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The superpotential in heterotic orbifold GUTs

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Le cœur a ses raisons que la raison ne connaît point.

Blaise Pascal

Abstract

We study in this work the phenomenology of heterotic orbifold compactifications. Exact and approximate R symmetries of the superpotential in the context of supersymmetric field theories are discussed. We further study symmetries, phenomenological implications and Yukawa couplings from superpotential contributions in extra dimensional theories. We apply the developed methods to models, which base on heterotic orbifolds.

Zusammenfassung

Wir studieren in dieser Arbeit die Phänomenologie von heterotischen Orbifoldkompaktifizierungen. Exakte und approximative R Symmetrien des Superpotentials im Kontext von supersymmetrischen Feldtheorien werden diskutiert. Weiterhin studieren wir Symmetrien, phänomenologische Konsequenzen und Yukawakopplungen von Superpotentialbeiträgen in Extradimensionen. Wir wenden die entwickelten Methoden auf Modelle, die auf heterotischen Orbifolds basieren an.

Contents

1 Introduction

Supersymmetry is probably the most common prejudice in particle physics today. It is the most discussed scenario for physics beyond the standard model (SM). The simplest version, the minimal supersymmetric standard model (MSSM) was used for exhaustive studies in the field of particle physics.

There have been two cornerstones of particle physics model building in the last decades: supersymmetry and Grand Unified Theories (GUTs). Apart from the mathematical beauty of both approaches [1], together they offer the possibility that gauge couplings unify at a higher unification scale without the need of introducing further particles [2,3]. In the past, one of the unsolved problems has been the compelling breaking of the GUT gauge group down to the SM gauge group below the GUT scale.

A decade ago, an elegant mechanism to break the GUT gauge group in field theory has been proposed [4]. This proposal is based on the projection conditions from additional discrete symmetries in extra dimensions. The extra dimensional space is compactified on an orbifold, which is a quotient space of a torus and a discrete symmetry. This discrete symmetry breaks the GUT gauge group to the SM gauge group at low energies. The same idea has been already known for some time in string theory [5]. It has been shown that such an approach can be realized in concrete models in heterotic string theory [6]. This offers a natural motivation for heterotic string theory from the need of a GUT breaking mechanism. On the other hand, the requirement for extending the SM is due to the experimental need to include for example neutrino masses, dark matter and a mechanism for baryogenesis. Several attempts to address these issues have been studied in extra dimensional theories. Neutrino masses can be explained by the seesaw mechanism [7]. In this scheme, leptogenesis can serve as a mechanism for baryogenesis. It has been shown that matter parity can be obtained in heterotic orbifold compactifications, which will lead to a stable supersymmetric particle as promising dark matter candidate [8]. Furthermore it was realized that flavor symmetries arise naturally in this class of models [9].

One of the major advantages of a string theory induced approach is that it provides a rigid and constrained framework. Even more, string theory offers the possibility to calculate free parameters of the MSSM or at least reduce the number of free parameters.

Let us briefly review the history and successes of heterotic string theory. It has been realized in 1974 that string theory includes a quantized version of gravity [10]. It was discovered ten years later that a supersymmetric string theory based on a SO(32) or $\mathrm{E}_8 \times \mathrm{E}_8$ gauge group is anomaly free in ten dimensions [11]. Based on this work, the so-called heterotic string theory which incorporates the $\mathrm{E}_8 \times \mathrm{E}_8$ gauge group was constructed [12, 13]. Shortly after that, it was shown that the heterotic string can consistently be compactified on orbifolds [14, 15]. This offered the possibility to build four dimensional models with the MSSM as low energy theory in a consistent theory which included all fundamental forces. In contrast to compactifications on Calabi-Yau manifolds [16], compactifications on orbifolds are simpler. This offers the possibility to calculate low energy observables directly from string theory.

One of the biggest problems in string model building so far is that string theory offers too many compactification possibilities and usually a too large particle content. Only quite recently, there have been found several models with exact MSSM spectra at low energy and only heavy exotics, which means no unobserved particles at the electroweak scale [6]. As outlined, these models have several appealing features regarding low energy phenomenology. This work will use these models as playground.

We will introduce mathematical techniques to handle the structure of these models in section 2. This is necessary, because of the typically large particle content of the discussed models. It will be possible to determine basic building blocks which fix the structure of the perturbative superpotential to all orders in perturbation theory.

In section 3 we will study approximate R symmetries which are naturally present in this kind of models. We will show how they can result in a suppressed vacuum expectation value of the superpotential. After that, we comment on the implications for moduli stabilization and the possibility to generate a μ term of the right scale. Moduli stabilization is an important topic in string theory, because the scale and the shape of the extra dimensions, as well as the size of the gauge coupling should be fixed within a dynamical process. At present, only models in type II string theory without the SM as low energy theory have been stabilized completely. Therefore, progress in this field is highly welcome and we will show that approximate R symmetries can serve as a fertile ingredient.

In section 4 and section 5 we try to constrain these models with phenomenological requirements in the quark sector. We will review how a large top Yukawa coupling can be obtained in extra dimensional theories through the origin of the Higgs field in the gauge multiplet. In contrast to type II string theory this mechanism can be embedded in heterotic string theory. In addition, we will discuss corrections through localized Fayet-Iliopoulos D-terms and the phenomenological implications. The detailed analysis of the quark mass hierarchies in a heterotic orbifold model will be given. We will show how constraints from low energy data as well as the rigid framework of string theory will allow it to disfavor this model.

Additional work which covers the explanation of dark matter and its experimental detection will only slightly be covered in this thesis.

Part of this work has been published in:

[17] **The Hilbert basis method for D-flat directions and the superpotential.**
Rolf Kappl, Michael Ratz, Christian Staudt. TUM-HEP-814-11, MPP-2011-99, CERN-PH-TH-2011-193, Aug 2011.
e-Print: **arXiv:1108.2154 [hep-th]**

[18] **New Limits on Dark Matter from Super-Kamiokande.**
Rolf Kappl, Martin Wolfgang Winkler. TUM-HEP-804-11, MPP-2011-37, Apr

2011.
Published in **Nucl.Phys.B850:505-521,2011.** e-Print: **arXiv:1104.0679 [hep-ph]**

[19] **String-derived MSSM vacua with residual R symmetries.**
Rolf Kappl, Bjoern Petersen, Stuart Raby, Michael Ratz, Roland Schieren, Patrick K.S. Vaudrevange. TUM-HEP-787-10, MPP-2010-169, NSF-KITP-10-160, LMU-ASC-106-10, OHSTPY-HEP-T-10-006, Dec 2010.
Published in **Nucl.Phys.B847:325-349,2011.** e-Print: **arXiv:1012.4574 [hep-th]**

[20] **Quark mass hierarchies in heterotic orbifold GUTs.**
Rolf Kappl. TUM-HEP-786-10, MPP-2010-170, Dec 2010.
Published in **JHEP 1104:019,2011.** e-Print: **arXiv:1012.4368 [hep-ph]**

[21] **Light dark matter in the singlet-extended MSSM.**
Rolf Kappl, Michael Ratz, Martin Wolfgang Winkler. TUM-HEP-772-10, MPP-2010-130, Oct 2010.
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[24] **Large hierarchies from approximate R symmetries.**
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2 Superpotential to all orders

One of the fundamental tasks to start with phenomenology in a given model is, to write down all possible interactions allowed by given symmetries. In this section we will show how this is efficiently possible even if the theory has many symmetries, gauge as well as discrete R and non-R symmetries and many particles.

2.1 The superpotential

We start with an U(1) gauge symmetry to find out to which mathematical problem we can reduce the physical task. In supersymmetric field theories the superpotential characterizes possible interactions of the theory. A field ϕ charged under a U(1) symmetry transforms like

$$\phi \to e^{iq\alpha}\phi \tag{2.1.1}$$

with charge q and α taking the transformation under the U(1) symmetry into account. A term in the superpotential is gauge invariant if

$$\mathcal{W} \supset \phi_1\phi_2\phi_3 \quad \Leftrightarrow \quad q_1 + q_2 + q_3 = 0. \tag{2.1.2}$$

q_i label the charges of the fields ϕ_i for the given U(1) symmetry. In a theory with m different U(1) gauge groups and M fields we define a charge matrix

$$Q = \begin{pmatrix} q_1^{(1)} & \dots & q_M^{(1)} \\ \vdots & & \vdots \\ q_1^{(m)} & \dots & q_M^{(m)} \end{pmatrix}. \tag{2.1.3}$$

A combination of fields is gauge invariant if

$$Q \cdot n = 0, \qquad n \in \mathbb{N}^M. \tag{2.1.4}$$

With the vector n we denote the exponents of the fields ϕ_i for an allowed superpotential term

$$\mathcal{W} \supset \phi_1^{n_1} \dots \phi_M^{n_M}, \qquad n = (n_1, \dots, n_M)^T. \tag{2.1.5}$$

The limitation $n \in \mathbb{N}^M$ is a consequence of the fact that fields cannot arise in the superpotential with a negative power. In other words, the superpotential has to be holomorphic.

Equation 2.1.4 is a homogeneous linear system of Diophantine equations. Such a system of equations can in general only be solved algorithmically. We describe the mathematical methods and various examples in the next section.

2.2 The Hilbert basis

2.2.1 A mathematical survey

To derive a basis of solutions for the matrix equation

$$Q \cdot n = 0, \qquad n \in \mathbb{N}^M \tag{2.2.1}$$

is non-trivial, because the natural numbers $\mathbb{N}$ only form a monoid and not a field. There is no analytical solution possible, but the problem is well known in the field of integer programming [25].

Definition 1 *The convex hull* $\mathrm{Conv}(Q)$ *of a matrix* $Q = (q_1, \ldots, q_M) \in \mathbb{Z}^{m\times M}$ *is given by*

$$\mathrm{Conv}(Q) = \left\{ \lambda_1 q_1 + \ldots + \lambda_M q_M \middle| \forall\ i\ : \lambda_i \geq 0, \sum_{j=1}^{M} \lambda_j = 1 \right\}. \tag{2.2.2}$$

We will review a theorem from [26].

Theorem 1 *If* $Q \in \mathbb{Z}^{m\times M}$ *the equation system* $Q\cdot x = 0$, $x \in \mathbb{N}^M$ *has a nonzero solution if and only if* $0 \in \mathrm{Conv}(Q)$.

We illustrate this with an example from [26]. We have

$$0 \notin \mathrm{Conv}(Q), \qquad Q = \begin{pmatrix} 4 & -1 & 5 & 2 & -2 \\ -3 & 5 & 2 & -1 & 3 \end{pmatrix}. \tag{2.2.3}$$

This can be graphically obtained from the visualization in figure 2.1. With this theorem

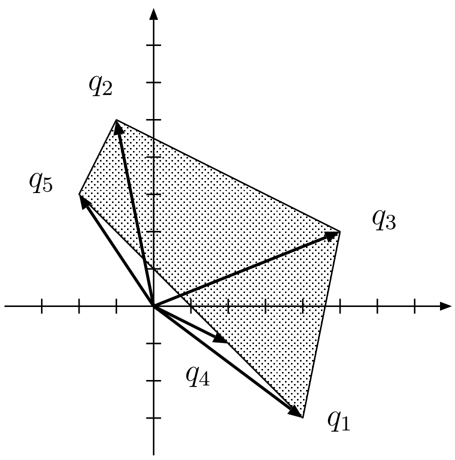

Figure 2.1: Visualization of equation 2.2.3.

it is possible to check if a given symmetry allows for given particles with well defined charges an interaction at all.

Definition 2 *A lattice* $\Lambda(Q) \subseteq \mathbb{Z}^M$ *with* $Q = (q_1, \ldots, q_M) \in \mathbb{Z}^{m \times M}$ *is defined by*

$$\Lambda(Q) = \{\lambda_1 q_1 + \ldots + \lambda_M q_M \,|\forall\ i\ : \lambda_i \in \mathbb{Z}\}\ . \tag{2.2.4}$$

For example $\Lambda = \ker_{\mathbb{Z}}(Q)$ with

$$\ker_{\mathbb{Z}}(Q) = \left\{x \,\middle|\, Q \cdot x = 0, Q \in \mathbb{Z}^{m \times M}, x \in \mathbb{Z}^M\right\} \tag{2.2.5}$$

is a lattice.

Definition 3 *A rational polyhedral cone* Cone(Q) *is given by*

$$\mathrm{Cone}(Q) = \left\{\lambda_1 q_1 + \ldots + \lambda_M q_M \,\middle|\forall\ i\ : \lambda_i \in \mathbb{R}_+, q_i \in Q \subseteq \mathbb{Z}^M\right\}\ . \tag{2.2.6}$$

We will now define the so-called Hilbert basis.

Definition 4 *Let* Cone$(Q) \subseteq \mathbb{R}^M$ *be a rational polyhedral cone and let* $\Lambda \subseteq \mathbb{Z}^M$ *be a lattice. We call the finite set*

$$\mathscr{H}(Q) = \{h_1, \ldots, h_N\} \subseteq \Lambda \cap \mathrm{Cone}(Q) \tag{2.2.7}$$

a Hilbert basis if for every $x \in \Lambda \cap \mathrm{Cone}(Q)$

$$x = \sum_{i=1}^{N} \lambda_i h_i, \quad \lambda_i \in \mathbb{N}. \tag{2.2.8}$$

This is close to the definition in [27]. The number of basis elements N cannot be determined analytically. For the special case $\Lambda = \ker_{\mathbb{Z}}(Q)$ we find

$$\Lambda \cap \mathrm{Cone}(Q) = \ker_{\mathbb{Z}}(Q) \cap \mathrm{Cone}(Q) = \ker_{\mathbb{N}}(Q) \tag{2.2.9}$$

which means that the Hilbert basis $\mathscr{H}(Q)$ in this case is the basis of all solutions to equation 2.2.1. The fact that such a basis exists is non-trivial. For our purposes it is sufficient how to compute it even for large matrices in reasonable time. We refer to computer packages like [28, 29] for this task.

2.2.2 Examples

A simple example

An example with two U(1) factors and four fields is given in [30]. The charge matrix is given by

$$Q = \begin{pmatrix} 2 & -2 & 1 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}. \tag{2.2.10}$$

The corresponding Hilbert basis $\mathscr{H}(Q)$ consists of the three elements

$$h_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \end{pmatrix}, \quad h_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \tag{2.2.11}$$

These solutions correspond to the gauge invariant monomials

$$\mathcal{M}_1 = \phi_1\phi_4^2, \quad \mathcal{M}_2 = \phi_1\phi_2^2\phi_3^2, \quad \mathcal{M}_3 = \phi_1\phi_2\phi_3\phi_4. \tag{2.2.12}$$

The complete superpotential is given by

$$\mathcal{W} = c_1\phi_1\phi_4^2 + c_2\phi_1\phi_2^2\phi_3^2 + c_3\phi_1\phi_2\phi_3\phi_4 + \dots \tag{2.2.13}$$

where all higher order terms are combinations of the monomials $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$. That means, we know the superpotential to all orders. The c_i are undetermined coupling coefficients independent of the fields ϕ_i.

An example with more fields

In this section we will discuss an example from [31]. We consider the example with the field content $\mathcal{S}_2 = \{X_0, \bar{X}_0^c, U_2, U_4, S_2, S_5, X_1^c, \bar{X}_1, Y_2^c, \bar{Y}_2, U_1^c, U_3, S_6, S_7\}$. There are six U(1) factors and the charge matrix rescaled to integers is given by

$$Q = \begin{pmatrix} 0 & 0 & 1 & -2 & 1 & -1 & 0 & 1 & 0 & 1 & -1 & 2 & -1 & 0 \\ -2 & 2 & 3 & -6 & -4 & 2 & 2 & 1 & -4 & -5 & -3 & -6 & 2 & -1 \\ 2 & 2 & -6 & 0 & -1 & -1 & 2 & 0 & -4 & -2 & -6 & 0 & -1 & -1 \\ 2 & -2 & 0 & 0 & 1 & 1 & 1 & -1 & 1 & -1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 \\ 10 & 10 & 0 & 0 & -5 & -5 & -10 & -10 & 0 & 0 & 0 & 0 & -5 & 5 \end{pmatrix}. \tag{2.2.14}$$

In [31] the Hilbert basis was not known and the solution space was only restricted to $\ker_{\mathbb{Z}}(Q)$. They found eight independent basis monomials Ω_i'. This is in agreement with the expectation because $\dim \ker_{\mathbb{Z}}(Q) = 8$. If we use our method instead and focus on $\ker_{\mathbb{N}}(Q)$ we find 13 monomials

$$\mathcal{M}_1 = \Omega_1' = \bar{X}_0^c S_2 S_6 \tag{2.2.15}$$
$$\mathcal{M}_2 = \Omega_2' = \bar{X}_0^c X_1^c Y_2^c \tag{2.2.16}$$
$$\mathcal{M}_3 = \Omega_3' = \bar{X}_0^c (S_5)^2 U_3 \tag{2.2.17}$$
$$\mathcal{M}_4 = \Omega_4' = X_0 \bar{X}_1 S_5 S_7 \tag{2.2.18}$$
$$\mathcal{M}_5 = \Omega_5' = X_0 \bar{X}_0^c X_1^c \bar{X}_1 U_1^c \tag{2.2.19}$$
$$\mathcal{M}_6 = \Omega_6' = X_0 \bar{X}_0^c \bar{X}_1 \bar{Y}_2 (S_5)^2 \tag{2.2.20}$$
$$\mathcal{M}_7 = \Omega_7' = X_0 \bar{X}_0^c (S_6)^2 \tag{2.2.21}$$
$$\mathcal{M}_8 = \Omega_8' = X_0 \bar{X}_0^c U_2 U_4 \tag{2.2.22}$$
$$\mathcal{M}_9 = \bar{X}_0^c S_2 S_6 \tag{2.2.23}$$
$$\mathcal{M}_{10} = \bar{X}_0^c S_5 S_6 U_3 \tag{2.2.24}$$
$$\mathcal{M}_{11} = \bar{X}_0^c (S_6)^2 U_3 \tag{2.2.25}$$
$$\mathcal{M}_{12} = X_0 \bar{X}_1 S_6 S_7 \tag{2.2.26}$$
$$\mathcal{M}_{13} = X_0 \bar{X}_0^c \bar{X}_1 \bar{Y}_2 S_5 S_6 \tag{2.2.27}$$

We can now see the power of our approach. The fields S_5 and S_6 in this example have the same charges. If we restrict ourselves only to $\ker_{\mathbb{Z}}(Q)$ a potential superpotential term is given for example by

$$\mathcal{W} \supset \frac{\Omega_3'\Omega_7'}{\Omega_6'} = \mathcal{M}_{11}. \tag{2.2.28}$$

We see that the appearance of negative powers in the framework dealing with the monomials Ω_i' makes the structure of the superpotential less transparent. If we deal with the Hilbert basis instead, we see that the superpotential is given by

$$\begin{aligned}\mathcal{W} = {} & c_1\mathcal{M}_1 + c_2\mathcal{M}_2 + c_3\mathcal{M}_3 + c_4\mathcal{M}_4 + c_5\mathcal{M}_5 + c_6\mathcal{M}_6 + c_7\mathcal{M}_7 + c_8\mathcal{M}_8 \\ & + c_9\mathcal{M}_9 + c_{10}\mathcal{M}_{10} + c_{11}\mathcal{M}_{11} + c_{12}\mathcal{M}_{12} + c_{13}\mathcal{M}_{13} + \ldots .\end{aligned} \tag{2.2.29}$$

This is the superpotential structure to all orders and shows the progress compared to [31] due to the Hilbert basis. The only drawback is that the number of monomials is usually larger than $\dim \ker_{\mathbb{Z}}(Q)$.

2.3 Extension to R symmetries

In the case of a continuous $\mathrm{U}(1)_R$ symmetry or a discrete $\mathbb{Z}_N^R$ symmetry we have to extend our method [17]. In contrast to equation 2.2.1, for R symmetries this equation will become inhomogeneous. Let us assume that the superpotential transforms with R charge -2 under an $\mathrm{U}(1)_R$ symmetry. If the fields ϕ_i have R charges q_i, we find that

$$\sum_i q_i n_i = -2 \tag{2.3.1}$$

has to be satisfied for allowed superpotential terms. n_i labels the number of fields ϕ_i present in the monomial. We introduce the dummy variable y and find

$$\sum_i q_i n_i + 2y = (q_1, \ldots, q_M, 2) \cdot \begin{pmatrix} n_1 \\ \vdots \\ n_M \\ y \end{pmatrix} = 0. \tag{2.3.2}$$

The extension to several $\mathrm{U}(1)_R$ symmetries results in the matrix equation

$$\underbrace{\begin{pmatrix} q_1^{(1)} & \ldots & q_M^{(1)} & 2 \\ \vdots & & \vdots & \vdots \\ q_1^{(m)} & \ldots & q_M^{(m)} & 2 \end{pmatrix}}_{=\tilde{Q}} \cdot \begin{pmatrix} n_1 \\ \vdots \\ n_M \\ y \end{pmatrix} = 0 \tag{2.3.3}$$

in the case of m different $\mathrm{U}(1)_R$ symmetries. All elements of the Hilbert basis $\mathcal{H}(\tilde{Q})$ with the last element $y = 1$ which we denote by $\mathcal{H}_{\text{inhom}}(\tilde{Q}) \subseteq \mathcal{H}(\tilde{Q})$ give rise to allowed superpotential terms. In addition we have homogeneous solutions with $y = 0$ which are elements of $\mathcal{H}_{\text{hom}}(\tilde{Q}) \subseteq \mathcal{H}(\tilde{Q})$. The exponents of every superpotential term can be written as

$$z = z_{\text{inhom}} + \sum_i \lambda_i z_{\text{hom}}^{(i)}, \qquad z_{\text{inhom}} \in \mathcal{H}_{\text{inhom}}(\tilde{Q}), \qquad \forall\ i\ : z_{\text{hom}}^{(i)} \in \mathcal{H}_{\text{hom}}(\tilde{Q}), \lambda_i \in \mathbb{N}. \tag{2.3.4}$$

To obtain monomials the last entry of z which is just a dummy number has to be skipped.

We will illustrate our approach with a small example. Let us consider two fields ϕ and ψ with R charges $q_\phi = 0$ and $q_\psi = -1$. We find

$$\tilde{Q} = \begin{pmatrix} 0 & -1 & 2 \end{pmatrix} \tag{2.3.5}$$

and the solutions

$$z_{\text{inhom}} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \qquad z_{\text{hom}}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{2.3.6}$$

These elements give rise to the monomials

$$\mathcal{M}_{\text{inhom}} = \psi^2, \qquad \mathcal{M}_{\text{hom}} = \phi \tag{2.3.7}$$

which results in a superpotential of the form

$$\mathcal{W} = \mathcal{M}_{\text{inhom}}(c_1 \mathcal{M}_{\text{hom}} + c_2(\mathcal{M}_{\text{hom}})^2 + \ldots) = \psi^2(c_1\phi + c_2\phi^2 + \ldots). \tag{2.3.8}$$

In [19] such a structure of the superpotential has been used to determine a fertile structure of minima of the scalar potential.

We can extend our approach further to discrete symmetries. For discrete $\mathbb{Z}_N^R$ symmetries we find

$$\sum_i q_i n_i = -2 \mod N \tag{2.3.9}$$

with the same notation as before. Without loss of generality we can assume that all q_i are non-positive. That enables us to introduce two dummy variables y and w with

$$\sum_i q_i n_i + 2y + Nw = 0. \tag{2.3.10}$$

We can now proceed as before with the additional requirement that we skip the last two entries of every solution. The same strategy can also be used for discrete non-R symmetries $\mathbb{Z}_N$. For the inclusion of non-Abelian symmetries and further details we refer to [17].

2.3.1 A complex example in heterotic orbifold compactifications

We base our example on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold example discussed in [19]. We consider the $\mathbb{Z}_4^R$ vacuum discussed there and construct the superpotential for the standard model singlets with R charges zero and two. The symmetries of this model and their origin from the heterotic string are discussed in [32]. At the orbifold point we find

$$G_{\text{symmetries}} = \mathrm{U}(1)^8 \times (\mathbb{Z}_2)^6 \times \left(\mathbb{Z}_4^R\right)^3 . \tag{2.3.11}$$

The non-Abelian symmetries have been eliminated by using techniques outlined in [17]. In [19] a $\mathbb{Z}_4^R$ symmetry forbids the superpotential at the perturbative level. This symmetry which is able to forbid all relevant proton decay operators is discussed in detail in [33, 34]. We break this symmetry in our example by giving the standard model singlet $\bar{\phi}_1$ a VEV. This is just an example to show how our method works and has no phenomenological relevance. The fields

$$\begin{aligned}&\bar{\phi}_1, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8, \phi_9, \phi_{10}, \phi_{11}, \phi_{12}, \phi_{13}, \phi_{14},\\ &x_1, x_2, x_3, x_4, x_5, \bar{x}_1, \bar{x}_3, \bar{x}_4, \bar{x}_5, y_3, y_4, y_5, y_6\end{aligned} \tag{2.3.12}$$

in the model from [19] will now obtain a VEV. The Hilbert basis for these 28 fields contains 15408 elements. We have used the program `rays` from the package [28] in combination with the program `normaliz` [29] to compute this basis. The superpotential starts at lowest order with four Hilbert basis monomials

$$\mathcal{W} = (x_4\bar{x}_4 + x_5\bar{x}_5 + \phi_9\phi_{13} + \phi_{10}\phi_{14})\bar{\phi}_1 + \dots . \tag{2.3.13}$$

We further considered the appearance of the proton decay operator $QQQL$. The lowest order $QQQL$ operator including first family quark and leptons occurs at order 11 in standard model singlets. An explicit example is given by

$$\mathcal{W} \supset Q_1Q_2Q_2L_1\bar{\phi}_1x_1x_2x_3x_4\bar{x}_3\phi_2\phi_4\phi_9\phi_{12}^2. \tag{2.3.14}$$

These examples show that with the help of the Hilbert basis complex examples with many fields can be handled. The method is powerful enough to solve computational problems which have been occurred in the past (see for example [35]).

3 Approximate R symmetries

In this section we study exact and approximate R symmetries of the superpotential. We will see that these symmetries are able to influence the superpotential VEV in a given minima of the scalar potential.

3.1 An exact $\mathrm{U}(1)_R$ symmetry

3.1.1 Global supersymmetry

We start with a superpotential of the form

$$\mathcal{W} = \sum c_{n_1 \dots n_M} \phi_1^{n_1} \dots \phi_M^{n_M} \tag{3.1.1}$$

with M different fields ϕ_i and different coefficients $c_{n_i} \in \mathbb{C}$. Assume that the superpotential $\mathcal{W}$ transforms under an exact $\mathrm{U}(1)_R$ symmetry with $\mathrm{U}(1)_R$ charge 2

$$\mathcal{W} \to e^{2i\alpha} \mathcal{W}. \tag{3.1.2}$$

The fields ϕ_i transform like

$$\phi_i \to \phi_i' = e^{i r_i \alpha} \phi_i \tag{3.1.3}$$

with charge r_i. The charges for each monomial in $\mathcal{W}$ have to sum up to 2. Let $\langle \phi_i \rangle$ denote a vacuum expectation value (VEV). If $\langle \phi_i \rangle$ is a solution to the F-term equations we have

$$F_i = \frac{\partial \mathcal{W}}{\partial \phi_i} = 0 \quad \text{for} \quad \phi_j = \langle \phi_j \rangle \quad \forall\, i, j. \tag{3.1.4}$$

An infinitesimal $\mathrm{U}(1)_R$ transformation results in

$$\mathcal{W}(\phi_i) \to \mathcal{W}(\phi_i') = \mathcal{W}(\phi_i) + \sum_j \frac{\partial \mathcal{W}}{\partial \phi_j} \Delta \phi_j \tag{3.1.5}$$

after a Taylor expansion. If the F-term equations are satisfied we have $\phi_i = \langle \phi_i \rangle$ which means that the second term in equation 3.1.5 vanishes and we get

$$\mathcal{W}(\langle \phi_i \rangle) \to \mathcal{W}(\langle \phi_i \rangle) \tag{3.1.6}$$

which is consistent with equation 3.1.2 only if $\mathcal{W}(\langle \phi_i \rangle) = \langle \mathcal{W} \rangle = 0$.

We will now present an algebraic way to obtain the same result. The $\mathrm{U}(1)_R$ symmetry implies

$$\sum_i n_i r_i = 2 \tag{3.1.7}$$

if the superpotential $\mathcal{W}$ transforms with $\mathrm{U}(1)_R$ charge 2. We can write the F-term equations as

$$F_i = \frac{\partial \mathcal{W}}{\partial \phi_i} = \sum n_i \phi_i^{-1} c_{n_1} \ldots c_{n_M} \phi_1^{n_1} \ldots \phi_M^{n_M}. \tag{3.1.8}$$

Let us consider the sum over the F-term equations with arbitrary polynomials $h_i(\phi_i)$ in the variables ϕ_i. This ansatz is inspired by algebraic geometry. We get

$$\sum_i h_i F_i = \sum_i \sum h_i n_i \phi_i^{-1} c_{n_1} \ldots c_{n_M} \phi_1^{n_1} \ldots \phi_M^{n_M} \tag{3.1.9}$$

$$= \sum_i \sum n_i r_i c_{n_1} \ldots c_{n_M} \phi_1^{n_1} \ldots \phi_M^{n_M} \tag{3.1.10}$$

$$= \mathcal{W} \tag{3.1.11}$$

if we specify $h_i = \frac{1}{2}\phi_i r_i$ and use the $\mathrm{U}(1)_R$ symmetry. We thus get

$$\mathcal{W} = \sum_i h_i F_i \tag{3.1.12}$$

and therefore $\mathcal{W} = 0$ whenever $F_i = 0 \quad \forall\, i$. This holds only when the superpotential exhibits an exact $\mathrm{U}(1)_R$ symmetry. In the language of algebraic geometry: the superpotential $\mathcal{W}$ is element of the ideal generated by the F-term equations when $\mathcal{W}$ exhibits an exact $\mathrm{U}(1)_R$ symmetry. We conclude that if the superpotential transforms under an $\mathrm{U}(1)_R$ symmetry it vanishes if the F-term equations are satisfied. This is especially true in a global supersymmetric minimum of the theory.

If the $\mathrm{U}(1)_R$ symmetry is spontaneously broken for $\phi_i = \langle \phi_i \rangle$ there is a massless Goldstone mode, called R axion.

3.1.2 Local supersymmetry

In local supersymmetry, also called supergravity $F_i = \partial_i \mathcal{W} = 0$ and $\mathcal{W} = 0$ imply

$$D_i \mathcal{W} = 0, \quad D_i = \partial_i + (\partial_i K) \tag{3.1.13}$$

with the Kähler potential K. Not all local supersymmetric minima have $\mathcal{W} = 0$. The F-term equations are not the same as in global supersymmetry but

$$D_i \mathcal{W} = \partial_i \mathcal{W} + (\partial_i K)\mathcal{W} = 0 \tag{3.1.14}$$

which can satisfied also by $\partial_i \mathcal{W} = -(\partial_i K)\mathcal{W}$. With the help of equation 3.1.12 we obtain

$$\mathcal{W} = \frac{1}{2}\sum_i \phi_i r_i \partial_i \mathcal{W} = -\frac{1}{2}\sum_i \phi_i r_i (\partial_i K)\mathcal{W}. \tag{3.1.15}$$

There are two solutions

$$\mathcal{W} = 0 \quad \text{or} \quad -\frac{1}{2}\sum_i \phi_i r_i (\partial_i K) = 1. \tag{3.1.16}$$

to fulfill the local F-term condition $D_i\mathcal{W} = 0$. We conclude that in a local supersymmetric minima a $\mathrm{U}(1)_R$ symmetry not necessarily implies $\mathcal{W} = 0$ in the minima as in the case of global supersymmetry.

In settings with a $U(1)_R$ symmetry the existence of supersymmetric but spontaneously R-breaking vacua is non-trivial (see for example [36]) and will be discussed in the next section.

3.1.3 The relation to the Nelson-Seiberg theorem

For global supersymmetric models of chiral superfields with generic superpotential the claim is [36]

1. if the model admits a supersymmetry-breaking global minima, then it possesses an exact $\mathrm{U}(1)_R$ symmetry (which may be broken or unbroken in the vacuum);
2. if the model possesses an exact $\mathrm{U}(1)_R$ symmetry and a global minima in which it is spontaneously broken, then the vacuum also breaks supersymmetry spontaneously.

The question arises if we can in our setup break the $\mathrm{U}(1)_R$ symmetry spontaneously and get a global supersymmetric minima which is in contradiction to the second statement. Let us review the Nelson-Seiberg argument and explain the possible loophole. Assume there are M chiral superfields ϕ_i with $\mathrm{U}(1)_R$ charges r_i. The superpotential $\mathcal{W}$ carries $\mathrm{U}(1)_R$ charge 2. Let ϕ_M break the $\mathrm{U}(1)_R$ symmetry, $\langle\phi_M\rangle \neq 0$ and $r_M \neq 0$. With a general function f we can write

$$\mathcal{W} = \phi_M^{\frac{2}{r_M}} f\left(\tilde{\phi}_1, \ldots, \tilde{\phi}_{M-1}\right). \tag{3.1.17}$$

The chiral superfields $\tilde{\phi}_i$ are defined as

$$\tilde{\phi}_i = \phi_i \phi_M^{-\frac{r_i}{r_M}}. \tag{3.1.18}$$

The global F-term equations are

$$\frac{\partial\mathcal{W}}{\partial\phi_i} = \phi_M^{\frac{2}{r_M}} \sum_{k=1}^{M-1} \frac{\partial f}{\partial\tilde{\phi}_k}\frac{\partial\tilde{\phi}_k}{\partial\phi_i} = \phi_M^{\frac{2-r_i}{r_M}} \frac{\partial f}{\partial\tilde{\phi}_j}, \quad \forall\, i = 1, \ldots M-1, \tag{3.1.19}$$

$$\frac{\partial\mathcal{W}}{\partial\phi_M} = \frac{2}{r_M}\phi_M^{\frac{2}{r_M}-1} f + \phi_M^{\frac{2}{r_M}} \sum_{k=1}^{M-1} \frac{\partial f}{\partial\tilde{\phi}_k}\phi_k \left(-\frac{r_k}{r_M}\right)\phi_M^{-\frac{r_k}{r_M}-1}. \tag{3.1.20}$$

To satisfy $F_i = 0$, we get the two conditions

$$f = 0, \qquad \frac{\partial f}{\partial\tilde{\phi}_i} = 0. \tag{3.1.21}$$

If f is a generic function these conditions have in general no solution because there are M equations for $M-1$ variables $\tilde{\phi}_M$. If there is no accidental solution this implies that supersymmetry is spontaneously broken or there is is no minimum.

The loophole is that f is not necessarily generic even if $\mathcal{W}$ is. If r_M is not an integer fraction of 2, a constant in f is not allowed. A constant term would be compatible with all symmetries but represent a non-polynomial piece in the superpotential. It is sufficient that f is quadratic in the fields $\tilde{\phi}_i$. Then the F-term equations are always satisfied for $\tilde{\phi}_i = 0$. Such a function f can be obtained in an effective theory where all massive modes are integrated out and the superpotential $\mathcal{W}$ starts with cubic terms.

A simple example consists of three chiral superfields X, Y and Z with $\mathrm{U}(1)_R$ charges $r_X = 3$, $r_Y = 1$ and $r_Z = -2$. The superpotential computable with this symmetry is

$$\mathcal{W} = Y^2 + XYZ + X^2Z^2 + Y^4Z + \dots . \tag{3.1.22}$$

The coefficients in front of the monomials have been set to one for simplicity. The F-term equations up to higher order corrections are

$$F_x = YZ + 2XZ^2 \stackrel{!}{=} 0 \tag{3.1.23}$$

$$F_Y = 2Y + XZ + 4Y^3Z \stackrel{!}{=} 0 \tag{3.1.24}$$

$$F_Z = XY + 2X^2Z + Y^4 \stackrel{!}{=} 0 \tag{3.1.25}$$

There are two different solutions. $Y = Z = 0$ or $X = Y = 0$. We will focus on the first one. For the first solution X is arbitrary, which means it is a flat direction. Along this flat direction, at any value $X \neq 0$ we find the $\mathrm{U}(1)_R$ symmetry spontaneously broken but supersymmetry is preserved as the F-term equations are satisfied. Because the potential must accommodate for a Goldstone boson and its complex partner whenever the $\mathrm{U}(1)_R$ symmetry is spontaneously broken there remains a supersymmetric flat direction.

Higher order corrections do not influence our argument because they are at least quadratic in Z, which for $Z = 0$ results in satisfied F-term equations.

We can rewrite the superpotential as

$$\mathcal{W} = X^{\frac{2}{3}} \underbrace{\left(\tilde{Y}^2 + \tilde{Y}\tilde{Z} + \tilde{Z}^2 + \tilde{Y}^4\tilde{Z}\right)}_{=f} \tag{3.1.26}$$

with

$$\tilde{Y} = YX^{-\frac{1}{3}}, \quad \tilde{Z} = ZX^{\frac{2}{3}}. \tag{3.1.27}$$

We notice that the function f in this example is not generic because a constant term is missing. The reason is that a term like $X^{\frac{2}{3}}$ would occur in the superpotential which is non-polynomial.

3.2 An approximate $\mathrm{U}(1)_R$ symmetry

3.2.1 Global supersymmetry

If the $\mathrm{U}(1)_R$ symmetry is broken by higher order terms in the superpotential we can write it like

$$\mathcal{W}(\phi_i) = \mathcal{W}_0(\phi_i) + \sum_j \mathcal{W}_j(\phi_i). \tag{3.2.1}$$

Here $\mathcal{W}_0(\phi_i)$ consists of monomials up to order $N-1$ which preserve the $\mathrm{U}(1)_R$ symmetry. This means that for all monomials in the superpotential with

$$\sum_i n_i \leq N-1 \tag{3.2.2}$$

the $\mathrm{U}(1)_R$ symmetry is exact. The terms $\mathcal{W}_j(\phi_i)$ are of higher order $\geq N$ and do not respect the $\mathrm{U}(1)_R$ symmetry. The transformation of the superpotential is

$$\mathcal{W}(\phi_i) \to e^{2i\alpha}\mathcal{W}_0(\phi_i) + \sum_j e^{iR_j\alpha}\mathcal{W}_j(\phi_i) \tag{3.2.3}$$

$$\approx \mathcal{W}(\phi_i) + i\alpha\left(2\mathcal{W}_0(\phi_i) + \sum_j R_j\mathcal{W}_j(\phi_i)\right) \tag{3.2.4}$$

with $R_j \neq 2$. A transformation of the fields ϕ_i results in

$$\mathcal{W}(\phi_i) \to \mathcal{W}\left(e^{ir_i\alpha}\phi_i\right) \approx \mathcal{W}(\phi_i) + i\alpha\sum_j \frac{\partial\mathcal{W}}{\partial\phi_j}r_j\phi_j. \tag{3.2.5}$$

Combing these expressions at order α in the Taylor expansion gives

$$2\mathcal{W}_0(\phi_i) + \sum_j R_j\mathcal{W}_j(\phi_i) = \sum_j \frac{\partial\mathcal{W}}{\partial\phi_j}r_j\phi_j. \tag{3.2.6}$$

If also the F-term equations are satisfied

$$\mathcal{W}(\langle\phi_i\rangle) = -\frac{1}{2}\sum_j(R_j-2)\mathcal{W}_j(\langle\phi_i\rangle). \tag{3.2.7}$$

In contrast to the case of an exact $\mathrm{U}(1)_R$ symmetry the superpotential no longer vanishes in the global supersymmetric vacuum. In the case of an approximate $U(1)_R$ symmetry there can be a suppressed VEV of the superpotential of the order $\mathcal{W}_j(\langle\phi_i\rangle)$. Because $\mathcal{W}_j$ consists only of higher order terms $\geq N$ the VEV of $\mathcal{W}$ is also induced by higher order terms

$$\mathcal{W}(\langle\phi_i\rangle) = \langle\mathcal{W}\rangle \sim \langle\phi_i\rangle^N. \tag{3.2.8}$$

Let us assume a mild hierarchy between the fundamental scale of the theory and a typical field VEV $\phi_i < M_{\mathrm{cutoff}}$. The suppression of the superpotential VEV $\langle\mathcal{W}\rangle$ is then enhanced by the Nth power of the field VEV suppression, similar to the Froggatt-Nielsen picture [37].

In contrast to the case of an exact $\mathrm{U}(1)_R$ symmetry, for an approximate symmetry the R axion η will get a mass term of the order

$$m_\eta \sim \langle\phi_i\rangle^{N-2} \tag{3.2.9}$$

where the two in the exponent results from the second derivative.

3.2.2 Local supersymmetry

We want to outline that $\langle \mathcal{W} \rangle \neq 0$ does not necessarily imply a anti-de Sitter (ADS) minimum. The scalar potential is given by

$$V = e^K (D_i \mathcal{W} D_{\bar{\jmath}} \bar{\mathcal{W}} K^{i\bar{\jmath}} - 3|\mathcal{W}|^2). \tag{3.2.10}$$

Consider a Kähler potential of the form [38, 39]

$$K = -3 \ln(T + \bar{T} - h(C_\alpha, \bar{C}_{\bar{\beta}})) + \widetilde{K}(S_n, \bar{S}_{\bar{m}}) \tag{3.2.11}$$

where T is the Kähler modulus and C_α and S_n are chiral superfields. The resulting scalar potential is

$$V = \frac{e^{\widetilde{K}}}{(T + \bar{T} - h)^3} \left(\partial_\alpha \mathcal{W} \partial_{\bar{\beta}} \bar{\mathcal{W}} h^{\alpha\bar{\beta}} \left(\frac{T + \bar{T} - h}{3} \right) + D_n \mathcal{W} D_{\bar{m}} \bar{\mathcal{W}} \widetilde{K}^{n\bar{m}} \right) \tag{3.2.12}$$

where $h^{\alpha\bar{\beta}}$ labels the inverse of $\partial_\alpha \partial_{\bar{\beta}} h$. Stationary points for V are

$$\partial_\alpha \mathcal{W} = D_n \mathcal{W} = 0. \tag{3.2.13}$$

These points can be supersymmetry breaking minima with $V = 0$ and $\mathcal{W} \neq 0$.

3.2.3 Examples

A simple example

Let us introduce two fields X and Y with $\mathrm{U}(1)_R$ charges $r_X = 2$ and $r_Y = 0$. The superpotential is given by

$$\mathcal{W} = X f(Y) \tag{3.2.14}$$

where f is an arbitrary function. Let us assume

$$\mathcal{W} = X \left(\lambda Y^2 + \frac{1}{M} Y^3 + \ldots \right) \tag{3.2.15}$$

with the coupling λ and the mass scale M. A solution to the F-term equations is

$$\langle X \rangle = 0, \quad \langle Y \rangle = -\lambda M. \tag{3.2.16}$$

Expanding around $\langle Y \rangle$ with $Y \to \langle Y \rangle + \delta Y$ leads to

$$\mathcal{W} = m X \delta Y + \ldots \tag{3.2.17}$$

with $m = -2\lambda^2 M$. m is now related to the fundamental scale and not to the scale of the supersymmetry breakdown. If we add a $\mathrm{U}(1)_R$ violating term we receive

$$\mathcal{W} = X \left(\lambda Y^2 + \frac{1}{M} Y^3 + \ldots \right) + \kappa Y^N. \tag{3.2.18}$$

The above minimum undergoes a small shift. If κ is small enough it will remain a minimum. The VEV of the superpotential is then of the order

$$\langle \mathcal{W} \rangle \sim \kappa (\lambda M)^N. \tag{3.2.19}$$

The gravitino mass is $m_{3/2} \sim \langle \mathcal{W} \rangle$ and the mass scale m can be much larger. If $\langle Y \rangle$ is slightly suppressed and N is large, $\langle \mathcal{W} \rangle$ and $m_{3/2}$ can be very small without tuning.

An example with moduli

Consider three matter fields X, Y and Z with charges $r_X = r_Y = 2$ and $r_Z = 0$. The superpotential is assumed to be

$$\mathcal{W} = X\left(\lambda_1(T)Z^2 + \frac{a_1}{M}Z^3 + \ldots\right) + Y\left(\lambda_2(T)Z^2 + \frac{a_2}{M}Z^3 + \ldots\right). \tag{3.2.20}$$

The couplings $\lambda_i(T)$ depend on an additional free field modulus T. The field content of the model consists of the matter fields X, Y and Z and the modulus T. The F-term equations are

$$F_X = \lambda_1(T)Z^2 + \frac{a_1}{M}Z^3 \tag{3.2.21}$$

$$F_Y = \lambda_2(T)Z^2 + \frac{a_2}{M}Z^3 \tag{3.2.22}$$

$$F_Z = X\left(2\lambda_1(T)Z + 3\frac{a_1}{M}Z^2\right) + Y\left(2\lambda_2(T)Z + 3\frac{a_2}{M}Z^2\right) \tag{3.2.23}$$

$$F_T = X\lambda_1'(T)Z^2 + Y\lambda_2'(T)Z^2. \tag{3.2.24}$$

There are two solutions, namely

$$\langle Z\rangle = 0 \quad \text{or} \quad \langle X\rangle = 0, \quad \langle Y\rangle = 0, \quad \langle Z\rangle = \frac{M}{a_1}\lambda_1(\langle T\rangle), \quad \frac{\lambda_1(\langle T\rangle)}{\lambda_2(\langle T\rangle)} = \frac{a_1}{a_2}. \tag{3.2.25}$$

We will focus on the second solution. In this example we will also derive the supersymmetric mass matrix

$$\mathcal{M} = \left(\frac{\partial^2\mathcal{W}}{\partial\phi_i\partial\phi_j}\right) \tag{3.2.26}$$

with the notation $\phi = (X, Y, Z, T)$. We get for our example

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 2\lambda_1 Z + \frac{3a_1Z^2}{M} & \lambda_1'(T)Z^2 \\ 0 & 0 & 2\lambda_2 Z + \frac{3a_2Z^2)}{M} & \lambda_2'(T)Z^2 \\ 2\lambda_1 Z + \frac{3a_1Z^2}{M} & 2\lambda_2 Z + \frac{3a_2Z^2}{M} & X\left(2\lambda_1 + \frac{6a_1Z}{M}\right) + Y\left(2\lambda_2 + \frac{6a_2Z}{M}\right) & 2X\lambda_1'(T)Z + 2Y\lambda_2'(T)Z \\ \lambda_1'(T)Z^2 & \lambda_2'(T)Z^2 & 2X\lambda_1'(T)Z + 2Y\lambda_2'(T)Z & X\lambda_1''(T)Z^2 + Y\lambda_2''(T)Z^2 \end{pmatrix}. \tag{3.2.27}$$

This matrix has full rank, even when the F-term equations are solved. Thus all fields including the modulus T are fixed in a supersymmetric minimum with $\langle\mathcal{W}\rangle = 0$. This gives a mechanism to stabilize free moduli via matter fields if the F-term equations of the matter fields have the right structure.

For $\lambda_i(T) = e^{-b_iT}$ we find

$$\langle T\rangle = \frac{1}{b_2 - b_1}\ln\frac{a_1}{a_2}. \tag{3.2.28}$$

The masses of the fields are not far below the fundamental scale of the model. Higher order terms have again the chance to break the $\mathrm{U}(1)_R$ symmetry and induce a VEV for the superpotential. We will come back to this example in section 3.2.5.

A generic example

Three fields X, Y and Z are charged under a $\mathbb{Z}_9 \times \mathbb{Z}_4$ symmetry in this model. The charges under these symmetries are given by $q_X = (1,0)$, $q_Y = (5,3)$ and $q_Z = (8,3)$. The first entry is the $\mathbb{Z}_9$ charge and the second entry is the $\mathbb{Z}_4$ charge. The superpotential induced by these symmetries is given by

$$\mathcal{W} = \lambda_5 XY^2Z^2 + \lambda_8 X^4Y^3Z^3 + \kappa_8 X^4Z^4 + \lambda_9 X^9 + \dots \tag{3.2.29}$$

up to higher order couplings. There is an approximate $\mathrm{U}(1)_R$ symmetry for this superpotential up to order 8. The last term $\lambda_9 X^9$ breaks this symmetry. The charges under the approximate symmetry are 0 for X and $\frac{1}{2}$ for Y and Z. A solution to the F-term equations is

$$\langle X \rangle = -\frac{(-2)^{\frac{2}{9}}}{3^{\frac{1}{3}}} \left(\frac{\lambda_5^3}{\lambda_8^2 \kappa_8} \right)^{\frac{1}{9}}, \tag{3.2.30}$$

$$\langle Y \rangle = -\frac{(-2)^{\frac{11}{18}}}{3^{\frac{5}{12}}} \frac{\lambda_5^{\frac{5}{12}} \lambda_9^{\frac{1}{4}}}{\lambda_8^{\frac{11}{18}} \kappa_8^{\frac{1}{18}}}, \tag{3.2.31}$$

$$\langle Z \rangle = \frac{(-2)^{\frac{5}{18}}}{3^{\frac{5}{12}}} \frac{\lambda_5^{\frac{5}{12}} \lambda_9^{\frac{1}{4}}}{\lambda_8^{\frac{5}{18}} \kappa_8^{\frac{7}{18}}}. \tag{3.2.32}$$

The superpotential VEV is given by

$$\langle \mathcal{W} \rangle = -\frac{4}{27} \frac{\lambda_5^3 \lambda_9}{\lambda_8^2 \kappa_8}. \tag{3.2.33}$$

Also in this example all masses for the fields in the supersymmetric minimum are non-vanishing. In the limit $\lambda_9 \to 0$ the $\mathrm{U}(1)_R$ symmetry is restored. In this case $\langle X \rangle$ remains finite but $\langle Y \rangle$ and $\langle Z \rangle$ go to zero. As it should be, the VEV of the superpotential $\langle \mathcal{W} \rangle$ vanishes in this limit term by term. In addition the fields Y and Z will become massless.

An example in heterotic orbifold compactifications

A more complex example is given by a heterotic orbifold compactification. The model is one of the models described in [8, 40]. It is based on a $\mathbb{Z}_{6-\mathrm{II}}$ orbifold geometry and described by the gauge shift V and the Wilson lines W_1 and W_2

$$V = \left(\frac{1}{3}, -\frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0, 0 \right) \left(\frac{1}{2}, -\frac{1}{6}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right), \tag{3.2.34}$$

$$W_1 = \left(\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right) \left(1, -1, -1, -1, -1, -\frac{1}{2}, -\frac{1}{2}, 2 \right), \tag{3.2.35}$$

$$W_2 = \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6} \right) \left(\frac{1}{3}, 0, 0, \frac{2}{3}, \frac{5}{3}, -2, 2, 2 \right). \tag{3.2.36}$$

The $E_8 \times E_8$ gauge group of the heterotic string is broken by the orbifold boundary conditions. The gauge group up to U(1) factors of the model is

$$\underbrace{(\mathrm{SU}(3)\times\mathrm{SU}(2))}_{\subset \mathrm{E}_8}\times\underbrace{(\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2))}_{\subset \mathrm{E}_8}. \tag{3.2.37}$$

We will only consider the fields

$$\phi = (s_1, s_6, s_7, s_{10}, s_{13}, s_{14}, s_{17}, s_{22}, s_{24}, s_{33}). \tag{3.2.38}$$

The superpotential is not only determined by the gauge symmetry but also by various other symmetries [40, 41]. There are 56 terms with 28 independent coefficients at order 11

$$\begin{aligned}
\mathcal{W} = {} & \frac{1}{288}\lambda_1 s_1 s_{22}^3 s_{33}^4\left(s_{10}^2+s_{17}^2\right)+\frac{1}{96}\lambda_2 s_1 s_{22}^2 s_{24} s_{33}^4\left(s_{10}^2+s_{17}^2\right)\\
&+\frac{1}{96}\lambda_3 s_1 s_{22} s_{24}^2 s_{33}^4\left(s_{10}^2+s_{17}^2\right)+\frac{1}{288}\lambda_4 s_1 s_{24}^3 s_{33}^4\left(s_{10}^2+s_{17}^2\right)\\
&+\frac{1}{2}\lambda_5 s_1 s_{22} s_{33}^2(s_{10}s_6+s_{13}s_{17})+\frac{1}{2}\lambda_6 s_1 s_{24} s_{33}^2(s_{10}s_6+s_{13}s_{17})\\
&+\frac{1}{144}\lambda_7 s_{22}^3 s_{33}^4(s_{10}s_7+s_{14}s_{17})+\frac{1}{48}\lambda_8 s_{22}^2 s_{24} s_{33}^4(s_{10}s_7+s_{14}s_{17})\\
&+\frac{1}{48}\lambda_9 s_{22} s_{24}^2 s_{33}^4(s_{10}s_7+s_{14}s_{17})+\frac{1}{144}\lambda_{10} s_{24}^3 s_{33}^4(s_{10}s_7+s_{14}s_{17})\\
&+\frac{1}{96}\lambda_{11} s_{10}s_{14}s_{17}s_{22}s_{33}^4 s_7(s_{10}s_7+s_{14}s_{17})+\frac{1}{96}\lambda_{12} s_{10}s_{14}s_{17}s_{24}s_{33}^4 s_7(s_{10}s_7+s_{14}s_{17})\\
&+\frac{1}{288}\lambda_{13} s_{14}s_{22}s_{33}^4 s_7\left(s_{13}^3 s_7+s_{14}s_6^3\right)+\frac{1}{288}\lambda_{14} s_{14}s_{24}s_{33}^4 s_7\left(s_{13}^3 s_7+s_{14}s_6^3\right)\\
&+\frac{1}{288}\lambda_{15} s_{14}s_{22}s_{33}^4 s_7\left(s_{10}^3 s_{14}+s_{17}^3 s_7\right)+\frac{1}{288}\lambda_{16} s_{14}s_{24}s_{33}^4 s_7\left(s_{10}^3 s_{14}+s_{17}^3 s_7\right)\\
&+\frac{1}{2}\lambda_{17} s_{22}s_{33}^2(s_{13}s_{14}+s_6 s_7)+\frac{1}{2}\lambda_{18} s_{24}s_{33}^2(s_{13}s_{14}+s_6 s_7)\\
&+\frac{1}{96}\lambda_{19} s_{13}s_{14}s_{22}s_{33}^4 s_6 s_7(s_{13}s_{14}+s_6 s_7)+\frac{1}{96}\lambda_{20} s_{13}s_{14}s_{24}s_{33}^4 s_6 s_7(s_{13}s_{14}+s_6 s_7)\\
&+\frac{1}{864}\lambda_{21} s_{22}s_{33}^4\left(s_{10}^3 s_7^3+s_{14}^3 s_{17}^3\right)+\frac{1}{864}\lambda_{22} s_{24}s_{33}^4\left(s_{10}^3 s_7^3+s_{14}^3 s_{17}^3\right)\\
&+\frac{1}{288}\lambda_{23} s_{13}s_{22}s_{33}^4 s_6\left(s_{13}s_7^3+s_{14}^3 s_6\right)+\frac{1}{288}\lambda_{24} s_{13}s_{24}s_{33}^4 s_6\left(s_{13}s_7^3+s_{14}^3 s_6\right)\\
&+\frac{1}{288}\lambda_{25} s_{10}s_{17}s_{22}s_{33}^4\left(s_{10}s_{14}^3+s_{17}s_7^3\right)+\frac{1}{288}\lambda_{26} s_{10}s_{17}s_{24}s_{33}^4\left(s_{10}s_{14}^3+s_{17}s_7^3\right)\\
&+\frac{1}{864}\lambda_{27} s_{22}s_{33}^4\left(s_{13}^3 s_{14}^3+s_6^3 s_7^3\right)+\frac{1}{864}\lambda_{28} s_{24}s_{33}^4\left(s_{13}^3 s_{14}^3+s_6^3 s_7^3\right).
\end{aligned} \tag{3.2.39}$$

Only 28 coefficients are independent because of a D_4 symmetry [9]. A solution of the F-term and D-term equations for this superpotential is given for example by

$$\begin{aligned}
&\langle s_1\rangle = -\frac{1}{10}, \quad \langle s_6\rangle = \frac{1}{10}, \quad \langle s_7\rangle = \frac{1}{10}, \quad \langle s_{10}\rangle = \frac{1}{10}, \quad \langle s_{13}\rangle = \frac{1}{10} \\
&\langle s_{14}\rangle = \frac{1}{10}, \quad \langle s_{17}\rangle = \frac{1}{10}, \quad \langle s_{22}\rangle = \frac{1}{10}, \quad \langle s_{24}\rangle = \frac{1}{10}, \quad \langle s_{33}\rangle = -\frac{\sqrt{1401}}{20\sqrt{70}}
\end{aligned} \tag{3.2.40}$$

with

$$\begin{aligned}
&\lambda_1 \approx 0.482, \quad \lambda_5 \approx -0.01, \quad \lambda_8 = 0.001, \quad \lambda_9 = 0.001 \\
&\lambda_{13} \approx -0.29, \quad \lambda_{14} \approx -0.22, \quad \lambda_{17} \approx -0.001, \quad \lambda_{18} = 0.001 \\
&\lambda_i = 0.01, \quad \forall\, i \notin \{1,5,8,9,13,14,17,18\}.
\end{aligned} \tag{3.2.41}$$

We have first determined the VEVs of the fields s_i and afterwards chosen some λ_i. A solution to the F-term equations was then found numerically. The VEV of s_{33} was chosen to cancel a Fayet-Iliopoulos D-term [42–44] of an anomoulos $U(1)$ which is present in this model.

This model exhibits an approximate $\mathrm{U}(1)_R$ symmetry up to order 10 in the superpotential. The VEV of the superpotential for the given solution is

$$\langle \mathcal{W}\rangle \approx 1.1 \cdot 10^{-12} \sim \langle \phi_i\rangle^N \tag{3.2.42}$$

as expected. The order of the $\mathrm{U}(1)_R$ symmetry breaking is $N = 11$ and the field VEVs are of the order $\mathcal{O}(0.1)$. The supersymmetric masses are

$$\begin{aligned}
m \approx \big(&1 \cdot 10^{-5}, 1 \cdot 10^{-5}, 1.4 \cdot 10^{-9}, 5.1 \cdot 10^{-10}, 2.7 \cdot 10^{-10}, \\
&2.2 \cdot 10^{-10}, 1 \cdot 10^{-10}, 3 \cdot 10^{-11}, 0, 0\big)\,.
\end{aligned} \tag{3.2.43}$$

The two massless fields will obtain masses from the D-term potential and the absorption of the Goldstone modes.

3.2.4 A link to computational algebraic geometry

In the last section we have discussed several examples with many fields. The task to solve polynomial, in general non-linear equations is a topic in computational algebraic geometry. It is possible to make some general statements of the solution structure in concrete examples with these techniques.

Definition 5 *The set of all polynomials in $\phi_1, \ldots, \phi_M$ with coefficients in $\mathbb{C}$ is denoted $\mathbb{C}[\phi_1, \ldots, \phi_M]$. $\mathbb{C}[\phi_1, \ldots, \phi_M]$ is called a polynomial ring.*

Definition 6 *Let F_i with $i \in \{1, \ldots, M\}$ be polynomials in $\mathbb{C}[\phi_1, \ldots, \phi_M]$. We call*

$$I = \langle F_1, \ldots, F_M\rangle = \left\{ \sum_{i=1}^{M} h_i F_i \;\middle|\; h_i \in \mathbb{C}[\phi_1, \ldots, \phi_M] \right\} \tag{3.2.44}$$

the ideal I generated by F_i.

Every ideal I is finitely generated by a given number of F_i. This is known as the Hilbert basis theorem [45].

Definition 7 *Let I be an ideal and F a polynomial. The radical of I is defined as*

$$\sqrt{I} = \{F \mid F^m \in I \text{ for some integer } m \geq 1\}. \tag{3.2.45}$$

If $\mathcal{W} \in \sqrt{I}$ where $\mathcal{W}$ is the superpotential we have $\mathcal{W}^m \in I$ for some m and can write

$$\mathcal{W}^m = \sum_{i=1}^{M} h_i F_i \tag{3.2.46}$$

if I is the ideal generated by the F-term equations. That would mean that at a common zero of F_i, which is a solution to the F-term equations we find $\mathcal{W} = 0$. If we want to have $\mathcal{W} \neq 0$ as in the case of an approximate $\mathrm{U}(1)_R$ symmetry we should thus find $\mathcal{W} \notin \sqrt{I}$. We further have to show that $\mathcal{W} \notin \sqrt{I}$ implies automatically $\mathcal{W} \neq 0$ for some common zeros of F_i.

Definition 8 *The variety V of an ideal $I \subset \mathbb{C}[\phi_1, \ldots, \phi_M]$ is defined as*

$$V(I) = \left\{\phi \in \mathbb{C}^M \;\middle|\; F(\phi) = 0 \;\forall\; F \in I\right\}. \tag{3.2.47}$$

That means the solution space of the F-term equations is a variety. On the other hand we can also define the ideal of a variety.

Definition 9 *The ideal $I(V)$ of a variety V is defined by*

$$I(V) = \{F \in \mathbb{C}[\phi_1, \ldots, \phi_M] \mid F(\phi) = 0 \;\forall\; \phi \in V\}. \tag{3.2.48}$$

Theorem 2 *The strong Nullstellensatz: if I is an ideal in $\mathbb{C}[\phi_1, \ldots, \phi_M]$ we have*

$$I(V(I)) = \sqrt{I}. \tag{3.2.49}$$

$V(I)$ is the solution space of the F-term equations that means all common zeros of all F_i. The ideal $I(V(I))$ is the ideal spanned by all polynomials which also vanish at all common zeros of all F_i. The strong Nullstellensatz states that this ideal is given by the radical $\sqrt{I}$. That means that if $\mathcal{W} \notin \sqrt{I}$ there have to be solutions where the F_i are all zero, but $\mathcal{W}$ is not, because otherwise it would be in $I(V(I))$. To test if $\mathcal{W} \in \sqrt{I}$ is known as the ideal membership problem and can be solved for a given polynomial and a given ideal algorithmically.

We have explicitly checked with the computer algebra system `Singular` [46] that for $\mathcal{W}$ given in section 3.2.3 $\mathcal{W} \notin \sqrt{I}$ if I is the ideal spanned by the F-term equations $I = \langle F_1, \ldots F_M \rangle$. That confirms our earlier numerical result that $\mathcal{W} \neq 0$ for broken $\mathrm{U}(1)_R$ symmetries exists with an algebraic method. It is furthermore possible to show that $\mathcal{W} \in I$ as expected if the $\mathrm{U}(1)_R$ symmetry is valid.

It is also possible to use methods based on Gröbner bases to compute exact solutions to the F-term equations with `Singular`. These techniques have been used to minimize the Higgs potential in [47] and more recently also in the string theory context [48]. It is possible to use Gröbner bases to compute the Hilbert basis of a given problem (see for example [49][p. 392]) which relates the mathematical topic of this section with the mathematical survey in section 2.2.1.

3.2.5 Moduli stabilization

One of the open questions in string theory, especially in heterotic string theory is moduli stabilization. How is it possible to determine and stabilize dynamically the value for the scale of the extra dimensions, the gauge coupling and additional degrees of freedom. This topic is often connected to supersymmetry breaking. In many attempts supersymmetry is broken by dimensional transmutation [50] for example with the help of a gaugino condensate [51]. This mechanism can only work if the gauge coupling which is in these settings given by the VEV of the dilaton is fixed. There have been several attempts for dilaton stabilization, namely race-track models [52], Kähler stabilization [53, 54] and an adaption of the KKLT scheme [55] from type II string theories. We will focus our attention on the KKLT idea. The original KKLT setup was used to stabilize the Kähler modulus T in type II string theories with a small constant c and a superpotential of the form

$$\mathcal{W} = c + Ae^{-bT}. \tag{3.2.50}$$

The origin of the small constant based on fluxes [56] and its scale is not fixed but can be small by accident. In heterotic string theory we have no fluxes at hand, but can use the small VEV of the perturbative superpotential $\langle\mathcal{W}_{\text{perturbative}}\rangle$ and a gaugino condensate for the non-perturbative part. In heterotic orbifold compactifications we can have a superpotential of the form

$$\mathcal{W} = \langle\mathcal{W}_{\text{perturbative}}\rangle + \underbrace{Ae^{-aS}}_{\mathcal{W}_{\text{non-perturbative}}}. \tag{3.2.51}$$

The exponential is determined by a hidden gauge group, S labels the dilaton and $\langle\mathcal{W}_{\text{perturbative}}\rangle$ is the VEV of the perturbative superpotential. This mechanism works if the VEVs of all singlet fields giving rise to $\langle\mathcal{W}_{\text{perturbative}}\rangle$ are already stabilized. We will assume that this is the case. As outlined in this study, we find a hierarchically small $\langle\mathcal{W}_{\text{perturbative}}\rangle$ due to an approximate $\mathrm{U}(1)_R$ symmetry. The minimum of the scalar potential for S is given by

$$|aSA \cdot e^{-aS}| \sim |\langle\mathcal{W}_{\text{perturbative}}\rangle|. \tag{3.2.52}$$

The small value for the superpotential VEV gives us a Dilaton VEV $\langle S\rangle$ of the right scale in this setup. This has been studied in a toy model in [57]. It is an interesting task for future work to study the outlined setup for Dilaton stabilization in a complete model. It would be possible to proceed as follows:

A superpotential like the one in equation 3.2.20 gives rise to F-term equations which lead to fixed values for the singlet VEVs $\langle\phi_i\rangle$ as well as the Kähler moduli T. A hidden sector will induce a gaugino condensate which breaks supersymmetry and will offer the possibility to further stabilize the Dilaton S. All moduli except the complex structure moduli U are fixed and supersymmetry is broken in such a setup. The complex structure moduli can maybe fixed by a mechanism like the one proposed in [58]. If it is possible to find such a setup in a realistic orbifold compactification is beyond the scope of this work.

3.3 Approximate $\mathrm{U}(1)_R$ symmetries and the μ term

The Higgsino mass term μ in the MSSM is related to the Z boson mass

$$|\mu|^2 = \frac{1}{\cos 2\beta}(m_{h_u}^2 \sin^2\beta - m_{h_d}^2 \cos^2\beta) - \frac{1}{2}M_Z^2. \tag{3.3.1}$$

m_{h_u} and m_{h_d} are the soft masses, $\tan\beta = \langle h_u \rangle / \langle h_d \rangle$ is the relation between the Higgs VEVs and M_Z is the Z boson mass. $\mu = 0$ is experimentally excluded because no Higgsinos have been found at colliders so far. On the other hand, equation 3.3.1 relates the μ term to the supersymmetry breaking soft masses. The μ term arises in the superpotential at the supersymmetric level and there is a priori no relation to the supersymmetry breaking scale. The issue of this section is to create a link between the μ term and the supersymmetry breaking scale with approximate $\mathrm{U}(1)_R$ symmetries.

3.3.1 An effective μ term

There are several mechanism known to provide a μ term of the supersymmetry breaking scale [59, 60]. Consider a superpotential and a Kähler potential of the form

$$\mathcal{W} = \mathcal{W}_0(\phi_i) + \widehat{\mu}(\phi_i)h_u h_d + \ldots \tag{3.3.2}$$
$$K = \mathcal{K}(\phi_i) + \mathcal{Y}_u(\phi_i)|h_u|^2 + \mathcal{Y}_d(\phi_i)|h_d|^2 + (\mathcal{Z}(\phi_i)h_u d_d + h.c.) + \ldots. \tag{3.3.3}$$

We ignore MSSM matter fields for the moment and focus on additional superfields ϕ_i which extend the MSSM. These fields are SM singlets. If $\langle \mathcal{W}_0 \rangle$ is real we find an effective μ term [61]

$$\mu = \frac{1}{\sqrt{\mathcal{Y}_d \mathcal{Y}_u}}\left(e^{\frac{1}{2}\mathcal{K}}\mathcal{W}_0 \mathcal{Z} + e^{\frac{1}{2}K} K^{ji} D_j \mathcal{W} \frac{\partial \mathcal{Z}}{\partial \bar{\phi}^i} + e^{\frac{1}{2}\mathcal{K}}\widehat{\mu} \right). \tag{3.3.4}$$

The first term is proportional to the gravitino mass $m_{3/2}$ which sets the supersymmetry breaking scale. The prefactor is generically

$$\mathcal{Z}\frac{1}{\sqrt{\mathcal{Y}_d \mathcal{Y}_u}} = \mathcal{O}(1). \tag{3.3.5}$$

The second term is proportional to the F-term and therefore of the order of the supersymmetry breaking scale [60]. The last term $\widehat{\mu}$ is unrelated and can in principle give a too large contribution to the effective μ term.

We can see that Kähler-Weyl transformations cannot cure the problem. Using a Kähler-Weyl transformation, it is always possible to absorb the $\widehat{\mu}$ term into the Kähler potential. We define the invariant function

$$G = K + \ln |\mathcal{W}|^2. \tag{3.3.6}$$

Instead of using $\mathcal{W}$ and K we can work with

$$\widetilde{\mathcal{W}} = \mathcal{W}e^{-f}, \qquad \widetilde{K} = K + f + \bar{f}, \qquad f = \frac{\widehat{\mu}}{\mathcal{W}_0}h_u h_d. \tag{3.3.7}$$

Expanding $\widetilde{K}$ results in

$$\widetilde{K} = \mathcal{K} + \mathcal{Y}_u |h_u|^2 + \mathcal{Y}_d |h_d|^2 + \left(\left(\mathcal{Z} + \frac{\widehat{\mu}}{\mathcal{W}_0}\right) h_u d_d + h.c.\right) + \dots . \tag{3.3.8}$$

As expected this does not relate $\widehat{\mu}$ to the supersymmetry breaking scale. The problem is just hidden in the Kähler potential.

Let us consider a superpotential where the combination $h_u h_d$ is a complete singlet under all symmetries

$$\mathcal{W} = \sum_a c_a \mathcal{M}_a(\phi_i) + h_u h_d \sum_a c'_a \mathcal{M}_a(\phi_i) + \dots . \tag{3.3.9}$$

We identify

$$\mathcal{W}_0 = \sum_a c_a \mathcal{M}_a(\phi_i), \quad \widehat{\mu} = \sum_a c'_a \mathcal{M}_a(\phi_i). \tag{3.3.10}$$

The $\mathcal{M}_a(\phi_i)$ are monomials in the additional fields ϕ_i. The monomials are singlets under all symmetries except possible discrete or continuous R symmetries. c_a and c'_a label numerical coupling coefficients.

If $c'_a = \lambda c_a$, we find [62]

$$\widehat{\mu} = \lambda \mathcal{W}_0. \tag{3.3.11}$$

The VEV $\langle \mathcal{W}_0 \rangle$ sets the gravitino mass and has to be of the correct size. We have shown in this work that this can be realized with an approximate $\mathrm{U}(1)_R$ symmetry. If we can explain that λ is $\mathcal{O}(1)$, approximate $\mathrm{U}(1)_R$ symmetries can explain the correct size of the μ term.

3.3.2 A μ term in an orbifold GUT

Let us assume a six dimensional theory on a $\mathbb{T}^2/\mathbb{Z}_2$ orbifold. The Kähler potential is

$$K = -\ln((T + \bar{T})(U + \bar{U}) - (h_u + \bar{h}_d)(h_d + \bar{h}_u)). \tag{3.3.12}$$

This form is induced by higher dimensional gauge invariance [63, 64] or by direct calculation in heterotic orbifold compactifications [65–68]. T is the Kähler modulus and U the complex structure modulus. h_u and h_d arise as extra components of six dimensional gauge fields. Gauge transformations mix h_u and $\bar{h}_d$ and enforce the structure of the Kähler potential in equation 3.3.12. An expansion of the Kähler potential gives

$$\begin{aligned} K &= -\ln((T + \bar{T})(Z + \bar{Z}) - (h_u + \bar{h}_d)(h_d + \bar{h}_u)) \\ &\approx -\ln((T + \bar{T})(U + \bar{u})) + \frac{1}{(T + \bar{T})(U + \bar{U})}(|h_u|^2 + |h_d|^2 + (h_u h_d + h.c.)) \\ &= -\ln((T + \bar{T})(U + \bar{U})) + \left(|\widehat{h}_u|^2 + |\widehat{h}_d|^2 + (\widehat{h}_u \widehat{h}_d + h.c.)\right). \end{aligned} \tag{3.3.13}$$

We introduced canonically normalized fields $\widehat{h}_u$ and $\widehat{h}_d$. Higher dimensional gauge invariance enforces the superpotential to be independent of $\widehat{h}_u \widehat{h}_d$

$$\mathcal{W} \not\supset \widehat{h}_u \widehat{h}_d. \tag{3.3.14}$$

This means that there are no monomials $\widehat{h}_u\widehat{h}_d$ in $\mathcal{W}$. In leading order in $\widehat{h}_u\widehat{h}_d$ this setting is equivalent to

$$\widetilde{\mathcal{W}} = e^{\widehat{h}_u\widehat{h}_d}\mathcal{W} \tag{3.3.15}$$

$$\widetilde{K} = -\ln((T+\bar{T})(U+\bar{U})) + \left(|\widehat{h}_u|^2 + |\widehat{h}_d|^2\right). \tag{3.3.16}$$

Expanding the superpotential shows that

$$\lambda = 1, \quad c'_a = c_a \tag{3.3.17}$$

which generates a μ term of the supersymmetry breaking scale. There have been two ingredients for this behavior. An approximate $\mathrm{U}(1)_R$ symmetry can induce a small superpotential VEV and thus a gravitino mass in the TeV range. Further the special form of the Kähler potential induced by the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold connects this scale with the μ term.

4 Gauge-top unification

We discuss in this section the top quark coupling in higher dimensional supersymmetric theories. We will review that orbifold theories can give rise to a top quark coupling which originates from higher dimensional gauge interactions [69]. We further discuss corrections and discuss the phenomenological consequences. In a final step we are able to relate low energy observables to the size of the extra dimensions.

4.1 Gauge-top unification at tree level

Assume a six dimensional orbifold GUT with SU(6) gauge group. The geometry of the model is $\mathbb{T}^2/\mathbb{Z}_2$ (see figure 4.1). The fixed points in the extra dimensions are labeled by

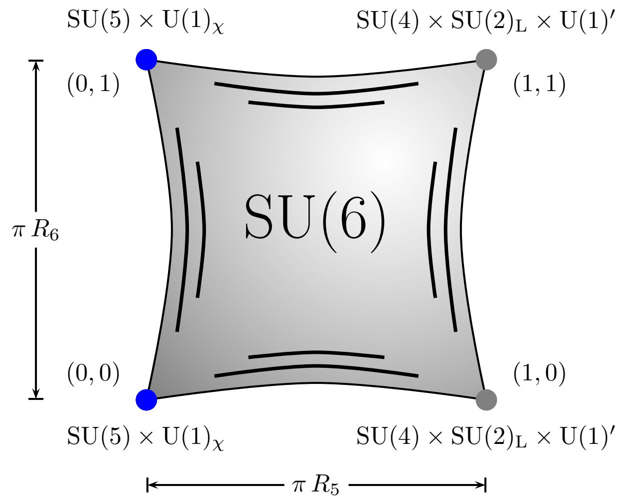

Figure 4.1: Sketch of the six dimensional GUT model.

two quantum numbers n_2 and n_2'. The distance between the fixed points are πR_5 and πR_6. The torus exhibits a rectangular lattice. The SU(6) gauge group gets broken by the orbifolding to $\mathrm{SU}(5) \times \mathrm{U}(1)_\chi$ for $n_2 = 0$ and to $\mathrm{SU}(4) \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)'$ for $n_2 = 1$. The gauge group in four dimensions is given by

$$\underbrace{\mathrm{SU}(3)_\mathrm{C} \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_Y}_{=G_\mathrm{SM}} \times \mathrm{U}(1). \tag{4.1.1}$$

With G_SM we denote the standard model gauge group. This model is inspired by [69,70].

The bulk supersymmetry corresponds to $N = 2$ supersymmetry from a four dimensional viewpoint. The six dimensional gauge multiplet (V, Φ) contains a four dimensional vector multiplet V and a chiral multiplet Φ. The orbifold boundary conditions

$$P_{n_2=0} = \mathrm{diag}(1,1,1,1,1,-1), \qquad P_{n_2=1} = \mathrm{diag}(1,1,1,-1,-1,1) \tag{4.1.2}$$

imply

$$\begin{aligned}\Phi &= \Phi^a T_a \\ &= \begin{pmatrix} \Phi^{(--)}_{(\mathbf{8},\mathbf{1})_0} - \frac{1}{\sqrt{15}}\Phi^{(--)}_Y + \frac{1}{2\sqrt{15}}\Phi^{(--)}_\chi & \frac{1}{\sqrt{2}}\Phi^{(-+)}_{(\mathbf{3},\mathbf{2})_{-\frac{5}{6}}} & \frac{1}{\sqrt{2}}\Phi^{(+-)}_{(\mathbf{3},\mathbf{1})_{-\frac{1}{3}}} \\ \frac{1}{\sqrt{2}}\Phi^{(-+)}_{(\bar{\mathbf{3}},\mathbf{2})_{\frac{5}{6}}} & \frac{1}{\sqrt{15}}\Phi^{(--)}_{(\mathbf{1},\mathbf{3})_0} + \frac{3}{2\sqrt{15}}\Phi^{(--)}_Y + \frac{1}{2\sqrt{15}}\Phi^{(--)}_\chi & \frac{1}{\sqrt{2}}\Phi^{(++)}_{(\mathbf{1},\mathbf{2})_{\frac{1}{2}}} \\ \frac{1}{\sqrt{2}}\Phi^{(+-)}_{(\bar{\mathbf{3}},\mathbf{1})_{\frac{1}{3}}} & \frac{1}{\sqrt{2}}\Phi^{(++)}_{(\mathbf{1},\mathbf{2})_{-\frac{1}{2}}} & -\frac{5}{2\sqrt{15}}\Phi^{(--)}_{(\mathbf{1},\mathbf{1})_0} \end{pmatrix}\end{aligned} \tag{4.1.3}$$

where the SU(6) generators are decomposed. The $+$ and $-$ signs label the orbifold parity in the two extra dimensions. We find two fields which can be identified as the MSSM Higgs doublets

$$h_u = \Phi^{(++)}_{(\mathbf{1},\mathbf{2})_{\frac{1}{2}}}, \quad h_d = \Phi^{(++)}_{(\mathbf{1},\mathbf{2})_{-\frac{1}{2}}}. \tag{4.1.4}$$

The normalization is chosen to be compatible with

$$\mathrm{tr}(T_a T_b) = \frac{1}{2}\delta_{ab}. \tag{4.1.5}$$

All fields are canonically normalized. There is also a hypermultiplet H that transforms as a **20**-plet under the SU(6) gauge group. In $N = 1$ supersymmetry we have

$$H = (\varphi, \varphi^c). \tag{4.1.6}$$

Here φ transforms as **20**-plet and φ^c as $\bar{\mathbf{20}}$-plet under SU(6). After orbifold projection the third generation quark doublet q_3 comes from φ. The up type quark $\bar{u}_3$ and the lepton singlet $\bar{e}_3$ originate from φ^c.

The two-component spinors in φ and φ^c are labeled with ξ and η. The scalar component of the chiral superfield Φ is given by

$$\phi = \frac{1}{\sqrt{2}}(A_5 + iA_6). \tag{4.1.7}$$

The top Yukawa coupling in the four dimensional Lagrangian is

$$y_t \bar{u}_3 q_3 h_u \subset \mathscr{L}_{4D}. \tag{4.1.8}$$

We will see that the origin of this coupling is the six dimensional gauge interaction. We have

$$\eta(\bar{\partial} + \sqrt{2} g_6 \phi)\xi \subset \mathscr{L}_{6D} \tag{4.1.9}$$

with the six dimensional gauge coupling g_6. The partial derivative is $\bar{\partial} = \partial_5 + i\partial_6$ (see for example [71]). With indices we find

$$\begin{aligned}&\frac{1}{3!}\eta_{i_1 i_2 i_3}\left(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3}\bar{\partial} + \sqrt{2}g_6\left(\phi^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_1}\phi^{i_2}_{j_2}\delta^{i_3}_{j_3} + \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\phi^{i_3}_{j_3}\right)\right)\xi^{j_1 j_2 j_3}\\ =&\frac{1}{3!}\eta_{i_1 i_2 i_3}\left(\delta^{i_3}_{j_3}\bar{\partial} + 3\sqrt{2}g_6\phi^{i_3}_{j_3}\right)\xi^{i_1 i_2 i_3}.\end{aligned} \tag{4.1.10}$$

q_3 is in the SU(5) **10**-plet of the SU(6) **20**-plet ξ

$$\xi^{ij6} = \mathbf{10}^{ij}. \tag{4.1.11}$$

For $\bar{u}_3$ out of the **10**-plet, which originates from the $\bar{\mathbf{2}}\mathbf{0}$-plet η we find

$$\eta_{ijk} = \frac{1}{2}\epsilon_{ijklm}(\mathbf{10}^c)^{lm}. \tag{4.1.12}$$

The fields q_3 and $\bar{u}_3$ come from different SU(5) multiplets. The final reduction gives

$$\begin{aligned}\frac{1}{2}\sqrt{2}g_6\eta_{ijk}\phi^k_6\xi^{ij6} &= \frac{1}{4}\sqrt{2}g_6\epsilon_{ijklm}(\mathbf{10}^c)^{ij}\mathbf{10}^{kl}\phi^m_6\\ &\supset \frac{1}{2}g_6\epsilon_{abc}\epsilon_{\alpha\beta}\epsilon^{abd}(\bar{u}_3)_d(q_3)^{c\alpha}(h_u)^\beta\\ &= g_6\bar{u}_3 q_3 h_u.\end{aligned} \tag{4.1.13}$$

The latin indices are $\mathrm{SU}(3)_C$ indices and the greek indices are $\mathrm{SU}(2)_L$ indices. We have seen that the top quark Yukawa coupling in our four dimensional theory originates from the six dimensional gauge multiplet. The Yukawa coupling is given by

$$y_t = g \tag{4.1.14}$$

at tree level. g is the four dimensional gauge coupling. The geometrical reduction has to be taken into account to relate g_6 and g.

4.2 Corrections

We will discuss several corrections to the tree level result in this section. The influence of localized Fayet-Iliopoulos D-terms will give the strongest corrections.

4.2.1 Localized states

The β-function in six dimensions is given by two contributions

$$\beta = b_6\mu^2 R_5 R_6 + b_4. \tag{4.2.1}$$

The dominant power-law contribution is universal for gauge and top coupling. In our setup only the third generation of quarks and leptons is located in the extra dimensions. This results in

$$b^t_6 = b^i_6 = -4, \quad \forall\ i = 1, 2, 3. \tag{4.2.2}$$

The subdominant logarithmic contribution is influenced by localized states sitting at the fixed points. b_4 is therefore not universal for the gauge and the top coupling. We estimate the effect as wave function renormalization

$$|\Delta y_t - \Delta g_i|_{\ln} \approx |\Delta g_i - \Delta g_j|_{\ln} \approx \left|\Delta b_4 \ln \frac{\Lambda}{M_{\mathrm{GUT}}}\right|. \tag{4.2.3}$$

Λ is the cut-off scale and Δb_4 is the difference between the β-functions. The corrections are of the order of MSSM threshold corrections. They originate from squarks and sleptons having mass splittings of the order $\mathcal{O}(1)$ to $\mathcal{O}(10)$.

4.2.2 Diagonalization effects

The up type Yukawa couplings are given by

$$Y_u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathcal{O}(g) \end{pmatrix} + \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}. \tag{4.2.4}$$

The entries y_{ij} in the second matrix are non-renormalizable couplings and couplings from localized states. The top Yukawa coupling after diagonalization is

$$y_t = \mathcal{O}(g) + \mathcal{O}(y_{ij}). \tag{4.2.5}$$

In our setup y_{33} is solely a non-renormalizable coupling and thus suppressed. To get a realistic Yukawa matrix the other couplings y_{ij} should also be suppressed which means that $\mathcal{O}(g)$ is the dominant contribution for y_t.

4.2.3 Influence of the Fayet-Iliopoulos D-term

Localized Fayet-Iliopoulos D-terms influence the coupling strengths in extra dimensional theories [72–74]. For a U(1) symmetry there can be

$$\mathrm{tr}(q_I) \neq 0, \quad I = (n_2, n_2') \tag{4.2.6}$$

even if the corresponding trace vanishes in the effective four dimensional theory which means the U(1) symmetry is non-anomalous in four dimensions. We follow in this section [74] and choose

$$\sum_I \xi_I = 0 \tag{4.2.7}$$

where ξ_I is the Fayet-Iliopoulos D-term. A generalization can be found in section 4.5.

$\mathrm{tr}(q_I) \neq 0$ induces localized Fayet-Iliopoulos D-terms. The zero mode wave function of a field charged under such an U(1) is in a six dimensional theory given by [74]

$$\varphi \approx \mathcal{N} \prod_I \left| \vartheta_1 \left(\frac{z - z_I}{2\pi} \middle| \tau \right) \right|^{\frac{1}{2\pi} g_6 q_\varphi \xi_I} \exp\left(-\frac{1}{8\pi^2 \tau_2} g_6 q_\varphi \xi_I (\mathrm{Im}(z - z_I))^2 \right). \tag{4.2.8}$$

This profile is the same for bosons and fermions and a subleading logarithmic contribution was neglected. $\mathcal{N}$ is a normalization constant. The coordinate system is defined by

$$z = \frac{1}{R_5}x^5 + \frac{\tau}{R_6}x^6. \tag{4.2.9}$$

τ is the modular parameter of the torus and in the rectangular case given by $\tau = i\tau_2 = iR_6/R_5$. The ϑ-function is given by [75]

$$\vartheta_1(z|\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi\tau\left(n+\frac{1}{2}\right)^2+2\pi i\left(n+\frac{1}{2}\right)\left(z+\frac{1}{2}\right)} \tag{4.2.10}$$

and

$$\xi_I = \frac{1}{16\pi^2}g_6\Lambda^2\left(\frac{1}{4}\,\mathrm{tr}(q) + \mathrm{tr}(q_I)\right) \tag{4.2.11}$$

is the localized Fayet-Iliopoulos D-term. q are the charges of the bulk fields under the $\mathrm{U}(1)$ and q_I the charges of the fields at the corresponding fixed point. Λ is the cut-off of the theory. We assume

$$\Lambda^2 = \frac{M_P}{\sqrt{V_{56}}},\quad M_P = 2.43\cdot 10^{18}\ \mathrm{GeV},\quad V_{56} = 2\pi^2 R_5 R_6,\quad g_6 = \sqrt{V_{56}}g. \tag{4.2.12}$$

The normalization condition on the orbifold is

$$\int_0^{\pi R_5} dx_5 \int_0^{2\pi R_6} dx_6 |\psi|^2 = 1. \tag{4.2.13}$$

The localization of the distorted wave function is visualized in figure 4.2. If q_3 and $\bar{u}_3$

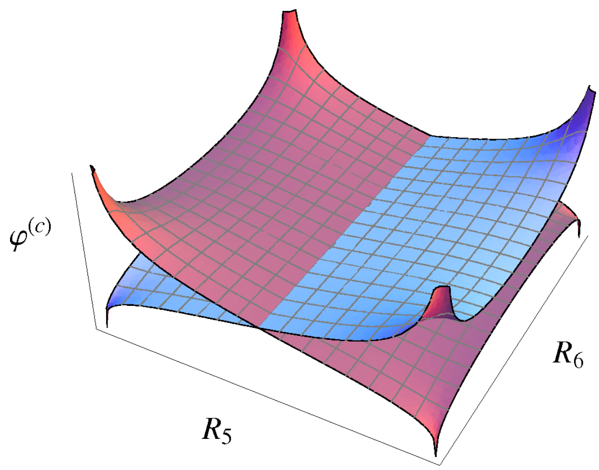

Figure 4.2: Localization of two wave function with opposite U(1) charges in the extra dimensions.

are charged under an U(1) with $\mathrm{tr}(q_I) \neq 0$, their wave function profiles are determined by equation 4.2.8. Different wave functions lead to different effective couplings (see for example [69]). The implication is a difference between y_t and g. The profiles for q_3 and $\bar{u}_3$ are inverse to each other because they come from φ and φ^c and have opposite U(1) charges.

For the gauge field A and the Higgs field h_u we find flat profiles and $\bar{u}_3^\dagger$ has the same profile than $\bar{u}_3$ because the wave function in equation 4.2.8 is real.

The top Yukawa coupling scales like the integral over $\bar{u}_3 q_3 h_u$ and the gauge coupling like $\bar{u}_3^\dagger \bar{u}_3 A$. We conclude that the top Yukawa coupling is suppressed. The strength of the suppression is depicted in figure 4.3. The suppression depends mainly on two

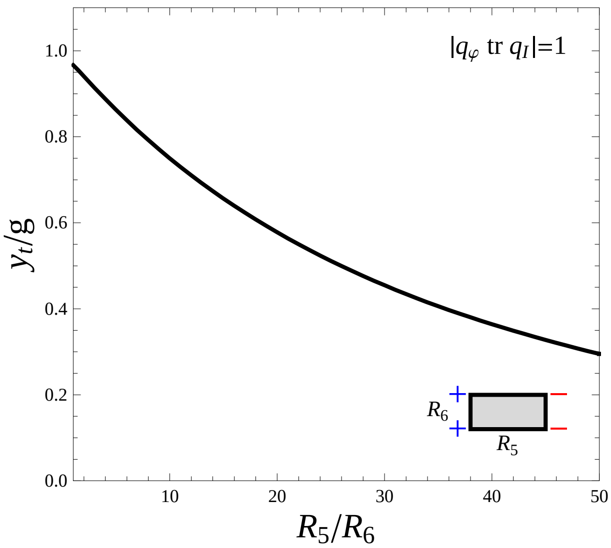

Figure 4.3: Suppression of the top Yukawa coupling y_t relative to the gauge coupling g.

different factors:

- The anisotropy R_5/R_6.
- The charges $q_\varphi \operatorname{tr} q_I$.

The charges $q_\varphi \operatorname{tr} q_I$ are model dependent and depend on the U(1) charges of the field q_φ and the strength of the localized Fayet-Iliopoulos D-term $\operatorname{tr} q_I$. We have chosen $\operatorname{tr} q_{(0,0)} = \operatorname{tr} q_{(0,1)} > 0$ and $\operatorname{tr} q_{(1,0)} = \operatorname{tr} q_{(1,1)} < 0$ as indicated by the inlay in figure 4.3. The gauge structure given in figure 4.1 restricts us to two possible sign pattern for $\operatorname{tr} q_I$.

The anisotropy R_5/R_6 can be linked to the GUT scale. The lightest Kaluza-Klein masses are given by $M = 1/(2R_5)$ or $M = 1/R_6$ respectively. We assume $R_5 > R_6$ as sketched in the inlay of figure 4.3. We identify the GUT scale with

$$M_{\text{GUT}} = \frac{1}{2R_5}, \qquad R_5 \approx 50. \tag{4.2.14}$$

The value for R_5 is fixed if we want to achieve $M_{\text{GUT}} \approx 2 \cdot 10^{16}$ GeV which seems to be the value where gauge couplings unify. Fixing the anisotropy R_5/R_6 is thus equivalent to fixing the radius R_6.

We conclude that in a concrete model where all charges are fixed, R_6 alone sets the suppression of y_t.

4.3 Phenomenological implications

It is possible to determine $\tan\beta$ with the soft masses and the top Yukawa coupling y_t at the GUT scale. The top quark mass is given by [76]

$$m_t = 173.1 \pm 1.3 \text{ GeV}. \tag{4.3.1}$$

The main influence on $\tan\beta$ is y_t. We plot y_t at the GUT scale versus $\tan\beta$ in figure 4.4. Because there is still a large uncertainty in the top quark mass we plot also the one

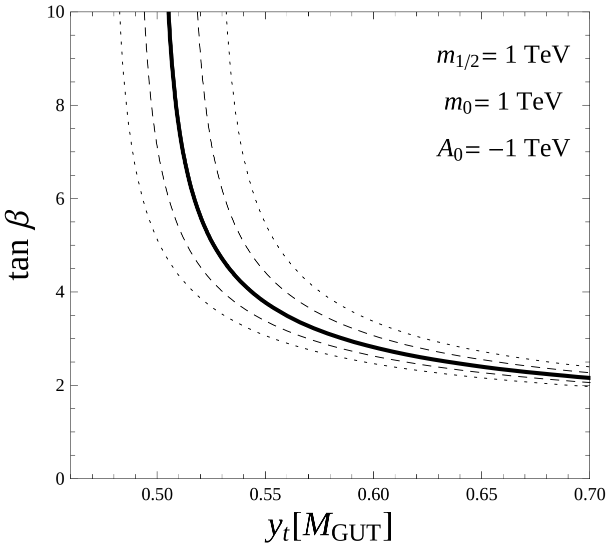

Figure 4.4: $\tan\beta$ as a function of the top Yukawa coupling $y_t(M_{\rm GUT})$ which is given at the GUT scale.

and two sigma deviation bounds for the top quark mass. For the boundary conditions at the GUT scale we choose the mSUGRA scenario with

$$m_0 = m_{1/2} = -A_0 = 1 \text{ TeV}. \tag{4.3.2}$$

The numerical analysis was performed with [77]. The influence of different boundary conditions at the GUT scale $M_{\rm GUT}$ is only a minor effect. The value of the unified gauge coupling g depends only mildly on the soft terms and is given by $g \approx 0.7$. The tree level relation $y_t = g$ would result in $y_t \approx 0.7$ at the GUT scale which would imply $\tan\beta \approx 2$ (see figure 4.4).

Searches for light Higgs bosons constrain the small $\tan\beta$ parameter space substantially [78]. The LEP bound on the SM Higgs mass is $m_h \geq 114.4$ GeV and can compared with the theoretical upper bound on the lightest Higgs mass. The maximal tree level SM Higgs mass is given by

$$m_h^2 \approx M_Z^2 \cos^2 2\beta \tag{4.3.3}$$

which vanishes for $\tan\beta = 1$. Radiative corrections will influence this result, but the lightest possible Higgs will remain for $\tan\beta \approx 1$ in the MSSM. For small $\tan\beta$ the LEP SM model Higgs bound is also applicable to the MSSM because the lightest CP-even Higgs boson couples to the Z boson with SM strength [79]. The "$m_h^{\max}$ scenario" in [79] which is designed to maximize the lightest Higgs boson mass for given parameters sets a lower bound of $\tan\beta > 2$. This shows that the naive tree level relation $y_t = g$ is already in mild conflict with experimental data. Taking the mSUGRA scenario and the top quark mass at the upper two sigma bound we obtain $\tan\beta > 3.3$ to be not in conflict with the LEP bound on the Higgs mass. We use this value to constrain the top Yukawa coupling at the GUT scale to be in the range of

$$0.48 < y_t < 0.6, \quad 0.69 < \frac{y_t}{g} < 0.86. \tag{4.3.4}$$

This shows that corrections to $y_t = g$ are welcome to avoid tension with the current experimental data in the MSSM. The second main conclusion is that given a value for $\tan\beta$ and a pattern of soft masses the anisotropy of the extra dimensions R_5/R_6 can be determined.

In figure 4.5 the ratio y_t/g in dependence of the charges and the anisotropy is shown.

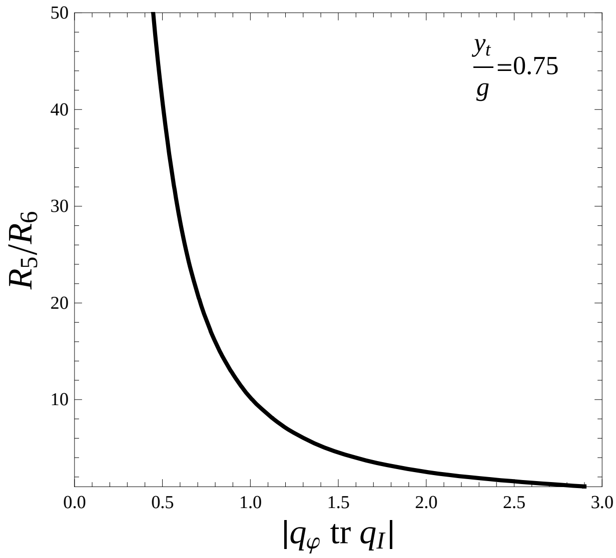

Figure 4.5: The anisotropy R_5/R_6 vs. the charges $q_\varphi \operatorname{tr} q_I$ for a realistic value of $y_t/g = 0.75$ is shown.

4.4 Examples

4.4.1 An example from heterotic orbifold compactifactions

The so-called benchmark model 1A of [8] possesses a six dimensional orbifold GUT. To be in the framework of [74] we consider only U(1) factors which are orthogonal to the

U(1) generated by t_{anom}. It is possible to choose an U(1) generator which contains then all local Fayet-Iliopoulos D-terms. The generator is given by

$$t = \frac{1}{\sqrt{105}} \left(-\frac{3}{2}, 0, 0, 0, 0, 0, 0, 0\right) \left(-\frac{11}{4}, \frac{21}{4}, 0, \frac{11}{4}, 0, \frac{11}{4}, 0, 0\right). \tag{4.4.1}$$

The normalization is chosen to be

$$|t_{\text{anom}}|^2 = \frac{1}{2}. \tag{4.4.2}$$

The charges of φ and φ^c are

$$q_\varphi = -q_{\varphi^c} = \frac{1}{\sqrt{105}} \frac{3}{4}. \tag{4.4.3}$$

We find localized Fayet-Iliopoulos D-terms because

$$\operatorname{tr} q_{(0,0)} = \operatorname{tr} q_{(0,1)} = \frac{32}{\sqrt{105}}, \quad \operatorname{tr} q_{(1,0)} = \operatorname{tr} q_{(1,1)} = -\frac{32}{\sqrt{105}}. \tag{4.4.4}$$

The resulting suppression of the top Yukawa coupling can be seen in figure 4.6. The

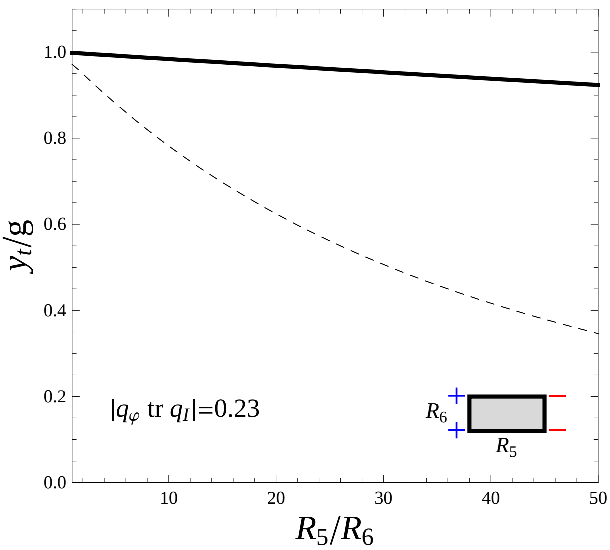

Figure 4.6: Suppression of the top Yukawa coupling y_t relative to the gauge coupling g in the so-called benchmark model 1A.

dashed line in the figure shows the influence of an increased cut-off Λ. The solid line is the six dimensional Planck scale as before, whereas the dashed line is given by an increased cut-off

$$\Lambda^2 = 4 \frac{M_P}{\sqrt{V_{56}}} \tag{4.4.5}$$

which is inspired by [80]. This shows another uncertainty of our approach. We want to remark that the suppression of the top Yukawa coupling in this model is only valid in the GUT limit with $R_5 > R_6$ and not in the limit $R_6 > R_5$ because of the sign of the localized $\operatorname{tr} q_I$. This consequence can also be seen in the gauge structure depicted in figure 4.1.

The comparison with the experimental Higgs boson search and the conclusion in equation 4.3.4 shows that without choosing

$$\frac{R_5}{R_6} > 50 \tag{4.4.6}$$

this model is in strong tension with the experimental situation. For a larger cut-off the situation rapidly changes and a slight anisotropy is favored.

4.4.2 A scan over several heterotic orbifold compactifications

A large subclass of 52 models from [8, 40] exhibit a six dimensional orbifold GUT and localized Fayet-Iliopoulos D-terms. The family structure is the same in every model and agrees also with the so-called benchmark model 1A discussed in section 4.4.1.

- The first and second SM family are located at the two fixed points with $n_2 = 0$.
- The first and second family originate from complete **16**-plets.
- The third family and the Higgs fields h_u and h_d are free to propagate everywhere in the six dimensional orbifold GUT.

The statistic of the different values for $q_\varphi \operatorname{tr} q_I$ can be found in figure 4.7. The values

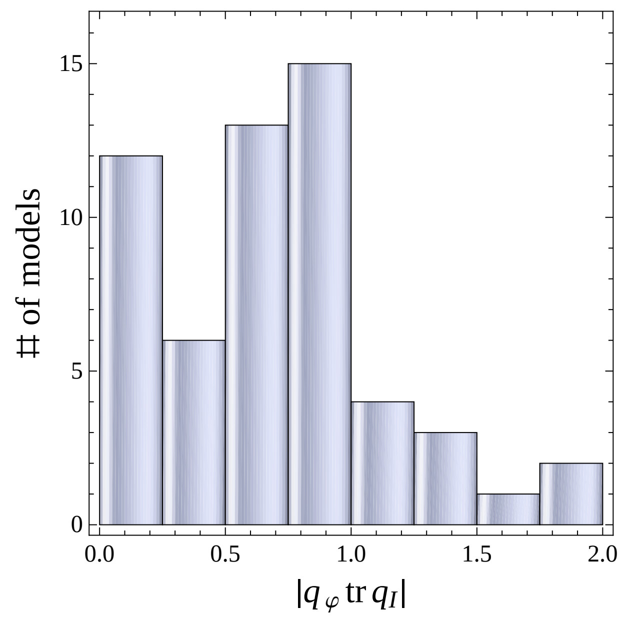

Figure 4.7: Statistic of $q_\varphi \operatorname{tr} q_I$ for 52 models of [40].

are distributed only in a very small range $q_\varphi \operatorname{tr} q_I < 2$. As outlined in section 4.4.1, a small value for $q_\varphi \operatorname{tr} q_I$ is in tension with the experimental data. The suppression for y_t for the mean value $q_\varphi \operatorname{tr} q_I = 0.75$ is shown in figure 4.8. We find a suitable value for y_t consistent with the current experimental situation for a light anisotropy $R_5/R_6 \approx 15$.

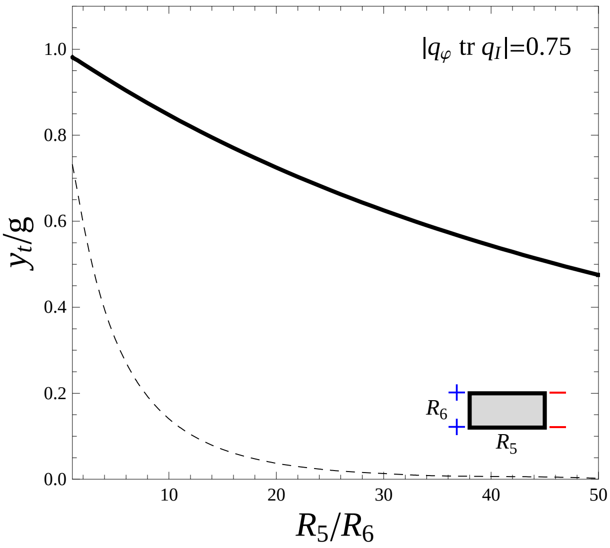

Figure 4.8: Suppression of y_t for the mean value $q_\varphi \operatorname{tr} q_I = 0.75$ of the 52 models under consideration.

4.5 A generalization

In this section we generalize [23] to the case

$$\sum_I \xi_I \neq 0. \tag{4.5.1}$$

We find the relevant condition for unbroken supersymmetry to be [74]

$$\langle D_3 \rangle = \langle F_{56} \rangle = \sum_I \left(\xi_I + g q_I |\langle \phi_I \rangle|^2 \right) \frac{4}{R_5^2} \delta^2(z - z_I) \tag{4.5.2}$$

if we neglect again the logarithmic divergent piece of the Fayet-Iliopoulos D-term and assume that the bulk fields do not break the U(1) symmetry. Repeating the analysis of [74] we find the disturbed wave function

$$\varphi \approx \mathcal{N} \prod_I \left| \vartheta_1 \left(\frac{z - z_I}{2\pi} \middle| \tau \right) \right|^{\frac{1}{2\pi} g_6 q_\varphi \aleph_I} \exp\left(-\frac{1}{8\pi^2 \tau_2} g_6 q_\varphi \aleph_I (\operatorname{Im}(z - z_I))^2 \right) \tag{4.5.3}$$

with

$$\aleph_I = \left(\xi_I + g q_I |\langle \phi_I \rangle|^2 \right). \tag{4.5.4}$$

The difference to the result in equation 4.2.8 depends on the values for the VEVs $\langle\phi_I\rangle$. These VEVs are given by the corresponding F-term equations and are therefore model dependent. Adopting the result for the improved φ to the phenomenological discussion and the results on the anisotropy R_5/R_6 in section 4.4 will affect the drawn conclusions. A more detailed study is therefore desirable, but beyond the scope of this work.

5 Yukawa couplings

Yukawa couplings in the MSSM are free parameters. They are a priori undetermined and the theory is lacking of an explanation of the coupling structure. There have been many attempts to explain the structure of coupling coefficients as well as their precise values with the localization of particles in extra dimensions (see for example [81]). This approach exhibits the beauty that the four visible dimensions determine the interaction of gravity, whereas the extra dimensions would determine the particle physics interactions.

We will focus in this section on heterotic orbifold compactifications. The advantage in contrast to approaches like [81] is that there are well defined rules how to compute the coupling strengths between particles localized in the extra dimensions. On the other hand, we are also able to focus on concrete models where the localization of the particles is determined by consistency constraints which lowers the number of free parameters.

5.1 Yukawa couplings in heterotic orbifold compactifications

Yukawa couplings in heterotic orbifold compactifications can be the consequence of the interaction between so-called twisted and untwisted strings. We will introduce the difference between twisted and untwisted strings first. After that, we discuss interactions of untwisted strings in heterotic string theory. We show how the top Yukawa coupling is calculated in heterotic string theory as a simple example. Finally we focus on the interactions between twisted strings which are the most relevant ones for phenomenology.

5.1.1 Twisted and untwisted strings

We have to distinguish between two different kinds of strings in heterotic string theory, twisted and untwisted strings. In field theories on orbifolds there are four dimensional sectors at the fixed points, sometimes called branes (these have nothing to do with D-branes in type II string theories) and a higher dimensional sector, called bulk [82]. How to couple these different sectors consistently in field theory, especially in supergravity is under discussion [83].

In heterotic orbifold compactifications this is well understood [84, 85]. States at the fixed points originate from twisted strings and states in the bulk from untwisted strings. Both live in different Hilbert spaces and the coupling between each other is well defined. Their origin is illustrated in figure 5.1. In heterotic string theory there exist only so-called closed strings. Every string has to form a loop as the solid black string in figure 5.1. Different points on an orbifold are identified with each other through a discrete

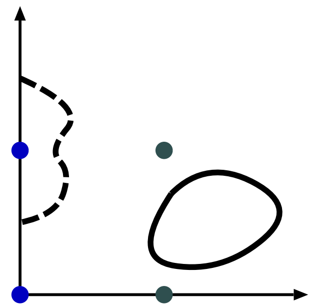

Figure 5.1: Two strings in a SO(4) torus. The solid string is a closed string on the torus and can give rise to a bulk state. The dashed string is only a closed string on the orbifold and is not allowed to move away from the fixed point.

symmetry. This makes it possible that also strings like the dashed one in figure 5.1 are closed strings on the orbifold. The discrete symmetry identifies the start point with the end point of the string. The string is not closed on the torus, but closed on the orbifold. As a consequence, the string is no longer allowed to move freely in the bulk. It is attached to the fixed point. Strings with this behavior are called twisted strings and result in the low energy theory in particles at the corresponding fixed point. There are also strings which are only closed on the torus, but not on the plane. These are so-called winding modes and have no analog in field theory. They are important for the interaction of twisted strings as we will see later.

We can rephrase the above statements in a mathematical way. A two dimensional torus $\mathbb{T}^2$ can be parameterized with two real coordinates x_5 and x_6. We find

$$x_5 = x_5 + 2\pi R_5, \qquad x_6 = x_6 + 2\pi R_6 \tag{5.1.1}$$

in the case of an rectangular torus with two periodicities $2\pi R_5$ and $2\pi R_6$. It is possible to extend this also to non-rectangular tori. Let us further introduce a discrete $\mathbb{Z}_2$ symmetry. Its generator θ acts as

$$\theta(x_5, x_6) = (-x_5, -x_6). \tag{5.1.2}$$

We can calculate the fixed points of this geometry and obtain as solution to the fixed point equations

$$x_5 = -x_5 + 2\pi R_5 n, \qquad x_6 = -x_6 + 2\pi R_6 m, \qquad n, m \in \mathbb{Z} \tag{5.1.3}$$

the four solutions

$$\zeta_1 = (0, 0), \quad \zeta_2 = (0, \pi R_6), \quad \zeta_3 = (\pi R_5, 0), \quad \zeta_4 = (\pi R_5, \pi R_6). \tag{5.1.4}$$

This is a description of the geometry in real coordinates. The same geometry has been discussed in complex coordinates in section 4. We want to define strings on this background. Let us define a bosonic string by $X(\sigma, \tau)$. σ parameterizes the position on the string, which is not a point particle but an extended object. Like a point particle, the

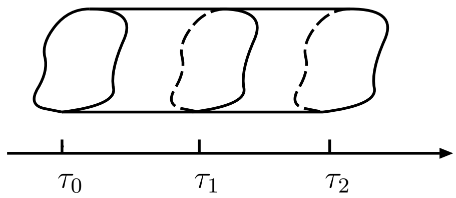

Figure 5.2: A closed string at three different times τ_0, τ_1 and τ_2. In the coordinate system (σ, τ), a freely moving closed string is forming a tube.

string is also moving in time (as depicted in figure 5.2), which makes it necessary to introduce another coordinate τ. The condition that the string is closed can be formulated as

$$X(\sigma + 2\pi, \tau) = X(\sigma, \tau). \tag{5.1.5}$$

The string is 2π periodic. A freely moving closed string is forming a tube in the (σ, τ) coordinate system (see also figure 5.2). This tube can be mathematically described as a Riemann surface and is called string world sheet in analogy to the world line of a point particle. The world sheet is most efficiently described via one complex coordinate z instead of the two real coordinates (σ, τ), because of the powerful mathematics of complex geometry. For that reason, we also use the notation $X(z)$ instead of $X(\sigma, \tau)$.

Let us return to the orbifold. The closed string condition for two real bosons X^5 and X^6 on the torus is given by

$$X^5(\sigma + 2\pi, \tau) = X^5(\sigma, \tau) + 2\pi R_5 n, \quad X^6(\sigma + 2\pi, \tau) = X^6(\sigma, \tau) + 2\pi R_6 m, \quad n, m \in \mathbb{Z}. \tag{5.1.6}$$

We see that it is natural to combine two real bosons X^5 and X^6 to a complex boson. Strings satisfying this condition are untwisted strings if $n = m = 0$ and otherwise winding modes. If the string is only closed on the orbifold we find

$$X^5(\sigma + 2\pi, \tau) = -X^5(\sigma, \tau) + 2\pi R_5 n, \quad X^6(\sigma + 2\pi, \tau) = -X^6(\sigma, \tau) + 2\pi R_6 m, \quad n, m \in \mathbb{Z}. \tag{5.1.7}$$

We want to comment that these strings are necessary, because otherwise modular invariance of the theory is broken [14, 15]. Modular invariance guarantees the absence of ultraviolet divergences, which makes it an essential symmetry of the theory.

We also see that twisted strings which are only closed on the orbifold are not able to move freely in the extra dimensional space but are attracted to the fixed point.

5.1.2 Interactions of untwisted strings

The interaction strength of untwisted strings can be obtained from a world sheet calculation in heterotic orbifold compactifications. We start with bulk states, which originate from untwisted strings. There is a vertex operator V_{Φ_i} for every bulk state Φ_i on the world sheet. This results from the so-called state-operator correspondence in conformal field theory (CFT) [86]. The interaction of several states Φ_i can be calculated on the world sheet using CFT. We illustrate the situation first graphically in various figures.

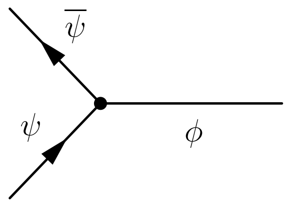

Figure 5.3: Yukawa coupling of two fermions ψ and a boson ϕ.

In figure 5.3 we depicted the usual interaction term in the effective theory. By matching, this term is related to the scattering of three strings of the fundamental theory in figure 5.4. We see in this figure that two strings on the left hand side are joining to one

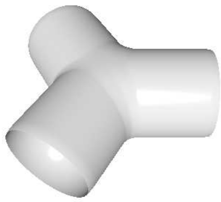

Figure 5.4: Interaction of three closed strings.

string on the right side if we interpret the timeline as in figure 5.2. We can interpret the three strings as external states. These external states are described by the vertex operators V_{Φ_i} which exist on the world sheet (see figure 5.5). For tree level amplitudes

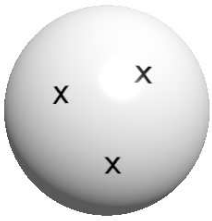

Figure 5.5: Every state is described by a cross $\times$ representing a vertex operator V_{Φ_i} on the world sheet.

the world sheet is always represented by a sphere. The two world sheets in figure 5.4 and figure 5.5 are topologically equivalent. The crosses in figure 5.5 represent punctures of the sphere. A one loop amplitude for example is always topologically equivalent to a torus. Higher genus amplitudes are described for example in [85]. The scattering amplitude of three strings can be written as

$$\mathcal{A} = \langle V_{\Phi_1} V_{\Phi_2} V_{\Phi_3} \rangle \tag{5.1.8}$$

where the brackets denote a CFT calculation. This calculation involves in principle world sheet integrals over the positions of the vertex operators V_{Φ_i}. Every vertex operator has a position z_i on the world sheet. This position is not fixed, because all positions are topologically equivalent. Conformal symmetry which manifests itself as a $\mathrm{PSL}(2,\mathbb{C})$ invariance of the world sheet allows to fix three vertex operator positions z_1, z_2 and z_3. This feature simplifies the case of three point interactions, because in this case all vertex operator positions on the world sheet are fixed and no integration is necessary. For the determination of N-point couplings there remain $N-3$ world sheet integrations. We can also give the general formula in that case

$$\mathcal{A} = \int d^2 z_4 \ldots \int d^2 z_N \, \langle V_{\Phi_1} \cdots V_{\Phi_N} \rangle \, . \tag{5.1.9}$$

In nearly all cases of interest it is not possible to solve the integrations over the world sheet analytically. We will see an example of an approximation in appendix A.1. We will further give an example for the simpler calculation of three untwisted strings in the next section.

5.1.3 Another look at the top Yukawa coupling

We showed in section 4 that in a six dimensional orbifold GUT the tree level relation $y_t = g$ can hold if the Higgs field is part of the higher dimensional gauge multiplet. We will recover this result in this section in heterotic orbifold compactifications. We will proceed along the lines of [87–89] and refer to [90, 91] for an introduction in the used techniques. We have to calculate

$$\mathcal{A}_u^{33} = \left\langle V_{q_3} V_{\bar{u}_3} V_{\bar{\phi}_1} \right\rangle \tag{5.1.10}$$

where we assume that all fields are bulk fields as in section 4. q_3 and $\bar{u}_3$ are the up quarks and $\bar{\phi}_1$ labels the up type Higgs field. The notation is the same as in [8]. This is an interaction of three untwisted strings as depicted in figure 5.4. The vertex operators are given by

$$V_{q_3}(z) = c(z)\bar{c}(\bar{z}) e^{-\frac{\phi(z)}{2}} \cdot u_\alpha S^\alpha(z) \cdot e^{i q_{q_3} \cdot H(z)} \cdot e^{i P_{q_3} \cdot Z(\bar{z})} \tag{5.1.11}$$

$$V_{\bar{u}_3} = c(z)\bar{c}(\bar{z}) e^{-\frac{\phi(z)}{2}} \cdot u_\beta S^\beta(z) \cdot e^{i q_{\bar{u}_3} \cdot H(z)} \cdot e^{i P_{\bar{u}_3} \cdot Z(\bar{z})} \tag{5.1.12}$$

and

$$V_{\bar{\phi}_1}(z) = c(z)\bar{c}(\bar{z}) e^{-\phi(z)} \cdot \mathbb{1} \cdot e^{i q_{\bar{\phi}_1} \cdot H(z)} \cdot e^{i P_{\bar{\phi}_1} \cdot Z(\bar{z})} \tag{5.1.13}$$

for the Higgs field vertex operator. We have separated each individual part by $\cdot$. The different parts are all referring to a separate CFT on the world sheet.

Let us briefly review the ingredients of the heterotic string. We have a ten dimensional supersymmetric sector consisting of five complex bosons and fermions

$$X^\mu(z,\bar{z}), \quad \psi^\mu(z), \quad \mu = 1, \ldots, 5. \tag{5.1.14}$$

The five complex fermions $\psi^\mu(z)$ will be bosonized to $H^\mu(z)$. In addition there is a sixteen dimensional gauge sector, described by eight complex bosons $Z^I(\bar{z}),\ I = 1, \ldots, 8$. These bosons have to live on a sixteen dimensional, even, self-dual lattice. For example the $\mathrm{E}_8 \times \mathrm{E}_8$ root lattice. Without this restriction, the theory would no longer be modular invariant and thus ultraviolet divergences would occur [92, p. 165].

We have in total a 26 dimensional anti-holomorphic sector consisting of $X^\mu(\bar{z})$ and $Z^I(\bar{z})$ forming a 26 dimensional bosonic string theory. There is also a ten dimensional superstring theory consisting of $X^\mu(z)$ and $\psi^\mu(z)$. The six dimensional internal part of this theory is forming a superconformal CFT by its own [93]. The splitting of $X^\mu(z, \bar{z})$ into an holomorphic and an anti-holomorphic part is always possible for tree level amplitudes [87].

There are two additional ghost systems in the usual covariant quantization. The so-called bc ghosts and the superconformal $\beta\gamma$ ghosts [91]. If one prefers the modern BRST approach, these ghosts are automatically part of the theory.

The first part of the vertex operators reflects the ghost contribution. The superconformal ghost system $\beta\gamma$ has been bosonized as usual (see for example [94] or [91, p. 17]). That means, we have introduced the free boson $\phi(z)$ on the world sheet to include this part of the theory. From the bc system we only have to deal with the c ghosts in the case of tree level amplitudes. A more detailed analysis of these systems can be found in [90, 91]. The second part, $u_\alpha S^\alpha(z)$ reflects the fermionic structure and is the four dimensional part of the ten dimensional fermionic CFT of the full theory. In other words, it is the part coming from two complex fermions $\psi^1(z)$ and $\psi^2(z)$. The scalar vertex operator of the Higgs field behaves as identity in this CFT. The internal part of the fermionic contribution was already bosonized to

$$e^{iq_i \cdot H(z)} \tag{5.1.15}$$

where $H(z)$ is a bosonic field and q_i is a three dimensional vector reflecting the six dimensional internal part of the so-called H-momentum (for details see for example [41, 95]). To be more concrete, we find for every complex fermion $\psi^3(z)$, $\psi^4(z)$ and $\psi^5(z)$ of the internal dimensions a complex bosonic partner $H^\mu(z)$ with $\mu = 3, 4, 5$.

The gauge degrees of freedom are encoded in $Z^I(\bar{z})$, which are eight complex bosonic fields on the sixteen dimensional $\mathrm{E}_8 \times \mathrm{E}_8$ root lattice. The spacetime bosonic part of the vertex operator, depending on $X^\mu(z, \bar{z})$ was neglected because we only will deal with contact interactions where the spacetime momentum can be set to zero.

As already outlined, the string theory amplitude 5.1.10 splits up into several pieces. Each part is reflecting an independent CFT. In every CFT we have to calculate the corresponding correlation functions. We will not show how to calculate all these correlators, but review all necessary correlation functions. For the c ghost correlators we find [90, p. 176]

$$\langle c(z_1)\bar{c}(\bar{z}_1)c(z_2)\bar{c}(\bar{z}_2)c(z_3)\bar{c}(\bar{z}_3)\rangle = |z_{12}|^2|z_{13}|^2|z_{23}|^2 \tag{5.1.16}$$

with the shortcut $z_{ij} = z_i - z_j$. The $\beta\gamma$ system gives after bosonization [94, 96]

$$\left\langle e^{-\frac{\phi(z_1)}{2}} e^{-\frac{\phi(z_2)}{2}} e^{-\phi(z_3)} \right\rangle = z_{12}^{-1/4} z_{13}^{-1/2} z_{23}^{-1/2}. \tag{5.1.17}$$

For the two spinors $S^\alpha(z)$ the amplitude results in [87, 93]

$$\left\langle S^\alpha(z_1) S^\beta(z_2) \right\rangle = C^{\alpha\beta} z_{12}^{-1/2} \tag{5.1.18}$$

where $C^{\alpha\beta}$ is a four dimensional charge conjugation matrix. This result is achieved by bosonization [93, 97]

$$S^\alpha(z) = e^{i\alpha\cdot H(z)}, \qquad S^{\dot\beta}(z) = e^{i\dot\beta\cdot H(z)}, \qquad \alpha = \left(\pm\frac{1}{2}, \pm\frac{1}{2}\right), \qquad \dot\beta = \left(\pm\frac{1}{2}, \mp\frac{1}{2}\right). \tag{5.1.19}$$

We only have to consider the first two entries of $H(z)$, namely $H^1(z)$ and $H^2(z)$. For two spinors of opposite chirality we find with the same technique [93]

$$\left\langle S^\alpha(z_1) S^{\dot\beta}(z_2)\right\rangle = 0. \tag{5.1.20}$$

The internal part of the fermionic sector gives [87]

$$\left\langle e^{iq_{q_3}\cdot H(z_1)} e^{iq_{\bar u_3}\cdot H(z_1)} e^{iq_{\bar\phi_1}\cdot H(z_1)}\right\rangle = z_{12}^{-1/4} z_{13}^{-1/2} z_{23}^{-1/2}. \tag{5.1.21}$$

This results from the masslessness of the involved particles [87] or can be obtained in a concrete example. Let us focus for the moment on the model 1A of the so-called minilandscape [8]. The R charges of the involved particles q_3, $\bar u_3$ and $\bar\phi_1$ are with $R_i = (R_1, R_2, R_3)^T$

$$R_{q_3} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \qquad R_{\bar u_3} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \qquad R_{\bar\phi_1} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}. \tag{5.1.22}$$

The internal H-momentum for the fermions is shifted by $(1/2, 1/2, 1/2)^T$ [41] and the H-momentum for the scalar $\bar\phi_1$ is just the R-charge which means

$$q_{q_3} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \qquad q_{\bar u_3} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \qquad q_{\bar\phi_1} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}. \tag{5.1.23}$$

This confirms the result in equation 5.1.21 in a concrete example. We can also consider the gauge degrees of freedom $Z^I(\bar z)$ and find [87, 88]

$$\left\langle e^{iP_{q_3}\cdot Z(\bar z_1)} e^{iP_{\bar u_3}\cdot Z(\bar z_2)} e^{iP_{\bar\phi_1}\cdot Z(\bar z_3)}\right\rangle = \bar z_{12}^{-1} \bar z_{13}^{-1} \bar z_{23}^{-1}. \tag{5.1.24}$$

The momentum vectors in this case are the sixteen dimensional $E_8 \times E_8$ root lattice vectors. This relation can also be computed in a concrete example with the gauge vectors P_i from [8]. We are ready to put all correlators together

$$\mathcal{A}_u^{33} = \left\langle V_{q_3} V_{\bar u_3} V_{\bar\phi_1}\right\rangle = u_\alpha u_\beta C^{\alpha\beta}. \tag{5.1.25}$$

As expected, all z_{ij} dependent pieces exactly cancel each other. That shows that it is due to the PSL$(2,\mathbb{C})$ invariance completely irrelevant which values we choose for z_1, z_2 and z_3. The overall normalization can be obtained with two different methods. It is possible to determine it by matching with the usual low energy Lagrangian of the MSSM or by use of the unitarity of the four point gauge boson amplitude. As outlined in [98], we can relate the string coupling g_s to the four dimensional gauge coupling

$$g = \sqrt{2} g_s. \tag{5.1.26}$$

The effective three point amplitude has to be proportional to $\sqrt{2} g_s$ [98] which shows that our string theory result is in agreement with the field theory reduction from section 4 (see also [87] for a more careful analysis). We have briefly reviewed how couplings in the low energy theory arise from the interaction of strings. The interaction of twisted strings is much more involved, because the CFT of twisted strings is more difficult to handle and correlation functions are not as easy to deduce as the ones in this section.

5.1.4 Interactions of twisted strings

Interactions between twisted strings in heterotic orbifold compactifications have been determined first in [84, 85]. These calculations have been extended in [99–105] to more complex examples. Several attempts [106, 107] tried to relate these calculations to phenomenology. We will follow this path and determine the phenomenology resulting from the Yukawa couplings in models on $\mathbb{Z}_{6-\mathrm{II}}$ orbifolds [8].

We want to briefly discuss how interactions between twisted strings are calculable first. Before we discuss the CFT of twisted strings and the corresponding vertex operators we start with a small illustration. How can strings from different twisted sectors interact at all with each other? We have depicted an example from [84] in figure 5.6. There are two

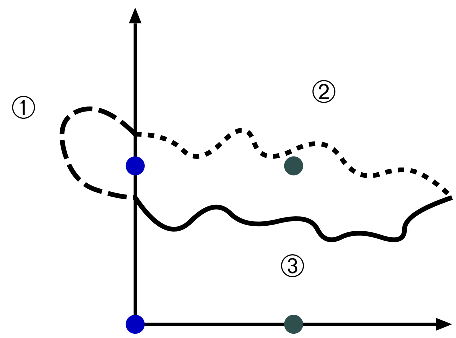

Figure 5.6: The interaction of three strings. The two twisted strings (① dashed and ② dotted) can interact via the exchange of a winding mode (③ solid).

twisted strings ① and ② at two different fixed points. They interact with each other via the exchange of a winding mode ③, a string which is only closed on the torus. To use CFT techniques, the vertex operators for twisted strings should be different. We get

$$V_{\Phi_i} = V_{\Phi_i} \cdot \mathbb{1} \quad \text{(untwisted state)} \tag{5.1.27}$$

for an untwisted state and

$$V_{\Phi_i}^{(f)} = V_{\Phi_i} \cdot \sigma_f \quad \text{(twisted state at fixed point } f\text{)} \tag{5.1.28}$$

for a twisted state at the fixed point labeled by f. The operator σ_f acts in the CFT of the bosons X^μ. In our examples so far, there have not been a contribution from the X^μ CFT, which means that the σ_f form their own CFT.

Let us give some technical details. Oscillator states of the theory are not covered with this simplified approach. They correspond to excited twist operators and are given by

$$V_{\Phi_i}^{(f)} = V_{\Phi_i} \cdot \tau_f \quad \text{(twisted oscillator state at fixed point } f\text{)}. \tag{5.1.29}$$

The excited twist operators τ_f include a bosonic part [85]

$$\partial X(z_1)\sigma_f(z_2) \sim z_{12}^{-1/2}\tau_f(z_2) + \dots \tag{5.1.30}$$

where the dots denote a non-singular part and with $\sim$ we label the operator product expansion (OPE) [86]. We will not further consider such correlators which are difficult to compute (see for example [108]).

The scattering amplitude for three twisted states without excited twist operators is given by

$$\mathcal{A} = \left\langle V_{\Phi_1}^{(f_1)} V_{\Phi_2}^{(f_2)} V_{\Phi_3}^{(f_3)} \right\rangle = \langle V_{\Phi_1} V_{\Phi_2} V_{\Phi_3} \rangle \cdot \langle \sigma_{f_1}\sigma_{f_2}\sigma_{f_3} \rangle \,. \tag{5.1.31}$$

The only location dependent piece of this correlators is covered in the $\langle \sigma_{f_1}\sigma_{f_2}\sigma_{f_3} \rangle$ term. This term alone is responsible for the influence of the location of the states. We will focus on such correlators later in concrete examples and in appendix A.1 and A.2. Detailed discussions of such correlators can be found in [84, 85].

5.2 Yukawa couplings in the minilandscape model 1A

In this section we will discuss the Yukawa couplings of a complete string theory model in detail. We focus on model 1A from [8]. We start with a brief discussion of the geometry, focus afterwards on the up quark and down quark mass hierarchies and discuss the influence of radiative effects.

5.2.1 Geometry of the minilandscape models

The concrete values of the Yukawa couplings in heterotic orbifold compactifications depend strongly on the internal, extra dimensional geometry. The minilandscape models base on a $\mathbb{Z}_{6-\mathrm{II}}$ orbifold. The six extra dimensions factorize in three two dimensional tori

$$\mathbb{T}^6 = \mathbb{T}^2_{\mathrm{G}_2} \times \mathbb{T}^2_{\mathrm{SU(3)}} \times \mathbb{T}^2_{\mathrm{SO(4)}}. \tag{5.2.1}$$

Each torus is described by a root lattice and an orbifold point group. There is a $\mathbb{Z}_2$ orbifold twist in the SO(4) torus, a $\mathbb{Z}_3$ twist in the SU(3) torus and a $\mathbb{Z}_6$ twist in the G_2 torus. Each torus can be treated separately. A sketch of the extra dimensions is displayed in figure 5.7, 5.8 and 5.9. The SO(4) torus is depicted with the gauge structure in figure 4.1. For a more detailed description of the geometry we refer to [41].

We want to remind that the six dimensional orbifold GUT used in section 4, is the six

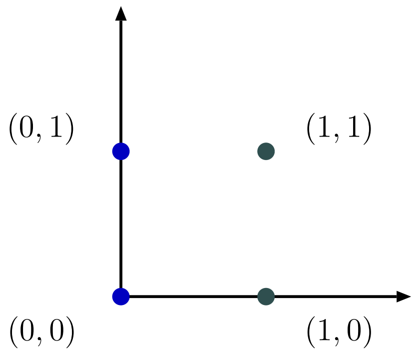

Figure 5.7: Sketch of the SO(4) torus. The labels of the fixed point correspond to (n_2, n_2').

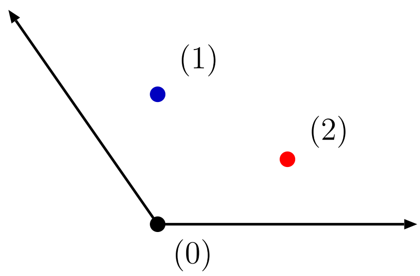

Figure 5.8: Sketch of the SU(3) torus. The fixed points are labeled with (n_3).

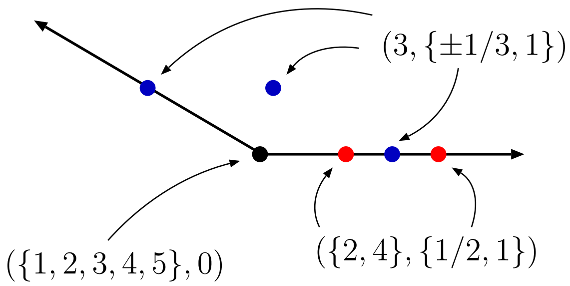

Figure 5.9: Sketch of the G_2 torus. The labels are (k, q_γ). A complete description of the complex G_2 geometry can be found in [41].

dimensional intermediate theory of this construction [70]. The ten dimensional space G_{10} needed for a consistent description in the framework of the heterotic string is

$$G_{10} = \underbrace{\mathcal{M}_4 \times \mathbb{T}^2_{\mathrm{SO}(4)}}_{\text{intermediate six dimensional GUT}} \times \mathbb{T}^2_{\mathrm{G}_2} \times \mathbb{T}^2_{\mathrm{SU}(3)}. \tag{5.2.2}$$

To obtain the right GUT scale, the compactification of the six extra dimensions has to be anisotropic [70, 80]. The SO(4) torus has to be larger than the other extra dimensions. Effects in the SO(4) torus are thus more relevant for the physical interactions at the electroweak scale.

There are several fixed points in each torus where twisted states are located. There are also various states which live in the bulk of one torus and at the fixed point in another torus. These states are also twisted states, but are twisted for example only in one two dimensional torus. The localization in the six dimensional internal space can be represented by five quantum numbers $(k, n_3, n_2, n_2', q_\gamma)$ [41]. An alternative description can be found in [109]. k and q_γ display the localization in the G_2 torus, n_3 labels the localization in the SU(3) torus and n_2 and n_2' mark the localization in the SO(4) torus. A star $(*)$ denotes no localization in the corresponding torus and thus refers to a bulk state in this torus. For example $n_3 = *$ means that the state is not localized in the SU(3) torus. Nevertheless it can be a twisted state in one of the other tori. An untwisted state is a state which is a complete bulk state and has the quantum number $k = 0$. There exist no twisted states which are not localized in the G_2 torus. Every state which is not localized in the G_2 torus is an untwisted state.

The factorization in three two dimensional tori has also consequences for the twist operators. We can factorize

$$\sigma_{(k,n_3,n_2,n_2',q_\gamma)} = \underbrace{\sigma_{(k,q_\gamma)}}_{\mathbb{T}^2_{\mathrm{G}_2}} \times \underbrace{\sigma_{(n_3)}}_{\mathbb{T}^2_{\mathrm{SU}(3)}} \times \underbrace{\sigma_{(n_2,n_2')}}_{\mathbb{T}^2_{\mathrm{SO}(4)}} . \tag{5.2.3}$$

We further find

$$\sigma_{(n_3=*)} = \mathbb{1}, \qquad \sigma_{(n_2=*,n_2'=*)} = \mathbb{1}. \tag{5.2.4}$$

5.2.2 The up quark sector

In this section we will discuss the up quark sector of model 1A from [8] in detail. In [8] the up quark matrix of the minilandscape model 1A was given by

$$Y_u = \begin{pmatrix} \tilde{s}^5 & \tilde{s}^5 & \tilde{s}^5 \\ \tilde{s}^5 & \tilde{s}^5 & \tilde{s}^6 \\ \tilde{s}^6 & \tilde{s}^6 & 1 \end{pmatrix} . \tag{5.2.5}$$

The top Yukawa coupling was already discussed in detail in section 4 and 5.1.3. All other couplings are induced by SM singlets $\tilde{s}$, where $\tilde{s}$ stand for several different singlets and the exponent labels the order in SM singlets at which the first couplings occur. In [8] it was shown that the matrix has full rank, but there was no detailed treatment of the different couplings. This is the topic of this section.

We start with the first generation up quark coupling. At order 5 in SM singlets we find 6 different couplings. The scattering amplitude is given by

$$\begin{aligned} \mathcal{A}_u^{11} = \int d^2 z_4 \ldots \int d^2 z_N \Big\langle & V_{q_1}^{(1,0,0,1,0)} V_{\bar{u}_1}^{(1,0,0,1,0)} V_{\bar{\phi}_1}^{(0,*,*,*,0)} V_{s_4^0}^{(1,0,0,0,0)} V_{s_5^0}^{(1,0,0,0,0)} V_{s_{26}^0}^{(4,0,*,*,0)} \\ & \Big(V_{h_1}^{(2,0,*,*,1/2)} V_{h_2}^{(2,0,*,*,1/2)} + V_{s_{17}^0}^{(2,0,*,*,1/2)} V_{s_{18}^0}^{(2,0,*,*,1/2)} + V_{s_{20}^0}^{(2,1,*,*,0)} V_{s_{22}^0}^{(2,2,*,*,0)} \\ & + V_{s_{21}^0}^{(2,1,*,*,1)} V_{s_{22}^0}^{(2,2,*,*,0)} + V_{s_{20}^0}^{(2,1,*,*,0)} V_{s_{23}^0}^{(2,2,*,*,1)} + V_{s_{21}^0}^{(2,1,*,*,1)} V_{s_{23}^0}^{(2,2,*,*,1)} \Big) \Big\rangle . \end{aligned} \tag{5.2.6}$$

The lower index labels the name of the particle according to the convention in [8]. The upper index displays the location of the particle in the extra dimensions with the help of the five introduced quantum numbers $(k, n_3, n_2, n_2', q_\gamma)$. To be more concrete, the induced Yukawa coupling Y_u^{11} is given after VEV assignment

$$\begin{aligned} Y_u^{11} = \left\langle s_4^0 \right\rangle \left\langle s_5^0 \right\rangle \left\langle s_{26}^0 \right\rangle \Bigg(& \langle h_1 \rangle \langle h_2 \rangle \int d^2 z_4 \dots \int d^2 z_N \\ & \left\langle V_{q_1}^{(1,0,0,1,0)} V_{\bar{u}_1}^{(1,0,0,1,0)} V_{\bar{\phi}_1}^{(0,*,*,*,0)} V_{s_4^0}^{(1,0,0,0,0)} V_{s_5^0}^{(1,0,0,0,0)} V_{s_{26}^0}^{(4,0,*,*,0)} V_{h_1}^{(2,0,*,*,1/2)} V_{h_2}^{(2,0,*,*,1/2)} \right\rangle \\ & + \left\langle s_{17}^0 \right\rangle \left\langle s_{18}^0 \right\rangle \int d^2 z_4 \dots \int d^2 z_N \\ & \left\langle V_{q_1}^{(1,0,0,1,0)} V_{\bar{u}_1}^{(1,0,0,1,0)} V_{\bar{\phi}_1}^{(0,*,*,*,0)} V_{s_4^0}^{(1,0,0,0,0)} V_{s_5^0}^{(1,0,0,0,0)} V_{s_{26}^0}^{(4,0,*,*,0)} V_{s_{17}^0}^{(2,0,*,*,1/2)} V_{s_{18}^0}^{(2,0,*,*,1/2)} \right\rangle \\ & + \left\langle s_{20}^0 \right\rangle \left\langle s_{22}^0 \right\rangle \int d^2 z_4 \dots \int d^2 z_N \\ & \left\langle V_{q_1}^{(1,0,0,1,0)} V_{\bar{u}_1}^{(1,0,0,1,0)} V_{\bar{\phi}_1}^{(0,*,*,*,0)} V_{s_4^0}^{(1,0,0,0,0)} V_{s_5^0}^{(1,0,0,0,0)} V_{s_{26}^0}^{(4,0,*,*,0)} V_{s_{20}^0}^{(2,1,*,*,0)} V_{s_{22}^0}^{(2,2,*,*,0)} \right\rangle \\ & + \left\langle s_{21}^0 \right\rangle \left\langle s_{22}^0 \right\rangle \int d^2 z_4 \dots \int d^2 z_N \\ & \left\langle V_{q_1}^{(1,0,0,1,0)} V_{\bar{u}_1}^{(1,0,0,1,0)} V_{\bar{\phi}_1}^{(0,*,*,*,0)} V_{s_4^0}^{(1,0,0,0,0)} V_{s_5^0}^{(1,0,0,0,0)} V_{s_{26}^0}^{(4,0,*,*,0)} V_{s_{21}^0}^{(2,1,*,*,1)} V_{s_{22}^0}^{(2,2,*,*,0)} \right\rangle \\ & + \left\langle s_{20}^0 \right\rangle \left\langle s_{23}^0 \right\rangle \int d^2 z_4 \dots \int d^2 z_N \\ & \left\langle V_{q_1}^{(1,0,0,1,0)} V_{\bar{u}_1}^{(1,0,0,1,0)} V_{\bar{\phi}_1}^{(0,*,*,*,0)} V_{s_4^0}^{(1,0,0,0,0)} V_{s_5^0}^{(1,0,0,0,0)} V_{s_{26}^0}^{(4,0,*,*,0)} V_{s_{20}^0}^{(2,1,*,*,0)} V_{s_{23}^0}^{(2,2,*,*,1)} \right\rangle \\ & + \left\langle s_{21}^0 \right\rangle \left\langle s_{23}^0 \right\rangle \int d^2 z_4 \dots \int d^2 z_N \\ & \left\langle V_{q_1}^{(1,0,0,1,0)} V_{\bar{u}_1}^{(1,0,0,1,0)} V_{\bar{\phi}_1}^{(0,*,*,*,0)} V_{s_4^0}^{(1,0,0,0,0)} V_{s_5^0}^{(1,0,0,0,0)} V_{s_{26}^0}^{(4,0,*,*,0)} V_{s_{21}^0}^{(2,1,*,*,1)} V_{s_{23}^0}^{(2,2,*,*,1)} \right\rangle \Bigg) . \end{aligned} \tag{5.2.7}$$

We want to clarify the notation. Whenever we mean the VEV of a field Φ_i, we write $\langle \Phi_i \rangle$. The CFT correlators $\langle V_{\Phi_1} \cdots V_{\Phi_N} \rangle$ is given by the same brackets, but operates on vertex operators V_{Φ_i}. We will see that it is not necessary to compute all correlators in detail. Instead, we will relate them to other entries of the Yukawa matrix Y_u.

Let us introduce the coupling between first and second generation up quarks. We find

$$\begin{aligned} \mathcal{A}_u^{12} = \int d^2 z_4 \dots \int d^2 z_N \Big\langle & V_{q_1}^{(1,0,0,1,0)} V_{\bar{u}_2}^{(1,0,0,0,0)} V_{\bar{\phi}_1}^{(0,*,*,*,0)} V_{s_4^0}^{(1,0,0,0,0)} V_{s_{11}^0}^{(1,0,0,1,0)} V_{s_{26}^0}^{(4,0,*,*,0)} \\ & \Big(V_{h_1}^{(2,0,*,*,1/2)} V_{h_2}^{(2,0,*,*,1/2)} + V_{s_{17}^0}^{(2,0,*,*,1/2)} V_{s_{18}^0}^{(2,0,*,*,1/2)} + V_{s_{20}^0}^{(2,1,*,*,0)} V_{s_{22}^0}^{(2,2,*,*,0)} \\ & + V_{s_{21}^0}^{(2,1,*,*,1)} V_{s_{22}^0}^{(2,2,*,*,0)} + V_{s_{20}^0}^{(2,1,*,*,0)} V_{s_{23}^0}^{(2,2,*,*,1)} + V_{s_{21}^0}^{(2,1,*,*,1)} V_{s_{23}^0}^{(2,2,*,*,1)} \Big) \Big\rangle . \end{aligned} \tag{5.2.8}$$

We already see that this amplitude differs only slightly from $\mathcal{A}_u^{11}$. We can also consider the VEVs of the SM singlets and obtain for the coupling

$$
\begin{aligned}
Y_u^{12} = \left\langle s_4^0 \right\rangle \left\langle s_{11}^0 \right\rangle \left\langle s_{26}^0 \right\rangle \Bigg(& \langle h_1 \rangle \langle h_2 \rangle \int d^2 z_4 \ldots \int d^2 z_N \\
& \left\langle V_{q_1}^{(1,0,0,1,0)} V_{\bar{u}_2}^{(1,0,0,0,0)} V_{\bar{\phi}_1}^{(0,*,*,*,0)} V_{s_4^0}^{(1,0,0,0,0)} V_{s_{11}^0}^{(1,0,0,1,0)} V_{s_{26}^0}^{(4,0,*,*,0)} V_{h_1}^{(2,0,*,*,1/2)} V_{h_2}^{(2,0,*,*,1/2)} \right\rangle \\
& + \left\langle s_{17}^0 \right\rangle \left\langle s_{18}^0 \right\rangle \int d^2 z_4 \ldots \int d^2 z_N \\
& \left\langle V_{q_1}^{(1,0,0,1,0)} V_{\bar{u}_2}^{(1,0,0,0,0)} V_{\bar{\phi}_1}^{(0,*,*,*,0)} V_{s_4^0}^{(1,0,0,0,0)} V_{s_{11}^0}^{(1,0,0,1,0)} V_{s_{26}^0}^{(4,0,*,*,0)} V_{s_{17}^0}^{(2,0,*,*,1/2)} V_{s_{18}^0}^{(2,0,*,*,1/2)} \right\rangle \\
& + \left\langle s_{20}^0 \right\rangle \left\langle s_{22}^0 \right\rangle \int d^2 z_4 \ldots \int d^2 z_N \\
& \left\langle V_{q_1}^{(1,0,0,1,0)} V_{\bar{u}_2}^{(1,0,0,0,0)} V_{\bar{\phi}_1}^{(0,*,*,*,0)} V_{s_4^0}^{(1,0,0,0,0)} V_{s_{11}^0}^{(1,0,0,1,0)} V_{s_{26}^0}^{(4,0,*,*,0)} V_{s_{20}^0}^{(2,1,*,*,0)} V_{s_{22}^0}^{(2,2,*,*,0)} \right\rangle \\
& + \left\langle s_{21}^0 \right\rangle \left\langle s_{22}^0 \right\rangle \int d^2 z_4 \ldots \int d^2 z_N \\
& \left\langle V_{q_1}^{(1,0,0,1,0)} V_{\bar{u}_2}^{(1,0,0,0,0)} V_{\bar{\phi}_1}^{(0,*,*,*,0)} V_{s_4^0}^{(1,0,0,0,0)} V_{s_{11}^0}^{(1,0,0,1,0)} V_{s_{26}^0}^{(4,0,*,*,0)} V_{s_{21}^0}^{(2,1,*,*,1)} V_{s_{22}^0}^{(2,2,*,*,0)} \right\rangle \\
& + \left\langle s_{20}^0 \right\rangle \left\langle s_{23}^0 \right\rangle \int d^2 z_4 \ldots \int d^2 z_N \\
& \left\langle V_{q_1}^{(1,0,0,1,0)} V_{\bar{u}_2}^{(1,0,0,0,0)} V_{\bar{\phi}_1}^{(0,*,*,*,0)} V_{s_4^0}^{(1,0,0,0,0)} V_{s_{11}^0}^{(1,0,0,1,0)} V_{s_{26}^0}^{(4,0,*,*,0)} V_{s_{20}^0}^{(2,1,*,*,0)} V_{s_{23}^0}^{(2,2,*,*,1)} \right\rangle \\
& + \left\langle s_{21}^0 \right\rangle \left\langle s_{23}^0 \right\rangle \int d^2 z_4 \ldots \int d^2 z_N \\
& \left\langle V_{q_1}^{(1,0,0,1,0)} V_{\bar{u}_2}^{(1,0,0,0,0)} V_{\bar{\phi}_1}^{(0,*,*,*,0)} V_{s_4^0}^{(1,0,0,0,0)} V_{s_{11}^0}^{(1,0,0,1,0)} V_{s_{26}^0}^{(4,0,*,*,0)} V_{s_{21}^0}^{(2,1,*,*,1)} V_{s_{23}^0}^{(2,2,*,*,1)} \right\rangle \Bigg).
\end{aligned}
\tag{5.2.9}
$$

We recognize that four fields ($\bar{u}_1$, $\bar{u}_2$, s_5^0 and s_{11}^0) differ between Y_u^{11} and Y_u^{12}. For the vertex operators we find

$$
V_{\bar{u}_2}^{(1,0,0,0,0)} V_{s_{11}^0}^{(1,0,0,1,0)} = V_{\bar{u}_2} V_{s_{11}^0} \cdot \sigma_{(1,0,0,0,0)} \sigma_{(1,0,0,1,0)} = V_{\bar{u}_1} V_{s_5^0} \cdot \sigma_{(1,0,0,0,0)} \sigma_{(1,0,0,1,0)} \tag{5.2.10}
$$

$$
= V_{\bar{u}_1}^{(1,0,0,1,0)} V_{s_5^0}^{(1,0,0,0,0)} \tag{5.2.11}
$$

where we used

$$
V_{\bar{u}_1} = V_{\bar{u}_2}, \qquad V_{s_5^0} = V_{s_{11}^0}. \tag{5.2.12}
$$

This is indeed true, because despite their localization property which is encoded in σ_f all quantum numbers of the states match [8]. For example

$$
q_{\bar{u}_1} = q_{\bar{u}_2}, \qquad P_{\bar{u}_1} = P_{\bar{u}_2}. \tag{5.2.13}
$$

The only fundamental difference between the quark flavors in these models is their different localization in the extra dimensions. This results in

$$
\mathcal{A}_u^{12} = \mathcal{A}_u^{11} \tag{5.2.14}
$$

and on the level of the induced Yukawa couplings we obtain

$$Y_u^{12} = \frac{\langle s_{11}^0 \rangle}{\langle s_5^0 \rangle} Y_u^{11}. \tag{5.2.15}$$

The singlet VEVs determine the ratio between the diagonal and the off-diagonal entry of the up quark Yukawa matrix. This is an appealing result, because it was not necessary to calculate all involved correlators of the different CFTs in detail.

We further find

$$Y_u^{21} = Y_u^{12}. \tag{5.2.16}$$

The reason for this symmetric behavior is a D_4 symmetry [9]. For the coupling of the second generation up quarks we get

$$\begin{aligned}\mathcal{A}_u^{22} = \int d^2 z_4 \ldots \int d^2 z_N \Big\langle & V_{q_2}^{(1,0,0,0,0)} V_{\bar{u}_2}^{(1,0,0,0,0)} V_{\bar{\phi}_1}^{(0,*,*,*,0)} V_{s_4^0}^{(1,0,0,0,0)} V_{s_5^0}^{(1,0,0,0,0)} V_{s_{26}^0}^{(4,0,*,*,0)} \\ & \Big(V_{h_1}^{(2,0,*,*,1/2)} V_{h_2}^{(2,0,*,*,1/2)} + V_{s_{17}^0}^{(2,0,*,*,1/2)} V_{s_{18}^0}^{(2,0,*,*,1/2)} + V_{s_{20}^0}^{(2,1,*,*,0)} V_{s_{22}^0}^{(2,2,*,*,0)} \\ & + V_{s_{21}^0}^{(2,1,*,*,1)} V_{s_{22}^0}^{(2,2,*,*,0)} + V_{s_{20}^0}^{(2,1,*,*,0)} V_{s_{23}^0}^{(2,2,*,*,1)} + V_{s_{21}^0}^{(2,1,*,*,1)} V_{s_{23}^0}^{(2,2,*,*,1)} \Big) \Big\rangle .\end{aligned} \tag{5.2.17}$$

In contrast to the first generation amplitude $\mathcal{A}_u^{11}$ the location dependent part is different. The difference is only in the SO(4) torus and we find

$$\left\langle \sigma_{(0,0)} \sigma_{(0,0)} \sigma_{(0,0)} \sigma_{(0,0)} \right\rangle \tag{5.2.18}$$

as part of the amplitude $\mathcal{A}_u^{22}$ in contrast to

$$\left\langle \sigma_{(0,1)} \sigma_{(0,1)} \sigma_{(0,0)} \sigma_{(0,0)} \right\rangle \tag{5.2.19}$$

as part of the amplitude $\mathcal{A}_u^{11}$. As shown in appendix A.1, it is possible to find an approximate relation between $\mathcal{A}_u^{11}$ and $\mathcal{A}_u^{22}$. There is

$$\left\langle \sigma_{(0,0)} \sigma_{(0,0)} \sigma_{(0,0)} \sigma_{(0,0)} \right\rangle \approx 1, \quad \left\langle \sigma_{(0,1)} \sigma_{(0,1)} \sigma_{(0,0)} \sigma_{(0,0)} \right\rangle \approx e^{-\frac{\pi R_6^2}{4\,\mathrm{Im}(z_4)}} \tag{5.2.20}$$

where z_4 is an undetermined vertex operator position on the world sheet. An integral approximation makes it possible to obtain

$$\mathcal{A}_u^{11} = e^{-\frac{\pi R_6^2}{4}} \mathcal{A}_u^{22} \tag{5.2.21}$$

where πR_6 is the distance between the two fixed points labeled with n_2' in the SO(4) torus.

The third family of quarks is very different from the two other families in model 1A. In contrast to the other two families it is not localized in the SO(4) torus. The up quarks and the Higgs fields are complete bulk fields everywhere and originate from untwisted strings. We find

$$\begin{aligned}\mathcal{A}_u^{13} = &\int d^2z_4 \ldots \int d^2z_N \Big\langle V_{q_1}^{(1,0,0,1,0)} V_{\bar{u}_3}^{(0,*,*,*,0)} V_{\bar{\phi}_1}^{(0,*,*,*,0)} V_{s_4^0}^{(1,0,0,0,0)} V_{s_6^0}^{(1,0,0,0,0)} V_{s_{11}^0}^{(1,0,0,1,0)} \\ &V_{s_{26}^0}^{(4,0,*,*,0)} \Big(V_{h_1}^{(2,0,*,*,1/2)} V_{h_2}^{(2,0,*,*,1/2)} + V_{s_{17}^0}^{(2,0,*,*,1/2)} V_{s_{18}^0}^{(2,0,*,*,1/2)} + V_{s_{20}^0}^{(2,1,*,*,0)} V_{s_{22}^0}^{(2,2,*,*,0)} \\ &+ V_{s_{21}^0}^{(2,1,*,*,1)} V_{s_{22}^0}^{(2,2,*,*,0)} + V_{s_{20}^0}^{(2,1,*,*,0)} V_{s_{23}^0}^{(2,2,*,*,1)} + V_{s_{21}^0}^{(2,1,*,*,1)} V_{s_{23}^0}^{(2,2,*,*,1)} \Big) \Big\rangle \\ &+ \int d^2z_4 \ldots \int d^2z_N \\ &\Big\langle V_{q_1}^{(1,0,0,1,0)} V_{\bar{u}_3}^{(0,*,*,*,0)} V_{\bar{\phi}_1}^{(0,*,*,*,0)} V_{s_{13}^0}^{(1,0,0,1,0)} \Big(V_{s_4^0}^{(1,0,0,0,0)} V_{s_{26}^0}^{(4,0,*,*,0)} \Big)^2 \Big\rangle .\end{aligned} \tag{5.2.22}$$

This amplitude consists of two different parts. The last one is a contribution at order five in SM singlets, containing the field s_{13}^0 which is an oscillator state (see the discussion in section 5.1.4). There is no easy way to relate this coupling to another coupling and we leave it as an open variable. The first contribution can be related to $\mathcal{A}_u^{11}$. Compared to $\mathcal{A}_u^{11}$ the twist operator part of the singlet s_6^0 is playing the role of s_5^0, whereas the location dependent part of s_{11}^0 is playing the role of $\bar{u}_1$. We obtain

$$Y_u^{13} = \frac{\langle s_6^0 \rangle \langle s_{11}^0 \rangle}{\langle s_5^0 \rangle} Y_u^{11} + B(\tilde{s}) \tag{5.2.23}$$

where the factor $B(\tilde{s})$ is the undetermined order five contribution proportional to s_{13}^0. We further find

$$Y_u^{31} = \langle s_9^0 \rangle Y_u^{11} \tag{5.2.24}$$

and

$$Y_u^{23} = \langle s_6^0 \rangle Y_u^{22}, \qquad Y_u^{32} = \frac{\langle s_9^0 \rangle \langle s_{11}^0 \rangle}{\langle s_5^0 \rangle} Y_u^{11}. \tag{5.2.25}$$

We set the top Yukawa coupling to $Y_u^{33} = y_t$ and remind the detailed discussion in section 4 and 5.1.3. The relations between all couplings become more obvious in the matrix notation. Inspired by equation 5.2.21 we introduce

$$Y_u^{11} = f(R_6) A(\tilde{s}), \qquad Y_u^{22} = A(\tilde{s}), \qquad f(R_6) = e^{-\frac{4R_6^2}{4}} \tag{5.2.26}$$

with the known function $f(R_6)$ and the undetermined quantity $A(\tilde{s})$. The complete up quark matrix in these variables is

$$Y_u = \begin{pmatrix} f(R_6)A(\tilde{s}) & \frac{\langle s_{11}^0 \rangle}{\langle s_5^0 \rangle} f(R_6)A(\tilde{s}) & \frac{\langle s_6^0 \rangle \langle s_{11}^0 \rangle}{\langle s_5^0 \rangle} f(R_6)A(\tilde{s}) + B(\tilde{s}) \\ \frac{\langle s_{11}^0 \rangle}{\langle s_5^0 \rangle} f(R_6)A(\tilde{s}) & A(\tilde{s}) & \langle s_6^0 \rangle A(\tilde{s}) \\ \langle s_9^0 \rangle f(R_6)A(\tilde{s}) & \frac{\langle s_9^0 \rangle \langle s_{11}^0 \rangle}{\langle s_5^0 \rangle} f(R_6)A(\tilde{s}) & y_t \end{pmatrix}. \tag{5.2.27}$$

Quark	Mass
up quark ($\bar{u}_1$)	$1.5 - 3.3$ MeV
charm quark ($\bar{u}_2$)	$1.27 \begin{pmatrix} +0.07 \\ -0.11 \end{pmatrix}$ GeV
top quark ($\bar{u}_3$)	$171.3 \pm 1.1 \pm 1.2$ GeV

Table 5.1: Up quark masses from [110].

This matrix depends on several free variables, but is also strongly constrained. The values for the singlet VEVs are determined by the F-term and D-term equations and are in principal no free parameters. We do not want to give a concrete solution to the F-term and D-term equations, it is sufficient to know that the order of the VEVs is set by the Fayet-Iliopoulos D-term which results in

$$\langle \tilde{s} \rangle = \mathcal{O}(0.1) \tag{5.2.28}$$

in natural units [22]. We conclude that the only way to create hierarchies in Y_u is an appropriate choice of $f(R_6)$. Let us approach quantitative results. We neglect the running of couplings, because it is beyond the accuracy of our approach [23]. The eigenvalues of Y_u should show the same hierarchy than the observed up quark masses. The experimental data are depicted in table 5.1. The approximate hierarchies are

$$\frac{m_u}{m_c} \overset{!}{\approx} \frac{1}{500}, \qquad \frac{m_c}{m_t} \overset{!}{\approx} \frac{1}{100}. \tag{5.2.29}$$

After the diagonalization of Y_u we find

$$\frac{m_u}{m_c} \approx f(R_6), \qquad \frac{m_c}{m_t} \approx A(\tilde{s}). \tag{5.2.30}$$

$B(\tilde{s})$ has been neglected for simplicity and the VEVs have been set to an unique value of 0.1. There are indeed two independent variables to obtain the correct quark mass ratios. The small value for $f(R_6)$ is natural due to the exponential character of the function $f(R_6)$. The value for the radius is around $R_6 \approx 2.8$ to result in $f(R_6) \approx 1/500$. If it is possible to tune $A(\tilde{s})$ is an open question and will not discussed any further. We will comment on R_6. As we have seen in section 4, it is possible to relate the top quark coupling to the anisotropy R_5/R_6. In the appropriate GUT limit where R_5 sets the GUT scale we have found $R_5 \approx 50$. Together with $R_6 \approx 2.8$ from the discussion of this section we find the prediction

$$\frac{R_5}{R_6} \approx 18 \tag{5.2.31}$$

in tension with the value from section 4.4.1 to obtain the correct top quark Yukawa coupling $R_5/R_6 > 50$. We see how constraint this setup is. The need to obtain the correct quark mass hierarchies in the up quark sector is already sufficient to overconstrain the size of two internal extra dimensions.

Quark	Mass
down quark (d_1)	$3.5 - 6$ MeV
strange quark ($\bar{d}_2$)	$105 \begin{pmatrix} +25 \\ -35 \end{pmatrix}$ MeV
bottom quark ($\bar{d}_3 - \bar{d}_4$)	$4.2 \begin{pmatrix} +0.17 \\ -0.07 \end{pmatrix}$ GeV

Table 5.2: Down quark masses from [110]. The labels for the quark masses are the same as in [8].

5.2.3 The down quark sector

The measured down quark masses are displayed in table 5.2. The observed hierarchies between the masses are smaller than in the up quark sector and are given by [111]

$$\frac{m_d}{m_s} \overset{!}{\approx} \frac{1}{19}, \qquad \frac{m_s}{m_b} \overset{!}{\approx} \frac{1}{40}. \tag{5.2.32}$$

The theoretical situation in the down quark sector in model 1A is also different to the up quark sector. The effective third generation is in fact a mixture between two split families $\bar{d}_3 - \bar{d}_4$. In contrast to [8], we will take even higher order couplings into account. We will go up to order nine in SM singlets to obtain a down quark matrix with full rank. This makes our approach more accurate. We find

$$Y_d = \begin{pmatrix} \tilde{s}^9 & \tilde{s}^5 & 0 \\ \tilde{s}^5 & \tilde{s}^9 & 0 \\ 0 & \tilde{s}^6 & \tilde{s}^8 \end{pmatrix}. \tag{5.2.33}$$

We start with the lowest contributions in SM singlets, the off-diagonal entries. We obtain

$$\begin{aligned} \mathcal{A}_d^{12} = \int d^2 z_4 \ldots \int d^2 z_N \Big\langle & V_{q_1}^{(1,0,0,1,0)} V_{\bar{d}_2}^{(1,0,0,0,0)} V_{\phi_1}^{(0,*,*,*,0)} V_{s_6^0}^{(1,0,0,0,0)} V_{s_9^0}^{(1,0,0,1,0)} V_{s_{26}^0}^{(4,0,*,*,0)} \\ & \Big(V_{h_1}^{(2,0,*,*,1/2)} V_{h_2}^{(2,0,*,*,1/2)} + V_{s_{17}^0}^{(2,0,*,*,1/2)} V_{s_{18}^0}^{(2,0,*,*,1/2)} + V_{s_{20}^0}^{(2,1,*,*,0)} V_{s_{22}^0}^{(2,2,*,*,0)} \\ & + V_{s_{21}^0}^{(2,1,*,*,1)} V_{s_{22}^0}^{(2,2,*,*,0)} + V_{s_{20}^0}^{(2,1,*,*,0)} V_{s_{23}^0}^{(2,2,*,*,1)} + V_{s_{21}^0}^{(2,1,*,*,1)} V_{s_{23}^0}^{(2,2,*,*,1)} \Big) \Big\rangle + \mathcal{O}(\tilde{s}^6). \end{aligned} \tag{5.2.34}$$

We already know this coupling structure from $\mathcal{A}_u^{11}$. We find for the twist operators in the SO(4) torus

$$\left\langle \sigma_{(0,1)} \sigma_{(0,0)} \sigma_{(0,0)} \sigma_{(0,1)} \right\rangle \approx e^{-\frac{\pi R_6^2}{4\,\mathrm{Im}(z_4)}} \tag{5.2.35}$$

and can use the same function $f(R_6)$ as in the up quark case to describe this behavior. The unknown part of the amplitude will be labeled by $D(\tilde{s})$. We get

$$Y_d^{12} = f(R_6) D(\tilde{s}) + \mathcal{O}(\tilde{s}^6) = \frac{1}{\tan\beta} \frac{\langle s_6^0 \rangle \langle s_9^0 \rangle}{\langle s_4^0 \rangle \langle s_5^0 \rangle} Y_u^{11} + \mathcal{O}(\tilde{s}^6). \tag{5.2.36}$$

The second equality is only true after electroweak symmetry breaking. This relation makes it in principle possible to relate the down quark mass to the up quark mass. We further find

$$Y_d^{21} = Y_d^{12} \tag{5.2.37}$$

where the reason for this feature as in the up quark case is a D_4 symmetry [9]. This symmetry holds to all orders in SM singlets $\tilde{s}$.

We will completely ignore the coupling $Y_d^{32} = X(\tilde{s})$, because the eigenvalues of the matrix Y_d which give the quark masses are independent of Y_d^{32}. The next contribution at order eight in SM singlets is given by

$$\begin{aligned}\mathcal{A}_d^{33} = \int d^2 z_4 \ldots \int d^2 z_N \Big\langle & V_{q_1}^{(1,0,0,1,0)} \left(V_{\bar{d}_3}^{(4,1,*,*,0)} - V_{\bar{d}_4}^{(4,1,*,*,1)} \right) V_{\phi_1}^{(0,*,*,*,0)} \\ & V_{h_2^0}^{(2,0,*,*,1/2)} V_{h_{10}^0}^{(4,0,*,*,1)} \left(V_{s_{13}^0}^{(1,0,0,1,0)} \right)^2 V_{s_{30}^0}^{(4,0,*,*,1/2)} \\ & \left(\left(V_{h_3^0}^{(3,*,0,0,-1/3)} \right)^2 + \left(V_{h_6^0}^{(3,*,0,1,-1/3)} \right)^2 \right) \left(V_{s_{20}^0}^{(2,1,*,*,0)} + V_{s_{21}^0}^{(2,1,*,*,1)} \right) \Big\rangle .\end{aligned} \tag{5.2.38}$$

This coupling is not related to other couplings and we parameterize it as $U(\tilde{s})$.

The two other diagonal couplings Y_d^{11} and Y_d^{22} show up at order nine in SM singlets. The couplings are quite involved. There is

$$\begin{aligned}\mathcal{A}_d^{11} = \int d^2 z_4 \ldots \int d^2 z_N \Big\langle & V_{q_1}^{(1,0,0,1,0)} V_{\bar{d}_1}^{(1,0,0,1,0)} V_{\phi_1}^{(0,*,*,*,0)} V_{s_4^0}^{(1,0,0,0,0)} \left(V_{s_{13}^0}^{(1,0,0,1,0)} \right)^2 \\ & V_{s_{26}^0}^{(4,0,*,*,0)} V_{s_{30}^0}^{(4,0,*,*,1/2)} \Big(V_{\chi_1}^{(3,*,0,0,-1/3)} V_{\chi_2}^{(3,*,0,0,-1/3)} V_{h_2}^{(2,0,*,*,1/2)} V_{h_4}^{(3,*,0,0,-1/3)} \\ & + V_{\chi_3}^{(3,*,0,1,-1/3)} V_{\chi_4}^{(3,*,0,1,-1/3)} V_{h_2^0}^{(2,0,*,*,1/2)} V_{h_4^0}^{(3,*,0,0,-1/3)} \\ & + V_{\chi_1}^{(3,*,0,0,-1/3)} V_{\chi_4}^{(3,*,0,1,-1/3)} V_{h_2^0}^{(2,0,*,*,1/2)} V_{h_6^0}^{(3,*,0,1,-1/3)} \\ & + V_{\chi_2}^{(3,*,0,0,-1/3)} V_{\chi_3}^{(3,*,0,1,-1/3)} V_{h_2^0}^{(2,0,*,*,1/2)} V_{h_6^0}^{(3,*,0,1,-1/3)} \\ & + V_{h_2}^{(2,0,*,*,1/2)} V_{h_3}^{(3,*,0,0,-1/3)} \left(V_{h_4^0}^{(3,*,0,0,-1/3)} \right)^2 \\ & + V_{h_2}^{(2,0,*,*,1/2)} V_{h_3}^{(3,*,0,0,-1/3)} \left(V_{h_6^0}^{(3,*,0,1,-1/3)} \right)^2 \\ & + V_{h_2}^{(2,0,*,*,1/2)} V_{h_4}^{(3,*,0,0,-1/3)} V_{h_5^0}^{(3,*,0,1,-1/3)} V_{h_6^0}^{(3,*,0,1,-1/3)} \Big) \Big\rangle .\end{aligned} \tag{5.2.39}$$

We further have

$$
\begin{aligned}
\mathcal{A}_d^{22} = \int d^2z_4 \dots \int d^2z_N \Big\langle & V_{q_2}^{(1,0,0,0,0)} V_{\bar{d}_2}^{(1,0,0,0,0)} V_{\phi_1}^{(0,*,*,*,0)} V_{s_4^0}^{(1,0,0,0,0)} \left(V_{s_{13}^0}^{(1,0,0,1,0)}\right)^2 \\
& V_{s_{26}^0}^{(4,0,*,*,0)} V_{s_{30}^0}^{(4,0,*,*,1/2)} \Big(V_{\chi_1}^{(3,*,0,0,-1/3)} V_{\chi_2}^{(3,*,0,0,-1/3)} V_{h_2}^{(2,0,*,*,1/2)} V_{h_4}^{(3,*,0,0,-1/3)} \\
& + V_{\chi_3}^{(3,*,0,1,-1/3)} V_{\chi_4}^{(3,*,0,1,-1/3)} V_{h_2^0}^{(2,0,*,*,1/2)} V_{h_4^0}^{(3,*,0,0,-1/3)} \\
& + V_{\chi_1}^{(3,*,0,0,-1/3)} V_{\chi_4}^{(3,*,0,1,-1/3)} V_{h_2^0}^{(2,0,*,*,1/2)} V_{h_6^0}^{(3,*,0,1,-1/3)} \\
& + V_{\chi_2}^{(3,*,0,0,-1/3)} V_{\chi_3}^{(3,*,0,1,-1/3)} V_{h_2^0}^{(2,0,*,*,1/2)} V_{h_6^0}^{(3,*,0,1,-1/3)} \\
& + V_{h_2}^{(2,0,*,*,1/2)} V_{h_3}^{(3,*,0,0,-1/3)} \left(V_{h_4^0}^{(3,*,0,0,-1/3)}\right)^2 \\
& + V_{h_2}^{(2,0,*,*,1/2)} V_{h_3}^{(3,*,0,0,-1/3)} \left(V_{h_6^0}^{(3,*,0,1,-1/3)}\right)^2 \\
& + V_{h_2}^{(2,0,*,*,1/2)} V_{h_4}^{(3,*,0,0,-1/3)} V_{h_5^0}^{(3,*,0,1,-1/3)} V_{h_6^0}^{(3,*,0,1,-1/3)} \Big) \Big\rangle .
\end{aligned}
\tag{5.2.40}
$$

The situation is the same as for the diagonal couplings in the up quark sector. The only difference between $\mathcal{A}_d^{11}$ and $\mathcal{A}_d^{22}$ is the localization of the quarks in the SO(4) torus. For the down quarks this localization dependent part is more involved. The correlators in the SO(4) torus are

$$\left\langle \sigma_{(0,1)}\sigma_{(0,1)}\sigma_{(0,0)}\sigma_{(0,1)}\sigma_{(0,1)} \left(2\sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,0)} + \sigma_{(0,1)}\sigma_{(0,1)}\sigma_{(0,0)} + 4\sigma_{(0,0)}\sigma_{(0,1)}\sigma_{(0,1)}\right)\right\rangle \tag{5.2.41}$$

for $\mathcal{A}_d^{11}$ and

$$\left\langle \sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,1)}\sigma_{(0,1)} \left(2\sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,0)} + \sigma_{(0,1)}\sigma_{(0,1)}\sigma_{(0,0)} + 4\sigma_{(0,0)}\sigma_{(0,1)}\sigma_{(0,1)}\right)\right\rangle \tag{5.2.42}$$

for $\mathcal{A}_d^{22}$. These correlators are discussed in more detail in appendix A.2. The result of this discussion is that the values in leading order are both equal to the twist correlators

$$\left\langle \sigma_{(0,1)}\sigma_{(0,1)}\sigma_{(0,0)}\sigma_{(0,0)} \right\rangle \approx e^{-\frac{\pi R_6^2}{4\,\mathrm{Im}(z_4)}}. \tag{5.2.43}$$

We thus find

$$Y_d^{11} = f(R_6)C(\tilde{s}), \qquad Y_d^{22} = Y_d^{11} = f(R_6)C(\tilde{s}) \tag{5.2.44}$$

with the same $f(R_6)$ as for the up quarks. In leading order the two entries are equal. The complete down quark Yukawa matrix is given by

$$Y_d = \begin{pmatrix} f(R_6)C(\tilde{s}) & f(R_6)D(\tilde{s}) & 0 \\ f(R_6)D(\tilde{s}) & f(R_6)C(\tilde{s}) & 0 \\ 0 & X(\tilde{s}) & U(\tilde{s}) \end{pmatrix}. \tag{5.2.45}$$

The three eigenvalues are given by $U(\tilde{s})$ and $f(R_6)(C(\tilde{s}) \pm D(\tilde{s}))$. $U(\tilde{s})$ can be used as free parameter to fit the bottom quark mass. We can obtain the right hierarchy between the down and the strange quark mass if

$$C(\tilde{s}) \overset{!}{\approx} \frac{9}{10} D(\tilde{s}). \tag{5.2.46}$$

We already discussed the localization behavior in the SO(4) torus which is similar between the diagonal and the off-diagonal couplings and can thus not give rise to an effect. The same is true for the SU(3) and the G_2 torus. In fact $C(\tilde{s})$ occurs at order five in SM singlets and $D(\tilde{s})$ results from an interaction with nine SM singlets. Even if we assume rather large VEVs $\langle\tilde{s}\rangle = 0.3$ [22], we find

$$\langle\tilde{s}\rangle^4 \approx \frac{1}{125}, \qquad C(\tilde{s}) \approx \frac{1}{125} D(\tilde{s}) \tag{5.2.47}$$

in contrast to the experimental data. Not all VEVs have to be of similar size, but to cure such large effects with a proper VEV assignment is rather unnatural. We find two almost degenerate mass eigenvalues

$$\frac{m_d}{m_s} \approx \frac{62}{63}. \tag{5.2.48}$$

We conclude that it is impossible to obtain a quark mass hierarchy as expected from experimental data in our setup. This further disfavors this model and shows how constraint the setup obtained from heterotic orbifold compactifications in practice is.

5.2.4 Radiative mass generation

It was proposed in [112] to generate Yukawa couplings via loop effects through supersymmetry breaking soft terms. This can be a mechanism to cure the problem of nearly degenerate quark masses in the down quark sector. This mechanism leads naturally to flavor changing neutral currents (FCNCs) which have to be compatible with experimental bounds (see for example [113] in the string theory context). If it is possible to fulfill these bounds in this class of models remains an open task for future work. Maybe there are also other mechanism which can give rise to essential corrections to the quark masses in the right way.

5.2.5 Discussion

We have discussed in the last sections the quark mass structure of model 1A in detail. We have used several approximations, but the need to obtain large hierarchies enabled us to draw robust conclusions. Before supersymmetry breaking the down quark masses are not realistic. On the other hand, the up quark masses can be realistic if the size of the two extra dimensions in the SO(4) torus can be stabilized at the right values. Nevertheless there is some tension between the required anisotropy in section 4 and the up quark masses for the light generations. We conclude that model 1A exhibits an unrealistic flavor structure. The discussion of the mixing and the lepton sector was therefore not carried out for this model, but it is in principle possible with the outlined techniques.

5.3 A comment on model 1B, model 2 and a $\mathbb{Z}_2 \times \mathbb{Z}_2$ model

Another vacuum configuration called model 1B was considered in [8]. This model has an unbroken D_4 family symmetry for the first two generations. The difference to model 1A is that the symmetry is not broken by the VEV assignment. The consequence is that all couplings for the first two generations are equal. As we have seen in the discussion of model 1A such a behavior is in conflict with observations. It seems to be impossible for such configurations to have a large hierarchy between the masses as required by experiment. As discussed, radiative quark mass generation after supersymmetry breaking can maybe avoid this problem.

Another model from the minilandscape, model 2 from [8] has a richer structure because of more Higgs fields. As a drawback there is no gauge-top unification (see section 4) and the top Yukawa coupling is already suppressed by two singlet VEVs. The other diagonal entries of the Yukawa matrix, as well as the first off-diagonal entries are zero which seems to indicate an unrealistic flavor structure right from construction. Therefore we have not considered the model in more detail. We conclude that none of the models presented in [8] exhibits an appealing flavor structure. It is a task for future work to apply the techniques outlined in this study to the MSSM models of [114] and the NMSSM models of [115] based on the same minilandscape geometry.

In [19] a model based on a $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold construction was discussed. The geometry of the extra dimensional structure of this model differs from the minilandscape geometry based on the $\mathbb{Z}_{6-\mathrm{II}}$ orbifold. A detailed understanding of the geometry is still under investigation. Even without this knowledge, the quark mass pattern is rather unrealistic. There seem to be wrong hierarchies in the up quark sector due to too many couplings at the trilinear level. Furthermore there are SU(5) relations between the first generations in the down quark and lepton sector.

6 Concluding remarks

We have discussed three main topics in this thesis.

- How to compute the structure of the superpotential for a model with given symmetries.
- Implications of exact and approximate R symmetries of the superpotential.
- Yukawa couplings, especially the top Yukawa coupling in heterotic orbifold compactifications.

Let us summarize the results.

We have successfully introduced the concept of Hilbert bases as a tool to determine if a given coupling in the superpotential is forbidden or allowed by gauge symmetries and discrete R or non-R symmetries. That was mainly a technical progress, which enabled us to study rather complex models, which naturally arise in the context of string theory. The Hilbert basis allowed us to construct building blocks of the perturbative superpotential. These building blocks are the ingredients of the superpotential and determine its structure to all orders in pertubation theory.

Exact and approximate R symmetries have been discussed in this work. We showed that a hierarchically small VEV for the superpotential as well as a realistic μ term in orbifold theories can be the consequence of approximate R symmetries. We discussed the mathematical background and further applications in the field of moduli stabilization, which is important in string theory. We gave several simple and quite complex examples. We outlined the possibility to stabilize the size of the extra dimensions and the value of the gauge coupling in heterotic orbifold compactifications with the help of such approximate symmetries.

We discussed the top Yukawa coupling in orbifold field theories. We showed that there are substantial corrections due to localized Fayet-Iliopoulos D-terms. In the usual Gauge-Higgs unification scenario they result in an adjustment which is welcome to be compatible with observations. We gave an example in the framework of heterotic orbifold compactifications and extend previous work. We further outlined that this mechanism makes it possible to relate the size of the extra dimensions to available low energy data.

The main part of this thesis was devoted to the discussion of the quark mass hierarchies and the structure of the Yukawa couplings in heterotic orbifold compactifications. We have been able to study several models in detail and showed how it is possible to derive quark mass hierarchies in explicit string theory constructions. It was possible to relate the quark mass hierarchies to the size of the extra dimensions. As outlined, the size of the extra dimensions influence also the value of the top quark coupling in these models.

These two effects make it possible to overconstrain these models. Another conflict with experimental data for the down quark mass hierarchies arised in a concrete model under investigation. We have been able to show that several models induced by heterotic string theory are in tension with experimental constraints.

The developed techniques are universal and can be applied to new constructed models as well. We would like to remind the reader that all results have been obtained on the supersymmetric level. As outlined, we cannot guarantee that supersymmetry breaking does not change the conclusions. At several points we have shown how this work can be extended.

The Hilbert bases for models with approximate R symmetries can be discussed, which should show an interesting behavior at the R symmetry breaking scale.

The application of approximate R symmetries to moduli stabilization, together with modern techniques like Gröbner bases can be used to discuss a complete string theory model.

The phenomenological consequences of the inclusion of all U(1) factors for the gauge-top unification scenario can be discussed.

A better understanding of the Yukawa couplings, especially of the interactions between excited twist operators is desirable. It will be interesting to compare the results for the Yukawa couplings with target space modular invariance as an additional symmetry of the superpotential.

It appears desirable to apply the techniques outlined in this thesis also to proton decay operators as well as to the lepton sector and the mixing matrices in the quark and lepton sectors.

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A N-point couplings for $\mathbb{Z}_2$ twists

A.1 N-point couplings for four $\mathbb{Z}_2$ twists

In this section we want to show how it is possible to find an approximate relation between the two factors in the SO(4) torus

$$\left\langle \sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,0)} \right\rangle \quad \text{and} \quad \left\langle \sigma_{(0,1)}\sigma_{(0,1)}\sigma_{(0,0)}\sigma_{(0,0)} \right\rangle . \tag{A.1.1}$$

The main problem will not be the discussion of these correlators, but the approximation of the remaining world sheet integrations. We need to calculate a N-point coupling with four $\mathbb{Z}_2$ twists. The calculation of a 4-point coupling with four $\mathbb{Z}_2$ twists has been calculated in [84,85]. Let us review here the result. There are two different contributions, one from the quantum part and one from the classical part

$$\left\langle \sigma_{(0,1)}\sigma_{(0,1)}\sigma_{(0,0)}\sigma_{(0,0)} \right\rangle = Z_{\text{qu}}(z_4) \cdot Z_{\text{cl}}(z_4). \tag{A.1.2}$$

The first is insensitive to the localization of the fields, whereas the last one is sensitive. The difference between the correlators is only influenced by the classical part. The quantum part is the same for both correlators. The result for the classical part is [85]

$$Z_{\text{cl}}(z_4) = \sum e^{-S_{\text{cl}}(z_4)} = \sum_{n_0,n_1\in\mathbb{Z}} e^{-\frac{\pi R_6^2}{\operatorname{Im}\tau(z_4)}\left|n_1+n_0\tau(z_4)+\frac{1}{2}(\epsilon_1+\epsilon_0\tau(z_4))\right|^2} \tag{A.1.3}$$

where we have

$$\tau(z_4) = \frac{i\,{}_2F_1\left(\frac{1}{2},\frac{1}{2},1,1-z_4\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2},1,z_4\right)} = \frac{i\,K(\sqrt{1-z_4})}{K(\sqrt{z_4})} \tag{A.1.4}$$

and ϵ_i labels the fixed points of the twist fields. With $K(z_4)$ we mean the complete elliptic integral of the first kind, which is given by

$$K(z_4) = \int_0^1 dt \frac{1}{\sqrt{(1-t^2)(1-z_4^2t^2)}}. \tag{A.1.5}$$

z_4 is a complex variable and can be interpreted as the position of one of the vertex operators on the world sheet. As we are dealing with a tree level amplitude, the world sheet is a sphere. The position z_4 cannot be fixed by $\mathrm{PSL}(2,\mathbb{C})$ invariance and has to be weighted by an integral in the end. That means, we have one free complex modulus of the Riemann surface describing the string interaction. We further have $\epsilon_i = 0$ for the fixed point $(n_2, n_2') = (0,0)$ and $\epsilon_i = 1$ for the fixed point $(n_2, n_2') = (0,1)$.

If all fields live at the same fixed point, which is true for

$$\left\langle \sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,0)} \right\rangle \tag{A.1.6}$$

we have $\epsilon_0 = \epsilon_1 = 0$. We get an unsuppressed amplitude, because the exponential breaks down. If the fields are localized at different fixed points, for example $\epsilon_0 = 0$ and $\epsilon_1 = 1$ we get a suppression going with the distance squared R_6^2. This is the case for the correlators

$$\left\langle \sigma_{(0,1)}\sigma_{(0,1)}\sigma_{(0,0)}\sigma_{(0,0)} \right\rangle . \tag{A.1.7}$$

The terms for $n_i \neq 0$ are further suppressed and come from strings winding several times around the torus. We can neglect their contribution, because it is subdominant and obtain

$$Z_{\text{cl}}(z_4) \approx \begin{cases} 1 & \text{all fields live at the same fixed point} \\ e^{-\frac{\pi R_6^2}{4\,\text{Im}\,\tau(z_4)}} & \text{fields live at different fixed points} \end{cases} . \tag{A.1.8}$$

This is the exact classical contribution in the SO(4) torus, because all other fields are untwisted in the SO(4) torus. We do not know the contribution in the other extra dimensions, namely the SU(3) and the G_2 torus in detail. We know that the additional contribution is depending on z_4. Because we have to integrate over z_4 in the end, we have to approximate the result.

Let us focus on the N-point coupling in more detail. We have a N-point coupling and therefore N vertex operator positions on the world sheet. We can fix three of them by $\text{PSL}(2,\mathbb{C})$ invariance. Integration over the remaining $N-3$ positions is necessary. At least one of these vertex operators is twisted in the SO(4) torus and we called his position z_4 (in fact it is also possible that all vertex operators in the SO(4) torus are undetermined by $\text{PSL}(2,\mathbb{C})$ invariance, but we will not consider this case here). The other vertex operators are untwisted in the SO(4) torus and we do not know their contribution exactly, but they depend on all $N-3$ variables. Their contribution is the same, regardless of their localization in the SO(4) torus. The complete contribution coming from this localization enters via equation (A.1.8). If we call the unknown part from the SU(3) and G_2 torus and the contributions not affected by the localization in the SO(4) torus $U(z_4,\dots,z_N)$, we get

$$\mathcal{A} = \int d^2 z_4 \dots \int d^2 z_N Z_{\text{cl}}(z_4) U(z_4,\dots,z_N) \tag{A.1.9}$$

for the complete amplitude. We can approximate the function $Z_{\text{cl}}(z_4)$ by setting $z_4 = \frac{1}{2}$ which is motivated by the shape of the function $S_{\text{cl}}(z_4)$ (see figure A.1). We obtain $\tau(z_4) = i$ for this choice. We get

$$\mathcal{A}_{\left\langle \sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,0)} \right\rangle} = \int d^2 z_4 \dots \int d^2 z_N U(z_4,\dots,z_N) \tag{A.1.10}$$

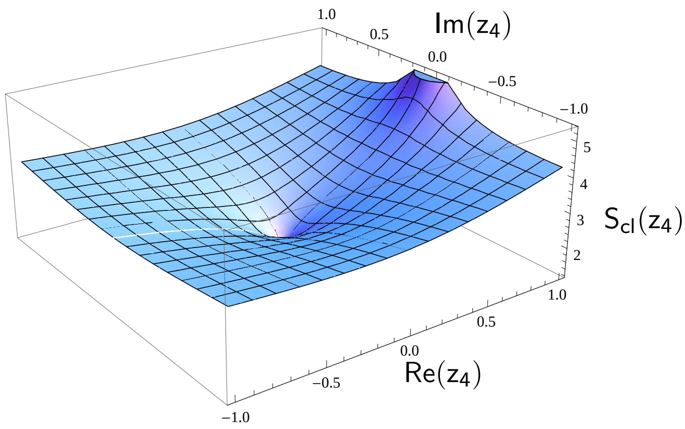

Figure A.1: The function $S_{\text{cl}}(z_4)$ for the values $\epsilon_0 = 0$, $\epsilon_1 = 1$ and $R_6 = 2$.

if all fields live at the same fixed point and the coupling is unsuppressed. In the case where the fields are localized at different fixed points we get

$$\mathcal{A}_{\langle \sigma_{(0,1)}\sigma_{(0,1)}\sigma_{(0,0)}\sigma_{(0,0)} \rangle} = e^{-\frac{\pi R_6^2}{4}} \int d^2 z_4 \dots \int d^2 z_N U(z_4, \dots, z_N). \tag{A.1.11}$$

We recognize that the integrals are after the approximation the same. We conclude that the required ratio between a suppressed and an unsuppressed coupling is

$$\frac{\mathcal{A}_{\langle \sigma_{(0,1)}\sigma_{(0,1)}\sigma_{(0,0)}\sigma_{(0,0)} \rangle}}{\mathcal{A}_{\langle \sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,0)} \rangle}} = \frac{e^{-\frac{\pi R_6^2}{4}} \int d^2 z_4 \dots \int d^2 z_N U(z_4, \dots, z_N)}{\int d^2 z_4 \dots \int d^2 z_N U(z_4, \dots, z_N)} = e^{-\frac{\pi R_6^2}{4}} \tag{A.1.12}$$

in a first approximation. The uncertainty of this approach is substantial, because the function $U(z_4, \dots, z_N)$ is unknown.

A.2 N-point couplings for eight $\mathbb{Z}_2$ twists

We also need to look at N-point couplings with eight $\mathbb{Z}_2$ twists. The computation of this amplitude is more complicated than the one reviewed in appendix A.1. General $\mathbb{Z}_2$ twist amplitudes have been considered in several papers [100, 116–120], also because they are of interest in the context of spin operators. The beautiful mathematics of hyperelliptic Riemann surfaces needed for these computations can be found for example in [121].

In the case of eight twist operators, at least five of them depend on unfixed moduli of the world sheet $z_4, \dots, z_8$. We will focus on the correlator

$$\left\langle \sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,1)}\sigma_{(0,1)} \right\rangle . \tag{A.2.1}$$

By symmetry reasons it is equal to

$$\left\langle \sigma_{(0,1)}\sigma_{(0,1)}\sigma_{(0,1)}\sigma_{(0,1)}\sigma_{(0,1)}\sigma_{(0,1)}\sigma_{(0,0)}\sigma_{(0,0)} \right\rangle . \tag{A.2.2}$$

The classical part can be written as

$$Z_{\text{cl}} = \sum e^{-S_{\text{cl}}(z_4,\ldots,z_8)}. \tag{A.2.3}$$

The dependence on R_6 is the same as for the correlators of four twist operators [100]. We find

$$Z_{\text{cl}} \approx e^{-\alpha_i R_6^2} \tag{A.2.4}$$

and thus again an exponential suppression. The constants $\alpha_i > 0$ are depending on the period matrix of the Riemann surface. In leading order, the contribution is equal to the contribution of four twist fields. The fact that four additional twisted strings live at one fixed point does not change the result substantially, because no additional winding modes are needed for the coupling to exist. We conclude that

$$\left\langle \sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,1)}\sigma_{(0,1)} \right\rangle \approx \left\langle \sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,1)}\sigma_{(0,1)} \right\rangle . \tag{A.2.5}$$

There is an additional contribution where four twist fields sit at one fixed point and four twist fields at the other fixed point

$$\left\langle \sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,0)}\sigma_{(0,1)}\sigma_{(0,q)}\sigma_{(0,1)}\sigma_{(0,1)} \right\rangle . \tag{A.2.6}$$

Nevertheless, this contribution is stronger suppressed [100] and thus only subleading.

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