AE-1

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Summary:

A systematic investigation is made of the eleven coordinate systems in which the reactor equation $\nabla^2 \phi + B^2 \phi = 0$ is separable. The fundamental solution and geometric buckling are given for those cases where the separated equations lead to known functions. It is especially shown that reactors of prolate and oblate spheroidal shape can be calculated in detail, and the results are given in extensive tables.

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Introduction

According to the one group neutron diffusion theory[1] the geometric buckling, B_g^2, for a reactor is defined as the lowest eigenvalue of the equation

$$\nabla^2 \phi + B^2 \phi = 0$$

with the boundary condition that the neutron flux, ϕ, is zero on the effective boundaries of the reactor. (In the following the above equation is referred to as the "reactor equation".) The geometric buckling is easily calculated for reactors of certain simple shapes, for instance parallelepipeds, spheres and circular cylinders. In the general case, however, only approximate solutions can be obtained.

On the other hand there is also a possibility to measure geometric bucklings in a model experiment using the pulsed neutron source technique[2] as the rate of decay of the neutron density in a small geometry in part is determined by the geometric buckling. This possibility has up to now not been used much, but it should be a valuable complement when calculations are difficult.

With exception of a short review in the Reactor Handbook[3] no systematic study has been made of the different reactor shapes which can be treated theoretically in a simple way. Therefore in this paper such an investigation is accomplished.

Separation possibilities

A necessary condition for obtaining an analytic solution to the reactor equation is that the equation is separable in the coordinate system to be used, and this system is in general determined by the boundary conditions. According to Eisenhart[4] there are eleven orthogonal coordinate systems in which the three-dimensional Schrödinger equation is separable, and in these the reactor equation can therefore also be separated. These eleven systems will be studied in detail here, and for completeness even the most wellknown cases are included.

1. Cartesian coordinates

Coordinate surfaces: Planes.

The reactor equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} + B^2 \phi = 0$$

We put

$$\phi(x, y, z) = X(x)Y(y)Z(z)$$

and obtain the equations

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0$$

$$\frac{d^2Y}{dy^2} + \beta^2 Y = 0$$

$$\frac{d^2Z}{dz^2} + \gamma^2 Z = 0$$

where α, β, and γ are separation constants and

$$\alpha^2 + \beta^2 + \gamma^2 = B^2$$

Solutions:

$$X = A_1 \cos \alpha x + A_2 \sin \alpha x$$

a. s. o.

Special case: Parallelepiped with the sides a, b, and c. Fundamental solution:

$$\phi = A \cos \frac{\pi x}{a} \cos \frac{\pi y}{b} \cos \frac{\pi z}{c}$$

$$B_g^2 = \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right)$$

2. Spherical coordinates

Coordinates:

$$x = r \sin \vartheta \cos \varphi \qquad 0 \leq r < \infty$$
$$y = r \sin \vartheta \sin \varphi \qquad 0 \leq \vartheta \leq \pi$$
$$z = r \cos \vartheta \qquad 0 \leq \varphi \leq 2\pi$$

Coordinate surfaces:

$$x^2 + y^2 + z^2 = r^2 \qquad \text{(spheres)}$$
$$x^2 + y^2 = z^2 \, tg^2 \vartheta \qquad \text{(cones)}$$
$$y = x \, tg \, \varphi \qquad \text{(planes)}$$

The reactor equation:

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \emptyset}{\partial r}\right) + \frac{1}{r^2 \sin \vartheta} \frac{\partial}{\partial \vartheta}\left(\sin \vartheta \frac{\partial \emptyset}{\partial \vartheta}\right) + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2 \emptyset}{\partial \varphi^2} + B^2 \emptyset = 0$$

We put

$$\emptyset(r, \vartheta, \varphi) = R(r) T(\vartheta) F(\varphi)$$

and obtain the equations

$$\frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \left(B^2 - \frac{\ell(\ell+1)}{r^2}\right) R = 0$$

$$\frac{1}{\sin \vartheta} \frac{d}{d\vartheta}\left(\sin \vartheta \frac{dT}{d\vartheta}\right) + \left(\ell(\ell+1) - \frac{m^2}{\sin^2 \vartheta}\right) T = 0$$

$$\frac{d^2 F}{d\varphi^2} + m^2 F = 0$$

where ℓ and m are separation constants (ℓ is not necessarily an integer).

Solutions:

$$R = \frac{A_1}{\sqrt{r}} J_{\ell+1/2}(Br) + \frac{A_2}{\sqrt{r}} J_{-\ell-1/2}(Br)$$

$$T = C_1 P_\ell^m (\cos \theta) + C_2 Q_\ell^m (\cos \theta)$$

$$F = D_1 \cos m \varphi + D_2 \sin m \varphi$$

Special cases:

Sphere with radius a. Fundamental solution:

$$\emptyset = \frac{A}{r} \sin \frac{\pi r}{a}$$

$$B_g^2 = \frac{\pi^2}{a^2}$$

Spherical sector limited by $\vartheta = \theta_\ell$ (cos θ_ℓ is the first zero of P_ℓ). Fundamental solution

$$\phi = \frac{A}{\sqrt{r}} J_{\ell+1/2} \left(\frac{\lambda_\ell r}{a}\right) P_\ell (\cos\theta) \qquad B_g^2 = \frac{\lambda_\ell^2}{a^2}$$

where λ_ℓ is the first zero of $J_{\ell+1/2}$.

Part of a spherical sector limited by $r = a$, $r = b$, $(b < a)$, and $\vartheta = \theta_\ell$, (cos θ_ℓ is the first zero of P_ℓ). Fundamental solution:

$$\phi = \frac{A}{\sqrt{r}} \left[J_{-\ell-1/2}(\beta a) J_{\ell+1/2}(\beta r) - J_{\ell+1/2}(\beta a) J_{-\ell-1/2}(\beta r) \right] P_\ell (\cos\vartheta)$$

$$B_g^2 = \beta^2$$

where β is determined by

$$J_{-\ell-1/2}(\beta a) J_{\ell+1/2}(\beta b) = J_{\ell+1/2}(\beta a) J_{-\ell-1/2}(\beta b)$$

Part of a spherical sector limited by $r = a$, $\vartheta = \theta_\ell^m$ (cos θ_ℓ^m is the first zero of P_ℓ^m) and $\varphi = \pm \frac{\pi}{2m}$. Fundamental solution:

$$\phi = \frac{A}{\sqrt{r}} J_{\ell+1/2} \left(\frac{\lambda_\ell r}{a}\right) P_\ell^m (\cos\vartheta) \cos m\varphi$$

$$B_g^2 = \frac{\lambda_\ell^2}{a^2}$$

where λ_ℓ is the first zero of $J_{\ell+1/2}$.

In this way solutions for even more complicated shapes can be obtained.

3. Circular cylinder coordinates

Coordinates: $x = r \cos \varphi$ $\quad 0 \leq r < \infty$

$y = r \sin \varphi$ $\quad 0 \leq \varphi \leq 2\pi$

$z = z$

Coordinate surfaces: $x^2 + y^2 = r^2$ (cylinders)

$y = x \operatorname{tg} \varphi$ (planes)

z (planes)

The reactor equation:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \varphi^2} + \frac{\partial^2 \phi}{\partial z^2} + B^2 \phi = 0$$

We put $\phi(r, \varphi, z) = R(r)F(\varphi)Z(z)$

and obtain the equations

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\beta^2 r^2 - \nu^2)R = 0$$

$$\frac{d^2F}{d\varphi^2} + \nu^2 F = 0$$

$$\frac{d^2Z}{dz^2} + \gamma^2 Z = 0$$

where β, γ, and ν are separation constants and

$$\beta^2 + \gamma^2 = B^2$$

Solutions:

$$R = A_1 J_\nu(\beta r) + A_2 Y_\nu(\beta r)$$

$$F = C_1 \cos \nu\varphi + C_2 \sin \nu\varphi$$

$$Z = D_1 \cos \gamma z + D_2 \sin \gamma z$$

Special cases:

Cylinder with radius a and height H. Fundamental solution:

$$\phi = A J_o\left(\frac{\lambda_o r}{a}\right) \cos \frac{\pi z}{H}$$

$$B_g^2 = \frac{\lambda_o^2}{a^2} + \frac{\pi^2}{H^2}$$

where λ_o is the first zero of J_o.

Cylinder sector limited by $\varphi = \pm \alpha$. Fundamental solution:

$$\phi = A\, J_\nu \left(\frac{\lambda_\nu r}{a}\right) \cos \nu\varphi \cos \frac{\pi z}{H}$$

$$B_g^2 = \frac{\lambda_\nu^2}{a^2} + \frac{\pi^2}{H^2}$$

where $\nu = \frac{\pi}{2\alpha}$ and λ_ν is the first zero of J_ν. Observe that ν need not be an integer as is assumed in the Reactor Handbook[3].

For the limiting case $\alpha = \pi$ we get $\lambda_{1/2} = \pi$, which corresponds to a cylinder with a black absorber in the halfplane $\varphi = \pi$.

Part of a cylinder sector limited by $r = a$, $r = b$ $(b < a)$, $\varphi = \pm \alpha$. Fundamental solution:

$$\phi = A \left[Y_\nu(\beta a)\, J_\nu(\beta r) - J_\nu(\beta a)\, Y_\nu(\beta r) \right] \cos \nu\varphi \cos \frac{\pi z}{H}$$

$$B_g^2 = \beta^2 + \frac{\pi^2}{H^2}$$

where β is determined by

$$Y_\nu(\beta a)\, J_\nu(\beta b) = J_\nu(\beta a)\, Y_\nu(\beta b)$$

4. Elliptic cylinder coordinates

Coordinates:

$$x = a \cosh \xi \cos \varphi \qquad 0 \leq \xi < \infty$$
$$y = a \sinh \xi \sin \varphi \qquad 0 \leq \varphi \leq 2\pi$$
$$z = z$$

Coordinate surfaces:

$$\frac{x^2}{a^2 \cosh^2 \xi} + \frac{y^2}{a^2 \sinh^2 \xi} = 1 \quad \text{(elliptic cylinders)}$$

$$\frac{x^2}{a^2 \cos^2 \varphi} - \frac{y^2}{a^2 \sin^2 \varphi} = 1 \quad \text{(hyperbolic cylinders)}$$

$$z \qquad \text{(planes)}$$

The reactor equation:

$$\frac{1}{a^2(\cosh^2\xi - \cos^2\varphi)} \left(\frac{\partial^2 \emptyset}{\partial \xi^2} + \frac{\partial^2 \emptyset}{\partial z^2} \right) + \frac{\partial^2 \emptyset}{\partial z^2} + B^2 \emptyset = 0$$

We put $\emptyset(\xi, \varphi, z) = X(\xi)Y(\varphi)Z(z)$

and obtain the equations

$$\frac{d^2X}{d\xi^2} + (\alpha^2 a^2 \cosh^2 \xi - \lambda)X = 0$$

$$\frac{d^2Y}{d\varphi^2} + (\lambda - \alpha^2 a^2 \cos^2 \varphi)Y = 0$$

$$\frac{d^2Z}{dz^2} + \beta^2 Z = 0$$

where α, β and λ are separation constants and

$$\alpha^2 + \beta^2 = B^2$$

The equations for X and Y lead to Mathieu functions. The case has been treated by Gast and Bournia[5], who give tables of the buckling as a function of the ratio (c) between the axes of the ellips. As a suitable approximation they give

$$B_g^2 = \frac{\pi^2}{H^2} + \frac{5,783}{m^2} \cdot \frac{1 + c^2}{2c^2}$$

where m is the semi-minor axis. The error in the last term is less than 2 % if $1 \leq c \leq 2$.

5. Parabolic cylinder coordinates

Coordinates:

$$x = \frac{u^2 - v^2}{2} \qquad 0 \leq u < \infty$$

$$y = \pm uv \qquad 0 \leq v < \infty$$

$$z = z$$

Coordinate surfaces:

$$y^2 = 2\,xv^2 + v^4 \qquad \text{(parabolic cylinders)}$$

$$y^2 = -2\,xu^2 + u^4 \qquad \text{(parabolic cylinders)}$$

$$z \qquad \text{(planes)}$$

The reactor equation:

$$\frac{1}{u^2 + v^2}\left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right) + \frac{\partial^2 \phi}{\partial z^2} + B^2 \phi = 0$$

We put $\phi(u, v, z) = U(u)V(v)Z(z)$
and obtain the equations

$$\frac{d^2 U}{du^2} + (\beta^2 u^2 - \gamma)U = 0$$

$$\frac{d^2 V}{dv^2} + (\beta^2 v^2 + \gamma)\,V = 0$$

$$\frac{d^2 Z}{dz^2} + \alpha^2 Z = 0$$

where α, β and γ are separation constants and

$$\alpha^2 + \beta^2 = B^2$$

In general U and V can not be expressed in known functions. For the special case of a parabolic cylinder with height H and limited by $u^2 = a$, $v^2 = a$, the symmetry gives $\gamma = 0$. Fundamental solution:

$$\phi = A\sqrt{uv}\; J_{-1/4}\left(\frac{\lambda_o u^2}{a}\right) J_{-1/4}\left(\frac{\lambda_o v^2}{a}\right) \cos \frac{\pi z}{H}$$

$$B_g^2 = \frac{\pi^2}{H^2} + \frac{4\lambda_o^2}{a^2}$$

where λ_o is the first zero of $J_{-1/4}$.

6. Rotation paraboloid coordinates

Coordinates:

$$x = uv\cos\varphi \qquad 0 \le u < \infty$$
$$y = uv\sin\varphi \qquad 0 \le v < \infty$$
$$z = \frac{1}{2}(u^2 - v^2) \qquad 0 \le \varphi \le 2\pi$$

Coordinate surfaces:

$$x^2 + y^2 = 2zv^2 + v^4 \quad \text{(rotation paraboloids)}$$
$$x^2 + y^2 = -2zu^2 + u^4 \quad \text{(rotation paraboloids)}$$
$$y = x\,\mathrm{tg}\,\varphi \quad \text{(planes)}$$

The reactor equation:

$$\frac{1}{u^2+v^2}\left\{\frac{1}{u}\frac{\partial}{\partial u}\left(u\frac{\partial \phi}{du}\right) + \frac{1}{v}\frac{\partial}{\partial v}\left(v\frac{\partial \phi}{\partial v}\right) + \left(\frac{1}{u^2}+\frac{1}{v^2}\right)\frac{\partial^2 \phi}{\partial \varphi^2}\right\} + B^2\phi = 0$$

We put $\phi(u,v,\varphi) = U(u)V(v)F(\varphi)$
and obtain the equations

$$u^2\frac{d^2U}{du^2} + u\frac{dU}{du} + (B^2u^4 + \gamma u^2 - \nu^2)U = 0$$

$$v^2\frac{d^2V}{dv^2} + v\frac{dV}{dv} + (B^2v^4 - \gamma v^2 - \nu^2)V = 0$$

$$\frac{d^2F}{d\varphi^2} + \nu^2 F = 0$$

where γ and ν are separation constants.

The equations for U and V can not in general be solved in known functions. For the special case of a reactor limited by the surfaces $u^2 = a$, $v^2 = a$, symmetry gives $\gamma = 0$, and ν is also $= 0$. Fundamental solution:

$$\phi = A\,J_o\left(\frac{\lambda_o u^2}{a}\right)J_o\left(\frac{\lambda_o v^2}{a}\right)$$

$$B_g^2 = \frac{4\lambda_o^2}{a^2}$$

where λ_o is the first zero of J_o.

7. Prolate spheroidal coordinates

Coordinates:

$$x = a\sqrt{\xi^2 - 1}\,\sqrt{1-\eta^2}\cos\varphi \qquad 1 \le \xi \le \infty$$

$$y = a\sqrt{\xi^2 - 1}\,\sqrt{1-\eta^2}\sin\varphi \qquad -1 \le \eta \le 1$$

$$z = a\,\xi\,\eta \qquad 0 \le \varphi \le 2\pi$$

Coordinate surfaces:

$$\frac{x^2+y^2}{a^2(\xi^2-1)} + \frac{z^2}{a^2\xi^2} = 1 \qquad \text{(prolate spheroids)}$$

$$\frac{x^2+y^2}{a^2(1-\eta^2)} - \frac{z^2}{a^2\eta^2} = -1 \qquad \text{(hyperboloids of two sheets)}$$

$$y = x \,\mathrm{tg}\,\varphi \qquad \text{(planes)}$$

The reactor equation:

$$\frac{\partial}{\partial\xi}\left[(\xi^2-1)\frac{\partial\phi}{\partial\xi}\right] + \frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\phi}{\partial\eta}\right] + \frac{\xi^2-\eta^2}{(\xi^2-1)(1-\eta^2)}\,\frac{\partial^2\phi}{\partial\varphi^2} + B^2a^2(\xi^2-\eta^2)' = 0$$

We put $\phi(\xi,\eta,\varphi) = J(\xi)S(\eta)F(\varphi)$

and obtain the equations

$$\frac{d}{d\xi}\left[(\xi^2-1)\frac{dJ}{d\xi}\right] - (A - h^2\xi^2 + \frac{m^2}{\xi^2-1})\,J = 0$$

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{dS}{d\eta}\right] + (A - h^2\eta^2 - \frac{m^2}{1-\eta^2})\,S = 0$$

$$\frac{d^2F}{d\varphi^2} + m^2F = 0$$

where $h = Ba$, and m and A are separation constants. The solutions are according to Stratton et al.[6]:

$$je_{m\ell}(h,\xi) = \left(\frac{\xi^2-1}{\xi^2}\right)^{\frac{m}{2}} \sum_n{}' \, a_n(h/m\ell)\, j_{n+m}(h\xi)$$

$$S_{m\ell}(h,\eta) = \sum_n{}' \, d_n(h/m\ell)\, P^m_{m+n}(\eta)$$

$$F(\varphi) = C_1 \cos m\varphi + C_2 \sin m\varphi$$

Here j_{n+m} is a spherical Bessel function defined as

$$j_k(x) = \sqrt{\frac{\pi}{2x}}\, J_{k+1/2}(x)$$

P^m_{m+n} are associated Legendre functions, m and ℓ integers, and the summations are made over even or odd integers n. a_n and d_n are tabulated in ref. 6 for different values of h, m and ℓ.

Special case: Prolate spheroid with semi-major axis $M_1 = a\xi_o$ and semi-minor axis $M_2 = a\sqrt{\xi_o^2 - 1}$. The excentricity e is then $1/\xi_o$ and the ratio of the axes

$$c = \frac{M_1}{M_2} = \frac{\xi_o}{\sqrt{\xi_o^2 - 1}}$$

In this case $m = \ell = 0$, and we therefore have to determine the first zero x_o of

$$je_{oo}(h,\xi) = \sum_n{}' \, a_n(h/oo)\, j_n(h\xi)$$

For different values of $h = Ba$, $x_o = h\xi_o$ has been computed numerically by use of tables of the expansion coefficients[6] and tables of spherical Bessel functions[7]. Table 1 gives the zeros x_o obtained for all tabulated values of h, and also the excentricity, the axis ratio, and $B_g^2M_2^2 = x_o^2 - h^2$. We see that for h = 0, that is a sphere, $B_g^2M_2^2 = \pi^2$. For $h \rightarrow \infty$ the spheroid approaches the shape of an infinite circular cylinder and $B_g^2M_2^2 \rightarrow (2.4048)^2$ as expected.[x)]

A good approximation for axis ratios, c, close to unity is

$$B_g^2M_2^2 = \frac{\pi^2}{3} \cdot \frac{2c^2+1}{c^2}$$

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x) The numerical calculations were performed by Mr. A. Rentze.

8. Oblate spheroidal coordinates

Coordinates:
$$x = b\sqrt{\xi^2+1}\,\sqrt{1-\eta^2}\,\cos\varphi \qquad 0 \le \xi < \infty$$
$$y = b\sqrt{\xi^2+1}\,\sqrt{1-\eta^2}\,\sin\varphi \qquad -1 \le \eta \le 1$$
$$z = b\,\xi\,\eta \qquad 0 \le \varphi \le 2\pi$$

Coordinate surfaces:
$$\frac{x^2+y^2}{b^2(\xi^2+1)} + \frac{z^2}{b^2\xi^2} = 1 \qquad \text{(oblate spheroids)}$$
$$\frac{x^2+y^2}{b^2(1-\eta^2)} - \frac{z^2}{b^2\eta^2} = 1 \qquad \text{(hyperboloids of one sheet)}$$
$$y = x\,\mathrm{tg}\,\varphi \qquad \text{(planes)}$$

The reactor equation:
$$\frac{\partial}{\partial\xi}\left[(\xi^2+1)\frac{\partial\phi}{\partial\xi}\right] + \frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\phi}{\partial\eta}\right] + \frac{\xi^2+\eta^2}{(\xi^2+1)(1-\eta^2)}\frac{\partial^2\phi}{\partial\varphi^2} + B^2b^2(\xi^2+\eta^2) = 0$$

We put $\phi(\xi,\eta,\varphi) = J(\xi)S(\eta)F(\varphi)$
and obtain the equations
$$\frac{d}{d\xi}\left[(\xi^2+1)\frac{dJ}{d\xi}\right] - \left(A - g^2\xi^2 - \frac{m^2}{\xi^2+1}\right)J = 0$$
$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{dS}{d\eta}\right] + \left(A + g^2\eta^2 - \frac{m^2}{1-\eta^2}\right)S = 0$$
$$\frac{d^2F}{d\varphi^2} + m^2F = 0$$

where $g = Bb$, and m and A are separation constants.

We see that these equations are very similar to those obtained for the prolate spheroidal coordinates. In fact, we get the oblate case from the prolate by the transformations

$\xi \longrightarrow -i\xi$
$a \longrightarrow ib$
$h \longrightarrow ig$

With the same notations as in the prolate case the solutions to the equations can therefore be written[6]:

$$je_{m\ell}(ig, -i\xi) = \left(\frac{\xi^2+1}{\xi^2}\right)^{\frac{m}{2}} \sum_n{}' a_n(ig/m\ell)\, j_{n+m}(g\xi)$$

$$S_{m\ell}(ig, \eta) = \sum_n{}' d_n(ig/m\ell)\, P^m_{n+m}(\eta)$$

$$F(\varphi) = C_1 \cos m\varphi + C_2 \sin m\varphi$$

Special case: Oblate spheroid with semi-major axis $M_1 = b\sqrt{\xi_o^2+1}$ and semi-minor axis $M_2 = b\,\xi_o$. The ratio of the axes is thus

$$c = \frac{M_1}{M_2} = \frac{\sqrt{\xi_o^2+1}}{\xi_o}$$

and the excentricity e is $1/\sqrt{\xi_o^2+1}$. As in the preceding case the buckling is obtained from the first zero x_o of the function

$$je_{oo}(ig, -i\xi) = \sum_n{}' a_n(ig/oo)\, j_n(g\xi)$$

For different values of $g = Bb$ $x_o = g\xi_o$ has been computed numerically by use of tables of the expansion coefficients[6] and tables of the spherical Bessel functions[7]. In table 2 the zeros x_o are listed for all given g-values, and also the excentricity, the axis ratio, and $B_g^2 M_2^2 = x_o^2$. We see that for $g = 0$ we get the buckling for a sphere, $B_g^2 M_2^2 = \pi^2$. For $h \longrightarrow \infty$ the spheroid approaches the shape of an infinite slab and $B_g^2 M_2^2 \longrightarrow \frac{\pi^2}{4}$ as expected[x].

A good approximation for axis ratios, c, close to unity is

$$B_g^2 M_2^2 = \frac{\pi^2}{3} \cdot \frac{c^2+2}{c^2}$$

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x) The numerical calculations were performed by Mr. A. Rentze.

9. General ellipsoidal coordinates

Coordinates: $$x^2 = \frac{(a^2+u)(a^2+v)(a^2+w)}{(a^2-b^2)(a^2-c^2)}$$

$$y^2 = \frac{(b^2+u)(b^2+v)(b^2+w)}{(b^2-c^2)(b^2-a^2)}$$

$$z^2 = \frac{(c^2+u)(c^2+v)(c^2+w)}{(c^2-a^2)(c^2-b^2)}$$

$$-u \leq c^2 \leq -v \leq b^2 \leq -w \leq a^2$$

Coordinate surfaces:

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1 \qquad \text{(ellipsoids)}$$

$$\frac{x^2}{a^2+v} + \frac{y^2}{b^2+v} + \frac{z^2}{c^2+v} = 1 \qquad \text{(hyperboloids of one sheet)}$$

$$\frac{x^2}{a^2+w} + \frac{y^2}{b^2+w} + \frac{z^2}{c^2+w} = 1 \qquad \text{(hyperboloids of two sheets)}$$

The reactor equation:

$$(v-w)K_u\frac{\partial}{\partial u}\left(K_u\frac{\partial\phi}{\partial u}\right) + (u-w)K_v\frac{\partial}{\partial v}\left(K_v\frac{\partial\phi}{\partial v}\right) + (u-v)K_w\frac{\partial}{\partial w}\left(K_w\frac{\partial\phi}{\partial w}\right) +$$

$$+\frac{B^2}{4}(u-v)(u-w)(v-w)\,\phi = 0$$

where $K_u = \sqrt{(a^2+u)(b^2+u)(c^2+u)}$ and corresponding for K_v and K_w.

We put $\phi(u,v,w) = U(u)V(v)W(w)$

and obtain the equation

$$K_u\frac{d}{du}\left(K_u\frac{dU}{du}\right) + \left(\frac{B^2u^2}{4} + \alpha u + \beta\right)U = 0$$

and corresponding for V and W. Here α and β are separation constants.

These equations belong to a general type of Lamé differential equations[8].

10. Confocal parabolic coordinates

Coordinates:

$$x = \frac{1}{2}(u+v+w-a-b)$$

$$y^2 = \frac{(u-a)(v-a)(a-w)}{b-a}$$

$$z^2 = \frac{(u-b)(b-v)(b-w)}{b-a}$$

$$w \leq a \leq v \leq b \leq u$$

Coordinate surfaces:

$$\frac{y^2}{u-a} + \frac{z^2}{u-b} = -2x+u \qquad \text{(elliptic paraboloid)}$$

$$\frac{y^2}{v-a} - \frac{z^2}{b-v} = -2x+v \qquad \text{(hyperbolic paraboloid)}$$

$$\frac{y^2}{a-w} + \frac{z^2}{b-w} = 2x-w \qquad \text{(elliptic paraboloid)}$$

With the notations

$$L_u = \sqrt{(u-a)(u-b)}$$

$$L_v = \sqrt{(v-a)(b-v)}$$

$$L_w = \sqrt{(a-w)(b-w)}$$

we obtain the reactor equation

$$(v-w)\,L_u \frac{\partial}{\partial u}\left(L_u \frac{\partial \phi}{\partial u}\right) + (u-w)\,L_v \frac{\partial}{\partial v}\left(L_v \frac{\partial \phi}{\partial v}\right) + (u-v)\,L_w \frac{\partial}{\partial w}\left(L_w \frac{\partial \phi}{\partial w}\right) +$$

$$+ \frac{B^2}{4}(u-v)(u-w)(v-w)\,\phi = 0$$

We put $\phi(u,v,w) = U(u)V(v)W(w)$
and obtain the equation

$$L_u \frac{d}{du}\left(L_u \frac{dU}{du}\right) + \left(\frac{B^2 u^2}{4} + \alpha u + \beta\right) U = 0$$

and corresponding for V and W. Here α and β are separation constants. The equation can also be written

$$(u-a)(u-b)\frac{d^2U}{du^2} + \left(u - \frac{a+b}{2}\right)\frac{dU}{du} + \left(\frac{B^2u^2}{4} + \alpha u + \beta\right) U = 0$$

The solution can not be expressed in known functions.

Special case: If a = b = 0 we get two rotation paraboloids with foci in the origin. The equation is then the same as in the previously treated case of rotation paraboloids.

11. Spherical - conical coordinates

Coordinates:

$$x = u\,\mathrm{dn}(v,k)\,\mathrm{sn}(w,k')$$
$$y = u\,\mathrm{sn}(v,k)\,\mathrm{dn}(w,k')$$
$$z = u\,\mathrm{cn}(v,k)\,\mathrm{cn}(w,k')$$
$$k^2 + k'^2 = 1$$

where sn, dn and cn are elliptic functions.

Coordinate surfaces: $x^2 + y^2 + z^2 = u^2$ (spheres)

$$\frac{x^2k^2}{\mathrm{dn}^2(v,k)} - \frac{y^2}{\mathrm{sn}^2(v,k)} + \frac{z^2}{\mathrm{cn}^2(v,k)} = 0 \qquad \text{(cones)}$$

$$-\frac{x^2}{\mathrm{sn}^2(w,k')} + \frac{k'^2y^2}{\mathrm{dn}^2(w,k')} + \frac{z^2}{\mathrm{cn}^2(w,k')} = 0 \qquad \text{(cones)}$$

The reactor equation:

$$\frac{\partial}{\partial u}\left(u^2\frac{\partial \phi}{\partial u}\right) + \frac{1}{k^2\mathrm{cn}^2(v,k) + k'^2\mathrm{cn}^2(w,k')}\left[\frac{\partial^2\phi}{\partial v^2} + \frac{\partial^2\phi}{\partial w^2}\right] + B^2u^2\phi = 0$$

We put $\phi(u,v,w) = U(u)V(v)W(w)$
and obtain

$$\frac{d}{du}\left(u^2\frac{dU}{du}\right) + (B^2u^2 + \alpha)U = 0$$

$$\frac{d^2V}{dv^2} - \left[\alpha k^2\mathrm{cn}^2(v,k) + \beta\right]V = 0$$

$$\frac{d^2W}{dw^2} - \left[\alpha k'^2\mathrm{cn}^2(w,k') - \beta\right]W = 0$$

where α and β are separation constants. The solution to the first equation is

$$U = \frac{A_1}{\sqrt{u}}J_\nu(Bu) + \frac{A_2}{\sqrt{u}}Y_\nu(Bu)$$

where $\nu = \frac{1}{2}\sqrt{1-4\alpha}$.

For the case $\alpha = 0$ we obtain the spherical case treated before with usual spherical coordinates.

The last two equations can be transformed to Lamé differential equations[8].

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Table 1. Prolate spheroids.

Values of x_o, excentricity, axis ratio, and buckling for h-values from 0 to 8.

h	x_o	$e = \frac{1}{\xi_o}$	$c = \frac{M_1}{M_2}$	$B_g^2 M_2^2$
0	3.14159	0	1.0000	9.8696
0.1	3.14265	0.031820	1.0005	9.8662
0.2	3.14584	0.063576	1.0020	9.8563
0.3	3.15116	0.095203	1.0046	9.8398
0.4	3.15861	0.126638	1.0081	9.8168
0.5	3.16822	0.157817	1.0127	9.7876
0.6	3.17999	0.188680	1.0183	9.7523
0.7	3.19396	0.219164	1.0249	9.7114
0.8	3.21013	0.249211	1.0326	9.6649
0.9	3.22852	0.278766	1.0413	9.6133
1.0	3.24917	0.307771	1.0510	9.5571
1.2	3.29729	0.363935	1.0736	9.4321
1.4	3.35462	0.417335	1.1004	9.2935
1.6	3.42125	0.467665	1.1313	9.1449
1.8	3.49718	0.514700	1.1664	8.9903
2.0	3.58231	0.558299	1.2053	8.8329
2.2	3.67642	0.598408	1.2481	8.6761
2.4	3.77919	0.635057	1.2946	8.5222
2.6	3.89021	0.668344	1.3444	8.3736
2.8	4.00897	0.698434	1.3973	8.2319
3.0	4.13493	0.725526	1.4531	8.0977
3.2	4.26752	0.749850	1.5115	7.9717
3.4	4.40615	0.771649	1.5722	7.8541
3.6	4.55027	0.791162	1.6351	7.7449
3.8	4.69933	0.808626	1.6997	7.6437
4.0	4.85285	0.824258	1.7661	7.5501
4.2	5.01036	0.838260	1.8340	7.4637
4.4	5.17145	0.850825	1.9031	7.3839
4.6	5.33574	0.862111	1.9735	7.3101
4.8	5.50290	0.872267	2.0449	7.2419
5.0	5.67264	0.881424	2.1172	7.1788

Table 1, cont.

h	x_o	$e = \frac{1}{\xi_o}$	$c = \frac{M_1}{M_2}$	$B_g^2 M_2^2$
5.2	5.84469	0.889696	2.1903	7.1204
5.4	6.01882	0.897186	2.2642	7.0662
5.6	6.19482	0.903981	2.3388	7.0158
5.8	6.37251	0.910159	2.4140	6.9689
6.0	6.55173	0.915789	2.4896	6.9252
6.2	6.73234	0.920928	2.5659	6.8844
6.4	6.91420	0.925631	2.6426	6.8462
6.6	7.09721	0.929943	2.7197	6.8104
6.8	7.28127	0.933903	2.7969	6.7769
7.0	7.46628	0.937549	2.8747	6.7453
7.2	7.65216	0.940911	2.9529	6.7156
7.4	7.83886	0.944015	3.0310	6.6877
7.6	8.02627	0.946891	3.1098	6.6610
7.8	8.21440	0.949552	3.1887	6.6364
8.0	8.40313	0.952026	3.2679	6.6126
∞	∞	1.0000	∞	5.7831

Table 2. Oblate spheroids.

Values of x_o, excentricity, axis ratio, and buckling for g-values from 0 to 8.

g	x_o	e	$c = \frac{M_1}{M_2}$	$B_g^2 M_2^2$
0	3.14159	0	1.0000	9.8696
0.1	3.14053	0.031826	1.0005	9.8629
0.2	3.13735	0.063619	1.0020	9.8430
0.3	3.13206	0.095347	1.0046	9.8098
0.4	3.12466	0.126978	1.0082	9.7635
0.5	3.11517	0.158477	1.0128	9.7043
0.6	3.10362	0.189808	1.0185	9.6325
0.7	3.09002	0.220938	1.0253	9.5482
0.8	3.07440	0.251828	1.0333	9.4519
0.9	3.05682	0.282436	1.0424	9.3442
1.0	3.03730	0.312726	1.0528	9.2252
1.2	2.99272	0.372169	1.0774	8.9564
1.4	2.94122	0.429789	1.1075	8.6508
1.6	2.88358	0.485183	1.1436	8.3150
1.8	2.82077	0.537932	1.1863	7.9567
2.0	2.75402	0.587610	1.2359	7.5846
2.2	2.68475	0.633822	1.2929	7.2079
2.4	2.61452	0.676238	1.3574	6.8357
2.6	2.54493	0.714633	1.4296	6.4767
2.8	2.47744	0.748929	1.5091	6.1377
3.0	2.41324	0.779186	1.5954	5.8237
3.2	2.35321	0.805620	1.6880	5.5376
3.4	2.29785	0.828528	1.7859	5.2801
3.6	2.24732	0.848284	1.8884	5.0505
3.8	2.20154	0.865276	1.9948	4.8468
4.0	2.16026	0.879881	2.1044	4.6667
4.2	2.12312	0.892453	2.2166	4.5076
4.4	2.08971	0.903301	2.3310	4.3669
4.6	2.05965	0.912689	2.4470	4.2422
4.8	2.03254	0.920845	2.5646	4.1312
5.0	2.00805	0.927961	2.6833	4.0323

Table 2, cont.

g	x_o	e	$c = \frac{M_1}{M_2}$	$B_g^2 M_2^2$
5.2	1.98585	0.934194	2.8030	3.9436
5.4	1.96568	0.939679	2.9235	3.8639
5.6	1.94728	0.944525	3.0447	3.7919
5.8	1.93046	0.948824	3.1665	3.7267
6.0	1.91503	0.952653	3.2888	3.6673
6.2	1.90084	0.956075	3.4116	3.6132
6.4	1.88774	0.959147	3.5347	3.5636
6.6	1.87561	0.961912	3.6582	3.5179
6.8	1.86437	0.964409	3.7819	3.4759
7.0	1.85391	0.966672	3.9060	3.4370
7.2	1.84416	0.968728	4.0303	3.4009
7.4	1.83506	0.970601	4.1547	3.3674
7.6	1.82653	0.972313	4.2794	3.3362
7.8	1.81853	0.973881	4.4042	3.3071
8.0	1.81101	0.975321	4.5292	3.2798
∞	1.57080	1.0000	∞	2.4674

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