 \le x < \frac{1}{2} \\ &= -1, \text{ if } \frac{1}{2} \le x < 1 \\ &= 0, \text{ otherwise} \end{aligned}$$

For $n \geq 1, \quad n = 2^j + k, \quad j \geq 0, \quad 0 \leq k < 2^j, \quad h_n(x)$, where

$$h_n(x) = 2^{j/2}\, h(2^j x - k) \ , \tag{2.1}$$

is called Haar orthonormal system. For a comprehensive account of this we refer to DeVore and Lucier [40] and Schipp, Wade and Simon [89].

Walsh orthonormal systems is defined as follows:

$$\begin{aligned}\varphi_0(x) &= 1 \text{ if } 0 \le x < \frac{1}{2} \\ &= -1 \text{ if } \frac{1}{2} \le x < 1 \\ \varphi_n(x) &= \varphi_0(2^n x), \quad \varphi_n(x+1) = \varphi_n(x)\end{aligned} \tag{2.2}$$

$\{\varphi_n(x)\}$ is known as the Rademacher system of functions. For $n = 2^{n_1} + 2^{n_2} + \ldots + 2^{n_j}$, $n_{i+1} < n_i$, $\psi_n(x)$ is defined as $\psi_n(x) = \varphi_{n_1}(x)\varphi_{x_2}(x)\ldots\varphi_{n_j}(x)$.

$\psi_n(x)$ is called the Walsh orthonormal system (see for example Schipp, Wade and *Simon [89] and Siddiqui [92]* for detailed information about the literature of this system).

Spline

Let I be an interval of R divided into a set of smaller subintervals. A function f on I is called a spline function of rder $m \ge 0$ if it is polynomial of degree less than or equal to $m+1$ on each of the given subintervals of I and if all of its derivatives upto order m are continuous on I.

The characteristic function of $[0,1]$ is the piecewise constant spline. The piecewise linear spline is given by

$$\varphi(x) = \begin{cases} 1 - |x|, & \text{if } 0 \le |x| \le 1 \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

The concept of B-splines (basic splines) was introduced by Curry and Schoenber which are splines with the smallest possible support. The B-spline of order 1 is the characteristic function of $[0,1]$ and the B-spline of degree $n > 1$ denoted by $B^n(x)$ is defined recursively by the convolution:

$$\begin{aligned}B^n(x) = B^1 * B^{n-1}(x) &= \int_{-\infty}^{\infty} B^{n-1}(x-t)\ B^1(t)dt \\ &= \int_0^1 B^{n-1}dt\end{aligned}$$

It can be seen that

$$B^n(x) = \sum_{j=0}^{n+1} \frac{(-1)^j}{n!} \binom{n+1}{j} \left[x + \frac{n+1}{2} - j\right]_+^n$$

where $[x]_+^n = \max\{0, x\}^n$ is the one sided power function of degree n. For details of splines we refer to Chui [10], Schoenberg [90], DeVore and Lorentz [38] and Unser and Aldroubi in [11, 91–122].

Function spaces: The function spaces, which have been studied in areas like Fourier analysis, partial differential equations, approximation theory, mathematical physics and more recently in the wavelet theory, are:
(i) L_p spaces (ii) Hölder (or Lipschitz class of functions) spaces (iii) Zygmund spaces (iv) Sobolev spaces (v) Lorentz spaces (vi) Calderon spaces ($L_{p,q}$ spaces) (vii) Orlicz–Sobolev spaces (ciii) I-M-S spaces (functions of $Lipj(t)$ class) (ix) Generalized Sobolev spaces (Sobolev spaces over metric spaces) (x) Besov spaces (vi) Hardy spaces (xii) Tent spaces (Coifman–Meyer–Stein spaces) (xiii) $F^s_{p,q}$ spaces (xiv) Spaces of functions of bounded oscillation - BMO spaces (xv) Atomic Hardy space.

All functions considered are real valued defined on R^n. However all these definitions can be extended for complex valued functions. Let $\Omega \subseteq R^n$.

$C(\Omega)$ denotes the set of real valued functions which are bounded and uniformly continuous functions in Ω, equipped with the norm:

$$\|f\|_\Omega = \sup_{x\in\Omega} |f(x)| \,.$$

$C(\Omega)$ is a Banach space (Ω has appropriate properties. In many cases we prefer one dimension setting to understand the notion in a more concise manner than is possible in higher dimension.

Let $k \in N$ (set of positive integers), then

$$C^k(\Omega) = \{f \in C(\Omega) | D^\alpha f \in C(\Omega) \ \text{ if } \ |\alpha| \le k\}$$

is a Banach space equipped with the norm

$$\|f\|_{k,\Omega} = \sum_{|\alpha|\le k} \|D^\alpha f\|_\Omega$$

where $\alpha = (\alpha_1, \alpha_2, \dots \alpha_n)$ with $\alpha_i \in N, |\alpha| = \sum_{j=1}^{h} \alpha_j$ and

$$D^\alpha f(x) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}(x)$$

[$C(\Omega)$ is also written as $C^0(\Omega)$].

For $0 < p < \infty$, L_p is the space of all those Lebesgue measurable functions such that $|f|^p$ is Lebesgue integrable, that is, $L_p = \{f | \int_\Omega |f|^p dx < \infty\}$ and L_∞ = space of essentially bounded functions. L_p is a Banach space for $1 < p \le \infty$ with the norm

$$\begin{aligned} \|f\|_{L_p} &= \left(\int_\Omega |f(x)|^p dx\right)^{2/p}, \quad 1 \le p < \infty \\ \|f\|_{L_\infty} &= \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| \end{aligned}$$

For $0 < p < 1$, L_p is a quasi-Banach space [101]. Let $\Delta_h f(x) = \Delta_h^1 f(x) = f(x+h) - f(x)$ and $\Delta_h^{k+1} = \Delta_h^1 \Delta_h^k$, k any natural number. Then

$$\Delta_h^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + h_j), \quad h \in \Omega, \quad x \in \Omega$$

where $\binom{k}{j}$ are binomial coefficients

$$Lip\lambda = \left\{ f \in C^k | \sup \frac{|f(x) - f(x)|}{|x-y|^\lambda} < \infty, 0 < \lambda < 1 \right\} \ k \in N$$

$Lip\lambda$ is a Banach space with respect to norm

$$\|f\|^k_{Lip\lambda} = \|f\|_{C(n)} + \sup \left\{ \frac{|f(x) - f(y)|}{|x-y|^\lambda} \right\}, \quad x \neq y$$

Let $s \in R$, then we write $s = [s] + \{s\} = [s]^- + \{s\}^+$ where $[s]$ and $[s]^-$ are integers, whereas $0 \leq \{s\} < 1$ and $0 < \{s\}^+ \leq 1$. Hölder space

$$H^s = \left\{ f \in C | \ \|f\|_{H^s} = \|f\|_C^{[s]} + \sum_{|\alpha|=[s]} \|D^\alpha f\| \ Lip\{s\} < \infty \right\}$$

Zygmund class: Let $s > 0$, then

$$Z^s = \left\{ f \in C | \ \|f\|_{Z^s} = \|f\|_C^{[s]^-} + \sum_{\substack{|\alpha|=[s]^- \\ h \neq 0 \in R^n}} \sup |h|^{-\{s\}^+} \|\Delta_h^2 D^\alpha f\|_C < \infty \right\}$$

The case $s = k$ or $s = 1$ $k \in N$ is generally known as the Zygmund class of functions. Let

$$\omega(t) = \omega(f,t) = \sup_{\substack{x,y \in \Omega \\ |x-y| \leq t}} |f(x) - f(y)|, \quad t \geq 0$$

$\omega(t) \to \omega(0)$, for $t \to 0$, ω is non-negative, non-decreasing, continuous on R_+ and subadditive $(\omega(t_1, +t_2) \leq \omega(t_1) + \omega(t_2)$. $\omega(f,t)$ is called the modulus of continuity of f.

r-th modulus of continuity of $f \in L_p(\Omega), 0 < p < \infty$ is defined as

$$\begin{aligned} \omega_r(f,t)_p &= \sup_{0<h\leq t} \|\Delta_h^r(f,\cdot)\|_{p,\Omega}, \quad t \geq 0 \\ \omega_1(f,t) &= \omega(f,t) . \end{aligned}$$

$|f\|_{Lip\alpha} = \sup_{t>0} (t^{-\alpha}\omega(f,t))$ is a semi-norm of $Lip\alpha$. For details see [38] and [101].

$$0 < p \leq \infty \qquad \begin{aligned} Lip(\alpha, L_p) &= \left\{ f \in L_p | \left[\int_\Omega |f(x+t) - f(x)|^p dx \right]^{1/p} \leq Mt^\alpha \right\} \\ |f|_{Lip(\alpha,L_p)} &= \sup_{t>0} (t^{-\alpha})\omega(f,t)_p) \end{aligned}$$

is a semi-norm on $Lip(\alpha,p), 0 < \alpha \le 1, p \ge 1$.
Generalized $Lip\alpha = \{f \in L_p | \int_\Omega |\Delta_t^r(f,x)|^p dx\}^{1/p} \le Mt^\alpha, t > 0$. See for details [38] and [101].

Sobolev spaces: $W_p^k = \{f \in L_p | D^\alpha f \in L_p, |\alpha| \le k\}$ is called Sobolev space of order k which is a Banach space with the norm

$$\|f\|_{W_p^k(\Omega)} = \left(\sum_{|\alpha|\le k} \|D^\alpha f\|_{L_p}^p\right)^{1/p}$$

It is a Hilbert space for $p = 2$ and $W_p^k = L_p$ for $k = 0$. For details see [1], [68] and [101] and references therein.

For Lorentz and Calderon spaces we refer to [38, pp.23–24].

Let $N(x) = \int_0^u M(t)dt$, where $M(t) \ge 0$ is increasing, with $M(0) = 0$. L_N the space of those functions f for which $\chi^N(af) \le 1$ for some $a > 0$, where

$$\chi^N(f) = \int_0^b N(|f(t)|)dt < \infty$$

is called Orlicz space

$$\chi^N(f) = L_p \quad \text{for} \quad M(t) = pt^{p-1}, p \ge 1$$

Orlicz–Sobolev space denote by $W_{L_N}^k$ is defined as $W_{L_N}^k = \{f \in L_N / D^\alpha f \in L_N\}$.

$W_{L_N}^k$ is a Banach space. It is a Sobolev space for $M(t) = pt, p \ge 1$. See Adams [1] and Trüdinger [102] for details of Orlicz–Sobolev spaces. For weighted Sobolev spaces one may see [68a].

I-M-S spaces: ($Lipj(t)$ class of functions): Let $j(t)$ be a positive and non-decreasing function defined on $(0,1)$ then

$$\begin{aligned}
Lipj(t) &= \left\{t \in C \middle| \sup_{t>0} \frac{|f(x+t) - f(x)|}{j(t)} \le M\right\} \\
(Lipj(t), L_p) &= \left\{f \in L_p \middle| \frac{\left(\int_0^1 |f(x+t) - f(x)|^p\right)^{1/p}}{j(t)} \le M\right\} \\
(Lipj(t), L_p, r) &= \{f \in L_p | \omega_r(f,t)_p \le j(t)\}
\end{aligned}$$

For details we refer to Izumi–Izumi [57].

Generalized Sobolev spaces: Let (X,d,μ) be a metric space (X,d) with finite diameter $\left(\dim X = \sup_{x,y\in X} d(x,y) < \infty\right)$ and a finite Borel measure μ. For $1 < p \le \infty$ we define

$L^{1,p}(X,d,\mu)$ and $W^{1,p}(X,d,\mu)$ as follows: $L^{1,p}(X,d,\mu) = \{f : X \to R | f$ is measurable and $\exists\ E \subset X, \mu(E) = 0$ and $\exists\ g \in L_p(\mu)$ such that $|f(x) - f(y)| \leq d(x,y)(g(x) + g(y))$ for all $x, y \in X \backslash E\}$

$$W^{1,p}(X,d,\mu) = \left\{f \in L^{1,p}(X,d,\mu) | f \in L_p\mu)\right\}$$

$W^{1,p}(X,d,\mu)$ is a generalized Sobolev space recently studied in [51, 52, 53].

Besov spaces: Besov spaces denoted by $B^{\alpha}_{pq}(R)$ are defined as:
For $0 < \alpha < r, 0 < p \leq \infty$

$$\begin{aligned}
B^{\alpha}_{pq}(R) &= \left\{f \in L_p \Big| \left(\int_0^{\infty} \left[t^{-\alpha}\omega_r(f,t)_p\right]^q \frac{dt}{t}\right)^{1/q} < \infty\right\} \\
B^{\alpha}_{p\infty}(R) &= \left\{f \in L_p \Big| \sup_{t \geq 0}\ t^{-\alpha}\omega_r(f,t)_p < \infty\right\} \\
\|f\|_{B^{\alpha}_{pq}} &= \|f\|_{L_p} + \left(\int_0^{\infty} [t^{-\alpha}\omega_r(f,t)_p]^q \frac{dt}{t}\right)^{1/q}, \quad 0 < q < \infty \\
&= \|f\|_{L_p} + \sup_{t \geq 0}\ t^{-\alpha}\omega_r(f,t)_p, \quad q = \infty
\end{aligned}$$

is a quasi-norm on $B^{\alpha}_{pq}(R)$ and for $1 \leq p < \infty, 1 \leq q < \infty$ it is a norm.

$Lip(\alpha, p) = B^{\alpha}_{p\infty}, 0 < \alpha < 1$ and Sobolev spaces W^{α}_{12} is equal to Besov spaces B^{α}_{22}. For more details and applications of Besov spaces we refer to [79, 101]. A good account of Hardy, Tent and F^s_{pq} spaces are presented in [101].

BMO spaces: Let g be a locally integrable function on R. Then g is called to have bounded mean oscillation, that is, $g \in BMO$ if

$$\|g\|_* = \sup_I \frac{1}{|I|} \int_I |g(x) - g_I| dx \ \text{ is finite}$$

where I is any finite subinterval of R and g_I is the average of g over I. It may be observed that $\|g\|_*$ is a seminorm on BMO which is a vector space.

Atomic Hardy space: The atomic Hardy space denoted by H'_{at} comprises all real valued functions f on R for which there omits atoms a_j and coefficients $\alpha_j (j = 1, 2, 2 \ldots)$ such that $j(x) = \sum_j \alpha_j a_j(x)$ at all those points whose measure is zero (a.e.) and $\sum_j |\alpha_j| < \infty$. The norm is defined by

$$\begin{aligned}
\|f\|_{H'_{at}} &= \inf \sum_j |\alpha_j| \\
f &= \sum_j \alpha_i a_j
\end{aligned}$$

where infinimum is taken over all such atomic representations of f. An atom is a real valued function on R for which there exists an interval I so that $|a| \leq |I|^{-1}\chi_I$ a.e., $\int_I a(x)dx = 0$, χ_I is the characteristic function of I and $|I|$ denote the length of I.

References about BMO spaces can be found in [79].

3 Wavelets and wavelet packets

A family of functions $\{\varphi_i\}$ in a Hilbert space H is called orthonormal if

$$\begin{aligned} \langle \varphi_i, \varphi_j \rangle &= 0 \text{ if } i \neq j \\ &= 1 \text{ if } i = 1 \end{aligned}$$

The family $\{\varphi_i\}$ in H is called a frame if there exist constants $A > 0$, $B < \infty$ so that for all f in H,

$$A\|f\|^2 \leq \sum_{i \in J} |\langle f, \varphi_i \rangle|^2 \leq B\|f\|^2$$

A and B are called the frame bounds. If $A = B$ then the frame is called the *light frame*. If $A = B = 1$ then the frame becomes an orthonormal basis. A system of functions $\{\varphi_i\}$ in H is called the Riesz basis or unconditional basis of H if

(a) for every $f \in H$ there are unqiue coefficients c_n such that

$$f(x) = \sum_n c_n \varphi_i$$

(b) there are positive constants A, B such that for each $f \in H$

$$A\|f\|^2 \leq \sum_n |c_n|^2 \leq B\|f\|^2$$

Every Riesz basis is a frame but the converse need not be true. Every orthonormal basis is a Riesz basis and hence a frame [see 30, 68b].

(a) can be replaced by the condition that $\{\varphi_i\}$ are linearly independent in H.

Definition 3.1 A function $\psi \in L_2 \subset R)$ is called a wavelet (or orthonormal wavelet) if the sytem $\psi_{j,k}(x) = 2^{j/2}\ \psi(2^j x - k), j, k \in Z$ (set of integers) is an orthonormal basis of $L_2(R)$, that is, the following conditions are satisfied:

$$\langle \psi_{j,k}(x), \psi_{\ell,m}(x) \rangle = \delta_{j\ell} \cdot \delta_{km}, \quad j, k, \ell, m \in Z, \tag{3.1}$$

where $\delta_{j\ell}$ is the Kronecker delta, and

$$f(x) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{j,k}\ \psi_{j,k} \tag{3.2}$$

where

$$c_{j,k} = \langle f, \psi_{j,k} \rangle \tag{3.3}$$

This definition can be generalized by replacing the orthonormal basis by a Riesz basis or furthermore by a frame.

Definition 3.2 The series given in the relation (3.2) is called the wavelet series and $c_{j,k}$ given in the relation (3.3) are called the wavelet coefficients.

Definition 3.3 Let $a \neq 0, b$ are arbitrary real numbers, then

$$\psi_{a,b}(x) = (|a|)^{-1/2}\, \psi\left(\frac{x-b}{a}\right), \quad \text{where } \psi \in L_2(R) \text{ and } \int_{-\infty}^{\infty} x\, \psi(u)dx = 0, \tag{3.4}$$

is called continuous wavelet.

The wavelet transform of f denoted by $W_{\psi_{(a,b)}}$ is defined as follows:

$$W_{\psi_{a,b}}(f) = \langle f, \psi_{a,b} \rangle = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} f(x)\, \psi\left(\frac{x-b}{a}\right) dx \tag{3.5}$$

It can be easily checked applying Parseval identity that

$$2\pi\, W_{\psi_{a,b}}(f) = \langle \hat{f}, \hat{\psi}_{a,b} \rangle \tag{3.6}$$

where

$$\hat{\psi}_{a,b}(\omega) = \frac{a}{\sqrt{|a|}}\, e^{-i\omega b}\, \hat{\psi}(a\omega) \tag{3.7}$$

It must be observed carefully that there is no relationship between Fourier coefficients and Fourier transformation but the following relation holds in the wavelet case

$$W_{\psi 2^{-j}, k2^{-j}}(f) = c_{j,k} \tag{3.8}$$

Definition 3.4 A *multiresolution analysis* of $L_2(R)$ is a sequence of its closed subspaces $V_j, j \in Z$ having the following properties:

1. $V_j \subset V_{j+1}$,
2. $v(x) \in V_j$ if and only if $v(2x) \in V_{j+1}$,
3. $v(x) \in V_0$ if and only if $v(x+1) \in V_0$,
4. $\bigcup_{j=-\infty}^{\infty} V_j$ is dense in $L_2(R)$ and $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$,
5. There exists a function $\varphi \in V_0$ such that $\{\varphi(x-k), k \in Z\}$ is an orthonormal basis of V_0.

$\varphi(x)$ is called the *scaling function.*

Since $\varphi \in V_0 \subset V_1$ there exists a sequence $\{h_k\} \in \ell_2$ such that the scaling function φ satisfies the equation

$$\varphi(x) = 2\sum_k h_k\, \varphi(2x - k) \tag{3.9}$$

Eq.(3.9) is known by several names, for example *the dilation equation, refinement equation* or two scale difference equation. It is easy to check that $\{\varphi_{i,j}\}$, where $\varphi_{ij}(x) = 2^{i/2}\,\varphi(2^i x - j)$ is an orthonormal basis of V_j.

We have the orthogonal complement of each in the next higher one on the ladder, that is

$$\begin{aligned} V_0 \oplus W_0 &= V_1, \quad W_0 \perp V_0 \\ V_1 \oplus W_1 &= V_2, \end{aligned}$$

In general, $V_i \oplus W_i = V_{i+1}, \quad V_i \perp W_i$.

Let W_0 be spanned by the integer translates of a function ψ, that is, translates of ψ are an orthonormal basis of W_0. The W_i is generated by $\{\psi_{i,j}\}$ where $\psi_{i,j}(x) = 2^{i/2}\psi(2^i x - j)$. Since $\psi \in W_0 \subset V_1$, we have

$$\psi(x) = 2\sum_{k\in Z} g_k\, \varphi(2^i x - k) \tag{3.10}$$

It can be verified that

$$g_k = (-1)^k h_{1-k}, k \in Z \tag{3.11}$$

$\{\psi_{i,j}\}$ is an orthonormal basis of $L_2(R)$.

The Fourier transform of φ is given by

$$\hat{\varphi}(\xi) = \frac{1}{2}\sum c_k\, e^{-i\pi k\xi}\, \hat{\varphi}(\xi/2) = m(\xi/2)\, \hat{\varphi}(\xi/2) \tag{3.12}$$

where

$$m(\xi) = \sum_k c_k\, e^{-i\pi k\xi} \tag{3.13}$$

This leads to the orthogonality on the c_k

$$\sum c_k\, \overline{c_{k-2j}} = 2\delta_{oj} \tag{3.14}$$

or on the function $m(\xi)$

$$|m(\xi)|^2 + \left|m\left(\xi + \frac{1}{2}\right)\right|^2 = 1 \tag{3.15}$$

There are two main methods to solve the dilation equation (3.9). One is by Fourier transform and the other is by matrix products. Both give φ as a limit not as an explicit function (see [25, 34, 35, 41, 61, 97(a),(b)] for a comprehensive account of the dilation equation).

Definition 3.5 (Wavelet packets). Let $\{h_k\}$ and $\{g_k\}$ be two sequences of ℓ_2 such that

$$\sum_{n\in Z} h_{n-2k}\, h_{n-2\ell} = \delta_{k\ell} \tag{3.16}$$

$$\sum_{n\in Z} h_n = \sqrt{2} \tag{3.17}$$

$$g_k = (-1)^k\, g_{1-k} \tag{3.18}$$

Furthermore, let $\varphi(x)$ be a continuous and compactly supported real valued function R that solves the equation

$$\varphi(x) = 2^{1/2} \sum_k h_k\, \varphi(2x-k) \tag{3.19}$$

with $\hat{\varphi}(0) = 1$.

Let $\psi(x)$ be an associated function defined by

$$\psi(x) = 2^{1/2} \sum_k g_k\, \varphi(2x-k) \tag{3.20}$$

A family of functions $w_n \in L_2(R), n = 0, 1, 2, \ldots$, defined recursively from φ and ψ as follows, is called the *wavelet packet*

$$\left.\begin{array}{llll} (i) & \omega_0(k) & = \varphi(x), \omega_1(x) = \psi(x) & 3.21(a) \\ (ii) & w_{2n}(x) & = 2^{1/2} \sum\limits_k h_k\, \omega_k(2x-k) & 3.21(b) \\ (iii) & w_{2n+1}(x) & = 2^{1/2} \sum_k g_k\, \omega_n(2x-k) & 3.21(c) \end{array}\right\} \tag{3.21}$$

$\psi(x)$ and $\varphi(x)$ are often called mother and father wavelet.

It has been proved that $\{\omega_n(x-k)\}$ is an orthonormal basis of $L_2(R)$ for all $n \geq 0$ where

$$\omega_n(x-k) = \frac{1}{\sqrt{2}} \sum_i h_{k-2i}\, \omega_{2n}\left(\frac{x}{2} - i\right) + \frac{1}{\sqrt{2}} \sum_i g_{k-2i}\, \omega_{2n+1}\left(\frac{x}{2} - i\right) \tag{3.22}$$

For $f \in L_2(R)$,

$$\sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{n,k}\, \omega_n(x-k) \tag{3.23}$$

where

$$c_{n,k} = \langle f, \omega_n(x-k)\rangle \tag{3.24}$$

is called the wavelet packet series and $c_{n,k}$ are called wavelet packet coefficients of f.

For proof one may see Wickerhauser [113].

Example 3.1 Haar function is a wavelet as $h_n(x)$ (Eq.(2.1)) is an orthonormal basis of

$L_2(R)$. The scaling function for this wavelet $\psi(x)$ is $\varphi(x)$, the characteristic function of $[0,1)$. $h_n = 1/\sqrt{2}$ for $n = 0, 1, \ldots$ and 0 otherwise

$$\begin{aligned} g_n &= (-1)^n\, h_{1-n} \\ \hat{\varphi}(\xi) &= e^{-\pi\xi i}\, \frac{\sin \pi\xi}{\pi\xi} = e^{-i\pi\xi/2} \cos \pi\xi/2\; \hat{\varphi}(\xi/2) \end{aligned}$$

The space V_0 consists of piecewise constant functions with possible jumps at integers.

Example 3.2 If $\varphi(x) =$ Shannon sampling function $\frac{\sin \pi x}{\pi x}$ then the corresponding wavelet is Shannon wavelet given

$$\begin{aligned} \psi_{Shannon}(x) &= \frac{\sin(2\pi x) - \sin(\pi x)}{\pi x} \\ \hat{\varphi}(\xi) &= \chi_{[-\frac{1}{2},\frac{1}{2}]}(\xi) = \text{characteristic function of } \left[-\frac{1}{2}, \frac{1}{2}\right] \\ \hat{\psi}(\xi) &= \begin{cases} (2\pi)^{-1/2}, & \pi \le |\xi| \le 2\pi \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Example 3.3 Let $\varphi(x)$ be piecewise linear spline (Eq.(2.3)) then $\psi(x) = \frac{\sqrt{3}}{2} \sum_n (g_{n+1} - 2g_n + g_{n-1})\, \varphi(2x - n)$ where g_n are Fourier coefficients of

$$\left[(1 - \sin^2 \xi/4)(1 + \cos^2 \xi/2)^{-1}(1 + \cos^2 \xi/4)^{-1}\right]^{1/2}$$

See Daubechies [28, 30] for more examples of higher order spline functions as scaling functions and the corresponding wavelets.

<u>Morlet wavelet</u> 3.4(a):

$$\psi(x) = \frac{1}{(\pi)^{1/4}}\, (e^{-iy_0 x} - e^{-y_0^2/2}) e^{-\frac{x^2}{2}}$$

where $y_0 = 5$.

Example 3.4 (The Meyer wavelet). The Meyer wavelet is the inverse of Fourier transform of $\hat{\psi}(y)$ where

$$\hat{\psi}(y) = \begin{cases} (2\pi)^{-1/2}\, e^{iy/2} \sin\left[\frac{\pi}{2}\, v\left(\frac{3}{2\pi}|y| - 1\right)\right], & \frac{2\pi}{3} \le |y| \le \frac{4\pi}{3} \\ (2\pi)^{-1/2}\, e^{iy/2} \cos\left[\frac{\pi}{2}\, v\left(\frac{3}{4\pi}|y| - 1\right)\right], & \frac{4\pi}{3} \le |y| \le \frac{8\pi}{3} \\ 0, & \frac{4\pi}{3} \le |y| \le \frac{8\pi}{3} \end{cases}$$

where

$$v(x) = \begin{cases} 0, & \text{if } x \le 0 \\ 1, & \text{if } x \ge 1 \end{cases}$$

Example 3.5 (Daubechies wavelet). These wavelets depend on an integer $N \geq 1$ that defines the support of $\varphi(x)$ and $\psi(x)$, namely $[0, 2N-1]$ and their regularity in the sense of Hölder. Let $\varphi(x)$ belong to C^{α}, where $\alpha = \alpha(N)$ and

$$\lim_{N \to \infty} \frac{\alpha(N)}{N} = \lambda > 0 \; (\lambda \simeq 1/5)$$

Let

$$P_N(t) = 1 - c_N \int_0^t (\sin u)^{2N-1} du = \sum_{|k| \leq N-1} \lambda_k \, e^{ikt}$$

with constant c_N chosen in such a way that $P_N(\pi) = 0$. There exists at least one finite *trigonometric sum* $m_0(t) = \frac{1}{\sqrt{2}} \sum_{0}^{2N-1} h_k \, e^{-ikt}$ *such that* $|m_0(t)|^2 = P_N(t)$ and $m_0(0) = 1$. φ is the solution of the dilation equation

$$\varphi(x) = \sqrt{2} \sum_{0}^{2N-1} h_k \, \varphi(2x - k), \quad \text{where} \quad \int_{-\infty}^{\infty} \varphi(x) = 1$$

From here we see that

$$\hat{\varphi}(\xi) = m_0(\xi_2) \, m_0(\xi/2^2) \, m_0(\xi/2^3) \ldots m_0(\xi/2^j) \ldots$$

where $\hat{\varphi}(\xi) = 0, (|\xi|^{-m})$ at infinity where $m = m(N) \to \infty$ as $N \to \infty$ supp $\varphi(x) \subset [0, 2N-1]$. $\varphi(x-k)$ is an orthonormal sequence. Let $m_1(t) = e^{i(1-2N)t/m_0(t+\pi)}$ then

$$\hat{\psi}(\xi) = m_1(\xi) \, \hat{\varphi}(\xi/2) \, m_1(\xi/2) \, m_0(\xi/2^2) \, m_0(\xi/2^3) \ldots m_0(\xi/2^j) \ldots$$

Inverse of $\hat{\psi}(\xi)$, that is, $\psi(x)$ is the desired wavelet. It may be observed that for $N = 1$, $\varphi(x)$ is the characteristic function of $[0,1)$ while $\psi(x) = 1$ on $[0, 1/2)$ and -1 on $[1/2, 1)$ and o elsewhere.

The orthonormal basis $2^{j/2}\psi(2^j x - k), j, k \in 2$, is then the Haar system.

Example 3.6 (Malvar wavelet) [See 80, pp.75-87]. Let $[a_i, a_{i+1}]$ be a sequence of closed *intervals on the real line* R *where* $\ldots a_{-2} < a_{-1} < a_0 < a_1 < a_2 < \ldots, \lim_{i \to \infty} a_i = \infty$ and $\lim_{i \to -\infty} a_i = -\infty$. Put $\ell_i = a_{i+1} - a_i$ and let $\alpha_i > 0$ be positive numbers such that $\ell_i \geq \alpha_i + \alpha_{i+1}$ for all $i \in Z$. Let $\omega_i(t)$ be the characteristic functions of the interval $[a_i, a_{i+1}]$ which must overlap if they are to be regular. More precisely it must satisfy the following conditions:

$0 \leq \omega_i(t) \leq 1$ for all $t \in R$

$\omega_i(t) = 1$ if $a_i + \alpha_i \leq t \leq a_{i+1} - \alpha_{i+1}$

$\omega_i(t) = 0$ if $t \leq a_i$ or $t \geq a_{i+1} + \alpha_{i+1}$

$\omega_i^2(a_i + \tau) + \omega_i^2(a_i - \tau) = 1$ if $|\tau| \geq \alpha_i$

$\omega_{i-1}(a_i + \tau) = \omega_i(a_i - \tau)$ if $|\tau| \leq \alpha_i$

It can be checked that $\sum\limits_{-\infty}^{\infty} (\omega_i(t))^2 = 1$

The Malvar wavelet is defined as follows:

(a)

$$
\begin{aligned}
\psi_{j,k}(t) &= \sqrt{\frac{2}{\ell_j}}\, \omega_j(t) \cos\left[\frac{\pi}{\ell_j}\left(k+\frac{\ell}{2}\right)(t-a_j)\right], \\
k &= 0,1,2,3\ldots \text{ and } j \in Z
\end{aligned}
$$

or (b)

$$
\begin{aligned}
\psi_{j,k}(t) &= \sqrt{\frac{2}{\ell_j}}\, \omega_j(t) \cos\frac{k\pi}{\ell_j}(t-a_j), \quad \text{for } j \in 2Z, k = 1,2,3\ldots \\
\psi_{j,k}(t) &= \sqrt{\frac{1}{\ell_j}}\, \omega_j(t), \quad \text{for } j \in 2Z, k = 0 \\
\psi_{j,k}(t) &= \sqrt{\frac{2}{\ell_j}}\, \omega_j(t) \sin\frac{k\pi}{\ell_j}(t-a_j), \quad \text{for } j \in 2Z+1 \text{ and } k = 1,2,3\ldots
\end{aligned}
$$

As seen above the Malvar wavelet has two distinct forms. Both forms are orthonormal basis of $L_2(R)$.

Example 3.7 (Wavelet packets). (i) The Walsh system $\omega_n(x)$ (see Section 2) is a wavelet packet where $h_k = \frac{1}{\sqrt{2}}, g_k = \frac{1}{\sqrt{2}}, k = 0,1, \omega_n(x), n \in N$ and $\omega_n(x-k), n \in N, k \in Z$ are orthonormal basis of $L_2(R)$. The Walsh system is an example of wavelet packet which is not a wavelet. Every wavelet can be treated as a wavelet packet where $\varphi = \psi^L_{0,\ldots 0}$ and $\psi = \psi^L_{1,0,\ldots 0}$ (for details see [61, 93, 94, 112]]).

Let

$$
m_0(\xi) = \frac{1}{\sqrt{2}} \sum_{k=0}^{N-1} h_k\, e^{-ik\xi}
$$

and

$$
m_1(\xi) = \frac{1}{\sqrt{2}} \sum_{k=0}^{2N-1} g_k\, e^{-ik\xi}, \qquad N \geq 1
$$

satisfying the conditions

$$
\begin{aligned}
g_k &= (-1)^{k+1}\, \bar{h}_{2N-1-k} \text{ or } m_1(\xi) = e^{i(2N-1)\frac{\xi}{m_0(\xi+\pi)}} \\
m_0(0) &= = 1, m_0(\xi) \neq 0 \text{ or } [-\pi/3, \pi/3]
\end{aligned}
$$

and

$$
|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1
$$

one possible choice may be

$$
|m_0(\xi)|^2 = 1 = c_N \int_0^{\xi} (\sin t)^{2N-1} dt, \quad \text{where}
$$

$$c_N \int_0^\pi (\sin t)^{2N-1} dt = 1$$

For $m_0 = \frac{1}{2}(e^{-i\xi}+1)$ and $m_1(\xi) = \frac{1}{2}(e^{-i\xi}-1)$ $\{h_k\}$ and $\{g_k\}$ can be calculated and we have a corresponding wavelet packet. We have a wavelet for h_0, h_1, h_2, h_3 defined as follows:

$$\begin{aligned}
\sqrt{2}\, h_0 &= \frac{1}{4}(1+\sqrt{3}), \quad \sqrt{2}\, h_1 = \frac{1}{4}(3+\sqrt{3}) \\
\sqrt{2}\, h_2 &= \frac{1}{4}(3-\sqrt{3}), \quad \sqrt{2}\, h_3 = \frac{1}{4}(1-\sqrt{3})
\end{aligned}$$

It may be observed that $\varphi(x) = \omega_0(x)$ appears as a fixed point of the operator $T : L_1(R) \to L_1(R)$ defined by

$$Tf(x) = \sqrt{2} \sum_0^{2N-1} h_k\, f(2x-k)$$

which becomes

$$(\widehat{Tf})(\xi) = m_0(\xi/2)\, \hat{f}(\xi/2)$$

by taking Fourier transform.

If f is normalized, that is, $\int_{-\infty}^{\infty} f(x)dx = 1$, the fixed point is unique and is given by

$$\hat{\varphi}(\xi) = m_0(\xi/2)\, m_0(\xi/2^2)\ldots, m_0(\xi/2^j)\ldots$$

We may remark that periodized wavelet packets and wavelet packets on interval can be obtained from the wavelet packets on R proceeding on the lines discussed in [30, 11, 79].

4 Wavelet packets in solution of differential equations

Let H and G be operators H and G on $\ell_2(Z)$ defined by the relation

$$\left.\begin{aligned}
Hf(i) &= \textstyle\sum_{k\in Z}\, h_{k-2i}\cdot f(k) \\
Gf(i) &= \textstyle\sum_{k\in Z}\, g_{k-2i}\cdot f(k)
\end{aligned}\right\} \tag{4.1}$$

The adjoint operators H^* and G^* of H and G respectively can be defined as follows:

$$\left.\begin{aligned}
H^*f(k) &= \textstyle\sum_{i\in Z}\, a_{k-2i}\cdot f(i) \\
G^*f(k) &= \textstyle\sum_{i\in Z}\, b_{k-2i}\cdot f(i)
\end{aligned}\right\} \tag{4.2}$$

and

$$H^*H + G^*G = I \tag{4.3}$$

Let Ω_n denote the linear span of integer translates of ω_n's:

$$\Omega_n = \left\{ f | f = \sum_{k\in Z} d_k^n\, \omega_n(t-k) \right\} \tag{4.4}$$

where $\{d_k^n\} \in \ell_2(Z)$ and ω_n denote a wavelet packet. It can be verified that

$$f(t) = \frac{1}{\sqrt{2}} \left\{ \sum_i Hf(i)\, \omega_{2n} \left(\frac{t}{2} - i \right) + \sum_i Gf(i)\, \omega_{2n+1} \left(\frac{t}{2} - i \right) \right\} \tag{4.5}$$

or $\sqrt{2}\ f(t) = p + q$ for $p \in \Omega_{2n}$ and $q \in \Omega_{2n+1}$.
If we define $\delta f(t) = \sqrt{2}\ f(2t)$, then

$$\delta \omega_n = \Omega_{2n} \oplus \Omega_{2n+1}$$

or more generally

$$\delta^k \Omega_n = \Omega_2^k n \oplus \ldots \oplus \Omega_2^k (n+1) - 1, k \geq 0 \tag{4.6}$$

Wickenhauser [112] has proved that for every partition P of the non-negative integers into the sets of the form $I_{kn} = \{2^k n, \ldots, 2^k(n+1) - 1\}$, the collection of functions $\{2^{k/2}\ \omega_n(2^k t - j) : I_{kn} \in P, j \in Z\}$ is an orthonormal basis of $L_2(R)$. Furthermore, the wavelet packet basis of $L_2(R)$ is an orthonormal basis selected from among the functions

$$\left\{ 2^{k/2}\ \omega_n(2^k t - j), j \in Z \right\} \tag{4.7}$$

Wavelet packets form a library of functions $\{2^{k/2}\omega_n(2^k t - j)\}$. Applying the filters '$a$' and '$b$', we get a binary tree with root $\Delta^j \Omega_0$ and leaves $\Omega_0, \Omega_1, \ldots, \Omega_2^j - 1$. As an example, we consider a function defined at points $x_0, x_1, \ldots x_7$. The wavelet packet coefficients of this function are shown in the following figure

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
			$a \swarrow$	$\searrow b$			
s_0	s_1	s_2	s_3	d_0	d_1	d_2	d_3
	$a \swarrow$	$\searrow$			$a \swarrow$	$\searrow b$	
ss_0	ss_1	ds_0	ds_1	sd_0	sd_1	dd_0	dd_1
$a \swarrow$	$\searrow b$	$a \swarrow$	$\searrow b$	$a \swarrow$	$\searrow b$	$a \swarrow$	$\searrow b$
sss	dss	sds	dds	ssd	dsd	sdd	ddd

Wavelet packet coefficients

Each row is computed from the row above it by applying the Haar filters,

$$a = \left\{ 1/\sqrt{2}, 1/\sqrt{2} \right\} \quad b = \left\{ 1/\sqrt{2}, -1/\sqrt{2} \right\}$$

which we indicate as "summing" (s) by the filter a and "differencing" (d) by the filter b, respectively. In particular

$$ss_0 = \frac{1}{\sqrt{2}}\,(s_0 + s_1), ds_0 = \frac{1}{\sqrt{2}}\,(s_0 - s_1) \quad \text{etc.} \tag{4.8}$$

The row number indicates the scale of wavelet packets whereas the column indices both the frequency and the position parameters. The bases of coefficients in the rectangle correspond to the decomposition of $\delta^3\Omega_0$ into the subspaces $\delta^k\Omega_n$, for $0 \le n < 2^{3-k}$. The top box corresponds to $\delta^3\Omega_0$, the bottom box corresponds to Ω_n, for $0 \le n < 2^3$

<table>
<tr><td colspan="8" align="center">$\delta^3\Omega_0$</td></tr>
<tr><td colspan="4" align="center">$\delta^2\Omega_0$</td><td colspan="4" align="center">$\delta^2\Omega_1$</td></tr>
<tr><td colspan="2" align="center">$\delta\Omega_0$</td><td colspan="2" align="center">$\delta\Omega_1$</td><td colspan="2" align="center">$\delta\Omega_2$</td><td colspan="2" align="center">$\delta\Omega_3$</td></tr>
<tr><td>Ω_0</td><td>Ω_1</td><td>Ω_2</td><td>Ω_3</td><td>Ω_4</td><td>Ω_5</td><td>Ω_6</td><td>Ω_7</td></tr>
</table>

We have many choices of representing $\delta^3\Omega_0$ as direct sum of orthonormal basis subsets. From the multiresolution analysis any function in $L_2(R)$ can be approximated by the piecewise constant functions from Ω_j provided j is large enough. Let us consider the boundary value problem:

$$\begin{aligned} -u''(x) + cu(x) &= f(x), \quad x \in \Gamma = (0,1) \\ u(0) &= u(1) = 0 \end{aligned} \tag{4.9}$$

where $c > 0$, a constant, $f \in L_2(\Gamma)$ and solve it for $u = u(x)$. In variational form, the solution $u \in W_2^1(\Gamma)$ of (4.9) satisfies

$$\int_\Gamma (u'v' + uv)dx = \int_\Gamma fv \ du \tag{4.10}$$

for all $v \in W_2^1(\Gamma)$.

To approximate u by the Galerkin's method, we choose a finite dimensional subspace of $W_2^1(\Gamma)$ which is a space spanned by the best basis of wavelet packet bases of $\delta^k\Omega_0 (k \ge 0)$ defined on the interval $[0,1]$. For getting numerical solution of (4.9) we choose a positive value m and approximate u by an element $u_m \in \delta^k\Omega_m$ that satisfies

$$\int_\Gamma (u'_m v' + u_m v)dx = \int_\Gamma f(x)v \ dx, \quad v \in \delta^k\Omega_m \tag{4.11}$$

where u_m can be written as

$$u_m = \sum_{k \in Z} d_k^n \ \omega_n(t-k) \tag{4.12}$$

We need now to determine d_k^n. By (4.9) and (4.12) we get

$$\sum_{k,k' \in Z} dk^n \int_\Gamma \{\omega'_n(t-k)\omega'_{n'}(t-k') + \omega_n(t-k)\omega_{n'}(t-k')\} \, dx = \int_\Gamma f\omega_{n'}(t-k')dx \tag{4.13}$$

or

$$L \, A = f$$

where

$$L = \int_\Gamma \{\omega'_n(t-k)\omega'_{n'}(t-k') + \omega_n(t-k)\omega_{n'}(t-k')\} \, dx \ ,$$

f is a vector with components $\int_\Gamma f\omega_{n'}(t-k')dx$ and $A = (d_k^n)$ $k \in Z$ is the coefficient vector of the unknown function.

5 Convergence of wavelet and wavelet packet series

The problem of convergence of the wavelet series has been studied by Meyer [79], Walter [109, 110] and Kelly, Kon and Raphael [67]. Meyer has proved that under certain regularity conditions on wavelets, the wavelet series of continuous functions converge everywhere. Kelly *et al.* have extended these results and have obtained results analogous to those obtained by Carleson in 1966 and Hunt in 1968 for the Fourier series. We present here three theorems concerning convergence of the wavelet packet series one generalizing Lemma 1 in [110] and others extending the results in Kelly *et al.* [67].

Let w_{jnk} be a wavelet packet of scale j, frequency index n, and position index k. The wavelet packets $\{w_{jnk} : k \in Z\}$ are basis for $\delta^j\Omega_n$. Since H and G satisfy (4.3) $\{w_{jnk}\}_{k\in Z}$ are orthonormal wavelet packets. There is a natural correspondence between dyadic subintervals and subspaces of L_2 namely $I_{jn} \leftrightarrow \partial^j\Omega_n$ where $I_{jk} = [k2^{-j}, (k+1)2^{-j}]$. We consider $L_2 = \sum_n \Omega_n$, and I, a collection of disjoint dyadic intervals of the type I_{jk} satisfying R^+ (positive half line) $= UI_{jk}$ such that $\{W_{jnk}\}$ is an orthonormal basis for $L_2(R)$ and has support $[k2^{-j}, (k+1)2^{-j}]$ of width 2^{-j}. If I is cover of R then $\{W_{jnk}\}$ is an orthonormal basis of $L_2(R)$.

A sequence $Q_m\ (\cdot, y)$ of functions in $L_1(R)$ with parameter y belonging to R having the following properties:

(i) there is a $c > 0$ such that

$$\int_{-\infty}^{\infty} |Q_m(x,y)|dx \le c, \quad y \in R, \quad m \in N \tag{5.1}$$

(ii) there is a $c > 0$ such that

$$\int_{y-c}^{y+c} Q_m(x,y)du \to 1 \text{ uniformly on compact subsets of } R, \text{ as } m \to \infty \tag{5.2}$$

(iii) for each $r > 0$m

$$\sup_{|x-y|\ge r} |Q_m(x,y)| \to 0 \text{ as } m \to \infty$$

is called a quasi-positive sequence.

Every positive quasi sequence $Q_m(x,y)$ converges to $\delta(x-y)$ as $m \to \infty$ [109]. Feyer kernel for trigonometrric system [110] and Nörlund kernel for Walsh orthonormal system [84] are quasi-positive sequences.

Lemma 5.1 The reproducing kernel of $\partial^j\Omega_n = V_j$ for the wavelet packet

$$q_j(r,t) = 2^j q(2^j x, 2^j t) \tag{5.3}$$

where

$$q(x,t) = \sum_k w_n(x-k)w_n(t-k) \tag{5.4}$$

is a quasi-positive sequence provided q has the properties:

$$\left.\begin{array}{ll} q(x+1,y+1)=q(x,y) & 5.5(i) \\ \left|\frac{\partial^{\alpha+\beta}}{\partial x^\alpha\,\partial y^\beta}\, q(x,y)\right| \le c_m(1+|x-y|)^{-m}, \quad 0\le\alpha,\beta\le r, m\in N & 5.5(ii) \\ \int_{-\infty}^{\infty} q(x,y)y^\alpha dy = x^\alpha, \quad 0\le\alpha\le r & 5.5(iii) \end{array}\right\} \qquad (5.5)$$

In (5.5(ii)) the derivative is considered either in the distributional sense or in the sense of dyadic derivative (see for example [89]).

Theorem 5.1 Every wavelet packet series of a function $f\in L_1$ converges at a point of continuity if the reproducing kernel of the wavelet packet satisfies (5.5).
Theorem 5.1 follows from Lemma 5.1 if we proceed on the lines Lemma 1 [110].

Proof of Lemma 5.1 (i) We have

$$\begin{aligned}\int_{-\infty}^{\infty}|q_j(x,t)|du &= \int_{-\infty}^{\infty} 2^j|q(2^jx,2^jt)|dx \\ &\le c\int_{-\infty}^{\infty}(1+|x-2^jt|)^{-2}dx = c\end{aligned}$$

(ii)

$$\int_{y-c}^{y+c} q_j(x,t) = \int_{t-2^jc}^{t+2^jc} q(x,t)dx \to 0\ 1 \quad \text{on} \quad I$$

(iii)

$$|q_j(x,t)| \le c\int_{t+2^jc}^{\infty}\frac{1}{1+(t-x)^2}\,dx \to 0 \quad \text{as} \quad j\to\infty$$

and this implies the desired result.
Following Kelly, Kon and Raphael [67] we can define the concepts of multiresolution expansion, scaling expansion for wavelet packets and we can prove partial wavelet packets analogue of Theorem 2.1 and Theorem 2.4 in [67]. These results can be stated as follows:

Theorem 5.2 The multiresolution expansion of wavelet packets and the wavelet packet series of functions $f\in L_2$ converge to f almost everywhere wavelet packets are in RB class (see [67, Definition 1.4]).

Theorem 5.3 Let

$$P_m(x,y)=\sum_{j<m} w_n(2^jx-k)\,w_n(2^jy-k), \quad n\ge 0, k\in Z$$

then

$$|P_j(x,y)|\le c2^jH(2^j|x-y|)$$

where $H(|\cdot|)\in RB$, provided $w_n(x)\log(2+|x|)\in RB$.

6 Wavelet packet coefficients

Interesting accounts of characterization of functions spaces in terms of wavelet coefficients are presented in Daubechies [30, 289–312, 58, 60a, 60, 79]. A few typical results are:

Theorem 6.1 [30]. $f \in L_p(R), 1 < p < \infty$ if and only if

$$\left[\sum_{jk} |\langle f, \psi_{jk}\rangle|^2 \, 2^{-j} \chi_{[2^j k, 2^j(k+1))}\right]^{1/2} \in L_p(R)$$

Theorem 6.2 [60]. $f \in Lip\alpha, 0 < \alpha < 1$ if and only if

$$|c_{j,k}| \leq c\, 2^{-(\frac{1}{2}+\alpha,j}$$

Throem 6.3 [58]. For all $\varepsilon < 0$ and all $f \in c^{\varepsilon}$, the condition:

$$|c_{jk}| \leq c\, 2^{-j(\alpha+1/2)}(1 + |2^j x_0 - k|^{\alpha}), \quad j \in R \ \text{ and } \ k \in Z$$

implies that, for $|x - x_0| \leq 1$,

$$|f(x) - f(x_0)| \leq c|x - x_0|^{\alpha} \log \frac{2}{|x - x_0|}$$

This estimate is best possible.

Theorem 6.3 means that the Hölder exponent (Lipschitz exponent) of f at a given point x_0 can be explicitly computed, upto a logarithmic factor, by size conditions on the wavelet coefficients of f.

More recently, Jaffard [60a] has proved the following result:

Theorem 6.4 If $\sup \sum_{i,k} |A_{j,k}|^p \leq c 2^{(m-ps-mp/2)j}, \quad s > 0$ then $g \in B^s_{p,\infty}(R^m)$, where $A_{j,k}$ wavelet coefficients, wavelets $\Phi(x) = \varphi(x_1)\phi(x_2)\ldots\varphi(x_m), \quad x = (x_1, x_2, \ldots x_m), \varphi(x)$ scaling function.

The following theorem can be proved proceeding on the lines of Theorem 1 [57].

Theorem 6.4 Let $j(t)$ be a positive and non-decreasing function defined on the interval $(0, 1)$, satisfying the following conditions:

$$\int_t^1 j(u)u^{-2}du \leq Aj(t)t^{-1} \ \text{ as } \ t \to 0$$

and

$$\int_0^t j(u)u^{-1}du \leq Aj(t) \ \text{ as } \ t \to 0$$

Then $f \in Lipj(t)$ if and only if

$$|c_{m,k}| \leq cj(2^{-(n+1)})$$

A beautiful description of numerical calculation of wavelet packet coefficient is given in Wickerhauser [113]. The problem of characterization of function spaces in terms of wavelet packet coefficients is open. To examine the regularity characterization of 15 function spaces given in Section 2 in terms of the orders of the wavelet packet coefficients will be a formidable task. For Walsh wavelet packet at least one result of this type is known (Siddiqi [91(b), Lemma 4.1]).

7 Approximation by wavelet packet polynomials

The projection of $f \in L_2(R)$ on the subspace V_j is defined by the relation

$$P_n(f) = \sum_{j<n} c_{j,k}\ \psi_{j,k}, \quad k \in Z \tag{7.1}$$

where $c_{j,k}$ are wavelet coefficients of the wavelet

$$\psi(t)\ (\psi_{j,k}(t) = 2^j \psi(2^j t - k))$$

The rate of convergence of the sequence $P_n(f)$ is of great interest as it provides the *computational cost of representing f to a prescribed accuracy using some f_j.* It follows immediately from a result of Strang and Fix [11] that

$$\|P_N(f) - f\| \le 2^{-jN} \frac{2}{4^{N!}}\ |f^{(N)}(u)| \tag{7.2}$$

where $f \in c^N[0,1], N \ge 0$. If the scaling function φ is regular in the sense that $|\partial\varphi(x)| \le c_m(1+|x|^{-m})$ for all integer $m \ge 0$ then

$$\lim_{n\to\infty} \|P_n(f) - f\|_{L_2} = 0$$

For $f \in W_p^s[0,1]$ we have

$$\|P_N(f) - f\|_{L_p} \le c2^{N s}\ \|f\|_{W_p^s} \tag{7.3}$$

Let Σ_n denote the set of elements of the form

$$P_N(f) = \sum_{j<N} c_{j,k}\ \psi_{j,k} \quad \text{for} \quad n \le N \tag{7.4}$$

and

$$En(f) = \inf_{P_N\in\Sigma_n} \|f - P_N\|_{L_p}, \quad 0 < p \le \infty \tag{7.5}$$

The important problems for investigation are

(i) How can we construct good or near best approximation from Σ_n?

(ii) For what class of functions we have a given error of approximation of order $O(n^{-\alpha})$?

(iii) Characterize those functions of $f \in L_p$ such that

$$\|\sigma_n(f) - f\|_p = O(n^{-1}) \tag{7.6}$$

where $\sigma_n(f)$ denote the Cesaro mean of order 1 of the wavelet series.

For problems (i) and (ii) we refer to DeVore and Lucier [40]. Problem (iii) has not been investigated.

These three problems in the setting of wavelet packet series in general are basically open. Only very recently it has been seen that [93] (7.2) and (7.3) holds for wavelet packets.

Let $\Sigma c_{j,k}\; w_n(x-k)$ be a wavelet packet series of $f \in L_2(0,1)$,

$$\Sigma_N = \left\{ P_N(f) | P_N(f) = \sum_{k<N} c_{j,k}\; w_n(t-k) \right\} \tag{7.7}$$

$$E_N(f, L_p) = \inf_{P\in\Omega_N} \|f - P\|_p \tag{7.8}$$

Walter [109] has proved the following results for wavelet series. We have for each $f \in L_2(R)$

$$f(x) = \sum_{k=-\infty}^{\infty} c_{j,k} 2^{j/2} \varphi(2^j x - k) + \sum_{i=j}^{\infty} \sum_{k=-\infty}^{\infty} \psi(2^i x - k) \tag{7.9}$$

where φ and ψ are father and mother wavelets respectively

$$= fj(x) + f^{\perp}j(x) \tag{7.10}$$

where $fj(x) \in V_j$ and $f^{\perp}j(x) \in V_j^{\perp}$ (orthogonal complement of V_j in V_{j+1}). We can write

$$fj(x) = \int_{-\infty}^{\infty} q_j(x,y)\; f(y) dy \tag{7.11}$$

where

$$q_j(x,y) = \sum_{x=-\infty}^{\infty} 2^j \varphi(2^j. \; - k)\varphi(2^j y - k)$$

$q_j(x,y) \to \delta(x-y)$ as $j \to \infty$, where $\delta(\cdot)$ is the Dirac delta function.

Theorem 7.1 For $f \in H_2^s, s > \frac{1}{2}, \|f - fj\|_\infty = 0(2^{-j(s-1/2)})$.

Theorem 7.1 can be studied for relevant function spaces discussed in Section 2. These two results due to Walter may be examined for wavelet packets.

For Walsh wavelet packet elegant approximation results have been obtained (see for example Móricz and Siddiqi [84]). How far these results can be extended to other wavelet packets or to a class of wavelet packets is a challenging task. The author along with his coworkers is engaged in obtaining appropriate results in this direction.

8 Conclusion

Essentially, problems of convergence of wavelet packet series, characterization of function spaces in terms of wavelet packet coefficients and approximation of functions by wavelet packet polynomials are yet to be studied. Since such results exist in a special wavelet packet, namely, Walsh orthonormal system, one may expect either natural extension or modified version. However new techniques like counter part of the dyadic group and addition modulo 2 operations may be required. Wickerhauser [113] has developed an excellent text book for understanding intricacies of wavelets and wavelet packets through computer analysis. It will be a boon for users of these concepts. Solutions of boundary value problems through wavelet packets are being investigated currently by different groups in the world and these techniques may prove superior in many respects to other existing techniques. There exists close connection between wavelet and fractal theory. The existence of a frame of the generalized Sobolev space (Sobolev spaces over a metric space say over the Cantor set) may be an interesting question to be explored.

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