An Introduction to the Neutron Transport Phenomena

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Abstract

The main goal of the present lecture is to is to give a short description of neutron transport phenomena limited to those definitions that are necessary to understand the approach to practical solution of the problem given in the second lecture on reactor lattice transport calculations. The discussion of the neutron cross sections has been skipped as other lecturers have treated this subject in detail.

1 Definitions

1.1 Description of the medium

The medium is described by its isotopic composition and its nuclear properties. The composition is given in terms of *number densities* of isotopes defined as the numbers of respective nuclei in a cubic centimetre:

$$ND^i = \frac{\rho^i \cdot N_A}{A^i}, \tag{1}$$

with the Avogadro Number N_A=6.022×10^{23}, ρ^i – density of the isotope i in grams per cubic centimetre, and A^i the Atomic Number.

The nuclear properties are described by *microscopic cross sections*, σ^x, for reactions of type x.

The *macroscopic cross sections*, $\Sigma^x(r,E)$ of an isotope i, is a product of its microscopic cross section and its number density, ND^i, at position r:

$$\Sigma^{x,i}(r,E) = ND^i(r) \cdot \sigma^{x,i}(E). \tag{2}$$

If the medium is composed of more than one isotope then the total macroscopic cross section of the medium is equal to the sum of cross sections for each isotope:

$$\Sigma^t{}_{medium}(r,E) = \sum_i \Sigma^{x,i}(r,E) = \sum_i (\sigma^{x,i}(E) \cdot ND^i(r)). \tag{3}$$

The sum of the partial cross sections for all possible types of neutron-nucleus collisions is the *total cross section*. It is defined as the total collision (or interaction) cross section of a neutron at position r having energy E (in the laboratory system). It is the probability of neutron interaction per unit distance travelled by a neutron and has the dimension of a reciprocal length, e.g., cm^{-1}:

$$\Sigma_m{}^{tot}(r,E) = \sum_x \Sigma_m{}^x(r,E). \tag{4}$$

A reciprocal of the total cross section, λ, is called a *mean free path* and is an average distance of neutron travel between two consecutive collisions.

$$\lambda = 1/\Sigma_m{}^{tot}(r,E). \tag{5}$$

A form of *differential cross section*,

$$\Sigma^x(r,E')f^x(r;\Omega',E' \rightarrow \Omega,E), \tag{6}$$

is defined for collisions, from which neutrons emerge, as the cross sections for initial direction Ω' and energy E' emerging in a collision in the interval $d\Omega$ about Ω with energy dE about E. The cross section for a reaction of type x, for neutrons of energy E', is Σ^x and $f^x(r;\Omega',E' \rightarrow \Omega,E)d\Omega\, dE$ is the probability that if a neutron of direction Ω' and energy E' has a collision of type x, there will emerge from the collision a neutron in the direction interval $d\Omega$ about Ω, with energy dE about E. For elastic scattering, integration of f^x over all directions and energies gives unity. For elastic scattering of neutrons from initially stationary nuclei f^x is

a function only of $\Omega' \cdot \Omega$ which is a cosine of the scattering angle between the directions of motion of the neutron before and after the collision. For fission it is a good approximation to assume that the neutrons are emitted isotropically in the laboratory system. Then it is possible to write:

$$f^x(r;\Omega',E' \rightarrow \Omega,E)d\Omega\, dE = 1/(4\pi)\cdot\kappa(r;E' \rightarrow E)\, d\Omega\, dE, \tag{7}$$

where $\cdot\kappa(r;E' \rightarrow E)dE$, referred to as the spectrum of the fission neutrons, is the probability that a fission caused by a neutron at r with energy E' will lead to a neutron within dE about E. It is normalised so that after integration over full angle it gives $\kappa(r;E')$, which is the average number of neutrons produced by a fission at r caused by a neutron of energy E'.

Instead of the neutron energy, E, the neutron velocity, v, may be used represented as:

$$\mathbf{v} = v\cdot\Omega,$$

where $v = |\mathbf{v}|$ is the neutron speed and is connected to the energy by a standard equation

$$E = mv^2/2.$$

with m the neutron mass.

The rate, in neutrons per unit volume and time at r and t, at which neutrons are transferred by interactions of type x into final directions within $d\Omega$ about Ω and final energies dE about E is:

$$v'\,\Sigma^x(r,E')f^x(r;\Omega',E' \rightarrow \Omega,E)\,N(r,\Omega',E',t)\,d\Omega' dE'\,d\Omega\,dE. \tag{8}$$

The total rate at which neutrons are transferred is obtained by integrating the above quantity over all initial neutron directions and energies, and summing over all reactions.

1.2 Description of neutrons

A population of neutrons is described by a quantity called the *neutron angular density* denoted by

$$N(r,\Omega,E,t),$$

and defined as the probable (or expected) number of neutrons at the position r with direction Ω and energy E at time t, per unit volume per unit solid angle per unit energy, e.g., per cm^3 per steradian per MeV. Thus

$$N(r,\Omega,E,t)dVd\Omega dE$$

is the expected number of neutrons in the volume element dV about r, having directions within $d\Omega$ about Ω and energies in dE about E at time t. Such a number of neutrons in an infinitesimal volume is sometimes referred to as a *packet* of neutrons.

In the definition the expression 'probable' or 'expected' number of neutrons is meant to imply that fluctuations from the mean neutron population are not taken into account.

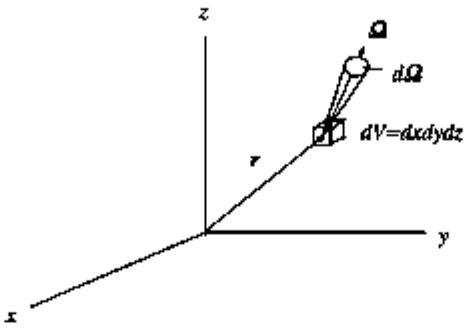

Figure 1. A packet of neutrons.

The integral of the neutron angular density over all directions is the energy dependent *neutron density n(r,E,t)*:

$$n(r,E,t) = \int_{4\pi} N(r,\Omega,E,t)d\Omega\,, \tag{9}$$

where the integral is taken over all directions. Hence $n(r,E,t)$ is the expected number of neutrons at r, with energy E at time t, per unit volume per unit energy.

The product of the neutron speed v and the neutron angular density is called the *neutron angular flux*:

$$\Phi(r,\Omega,E,t)=v\cdot N(r,\Omega,E,t), \tag{10}$$

The integral over all directions of the neutron angular flux is called *total neutron flux*, often referred to simply as the *neutron flux*, and is equal to:

$$\phi(r,E,t)=v\cdot n(r,E,t). \tag{11}$$

The neutron flux has been introduced as a quantity much more useful for the description of reactor properties than the neutron number density. The product of the neutron flux and the macroscopic cross section:

$$[RR]^x=\Sigma^x(r,E)\cdot\phi(r,E,t) \tag{12}$$

gives, by definition, the number of reactions suffered by neutrons per second, per cubic centimetre and per eV. This quantity is called a *reaction rate* of type x. In particular, the fission reaction rate integrated over energy is used to calculate the energy production.

The net number of neutrons crossing a surface element per unit energy in unit time is called the neutron current:

$$J(r,E,t) = v\int_{4\pi} \Omega N(r,\Omega,E,t)d\Omega\,. \tag{13}$$

The quantities defined above are expressed in units given in Table 1.

Table 1. Summary of introduced quantities.

Quantity	Symbol	Unit
Neutron angular density	N	neutrons/(cm^3·steradian·eV)
Neutron density	n	neutrons/(cm^3·eV)
Neutron angular flux	Φ	neutrons/(cm^2·steradian·eV·sec)
Neutron flux	ϕ	neutrons/(cm^2·eV·sec)
Neutron net current	J	neutrons/(cm^2·eV·sec)

It can be seen that the neutron current and the neutron total flux have the same units. The difference between these two quantities can be better understood if we compare their definitions, derived from Eqs. (9-11) and Eq.(13), but expressed in terms of the angular flux:

$$\phi(r,E,t) = \int_{4\pi} \Phi(r,\Omega,E,t)d\Omega\,,$$
$$J(r,E,t) = \int_{4\pi} \Omega\cdot\Phi(r,\Omega,E,t)d\Omega\,. \tag{14}$$

Equations (14) show that the neutron flux and current are, respectively, the zero's and first moment of the neutron angular flux.

By the *neutron sources* we understand neutrons which emerge in the system from events other than neutron collision and, therefore, they are independent of the neutron density. They are usually denoted by $Q(r,\Omega,E,t)$, which expresses the probability per unit time that a neutron of energy E will appear at r per unit volume per unit solid angle per unit energy. Sometimes they are referred to as external or independent neutron sources.

2 The neutron transport equation

2.1 *Two basic forms of the neutron transport equation*

The neutron transport equation is derived from the neutron balance inside a neutron packet (cf. Fig.1). It takes into account the number of neutrons remaining in the packet, the number of neutrons entering the packet as a result of collision and the number of neutrons entering the packet from external sources. The final result is:

$$\frac{\partial N}{\partial t} + v\Omega\nabla N + \Sigma vN = \iint \Sigma' f v' N' dE'd\Omega' + Q\,, \tag{15}$$

where $N=N(r,\Omega,E,t)$, $N'=N(r,\Omega',E',t)$, $\Sigma=\Sigma(r,E)$, $\Sigma' = \Sigma(r,E')$, $f = f(r;\Omega',E'\to\Omega,E)$, $Q = Q(r,\Omega,E,t)$.

The neutron transport equation may be also expressed in terms of the angular flux Φ, which is equal to v·N. By a direct substitution of its definition one gets:

$$\frac{1}{v}\frac{\partial \Phi}{\partial t} + \Omega \nabla \Phi + \Sigma \Phi = \iint \Sigma' f \Phi dE' d\Omega' + Q \quad . \tag{16}$$

2.2 *Interface conditions*

The solutions to the neutron transport equation are frequently sought in spatial regions including interfaces between different materials. At such interfaces, the cross sections are discontinuous and it is necessary to consider how the transport equation is to be used in these circumstances. The number of neutrons in a packet is not changed, merely by crossing a physical interface. This means that the neutron angular density must be continuous in r as the interface is crossed. Thus the continuity condition is to be used at the interface. This refers to the neutron angular density but is equally applicable to the angular flux.

2.3 *Boundary conditions*

In general the region of interest is surrounded by a convex surface. A neutron leaving the region through the surface cannot intersect the surface again. If neutrons enter the region from external sources, then the incoming neutron flux must be specified. If no neutrons enter from external sources and if a neutron, once it leaves the surface, cannot return, then the surface is called a free surface and we have the condition:

$$N(r,\Omega,E,t) = 0 \qquad \text{if} \quad n \cdot \Omega < 0, \tag{17}$$

where n is a unit vector in the direction of the outward normal at a position r on the surface. Such situation appears, if the region is surrounded by vacuum or a perfect absorber.

In practical applications we deal quite often with another type of boundary conditions. We speak about a specular reflection boundary condition if all the neutrons approaching the boundary from within are reflected back to the region with angles preserving the general reflection rule, and about a white boundary condition if they are reflected back with an isotropic distribution. The schematic explanation of the difference between the two boundary conditions is given in Fig. 2. It can be seen that if a packet of neutrons born in the middle of square region reaches the consecutive boundaries without collision they have a good chance to reach the central region under both types of boundary conditions. The same packet born in the middle of a circular region has a good chance to reach the central region under the white boundary condition but practically no chance under the reflective one. This fact will be used in the application of the transport theory to the solution of practical reactor problems.

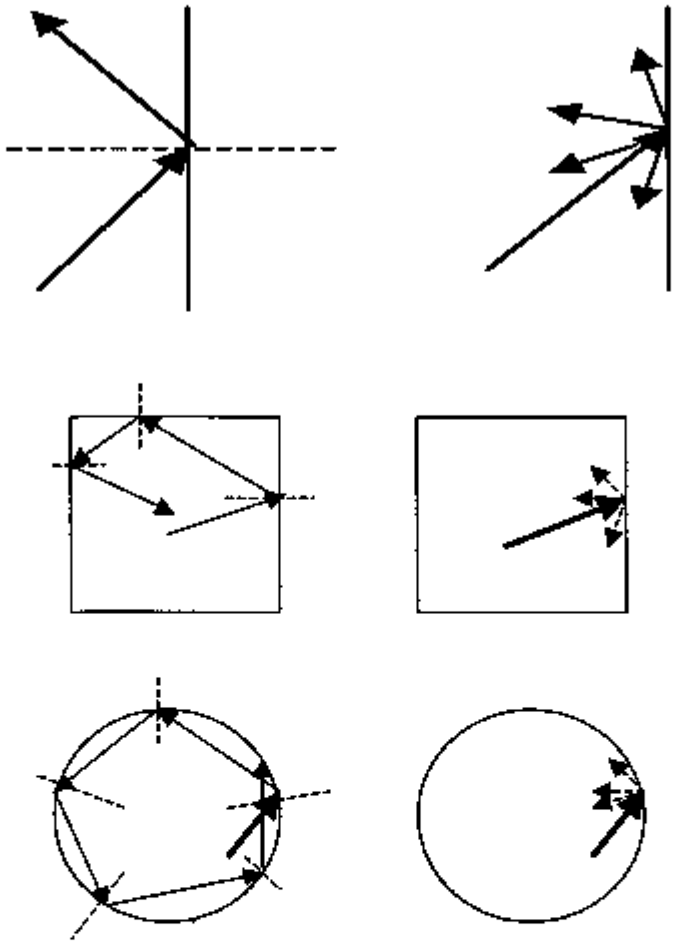

Figure 2. Effect of reflective (on the left hand side) and white (on the right hand side) boundary conditions for plane, square and circular boundaries

2.4 *Neutron conservation*

The neutron transport equation is simply a statement of neutron conservation as applied to an infinitesimal element of volume, direction and energy. If it is integrated over all directions, the result will be a statement of neutron conservation for a small element of volume and energy. Integration of the neutron transport equation (15) over all values of Ω gives with the previous notation (cf. Eqs. (9) and (13)):

$$\frac{\partial n}{\partial t} + \nabla J + \Sigma v n = \int \Sigma(r; E' \rightarrow E) v' n' dE' + Q\,, \tag{18}$$

where also Q and Σ have been integrated over the angle and

$$\Sigma(r;E'\rightarrow E)=\int \Sigma(r;E')f(r;\Omega',E'\rightarrow\Omega,E)d\Omega\,, \tag{19}$$

which is the cross section at r for collisions which result in a neutron of energy E' being replaced by one of energy E.

Integration over a finite region of volume, V, and energy gives:

$$\begin{aligned}&\frac{\partial\iint ndVdE}{\partial t}+\iint \nabla JdVdE+\iint vn\Sigma dVdE\\&=\iiint \Sigma(r;E'\rightarrow E)v'n'dE'dVdE+\iint QdVdE\end{aligned}\,. \tag{20}$$

Each of the five terms in the equation has clear physical meaning. The quantity

$$\iint ndVdE$$

is the total number of neutrons in the space-energy region under consideration. Hence the first term is the time rate of change of the total number of neutrons in this region. The second term can be written with application of the divergence theorem:

$$\iint_V \nabla J\cdot dVdE=\iint_A Jn\cdot dAdE\,, \tag{21}$$

where dA refers to an element of area, on the boundary surface of the region, V, under consideration and n is a unit normal to the surface element, directed outward from the region. Hence, the second term is the net number of neutrons flowing out of the space-energy region per unit time. The third term

$$\iint v\Sigma ndVdE$$

is the rate at which neutrons are entering into collisions in the given region, i.e., the total collision rate, and the fourth term

$$\iiint \Sigma(r;E'\rightarrow E)v'n'dE'dVdE$$

is the rate at which they emerge from these collisions. The fifth term gives the rate at which neutrons from independent sources are introduced into the region.

Thus the direct representation of particular terms gives:

Rate of change of neutrons = Net rate of generation of neutrons in collisions
+ Rate of introduction of source neutrons
- Net rate of outflow of neutrons.

2.5 *Integral equation for neutron transport*

The neutron transport equation is an integro-differential one for the neutron angular density or flux. By the application of the method of characteristics to the neutron transport equation, it can be converted into an integral equation:

$$\Phi(r,\Omega,E,t)=\int_0^{\infty} exp\left[-\int_0^{s'}\Sigma(r-s''\Omega;E)ds''\right]q(r-s'\Omega;\Omega,E,t-\frac{s'}{v})ds' \quad (22)$$

with

$$q(r;\Omega,E,t)=$$

$$=\int dE'\int\Sigma(r;E')f(r;\Omega',E'\to\Omega,E)\Phi(r,\Omega,\Omega',E)d\Omega'+Q(r,\Omega,E,t)$$

Thus q is the total rate with which neutrons appear at r, Ω, E and t as a result of both collisions and the independent source.

The integral equation in the simple case of the total cross section independent of position, isotropic scattering and source and no time dependence of the neutron flux becomes:

$$\Phi(r,E)=\int\frac{e^{-\Sigma(E)R}}{4\pi\cdot R^2}dV'\left[-\int\Sigma(r';E'\to E)\Phi(r',E')dE'+Q(r',E)\right], \quad (23)$$

with $R=|r-r'|$.

The assumption of a cross section independent of the spatial variable is not fulfilled in any realistic reactor system. However, if the system can be divided into subregions with constant material properties Eq. (23) can be used for effective reactor calculations.

2.6 *Multigroup approach to the neutron transport equation solution*

There is no possibility of obtaining exact solutions to the energy-dependent neutron transport equation for general reactor problems. It is necessary, therefore, to adopt approximate methods for solving the transport equation. The most important are the multigroup methods in which the neutron energy interval of interest is divided into a finite number of intervals, ΔE_g (called groups). It is then assumed that the cross section in each group is constant, e.g., equal to an average over energy. Within each group it is then independent of energy, although arbitrarily dependent on position:

$$\Sigma^x(r,E)\Rightarrow\Sigma^x_g(r), \quad \text{for } g=1,2,\ldots G,$$

$$\int_{\Delta E_{g'}}\int_{\Delta E_g}f(r;\Omega',E'\to\Omega,E)dE'dE=c_g(r)\cdot f_{g'g}(r;\Omega'\to\Omega). \quad (24)$$

The neutron angular flux and sources are integrated over respective energy intervals of groups:

$$\Phi_g(r,\Omega) = \int_{\Delta E_g} \Phi(r,\Omega,E)dE$$
$$Q_g(r,\Omega) = \int_{\Delta E_g} Q(r,\Omega,E)dE \qquad (25)$$

For a time independent case, using definitions from Eqs. (24,25), the energy dependent equation is replaced by a set of coupled one-speed equations which are then solved by approximate methods

$$\Omega\nabla\Phi_g(r,\Omega) + \Sigma_g(r)\Phi_g(r,\Omega) = \\ = c_g(r)\sum_{g'}\Sigma_{g'}(r)\int f_{g'g}(r;\Omega'\to\Omega)\Phi_{g'}(r,\Omega')d\Omega' + Q_g(r,\Omega) \qquad (26)$$

The quantity $c_g(r)$ introduced in Eq. .(24) has a meaning of the mean number of neutrons with energy in ΔE_g emerging per collision at r. For scattering collision $c = 1$ and for fission $c = \kappa$ (cf. Eq. (7)).

3 Criticality

From physical consideration, it is to be expected that system containing fissile nuclides can be subcritical, critical or supercritical, based on the behaviour of the neutron population as a function of time.

A system is said to be subcritical if for any nonzero initial neutron population, the expected population dies out with time unless it is sustained by an external neutron source.

A system is said to be supercritical when the expected neutron population diverges with time, starting from any nonzero population.

A system is said to be critical as one in which a steady, time independent neutron population can be maintained in the absence of a source.

The neutron transport equation with boundary conditions defines an initial value problem. If the neutron angular density at $t = 0$ is given, the expected density at any later time can be found, in principle, by solving the neutron transport equation. It has been shown that such a solution exists and is unique, provided some mathematical conditions are satisfied for actual physical situations.

The homogeneous (source free) neutron transport equation may be written in the operator form:

$$\frac{\partial N}{\partial t} = -v\Omega\nabla N - \Sigma vN + \iint \Sigma' fv' N' dE' d\Omega' = LN\,, \qquad (27)$$

where L is the transport operator. The boundary condition of no incoming neutrons is assumed. We consider the solution of the equation expressed in the form $N = N(r,\Omega,E)e^{\omega t}$

from which $\alpha N(r,\Omega,E) = LN(r,\Omega,E)$. There may exist many eigenvalues α of the operator L, represented by α_j with corresponding eigenfunctions N_j:

$$\alpha_j N_j = L N_j. \tag{28}$$

In practical cases there exist a real eigenvalue greater than the real part of any other eigenvalue. It will be denoted α_0 and the eigenfunction associated with it $N_0(r,\Omega,E)$. If the sign of α_0 is negative the solution of Eq. (27) will decrease asymptotically and the system is subcritical. If the sign is positive the solution will tend asymptotically to infinity and the system is supercritical. More rigorous considerations consisting in applying the Laplace transform lead to the asymptotic solution in the form:

$$N(r,\Omega,E,t) = A\ exp(\alpha_o t)\ N_o\ (r,\Omega,E), \quad \text{as } t\to\infty, \tag{29}$$

where A is a constant determined by the initial conditions. Thus, the criticality problem is that of finding the conditions for which $\alpha_o = 0$. A rigorous analysis has shown that for certain conditions (satisfied in practice) on the scattering kernel there is at least one discrete eigenvalue.

The homogeneous neutron transport equation will have a time independent solution when $\alpha_o = 0$ or the system is critical:

$$LN_0 = 0.$$

Introducing auxiliary characteristic values may approach the criticality problem. In particular, the spectrum of the fission neutrons $\kappa(r,E'\to E)$ we replace by $\kappa(r,E'\to E)/k$, and k can be varied to obtain the stationary solution, with $k = k\text{-}eff$, called effective multiplication factor. This amounts to multiplying the number of neutrons emitted per fission by the factor $1/k\text{-}eff$.

By definition $k\text{-}eff$ is a characteristic value of the equation:

$$\begin{aligned} &v\Omega\nabla N_k + \Sigma v N_k = \\ &= \iint \sum_{x\neq f} \Sigma^x{}' f^x \nu' N_k{}' \, d\Omega' \, dE' + \frac{1}{k_{eff}} \iint \frac{1}{4\pi} \kappa(r; E'\to E) \Sigma^f{}' \nu' N_k{}' \, d\Omega' \, dE' \end{aligned} \tag{30}$$

where the summation over x unequal f refers to collisions other than fission in which neutrons are produced and N_k are eigenfunctions independent of time.

In elementary reactor theory $k\text{-}eff$ is treated as a ratio between the numbers of neutrons in successive generations, with the fission process being regarded as the birth event which separates generations of neutrons.

For a critical system, i.e., when $\alpha_o = 0$ and $k\text{-}eff = 1$, the corresponding eigenfunctions satisfy the same equation, for any other system, however, the two eigenfunctions are different. This can be seen when writing the homogeneous eigenvalue equation in the form:

$$v\Omega\nabla N_{\alpha_o} + \left(\Sigma + \frac{\alpha_o}{v}\right) v N_{\alpha_o} = \iint \Sigma' f \nu' N'_{\alpha_o} \, d\Omega' \, dE'. \tag{31}$$

The term α_o/v appears as an additional absorption and it is sometimes referred to as 'time absorption' (or production).

4 Solution of the one-speed transport equation by the spherical harmonics method

4.1 Limitation to a time independent one-speed transport equation

The method is demonstrated for the time-independent, one-speed neutron transport equation in plane geometry. It has been shown that the application of the multigroup approach leads to a set of coupled one speed equations and, therefore, the assumption of one speed is not a real limitation of the method.

For a one speed case scattering is a function only of the cosine of the scattering angle, i.e., $\mu_0 = \Omega \cdot \Omega'$, where Ω' and Ω are the neutron directions before and after scattering, respectively. A quantity $\Sigma_s(r,\Omega,\Omega')$ is then defined by:

$$\Sigma_s(r,\Omega,\Omega') = \Sigma(r)c(r)f(r;\ \Omega \rightarrow \Omega'), \tag{32}$$

which will be referred to as scattering cross section. With this notation the one-speed Eq. (26) may be written:

$$\Omega \nabla \Phi(r,\Omega) + \Sigma(r)\Phi(r,\Omega) = \int \Sigma_s(r,\Omega \cdot \Omega')\Phi(r,\Omega')d\Omega' + Q(r,\Omega). \tag{33}$$

4.2 Choice of geometry

To apply any effective method of solution to the neutron transport equation one has to specify the streaming term given by Eq. (21), i.e., to have expressions for the quantity $\Omega \nabla N$ or $\Omega \nabla \Phi$. For that purpose a system of co-ordinates has to be chosen and the geometry defined. The expression can be derived for co-ordinate systems where the position vector r is given in terms of rectangular, spherical, or cylindrical co-ordinates. Two angular co-ordinates are required to specify the neutron direction and these are chosen to be polar and azimuthal angles. Here the method is demonstrated using the simplest possible geometry, i.e., the plane geometry, for which spherical harmonics reduce to Legendre polynomials.

For plane geometry, in which the neutron angular density (for a specific energy) is a function of x and the azimuthal angle, θ, the streaming term can be expressed:

$$\Omega \nabla \Phi = \frac{d\Phi}{ds} = \frac{\partial \Phi}{\partial x}\frac{dx}{ds} = \frac{\partial \Phi}{\partial x}\cos\theta = \mu \frac{\partial \Phi}{\partial x} \tag{34}$$

with $\mu = \cos\theta$, where θ is the azimuthal angle corresponding to the direction Ω.

Hence Eq. (32) becomes:

$$\mu \frac{\partial \Phi(x,\mu)}{\partial x} + \Sigma(x)\Phi(x,\mu) = \int_0^{2\pi} d\theta' \int_{-1}^{1} \Sigma_s(x,\mu_0)\Phi(x,\mu')d\mu' + Q(x,\mu), \tag{35}$$

4.3 Expansion of the scattering cross section

The scattering cross section is expanded in Legendre polynomials and associated Legendre functions:

$$\Sigma_s(x,\mu_0) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \Sigma_{sl}(x) P_l(\mu_0) \tag{36}$$

and then $P_l(\mu_0)$ is expressed in terms of Legendre polynomials and associated Legendre functions of the directional cosines μ and μ'. The integration over the azimuthal angle is carried out giving:

$$\mu \frac{\partial \Phi(x,\mu)}{\partial x} + \Sigma(x)\Phi(x,\mu) = \sum_{l=0}^{\infty} \frac{2l+1}{2} \Sigma_{sl}(x) P_l(\mu) \int_{-1}^{1} \Phi(x,\mu') P_l(\mu') d\mu' + Q(x,\mu) \tag{37}$$

4.4 Expansion of the flux

The next step is to represent the angular dependence of the neutron flux as an expansion in terms of Legendre polynomials, $P_m(\mu)$:

$$\Phi(x,\mu) = \sum_{m=0}^{\infty} \frac{2m+1}{4\pi} \phi_m(x) P_m(\mu) \tag{38}$$

where $\phi_m(x)$ are the expansion coefficients dependent on x. Because of the orthogonality of $P_m(\mu)$ the latter are given by

$$\phi_m(x) = \int \Phi(x,\mu) P_m(\mu) d\Omega = 2\pi \int_{-1}^{1} \Phi(x,\mu) P_m(\mu) d\mu . \tag{39}$$

If the series is truncated after $N+1$ terms, the result is referred to as a P_N approximation.

For $m = 0$, $P_0(\mu) \equiv 1$; hence $\phi_0(x)$ is simply the total flux at x. For $m = 1$, $P_1(\mu) = \mu$ and Eq. (39) gives

$$\phi_1(x) = 2\pi \int_{-1}^{1} \mu \Phi(x,\mu) d\mu , \tag{40}$$

which is the net current at x in the positive direction.

The general form of equations obtained by substituting expansion of Eq. (38) into Eq. (37) is:

$$(n+1)\frac{d\phi_{n+1}(x)}{dx} + n\frac{d\phi_{n-1}}{dx} + (2n+1)\Sigma_n(x)\phi_n(x) = (2n+1)Q_n(x) \tag{41}$$

$$n = 0, 1, 2, \ldots$$

where $\Sigma_n(x) = \Sigma(x) - \Sigma_{sn}(x)$ and the expansion coefficients are given by the orthogonality relations:

$$\phi_n(x) = 2\pi \int_{-1}^{1} \Phi(x,\mu) P_n(\mu) d\mu, \tag{42a}$$

$$Q_n(x) = 2\pi \int_{-1}^{1} Q(x,\mu) P_n(\mu) d\mu. \tag{42b}$$

4.5 The P_1 approximation

It is easy to see that the first two equations of the system (41), for $n = 0$ and $n = 1$ are:

$$\frac{dJ(x)}{dx} + \Sigma_0(x)\phi(x) = Q_0(x), \tag{43a}$$

$$\frac{d\phi(x)}{dx} + 3\Sigma_1(x)J(x) = 3Q_1(x), \tag{43b}$$

with appropriate definitions of $Q_0(x)$ and $Q_1(x)$.

By definition $\Sigma_a(x) = \Sigma(x) - \Sigma_{s0}(x)$ and, therefore, is equal to the absorption cross section while Σ_1 is the transport cross section..

4.6 Diffusion approximation

If the source is isotropic, $Q_1(x) = 0$ and Eq. (43b) becomes a so called Fick's law:

$$\phi_1(x) = J(x) = -D\frac{d\phi(x)}{dx}, \tag{44}$$

where $D(x)=1/3\Sigma_1(x)$ is the *diffusion coefficient*.

Equation (44) combined with Eq. (43a) gives the *diffusion equation*:

$$-\frac{d}{dx}\left[D(x)\frac{d\phi(x)}{dx}\right] + \Sigma_0(x)\phi(x) = Q_0(x) \tag{45}$$

With anisotropic scattering the equivalent equation can be obtained but with the diffusion coefficient defined as

$$D=(3\Sigma_s(1-\mu_0))^{-1} \tag{46}$$

The quantity $(1-\mu_0)\Sigma_s = \Sigma_{tr}$ is the transport cross section corrected for the first order of anisotropy.

5 Multigroup equations

5.1 P_l equations

The general form of the P_l equations in the multigroup approximation with the characteristic value introduced in section 3 is:

$$\nabla J_g(r) + \Sigma_{0,g}(r)\phi_g(r) = \sum_{g'} \Sigma^s_{g'\to g}(r)\phi_{g'}(r) + \frac{1}{k}\sum_{g'} \kappa\cdot\Sigma^f_{g'\to g}(r)\phi_{g'}(r)$$

$$\nabla \phi_g(r) + 3\Sigma_{0,g}(r)J_g(r) = 3\sum_{g'} \Sigma_{1,g'\to g}(r)J_{g'}(r); \qquad g,g'=1,2,\ldots,G. \tag{47}$$

In Eqs. (47) the indices g and g' refer to the group number and represent the energy dependence, while the variable r refers to the spatial dependence.

5.2 Diffusion equations

With the same notation the multigroup diffusion equations are:

$$-\nabla D_g(r)\nabla\phi_g(r) + \Sigma_{0,g}\phi_g(r) = \sum_{g'} \Sigma^s_{0,g'\to g}(r)\phi_{g'}(r) + \frac{1}{k}\sum_{g'} \kappa\cdot\Sigma^f_{g'\to g}(r)\phi_{g'}(r) \tag{48}$$
$$g=1,2,\ldots,G,\ g'=1,2,\ldots,G$$

6 The B_N approximation

6.1 Assumption on the spatial shape of the neutron flux

The method is demonstrated for the time independent neutron transport equation in plane geometry. The basis of the B_N method is that the spatial dependence of the angular flux can be often approximated by a cosine or exponential term. Thus, by assuming spatial distribution independent of neutron energy it is possible to write:

$$\Phi(x,\mu,E) = e^{-iBx}\Psi(\mu,E), \tag{49}$$

where B^2 for a bare reactor is the lowest eigenvalue of the wave equation, i.e., $\nabla^2\Phi = B^2\Phi$ with the zero flux boundary condition. For a reflected reactor, B is expected to be a real number in the core and an imaginary number in the reflector.

6.2 Expansion of the scattering cross section

If Eq. (49) is inserted to the neutron transport equation with scattering cross section expanded in Legendre polynomials (Eq.(37) generalized to the energy dependent case), we get:

$$\Sigma\left(1-\frac{iB\mu}{\Sigma}\right)\Psi(\mu,E)= \\ =\sum_{l=0}^{\infty}\frac{2l+1}{2}P_l(\mu)\int\Sigma_l^s(E'\to E)\int_{-1}^{1}\Psi(\mu',E')P_l(\mu')d\mu'dE'+\frac{1}{2}F(E) \tag{50}$$

where $Q(x,\mu,E)$ has been replaced by an isotropic fission source, $\frac{1}{2}F(E)e^{-iBx}$.

6.3 *Algebraic transformations*

Equation (50) is divided by $1-(iB\mu/\Sigma)$, multiplied by $P_n(\mu)$, and then integrated to obtain for $n = 0, 1, 2, \ldots$

$$\Sigma(E)\phi_n(E)=\sum_{l=0}^{\infty}(2l+1)A_{ln}(E)\int\Sigma_l^s(E'\to E)\phi_l(E')dE'+A_{0n}(E)F(E) \tag{51}$$

with

$$A_{ln}(E)=\frac{1}{2}\int_{-1}^{1}\frac{P_l(\mu)P_n(\mu)}{1-\dfrac{iB\mu}{\Sigma(E)}}d\mu, \tag{51a}$$

$$\phi_n(E)=\int_{-1}^{1}\Psi(\mu,E)P_n(\mu)d\mu. \tag{51b}$$

The coefficients A_{ln} can be found by the fact that they satisfy the recurrence relation:

$$\frac{1}{y}(2l+1)A_{j,l}(y)-(l+1)A_{j,l+1}-lA_{j,l-1}=\frac{\delta_{jl}}{y} \tag{52}$$

where $y=\dfrac{iB}{\Sigma(E)}$.

Furthermore, $A_{jl}=A_{lj}$ and $A_{00}=\dfrac{\tanh^{-1}y}{y}$.

The set of coupled equations (51) can be solved numerically for ϕ_n provided the sum on the right-hand side is truncated. If the series is terminated by assuming $\phi_l = 0$ for $l>N$ the result is the B_N approximation. In practical lattice calculations the most often used is the B_1 approximation.

7 Leakage in diffusion approximation

In every realistic reactor system there exists the neutron leakage through the outer boundary. This leakage can be accounted for in an approximate way by applying the formulas based on the Fick's law. Let us consider in infinitesimal cube defined in Fig.3.

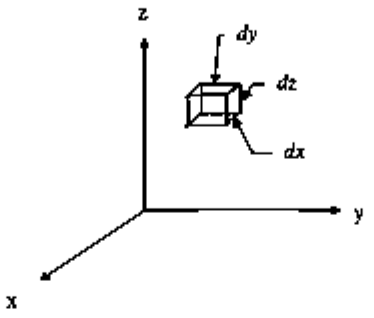

Figure 3. Infinitesimal cube in rectangular co-ordinate system

Through each of the cube boundaries there exists a neutron flow connected with the net current:

$$J_{net} = J_+ - J_- \tag{53}$$

where the components, J_+ and J_-, describe the partial currents in positive and negative directions. Using the definition of the diffusion coefficient introduced in Eqs. (44,46):

$$J = D \cdot \frac{\partial \phi}{\partial z}$$

$$D = \lambda_{tr}/3,$$

the following formulas can be derived for the magnitude of the partial currents in the z direction:

$$\begin{aligned} J_+ &= \frac{\phi}{4} - \frac{D}{2} \cdot \frac{\partial \phi}{\partial z} \\ J_- &= \frac{\phi}{4} + \frac{D}{2} \cdot \frac{\partial \phi}{\partial z} \end{aligned} \tag{54}$$

Hence for the z direction:

$$J_z = -D \frac{\partial \phi}{\partial z} = -\frac{\lambda_{tr}}{3} \frac{\partial \phi}{\partial z} \tag{55}$$

and

$$(J_{z+dz} - J_z)dxdy = -D\left[\left(\frac{\partial\phi}{\partial z}\right)_{z+dz} - \left(\frac{\partial\phi}{\partial z}\right)_z\right]dxdy = \\ = -D\frac{\partial^2\phi}{\partial z^2}dxdydz = -D\frac{\partial^2\phi}{\partial z^2}dV \tag{56}$$

Similar expressions are obtained for x and y.

The leakage of neutrons out of an arbitrary volume will be composed of those in all partial directions and can be written as $-D\nabla^2\phi$. Assuming the spatial independence of the diffusion coefficient $D(x)=D$ in Eq. (45) the diffusion equation can be written:

$$D\nabla^2\phi - \Sigma^a\phi + Q = 0\,, \tag{57}$$

which expresses the neutron balance in diffusion approximation: leakage out of the system plus absorption equals the total sources, Q, including the external as well as internal sources.

8 The boundary with vacuum

The typical boundary condition, introduced together with the diffusion approximation, is the neutron flux going to zero at some distance from the outer boundary of the system considered. Let us consider an idealised case of an infinite plane reactor core surrounded by vacuum. The distance at which the flux drops off to zero is called then the *extrapolation distance*, and it is shown in Fig. 4.

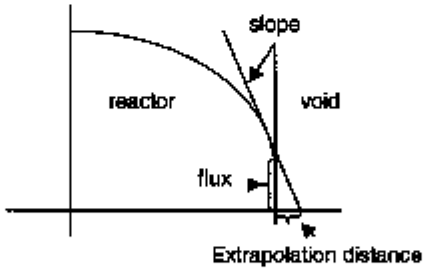

Figure 4. Extrapolation distance.

By diffusion theory, using Eqs. (55) the extrapolation distance, d, is found to be equal to:

$$d = \frac{2}{3}\lambda_{tr}\,.$$

A calculation based on the transport theory gives approximately:

$$d = -\frac{\phi}{d\phi / dx} = 0.71\,\lambda_{tr}\,. \tag{58}$$

where λ_{tr} is the transport mean free path.

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