Hilbert schemes of points and Heisenberg algebras

Geir Ellingsrud and Lothar Göttsche[†]

[†] *The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.*

Lecture given at the

School on Algebraic Geometry

Trieste, 26 July – 13 August 1999

LNS001002

*gottsche@ictp.trieste.it

Abstract

Let $X^{[n]}$ be the Hilbert scheme of n points on a smooth projective surface X over the complex numbers. In these lectures we describe the action of the Heisenberg algebra on the direct sum of the cohomologies of all the $X^{[n]}$, which has been constructed by Nakajima. In the second half of the lectures we study the relation of the Heisenberg algebra action and the ring structures of the cohomologies of the $X^{[n]}$, following recent work of Lehn. In particular we study the Chern and Segre classes of tautological vector bundles on the Hilbert schemes $X^{[n]}$.

Contents

1. Introduction

In these notes X will be a smooth and projective surface over the complex numbers. The object of our interest will be the Hilbert scheme of points on X. For any nonnegative integer n there is such a Hilbert scheme $X^{[n]}$ which parameterizes finite subschemes of X of length n.

If $W \subset X$ is a finite subscheme of length n, we shall also denote the corresponding point in $X^{[n]}$ by W.

There is a universal subscheme $Z_n \subset X^{[n]} \times X$ whose underlying set is given as $Z_n = \{(W, P) \mid P \in W\}$. The first projection from $Z_n \subset X^{[n]} \times X$ onto $X^{[n]}$ induces a finite and flat map $\pi : Z_n \to X^{[n]}$. Let $\mathcal{O}^{[n]} := \pi_*(\mathcal{O}_{Z_n})$. It is a locally free sheaf on $X^{[n]}$ of rank n.

The Hilbert scheme $X^{[n]}$ enjoys several nice geometric properties, the most basic one being:

Theorem 1.1. *The Hilbert scheme $X^{[n]}$ is smooth, connected and of dimension $2n$.*

The first proof of this result was given in [6]. Once connectedness is established, that the dimension of $X^{[n]}$ is $2n$, is clear: Each of the n points has two degrees of freedom.

Any subscheme $W \in X^{[n]}$ can be written as $W = \bigcup_i W_i$ where the W_i are mutually disjoint subschemes each having support in just one point. If $\operatorname{Supp} W_i = \{P_i\}$, we may define the 0-cycle

$$\rho(W) := \sum_i (\operatorname{length} W_i) P_i = \sum_{W \in X} (\operatorname{length} \mathcal{O}_{W,P}) P.$$

This 0-cycle is an element of the symmetric power $X^{(n)} := X^n / \mathfrak{S}_n$; the quotient of X^n by the symmetric group $\mathfrak{S}_n$ acting on X^n by permutation. In this way we get a map $\rho : X^{[n]} \to X^{(n)}$, which turns out to be a morphism (see [6]). It is called the Hilbert-Chow morphism.

Contrary to $X^{[n]}$ the symmetric power $X^{(n)}$ is singular. Along the diagonals, where two or more points come together, the action of the symmetric group has nontrivial isotropy, and because this happens in codimension two or more, the quotient will be singular.

It is easy to see that the Hilbert-Chow morphism is birational; indeed it is an isomorphism between the set of reduced subschemes in $X^{[n]}$ and the subset of $X^{(n)}$ consisting of 0-cycles all whose points are different.

Theorem 1.2. *The Hilbert-Chow morphism is a resolution of the singularities. In fact it is even a semismall resolution; which means that*

$$\operatorname{codim}\{z| \dim \rho^{-1}(z) \geq r\} \geq 2r$$

for any natural number r.

For any point $P \in X$ we let the closed subscheme $M_n(P) \subset X^{[n]}$ be the set of subschemes whose support is the single point P. In other words

$$M_n(P) = \{W \in X^{[n]} \mid \operatorname{Supp}(W) = \{P\}\}.$$

This is set-theoretically the same as $\rho^{-1}(nP)$, and $M_n(P)$ is indeed closed. We also give a name to the closed subset of $X^{[n]}$ whose elements are the subschemes with support in one (unspecified) point, and define

$$M_n := \{W \in X^{[n]} \mid \operatorname{Supp}(W) \text{ contains just one point}\}.$$

There is an obvious map $M_n \to X$ which sends a one-point-supported subscheme to the point where it is supported. The following is a basic result which now has several proofs. The first one was given by Briançon in [2], For other proofs see [4] or [5].

Theorem 1.3. *$M_n(P)$ is irreducible of dimension $n-1$, and M_n is irreducible of dimension $n+1$.*

When studying the Hilbert schemes $X^{[n]}$ of points, it is often a good idea to look at all the $X^{[n]}$ at the same time, because they are all related and therefore there is hope that a new structure emerges. One instance of this is the fact that there is a nice generating function for all the Betti numbers of all the $X^{[n]}$. We shall see that this is a reflection of the fact that the direct sum of all the cohomologies of all the $X^{[n]}$ has an additional structure. It is an irreducible module for a Heisenberg algebra action. This has been shown by Nakajima [13]. This Heisenberg action is constructed by means of correspondences between the Hilbert schemes, and the varieties M_n and $M_n(P)$ play a big role. In fact the idea is that one can go from the cohomology of $X^{[k]}$ to that of $X^{[k+n]}$, by adding subschemes of length n supported in one point of X.

In the second part of these lecture notes we will investigate how this Heisenberg action is related to the ring structure of the cohomology rings of the Hilbert schemes. Here we follow the work [11] of Lehn. We are particularly interested in the Chern classes of so-called tautological vector bundles on the Hilbert schemes. For every vector bundle V on X one has an associated tautological vector bundle $V^{[n]}$ on $X^{[n]}$ whose fibers over the points $W \in X^{[n]}$ are naturally identified with $H^0(W, V|_W)$. In particular, if V has rank r, then $V^{[n]}$ is a vector bundle of rank nr. The Chern classes and Chern numbers of these tautological bundles have interesting geometrical and enumerative interpretations.

We study the operators of multiplication with the Chern classes of the tautological sheaves, and express them in terms of the operators of the Heisenberg algebra action. It is easy to see that the Heisenberg algebra action induces an action of a Virasoro algebra and an important step in the argument is a geometric interpretation of the Virasoro operators. Finally, we restrict to the case of tautological vector bundles associated to a line bundle L on X. We find a generating function for all the Chern classes in terms of the Heisenberg operators and, at least conjecturally, a generating function for the top Segre classes of the $L^{[n]}$.

2. The Betti numbers of $X^{[n]}$

If one is interested in the cohomology of $X^{[n]}$, the first question to ask is what are the Betti numbers of $X^{[n]}$; i.e., what are the dimensions $b_i(X^{[n]}) := \dim H^i(X^{[n]})$? (In these notes we will only be interested in homology and cohomology with coefficients in $\mathbb{Q}$, so for any space Y we write $H^i(Y)$ for $H^i(Y, \mathbb{Q})$ and $H_i(Y)$ for $H_i(Y, \mathbb{Q})$.)

The Betti numbers of the Hilbert schemes $X^{[n]}$ were determined in [8]. There the following generating series for the Betti numbers was obtained:

Theorem 2.1.

$$\sum_{n\geq 0, i\geq 0} b_i(X^{[n]})\, t^i q^n = \prod_{m>0, i\geq 0} (1 - (-1)^i t^{2m-2+i} q^m)^{(-1)^{i+1} b_i(X^{[n]})}.$$

There are several proofs of this formula. The original proof is by using the Weil-conjectures and counting subschemes over finite fields. A second proof, based on intersection cohomology, was given by Göttsche and Soergel in [9], and finally in [3] Cheah gave a third proof using the so-called virtual Hodge polynomials. In addition to the Betti numbers, the last two proofs also give the Hodge numbers of the Hilbert schemes.

If one puts $t = -1$ in Theorem 2.1, one gets an expression for the topological Euler characteristic $e(X^{[n]})$ of the spaces $X^{[n]}$:

$$\sum_{n\geq 0} e(X^{[n]})\, q^n = \prod_{m>0} (1 - q^m)^{-e(X)}$$

and by putting $t = 1$ one gets the generating series for the total dimensions of the cohomology of $X^{[n]}$:

$$\sum_{n\geq 0} \dim_{\mathbb{Q}} H^\bullet(X^{[n]})\, q^n = \prod_{m>0} \frac{(1+q^m)^{d_-}}{(1-q^m)^{d_+}}. \tag{1}$$

Here d_- and d_+ are respectively the dimensions of the even and odd part of $H^\bullet(X)$, i.e.,

$$d_+ = \sum_i \dim H^{2i}(X), \quad d_- = \sum_i \dim H^{2i+1}(X).$$

Later in these notes we shall come back to these formulas and give indications on how one can prove them.

One should note that we got nice generating functions for the Betti numbers and Euler numbers by looking at all the Hilbert schemes $X^{[n]}$ at once. This is a first indication that one should also look at all the cohomologies of the Hilbert schemes at the same time.

3. The Fock space and the current algebra

Let

$$\mathbb{H}(X) = \bigoplus_{n\geq 0} H^\bullet(X^{[n]})$$

be the direct sum of all the cohomologies of all the Hilbert schemes $X^{[n]}$. This is a bigraded vector space over $\mathbb{Q}$ whose homogeneous parts are the cohomology groups $H^i(X^{[n]})$ for $n \geq 0$ and $i \geq 0$. For any class $\alpha \in H^i(X^{[n]})$ we will call n the *weight* of α and i the *cohomological degree* or for short the *degree* of α. Sometimes we will write $\deg \alpha = (n, i)$.

The Hilbert scheme $X^{[0]}$ is just one point — the empty set is the only subscheme of length zero. Hence $H^\bullet(X^{[0]}) \cong \mathbb{Q}$ in a canonical way. We let $\mathbf{1}$ denote the fundamental class $[X^{[0]}]$. It corresponds to $1 \in \mathbb{Q}$, and we call it the *vacuum vector*.

The space $\mathbb{H}(X)$ has a parity structure, or a super structure as many call it: There is a decomposition

$$\mathbb{H}(X) = \mathbb{H}^+(X) \oplus \mathbb{H}^-(X)$$

where $\mathbb{H}^+(X)$ and $\mathbb{H}^-(X)$ are respectively the sums of the even and odd part of the cohomology $H^\bullet(X^{[n]})$; that is

$$\mathbb{H}^+(X) = \bigoplus_{n\geq 0, i\geq 0} H^{2i}(X^{[n]}),$$
$$\mathbb{H}^-(X) = \bigoplus_{n\geq 0, i\geq 0} H^{2i+1}(X^{[n]}).$$

The intersection form

$$\int_{X^{[n]}} \alpha\beta =: \langle \alpha, \beta \rangle$$

induces an intersection form on $\mathbb{H}(X)$ which respects the parity structure, which means that it is symmetric on $\mathbb{H}^+(X)$ and antisymmetric on $\mathbb{H}^-(X)$, and that the two spaces $\mathbb{H}^+(X)$ and $\mathbb{H}^-(X)$ are orthogonal.

The Poincaré series of $\mathbb{H}(X)$ with respect to the weight-grading is given by Göttsche's formula with $t = 1$ as in (1):

$$\sum_{n\geq 0} \dim_{\mathbb{Q}} H^\bullet(X^{[n]})\, q^n = \prod_{m>0} \frac{(1+q^m)^{d_-}}{(1-q^m)^{d_+}}.$$

This series also appears naturally in a construction in the theory of Lie algebras: Let V be a $\mathbb{Q}$-vector space with a parity structure, or a super space if you want; that is a decomposition $V = V^+ \oplus V^-$ of V into an odd and an even part. Assume that V comes equipped with a bilinear form $\langle\ ,\ \rangle$ respecting the parity structure. The cohomology $H^\bullet(X)$ with the pairing $\int_X \alpha \cdot \beta$ is our prototype of such a V.

Associated to V one constructs *the Fock space* $\mathbb{F}(V)$ in the following way: First we take a look at $V \otimes_{\mathbb{Q}} t\,\mathbb{Q}[t]$. A typical element of this space looks like $\sum_{i=1}^m v_i \otimes t^i$. Let T be the full tensor algebra on $V \otimes_{\mathbb{Q}} t\,\mathbb{Q}[t]$. To construct $\mathbb{F}(V)$ we impose in T the (super-)commutation relations:

$$(2) \qquad [u\otimes t^i, v\otimes t^j] := (u\otimes t^i)(v\otimes t^j) - (-1)^{p(u)p(v)}(v\otimes t^j)(u\otimes t^i) = 0$$

where u and v are any homogeneous elements in V, i.e., elements either in V^+ or V^-, and where $i \geq 1$ and $j \geq 1$ are any integers. By $p(w)$ we mean the parity of a homogeneous element w, i.e., $p(w) = 0$ when $w \in V^+$ and $p(w) = 1$ when $w \in V^-$. In order not to get confused with having two different $\otimes$-signs around, one from $V \otimes_{\mathbb{Q}} t\,\mathbb{Q}[t]$ and one from T, we have suppressed the $\otimes$-signs from the tensor algebra T in equation (2).

The formal way to impose the relations above, is to divide T by the two-sided ideal generated by the relations in (2). Clearly $\mathbb{F}(V)$ is an algebra. The unit element $\mathbf{1} \in \mathbb{F}^0(V)$ is called the *vacuum vector*.

There is a natural grading on $V \otimes_{\mathbb{Q}} t\,\mathbb{Q}[t]$ for which the degree of $v \otimes t^i$ is i. This grading induces, in an obvious way, a grading on the tensor algebra T. As the relations (2) are homogeneous of degree $i + j$, the Fock space $\mathbb{F}(V)$ is graded.

The elements of $\mathbb{F}(V)$ are linear combinations of monomials of the form

$$(v_1 \otimes t^{j_1})\,(v_2 \otimes t^{j_2}) \dots (v_p \otimes t^{j_p})$$

where each v_m is either an even or an odd element. The degree of such a monomial is $\sum j_m$. The Fock space also has a parity structure. A monomial as the one above is even (resp. odd) if the number of odd v_m's is even (resp. odd).

One may then easily check that there is an isomorphism of graded vector spaces

$$\mathbb{F}(V) \cong \bigotimes_{m=0}^{\infty} S(V^+ \otimes t^m) \otimes \Lambda(V^- \otimes t^m).$$

Here

$$S(V) := \bigoplus_{i \geq 0} S^i(V), \quad \Lambda(V) := \bigoplus_{i \geq 0} \Lambda^i(V),$$

are the symmetric and alternating algebra on V.

From this the Poincaré series of $\mathbb{F}(V)$ is readily found to be

$$\sum_{m \geq 0} \dim_{\mathbb{Q}} \mathbb{F}^m(V) = \prod_{m > 0} \frac{(1 + q^m)^{\dim V^-}}{(1 - q^m)^{\dim V^+}}.$$

There is another algebra one may associate to V called the *current algebra*. To construct this we start by setting $V[t, t^{-1}] = V \otimes \mathbb{Q}[t, t^{-1}]$. The elements of $V[t, t^{-1}]$ are linear combinations of the elements $q_i[v] := v \otimes t^i$ for $v \in V$ and $i \in \mathbb{Z}$.

Let now T be the full tensor algebra on $V[t, t^{-1}]$. Elements of T are linear combinations of monomials $q_{i_1}[v_1]\,q_{i_2}[v_2] \dots q_{i_p}[v_p]$ where we again suppress the $\otimes$-signs.

By declaring the degree (or weight) of $q_i[v]$ to be i, we get a grading on T. There is also a parity structure on T: We declare $q_i[v]$ to be even if v is even and odd if v is odd; and a monomial $q_{i_1}[v_1]\,q_{i_2}[v_2] \dots q_{i_p}[v_p]$ is even (resp. odd) if it contains an even (resp. odd) number of odd $q_i[v]$'s.

We get the current algebra $\mathbb{S}(V)$ by imposing the following relations in T:

$$\big[q_i[u], q_j[v]\big] = i\delta_{i+j}\langle u, v\rangle \mathbf{e} \tag{3}$$

where $\mathbf{e}$ is the unit element in $T^0V[t, t^{-1}] = \mathbb{Q}$, and where u and v are any elements either in V^+ or in V^-. The bracket is the supercommutator

$$[A, B] = AB - (-1)^{p(A)p(B)}BA.$$

We also use the convention that $\delta_m = 0$ if $m \neq 0$ and $\delta_0 = 1$.

The current algebra $\mathbb{S}(V)$ acts on the Fock space $\mathbb{F}(V)$ in the following way. Recall that the Fock space is an algebra.

If $i > 0$, we let the element $q_i[u]$ act as multiplication by $u \otimes t^i$ in the algebra $\mathbb{F}(V)$, i.e., $q_i[u]\,w = (u \otimes t^i)\,w$ for any $w \in \mathbb{F}(V)$. In particular $q_i[u]\mathbf{1} = u \otimes t^i$.

For $i = 0$, we simply put $q_0[u]\,w = 0$ for any u and w.

To define the action of the operators $q_{-i}[u]$, with $i > 0$, it is sufficient to state that $q_{-i}[u]\mathbf{1} = 0$ for any $i > 0$ and any u. Indeed by the relations (3) we get

$$\begin{aligned} q_{-i}[u]\,(v \otimes t^j) &= q_{-i}[u]\,q_j[v]\mathbf{1} \\ &= \pm q_j[v]\,q_{-i}[u]\mathbf{1} - i\delta_{j-i}\langle u, v\rangle \mathbf{1} \\ &= -i\delta_{j-i}\langle u, v\rangle \mathbf{1}. \end{aligned}$$

Thus the action is given by the formula

$$q_{-i}[u]\,(v \otimes t^j) = -i\delta_{j-i}\langle u, v\rangle \mathbf{1}. \tag{4}$$

We call the operators $q_i[u]$ *creation* operators if $i > 0$ and *annihilation* operators if $i < 0$. One has the following lemma:

Lemma 3.1. *If the pairing $\langle\ ,\ \rangle$ is non-degenerate, the $\mathbb{S}(V)$-module $\mathbb{F}(V)$ is irreducible, i.e., there is no proper, nonzero subspace invariant under $\mathbb{S}(V)$.*

Proof. It is clear that the vacuum vector $\mathbf{1}$ is a generator for $\mathbb{F}(V)$ as a module over $\mathbb{S}(V)$. On the other hand, by applying an appropriate sequence of annihilation operators $q_{-i}[u]$ to any element w of $\mathbb{F}(V)$, we may bring it back to the vacuum $\mathbf{1}$. Indeed if $\{v_\bullet\}$ and $\{v'_\bullet\}$ are dual bases for V, then by equation (4) above we get

$$\begin{aligned} q_{-i_p}[v'_{i_p}]\,q_{-i_{p-1}}[v'_{i_{p-1}}]\dots q_{-i_1}[v'_{i_1}](v_{i_1} \otimes t^{i_1})\,(v_{i_2} \otimes t^{i_2})\dots(v_{i_p} \otimes t^{i_p}) &= \\ = (-1)^p i_1 \cdot i_2 \cdot \ldots \cdot i_p\,\mathbf{1} \end{aligned}$$

where the v_i's and the v_i''s are elements from the bases $\{v_\bullet\}$ and $\{v'_\bullet\}$. The operator
$q_{-i_p}[u_{i_p}]q_{-i_{p-1}}[u_{i_{p-1}}]\dots q_{-i_1}[u_{i_1}]$ kills any other monomial made from elements in $\{v_\bullet\}$, again by the relation (4). Hence any nonzero and invariant subspace contains the vacuum, and consequently equals $\mathbb{F}(V)$ because the vacuum generates $\mathbb{F}(V)$ as an $\mathbb{S}(V)$-module. □

4. The Nakajima operators

We now come back to our space $\mathbb{H}(X)$. It has the same Poincaré series as the Fock space modelled on the cohomology $H^\bullet(X)$ of X. The aim of this section is to define an action of the current algebra $\mathbb{S}\big(H^\bullet(X)\big)$ on the space $\mathbb{H}(X)$ in a geometric way, making $\mathbb{H}(X)$ and $\mathbb{F}\big(H^\bullet(X)\big)$ isomorphic as $\mathbb{S}\big(H^\bullet(X)\big)$-modules.

We need to define operators $q_i[u]$ for $i \in \mathbb{Z}$ and $u \in H^\bullet(X)$ satisfying the relations (3). The operator $q_i[u]$ changes the weight by i, hence is given by a map $H^\bullet(X^{[n]}) \to H^\bullet(X^{[n+i]})$ for any $n \geq 0$. In order to define these maps, we introduce the *incidence scheme* $X^{[n,n+i]} \subset X^{[n]} \times X^{[n+i]}$, where now $i \geq 0$. It is defined as

$$X^{[n,n+i]} := \big\{(W,W') \bigm| W \subset W', W \in X^{[n]} \text{ and } W' \in X^{[n+i]}\big\}$$

Here, as also in future $W \subset W'$ means that W is a subscheme of W'. This is easily seen to be a closed subset of the product, and we give it the reduced scheme structure.

The two projections induce two maps $p_n : X^{[n,n+i]} \to X^{[n]}$ and $q_{n+i} : X^{[n,n+i]} \to X^{[n+i]}$. There also is a morphism $\rho : X^{[n,n+i]} \to X^{(i)}$ which is a variant of the Hilbert-Chow-map. If $W \subset W'$, then for the ideals I_W and $I_{W'}$ of I_W and $I_{W'}$, we do have the inclusion $I_{W'} \subset I_W$, and the quotient $I_W/I_{W'}$ is an $\mathcal{O}_X$-module of finite length which is supported at the points where the two subschemes W and W' differ. We define

$$\rho(W,W') := \sum_{P \in X} \operatorname{length}(I_W/I_{W'})\, P \in X^{(i)}.$$

One may show that ρ is a morphism.

Inside $X^{(i)}$ there is the small diagonal $\Delta = \big\{iP \bigm| P \in X\big\}$ which is isomorphic to X.

We have the following diagram:

$$
\begin{array}{ccccccc}
X = & \Delta & \subset & X^{(i)} & & \\
& f\uparrow & & \rho\uparrow & & \\
& Z_{n,i} & \subset & X^{[n,n+i]} & \xrightarrow{q_{n+i}} & X^{[n+i]} \\
& & & p_n\downarrow & & \\
& & & X^{[n]} & &
\end{array} \tag{5}
$$

where $Z_{n,i}$ is the component[1] of

$$\rho^{-1}(\Delta) = \{(W, W') \mid W \subset W', I_W/I_{W'} \text{ is supported in one point}\}$$

which is the closure of the subset where Supp $(I_W/I_{W'})$ is disjoint from W. We give it the reduced scheme structure.[2] One easily checks that

$$\dim Z_{n,i} = 2n + i + 1, \tag{6}$$

indeed W is arbitrary in $X^{[n]}$, but $W' - W$ is confined to M_i.

We may pull back any class $u \in H^\bullet(X)$ along f to get a cohomology class f^*u on $Z_{n,i}$. Applying this to the fundamental class $[Z_{n,i}]$, we get the homology class $f^*u \cap [Z_{n,i}]$. This in turn we may push forward to $X^{[n,n+i]}$ via the inclusion $j : Z_{n,i} \to X^{[n,n+i]}$, and in this way we get the homology class

$$Q_{n,i}(u) := j_*(f^*u \cap [Z_{n,i}])$$

on $X^{[n,n+i]}$.

Now we are ready to define the Nakajima *creation operators*; i.e., the operators $q_i[u]$ with $i \geq 0$. We define their action on an element $\alpha \in H^\bullet(X^{[n]})$ by

$$q_i[u]\,\alpha := q_{n+i\,*}(p_n^*\,\alpha \cap Q_{n,i}(u)),$$

which we regard as an element in $H^\bullet(X^{[n+i]})$ by Poincaré duality.

This definition is similar to the classical way of defining the *correspondence* between $X^{[n]}$ and $X^{[n+i]}$ associated to a class on their product — if one insists on $q_i[u]$ being a correspondence, one has

$$q_i[u]\,\alpha = pr_{2\,*}(pr_1^*\,\alpha \cap \nu_* Q_{n,i}(u))$$

where $\nu : Z_{n,i} \to X^{[n]} \times X^{[n+i]}$ is the inclusion map, and where pr_1 and pr_2 are the two projections.

---

[1] To our knowledge it is unknown whether $\rho^{-1}(\Delta)$ is irreducible or not.

[2] The scheme-theoretical inverse image $\rho^{-1}(\Delta)$ is not reduced.

In order to get some geometric feeling for what these operators do, we assume that u and α are represented by submanifolds $U \subset X$ and $A \subset X^{[n]}$. Then $q_i[u]\,\alpha$ is represented by the subspace

$$(7)\quad \big\{W' \in X^{[n+i]} \;\big|\; \text{there is a } W \in A \text{ with } W \subset W',\ W \text{ and } W' \text{ such that they differ in one point in } U\big\}.$$

To put it loosely, the creation operator $q_i[u]$ sends A to the set of subschemes which we obtain by adding a subscheme of length i supported in just one point from U to a subscheme in A. As an illustration we prove the following lemma

Lemma 4.1.

$$q_i[\mathrm{pt}]\mathbf{1} = [M_i(P)].$$
$$q_i[X]\mathbf{1} = [M_i].$$

Proof. To explain the first equality, we observe that $\mathbf{1}$ is represented by the empty set. Hence by (7) the class $q_i[\mathrm{pt}]\mathbf{1}$ is represented by

$$\big\{W' \in X^{[i]} \;\big|\; \emptyset \subset W', \emptyset \text{ and } W' \text{ differ only in } P\big\},$$

where P is any point in X, and this is clearly $M_i(P)$; we are just adding subschemes supported at P to the empty set.

The second equality is similar. We add subschemes of length i supported in one point to the empty set, but this time without any constraint on the point. □

We now come to the definition of the Nakajima *annihilation operators* $q_{-i}[u]$, where $i > 0$. We shall, except for a sign factor, literally go the other way around in the diagram (5). For any class $\beta \in H^\bullet(X^{[n+i]})$ we define

$$q_{-i}[u]\beta := (-1)^i p_{n\,*}\big(q_{n+i}^*\,\beta \cap Q_{n,i}(u)\big).$$

The geometrical interpretation of these annihilation operators is analogous to that of the creation operators. If the class β is represented by a submanifold $B \subset X^{[n+i]}$, then $q_{-i}[u]\beta$ will be represented by the subspace

$$(8)\quad \big\{W \in X^{[n]} \;\big|\; \text{there is a } W' \in B \text{ with } W \subset W' \text{ such that they differ in just one point in } U\big\}.$$

In other words, the annihilation operator $q_{-i}[u]$ sends B to the set of the subschemes we get by throwing away subschemes supported in one point in U from subschemes in B. Of course this is possible only for some of the subschemes in B.

We will give one example. Let $C \subset X$ be a smooth curve, and let $\sigma = [C]$ be its fundamental class in $H^2(X)$. For every $n \geq 0$ the symmetric product $C^{(n)}$ is naturally embedded in the Hilbert scheme $X^{[n]}$. Put $\sigma^n = [C^{(n)}]$. Let C' be another smooth curve, and assume that $\langle C, C' \rangle = a$. Let $\sigma' = [C']$.

Lemma 4.2.

$$q_{-i}[\sigma']\sigma^n = (-1)^i a \sigma^{n-i}$$

Proof. We assume for simplicity that C and C' intersect transversally in just one point. Because C is smooth, a subscheme $W \subset C$ is uniquely determined by the associated 0-cycle $\sum_{P \in C} \operatorname{length}(W_P) P$. Hence there is just one subscheme W' of length i in $C^{(i)}$, whose support is $C \cap C'$. Splitting off W' from the subschemes in $C^{(n)}$ containing it, obviously gives an isomorphism from $\{W \cup W' \in C^{(n)} \mid W \in C^{(n-i)}\}$ to C^{n-i}. This concludes the proof. □

The operators $q_i[u]$ and $q_{-i}[u]$ behave very well with respect to the intersection pairings on $X^{[n]}$ and $X^{[n+i]}$:

Lemma 4.3. *For classes $\alpha \in H^\bullet(X^{[n]})$ and $\beta \in H^\bullet(X^{[n+i]})$ we have the equality*

$$(-1)^i \int_{X^{[n]}} \alpha \cdot q_{-i}[u]\beta = \int_{X^{[n+i]}} (q_i[u]\alpha) \cdot \beta.$$

Proof. By the definition of the operators and the projection formula, both are equal to

$$\int_{X^{[n,n+i]}} p_n^* \alpha \cdot q_{n+i}^* \beta \cap Q_{n,i}(u).$$

□

The following lemma is easily deduced from the definition of the Nakajima operators

Lemma 4.4. *The operator $q_i[u]$ is of bidegree $(i, \deg u + 2(i-1))$.*

5. The relations

The basic result of Nakajima in [13] is that his creation and annihilation operators satisfy the relations of the current algebra. Below we shall sketch a proof of that, closely following the proof that Nakajima gave in [14].

Theorem 5.1. *(Nakajima, Grojnowski) For all integers i and j and all classes u and v in $H^\bullet(X)$ the following relation holds*

$$\big[q_i[u], q_j[v]\big] = i\delta_{i+j}\langle u, v\rangle \mathrm{id}.$$

The proof is in two steps. The first is to establish

Proposition 5.2. *There are universal non-zero constants c_i such that*

$$\big[q_i[u], q_j[v]\big] = c_i \delta_{i+j} \langle u, v\rangle \mathrm{id}.$$

Here by universal we mean that the c_i's neither depend on u or v nor on the surface X. A sketch of the proof of this proposition, will occupy section 6. The next step is — naturally enough — to establish

Proposition 5.3. $c_i = i$.

The last proposition can be proved in two different ways. The constants c_i have a natural interpretation as intersection numbers on the Hilbert scheme. Recall that $\dim M_i = i+1$ and $\dim M_i(P) = i-1$. Therefore M_i and $M_i(P)$ are of complementary dimension, and their intersection gives a number. However $M_i(P) \subset M_i$ so they do not intersect properly and $\int_{X^{[i]}} [M_i(P)][M_i]$ is not entirely obvious to compute. By induction one may prove (see [5]):

Proposition 5.4. *(Ellingsrud–Strømme)*

$$\int_{X^{[i]}} [M_i(P)][M_i] = (-1)^{i-1} i.$$

The following lemma then proves Proposition 5.3.

Lemma 5.5. *If $i > 0$ then $c_i = (-1)^{i-1} \int_{X^{[i]}} [M_i(P)][M_i]$.*

Proof. Recall that by Lemma 4.1 we have $[M_i(P)] = q_i[\mathrm{pt}]\mathbf{1}$ and $[M_i] = q_i[X]\mathbf{1}$. The Nakajima relation for the operators $q_{-i}[X]$ and $q_i[X]$ reads

$$q_i[X]\, q_{-i}[\mathrm{pt}] - q_{-i}[\mathrm{pt}]\, q_i[X] = c_i \cdot \mathrm{id}.$$

When we apply this to the vacuum vector, we obtain

$$q_{-i}[\mathrm{pt}]\, q_i[X]\mathbf{1} = -c_i$$

because any annihilation-operator kills the vacuum. Now, by Lemma 4.3, we get

$$\begin{aligned}
\int_{X^{[i]}} [M_i(P)][M_i] &= \int_{X^{[i]}} q_i[\mathrm{pt}]\mathbf{1} \cdot q_i[X]\mathbf{1} = \\
&= (-1)^i \int_{X^{[0]}} \mathbf{1} \cdot q_{-i}[\mathrm{pt}] q_i[X]\mathbf{1} = \\
&= (-1)^i \int_{X^{[0]}} (-c_i)\mathbf{1} = (-1)^{i-1} c_i.
\end{aligned}$$

□

There is also another and very elegant approach to Proposition 5.3 due to Nakajima where he uses vertex operators. We shall give this later on.

The main consequence of the Nakajima-Grojnowski theorem is the following:

Theorem 5.6. *The space* $\mathbb{H}(X)$ *and the Fock-space* $\mathbb{F}(H^\bullet(X))$ *are isomorphic as* $\mathbb{S}(H^\bullet(X))$*-modules.*

Proof. There is a map as $\mathbb{S}(H^\bullet(X))$-modules from $\mathbb{F}(H^\bullet(X))$ to $\mathbb{H}(X)$ defined by sending $u \otimes t^i$ to $q_i[u]\mathbf{1}$. The two spaces have the same Poincaré series, and $\mathbb{F}(H^\bullet(X))$ is an irreducible $\mathbb{S}(H^\bullet(X))$-module. □

6. Indication of how to get the relations

In this section we explain in a sketchy way why the commutation relations in Theorem 5.1 hold.

We will first treat the case when i and j have the same sign, for example both are positive. This is the case of the composition of two creation-operators.

Then $\delta_{i+j} = 0$, and we have to prove that $q_i[u]$ and $q_j[v]$ commute up to the correct sign. For simplicity we also assume that $u = [U]$ and $v = [V]$ where U and V are submanifolds of X intersecting transversally.

In the definition of the Nakajima operators we made use of the subvariety $Z_{n,i} \subset X^{[n]} \times X^{[n+i]}$. Recall that it was given as

$$Z_{n,i} = \{(W, W') \mid W \subset W' \text{ differ in one point}\}.$$

We are going to compare the two operators $q_j[v]q_i[u]$ and $q_i[u]q_j[v]$, which both map the cohomology of $X^{[n]}$ to the cohomology of $X^{[n+i+j]}$. The natural place to describe the operator $q_j[v]q_i[u]$, which is the composition of two correspondences, is on the product $X^{[n]} \times X^{[n+i]} \times X^{[n+i+j]}$. In the description the following subvariety of this product will play a role:

$$Z_1 = p_{12}^{-1}(Z_{n,i}) \cap p_{23}^{-1}(Z_{n+i,j}). \tag{9}$$

It consists of triples (W, W', W'') of nested subschemes — i.e., $W \subset W' \subset W''$ — such that W and W' just differ in one point which we call P, and at the same time W' and W'' are different only in one point that we call Q. The quotient $I_W/I_{W'}$ has support $\{P\}$ and satisfies length $I_W/I_{W'} = i$. Similarly, the quotient $I_{W'}/I_{W''}$ has support $\{Q\}$ and is of length j.

There is a map $f_1 : Z_1 \to X \times X$ sending the triple (W, W', W'') to the pair (P, Q).

In a similar manner we let $Z_2 \subset X^{[n]} \times X^{[n+j]} \times X^{[n+i+j]}$ be the subvariety given by

$$Z_2 = p_{12}^{-1} Z_{n,j} \cap p_{23}^{-1} Z_{n+j,i}. \tag{10}$$

Its elements are the triples (W, W', W'') of nested subschemes with $I_W/I_{W'}$ and $I_{W'}/I_{W''}$ both having one-point-support in, say, Q and P respectively; the first one of length j and the other one of length i. As above there is a morphism $f_2 : Z_2 \to X \times X$, sending the triple (W, W', W'') to the pair (Q, P).

Lemma 6.1. *Let α be a class on $X^{[n]}$.*

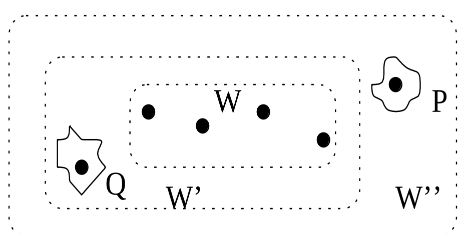
$$q_i[u]\, q_j[v]\alpha = p_{3*}\big(p_1^*\alpha \cdot f_2^*(v \times u) \cap [Z_2]\big), \tag{11}$$

$$q_j[v]\, q_i[u]\alpha = p_{3*}\big(p_1^*\alpha \cdot f_1^*(u \times v) \cap [Z_1]\big), \tag{12}$$

where p_i denotes the restriction of the i-th projection to Z_1 in the first line, and of Z_2 in the second.

Proof. This is just the formula for composing correspondences; the only point to check is that the intersections in (9) and (10) are both proper. □

Let $Z_1' \subset Z_1$ and $Z_2' \subset Z_2$ be the two open subsets where the two points P and Q are different. A typical element of Z_1', for example, may be drawn as



It has a 'central' part W and two 'fuzzy' ends, one in P and one in Q. The 'fuzzy' end at P is a subscheme of length i supported there, and the other 'fuzzy' end is a subscheme supported at Q of length j. The subscheme W' is the union of the 'central' part and the 'fuzzy' end at P. Of course P or Q may belong to the central part, but still the above statement makes sense if interpreted in the right way.

The drawing above might as well represent a typical element in Z_2'. The only difference being that in that case the 'fuzzy' part of length j at Q would belong to W' instead of the one of length i at P. Hence to any nested triple (W, V, W'') in Z_1' we may associate the triple (W, V', W'') where we get V'

from V by swapping the 'fuzzy' parts at P and Q. With a little thought one may convince oneself that this swapping is well defined even if the 'central' part touches P or Q. In this way we get an isomorphism $g : Z_1' \cong Z_2'$.

Clearly this isomorphism respects both p_1 and p_3 — it doesn't change the extreme subschemes W and W'' — and up to permutation of the two factors of $X \times X$, it respects f_1 and f_2. By the projection formula we therefore get the following equality

$$g_*\big(p_1^*\alpha \cdot f_1^*(u \times v) \cap [Z_1']\big) = (-1)^{\deg u \deg v} p_2^*\alpha \cdot f_2^*(v \times u) \cap [Z_2'].$$

The sign comes from the following: $u \times v = pr_1^*u \cdot pr_2^*v$ and via g^* this is mapped to $pr_2^*u \cdot pr_1^*v = (-1)^{\deg u \deg v} v \times u$.

It only remains to see that there is no contribution from the boundaries, i.e., when $P = Q$. The easy case is when $U \cap V = \emptyset$, then the boundary is empty — indeed $P \in U$ and $Q \in V$.

In general, a dimension estimate will show that all components of the boundary are — with good margin — of too small dimension to contribute. We shall need

$$\dim Z_1' = \dim Z_2' = 2n + i + j + 2.$$

Indeed, the n points in the 'central' part each have 2 degrees of freedom, and we are free to choose the 'fuzzy' ends from M_i and M_j, and these two varieties are of dimension $i+1$ and $j+1$ respectively.

By the transversality of U and V we know that

$$\dim_{\mathbb{R}} U \cap V = \dim_{\mathbb{R}} U + \dim_{\mathbb{R}} V - 4$$

We now give the dimension count for $f^{-1}(U \times V) \cap (Z - Z')$, where we have suppressed the indices and only write f, Z, Z'; the suppressed index can be either 1 or 2. The 'central' part is of length n and gives a contribution of $4n$ to the (real) dimension. Now $P = Q$, so the two 'fuzzy' parts live at the same point. If they could be chosen independently, their contribution to the dimension would be

$$\dim_{\mathbb{R}}(M_i(P) \times M_j(P) = 2(i-1) + 2(j-1)$$

as long as P is fixed, and P can only move in $U \cap V$. As this gives an upper bound of their contribution, we get

$$\begin{aligned}\dim_{\mathbb{R}}(f^{-1}(U \times V) \cap (Z - Z')) &\le \dim_{\mathbb{R}} M_i(P) \times M_j(P) + \dim_{\mathbb{R}} U \cap V \\ &\le 4n + 2i + 2j + \dim_{\mathbb{R}} U + \dim_{\mathbb{R}} V - 8 \\ &< 4n + 2i + 2j + \dim_{\mathbb{R}} U + \dim_{\mathbb{R}} V - 4.\end{aligned}$$

The class $f^*(u \times v) \cap [Z]$ lives in $H_r(Z)$ where

$$\begin{aligned} r &= \dim_{\mathbb{R}} Z - (4 - \dim_{\mathbb{R}} U) - (4 - \dim_{\mathbb{R}} V) \\ &= 4n + 2i + 2j + \dim_{\mathbb{R}} U + \dim_{\mathbb{R}} V - 4. \end{aligned}$$

After the dimension count, we know that the map $H_r(Z - Z') \to H_r(Z)$ induced by the inclusion is an isomorphism. Hence

$$g_* f_1^*(u \times v) \cap [Z_1] = (-1)^{\deg u \deg v} f_2^*(u \times v) \cap [Z_2],$$

and we are done.

Now we shall treat the perhaps more interesting — at least more subtle — case of the composition of one creation and one annihilation operator. That is, the composition of one operator of the form $q_{-i}[u]$ and one of the form $q_j[v]$ where $i \geq 0$ and $j \geq 0$.

We have to explain why

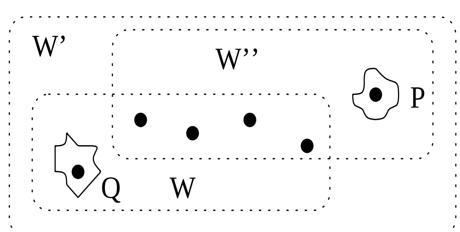
$$q_{-i}[u]\, q_j[v] + (-1)^{\deg u \deg v} q_j[v]\, q_{-i}[u] = -i\langle u, v\rangle \delta_{j-i} \mathrm{id},$$

and we start by examining the composition $q_{-i}[u]\, q_j[v]$. For any $n \geq 0$ it induces a map from $H^\bullet(X^{[n]})$ to $H^\bullet(X^{[n+j-i]})$. As in the preceding case, it is natural to look at the subvariety

$$Z_1 = p_{12}^{-1} Z_{n,j} \cap p_{23}^{-1} Z_{n+j-i,i} \subset X^{[n]} \times X^{[n+j]} \times X^{[n+j-i]}.$$

It may be described as the variety of triples $(W, W', W'') \in X^{[n]} \times X^{[n+j]} \times X^{[n+j-i]}$ with $W \subset W'$ and $W'' \subset W'$ — this time the one in the middle is bigger than the two on the sides — such that W' and W'' differ in just one point, and at the same time W' and W'' also differ only in one point. Call those points P and Q respectively.

The picture now looks like



This time the big one in the middle — W' — is the whole subscheme. The one to the left — W — is the whole except the 'fuzzy' part at P, and the one to the right — W'' — is the whole except the 'fuzzy' part supported at Q. As before there is a map $f_1 : Z_1 \to X \times X$ sending a triple to the two points (P, Q) and there is the lemma

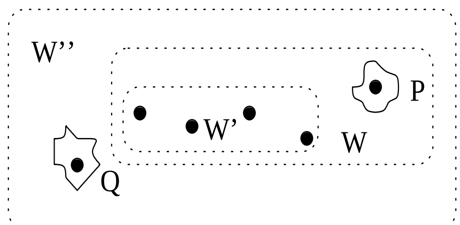
Lemma 6.2.

$$q_{-i}[u]\, q_j[v]\alpha = p_{3*}\big(p_1^*\alpha \cdot f_2^*(v \times u) \cap [Z_1]\big).$$

To understand the composition $q_{-i}[u]\, q_j[v]$, we introduce the subvariety

$$Z_2 = p_{12}^{-1} Z_{n-i,i} \cap p_{23}^{-1} Z_{n-i,j} \subset X^{[n]} \times X^{[n-i]} \times X^{[n+j-i]}.$$

This time the points in Z_2 are triples (W, W', W'') of subschemes with $W' \subset W$ and $W' \subset W''$ — the one in the middle is smaller than the other two — and as usual W' and W are different only at a point P and W' and W'' differ only at a point Q. The picture looks like



The little one in the middle — W' — is the 'central' part, and the two extremes — W and W'' — are subschemes we get by adding the 'fuzzy' part located at P respectively Q.

Just as before one checks that

$$\dim Z_1 = \dim Z_2 = 2n + i + j + 2,$$

for the complex dimensions, and there is the usual map $f_2 : Z_2 \to X \times X$.

We follow the same track as in the creation-creation process, and define $Z' \subset Z$ — where the missing index is either 1 or 2 — as the open subsets where $P \neq Q$. Then there is an isomorphism $g : Z_1' \cong Z_2'$. Indeed we keep the two extremes and exchange the smallest 'central' part with the whole. Writing W_P for the part of W supported at P and similarly for Q and W', W'', this amounts to sending the biggest one, W', to $(W' \setminus W_P' \setminus W_Q') \cup W_P \cup W_Q''$ which has a meaning as long as $P \neq Q$. In the same way, it is easy to write down the inverse of g.

Lemma 6.3.

$$g_*\big(p_1^*\alpha \cdot f_1^*(u \times v) \cap [Z_1']\big) = (-1)^{\deg u \deg v} p_2^*\alpha \cdot f_2^*(v \times u) \cap [Z_2'].$$

Now we come to the more subtle point of analyzing the boundaries where $P = Q$. Because when we compute the composition, we apply p_{13*}, what really matters is the dimension of $p_{13}(Z \setminus Z')$ — for missing index equal 1 and

2. In the case of $p_{13}(Z_2 \setminus Z_2')$ everything works as in the creation-creation case, and there will be no contribution from the boundary, so let us turn our attention to the subtle case $p_{13}(Z_1 \setminus Z_1')$. The case $U \cap V = \emptyset$ gives no boundary at all, but if $U \cap V = \{P\}$ something happens. If in addition $i = j$ we may take $W = W''$. There always exists a subscheme of length $n + j$ containing any subscheme of length n which is supported at p. Hence in this case $p_{13}(Z_1 \setminus Z_1')$ will be supported along the diagonal in $X^{[n]} \times X^{[n]}$. One may check by dimension count as before that this is the only possible contribution from the boundary. It follows that

$$\big[q_{-i}[u], q_i[v]\big] = \mu \,\mathrm{id}$$

for some number μ.

7. Vertex operators and Nakajimas computation of the constants

For any class $u \in H^\bullet(X)$ and any sequence $\mathbf{d} = \{d_m\}_{m \geq 0}$ of numbers we introduce the following operator, often called a vertex operator,

$$E_{\mathbf{d},u}(z) = \exp\big(\sum_{m>0} d_m q_m[u] z^m\big) = \exp(P(z)).$$

where $P(z) = \sum_{m>0} d_m q_m[u] z^m$. When we apply $E_{\mathbf{d},u}(z)$ to the vacuum vector, we obtain a sequence $\{\alpha_m\}_{m \geq 0}$ of classes in $\mathbb{H}(X)$, with α_m of weight m and $\alpha_0 = \mathbf{1}$, which are defined by the expression

$$\sum_{m \geq 0} \alpha_m z^m := \exp\big(\sum_{m>0} d_m z^m q_m[u]\big)\mathbf{1} = \exp(P(z)) \cdot \mathbf{1}.$$

We have

Proposition 7.1. *For any two classes u, v in $H^\bullet(X)$, and any natural number i, the element $\exp(P(z)) \cdot \mathbf{1}$ is an eigenvector for $q_i[v]$ with eigenvalue $-c_i d_i (\int_X u \cdot v) z^i$. That is, for $m \geq 0$, we have the equality*

$$q_i[v]\alpha_m = -c_i d_i \Big(\int_X u \cdot v \Big) \alpha_{m-i}.$$

In the proof of the proposition we shall need the following easy lemma:

Lemma 7.2. *If A and B are two operators commuting with their commutator, then for any $p \geq 1$*

$$[A, B^p] = p[A, B]B^{p-1}.$$

Furthermore

$$[A, \exp B] = [A, B] \exp B.$$

Proof. Exercise. □

To prove Proposition 7.1 we do the following computation:

$$\begin{aligned} q_{-i}[v]\exp(P(z))\cdot \mathbf{1} &= \big[q_{-i}[v], \exp(P(z))\big]\mathbf{1} && \text{ann. oper. kill vacuum} \\ &= [q_{-i}[v], P(z)]\exp(P(z))\cdot \mathbf{1} && \text{Lemma 7.2} \\ &= \left(\sum_{m>0} d_m \left[q_{-i}[v], q_m[u]\right] z^m\right)\exp(P(z))\cdot \mathbf{1} && \text{definition of } P(z) \\ &= -\, d_i c_i (\int_X uv) z^i \exp(P(z))\cdot \mathbf{1} && \text{Nakajima relations.} \end{aligned}$$

By the definition of $\{\alpha_m\}$, this completes the proof.

The property in Proposition 7.1 is very strong. In fact, it determines the sequence α_m completely.

Lemma 7.3. *Let the two sequences $\{\alpha_m\}$ and $\{\beta_m\}$ from $\mathbb{H}(X)$ be given, with α_m and β_m both of weight m and $\alpha_0 = \beta_0 = \mathbf{1}$. Assume that for any $i > 0$ and any class v in $H^\bullet(X)$, there is a number $e_{i,v}$ such that both α_m and β_m satisfy the equation*

$$q_i[v]x_m = e_{i,v}x_{m-i}$$

for all $n \geq 0$. Then $\alpha_m = \beta_m$ for all $m \geq 1$.

Proof. The proof goes by induction on m. We assume that $\alpha_j = \beta_j$ for $j < m$. Then for any $i \geq 0$ and any class v on X we have

$$q_{-i}[v](\alpha_m - \beta_m) = e_{i,v}(\alpha_{m-i} - \beta_{m-i}) = 0$$

by induction. Hence $\mathbb{S}(H^\bullet(X))(\alpha_m - \beta_m)$ will be a sub $\mathbb{S}(H^\bullet(X))$-module all of whose elements are of weight greater than or equal to m. Now if $m \geq 1$, the vacuum, being of weight 0, cannot be in this module which consequently must be trivial, since $\mathbb{H}(X)$ is an irreducible $\mathbb{S}(H^\bullet(X))$-module. Hence $\alpha_m = \beta_m$, and we are done. □

We shall need the following variant of the above lemma:

Lemma 7.4. *Let $\{\alpha_m\}$ and $\{\beta_m\}$ be two sequences in $\mathbb{H}(X)$ with α_m and β_m both of weight m and $\alpha_0 = \beta_0$. Assume that for all $i \geq 0$ and all classes v in $H^\bullet(X)$ there are numbers $e_{i,v}$ with $e_{i,v} = 0$ if $\deg v < 2$, such that the following two conditions are satisfied.*

1. $q_{-i}[v]\beta_m = e_{i,v}\beta_{m-i}$ *for all* $i \geq 0$ *and all classes* v *in* $h^\bullet(X)$,
2. $\deg \alpha_m = 2m$ *and*

$$q_{-i}[v]\alpha_m = e_{i,v}\alpha_{m-i}$$

whenever $\deg v \geq 2$ *and* $i > 0$.

Then $\alpha_m = \beta_m$ *for all* $m \geq 0$.

Proof. Again we use induction on m and assume that $\alpha_{m-i} = \beta_{m-i}$ for all $i > 0$. Just as in the proof above, it is sufficient to see that the vacuum vector is not contained in the $\mathbb{S}(H^\bullet(X))$-module spanned by $\alpha_m - \beta_m$. In other words we must check that any sequence of 'backwards' moves kills $\alpha_m - \beta_m$; to that end let

$$z = q_{-i_1}[v_1]q_{-i_2}[v_2]\dots q_{-i_p}[v_p](\alpha_m - \beta_m)$$

be the result of p 'backwards' moves applied to $\alpha_m - \beta_m$. If one of the v_i's is of degree greater than or equal to 2, we know that $z = 0$. Indeed, this follows by induction from two conditions in the lemma since the annihilation operators involved all commute — we can move the annihilation $q_{i_j}[v_j]$ with $\deg v_j \geq 2$ to the right in the 'backwards' sequence. Hence we may assume that all the v_i's are of degree less than 2. Then by condition 1. in the lemma, we have $q_{-i_1}[v_1]q_{-i_2}[v_2]\dots q_{-i_p}[v_p]\beta_m = 0$ and hence

$$z = q_{-i_1}[v_1]q_{-i_2}[v_2]\dots q_{-i_p}[v_p]\alpha_m.$$

We want to see that the case $z = \mathbf{1}$ cannot happen. Indeed, if $z = \mathbf{1}$, then $\sum_{j=1}^p i_j = m$. By computing the degree of z from the expression above, we obtain

$$\begin{aligned}\deg z =& \deg \alpha_m + \sum \deg\bigl(v_j - 2(i_j + 1)\bigr) = \\ =& 2m + \sum(\deg v_j - 2) - 2\sum i_j = \\ =& \sum(\deg v_i - 2),\end{aligned}$$

from which it follows that $\deg z < 0$, and thus $z = 0$. □

Let now $C \subset X$ be a smooth curve whose class in $H^\bullet(X)$ is σ. Let σ_n denote the class of the n-th symmetric power $C^{(n)}$ of C in $X^{[n]}$. The classes σ_n may be computed in terms of the Nakajima creation operators as in the following theorem which appeared in [13] and [10].

Theorem 7.5. *(Nakajima, Grojnowski)*

$$\sum_{n\geq 0}\sigma_n z^n = \exp\Big(\sum_{m>0}\frac{(-1)^{m-1}}{c_m}q_m[\sigma]z^m\Big)\cdot\mathbf{1}.$$

Proof. By Proposition 7.1 we know that the sequence $\{\alpha_m\}$ defined by the identity

$$\sum_{n\geq 0}\alpha_n z^n = \exp\Big(\sum_{m>0}\frac{(-1)^{m-1}}{c_m}q_m[\sigma]z^m\Big)\cdot\mathbf{1}$$

satisfies

$$q_i[v]\alpha_m = (-1)^i\Big(\int_X \sigma v\Big)\alpha_{m-i}$$

for all $i>0$ and all $v\in H^\bullet(X)$. From Lemma 4.2 we know that $q_{-i}[v]\sigma^n = (-1)^i a\sigma^{n-i}$ for any curve class v satisfying $\int_X \sigma v = a$. It is also clear that if $v=[V]$ for V a submanifold of X with $C\cap V=\emptyset$, then $q_{-i}[v]\sigma^n = 0$; hence we know that

$$q_i[v]\sigma^n = (-1)^i(\int_X v\sigma)\sigma^{n-i}$$

holds for all $i>0$ and all classes v on X of degree 2 or more. The theorem then follows from Lemma 7.4. □

Finally we will give the second computation — due to Nakajima — of the constants c_i as we promised. We start by computing derivatives in Theorem 7.5 to obtain

$$\begin{aligned}\sum_{n\geq 1} n\sigma^n z^{n-1} &= \Big(\frac{d}{dz}\exp P(z)\Big)\mathbf{1} = \big(\frac{d}{dz}P(z)\big)\exp(P(z))\cdot\mathbf{1} = \\ &= \Big(\Big(\sum_{m>0}\frac{(-1)^{m-1}m}{c_m}q_m[\sigma]z^{m-1}\Big)\cdot\sum_{n=0}^{\infty}\sigma^n z^n\Big)\cdot\mathbf{1}.\end{aligned}$$

From this we obtain

$$n\sigma_n = \sum_{m=1}^{n}\frac{(-1)^{m-1}m}{c_m}q_m[\sigma]\sigma_{n-m}. \tag{13}$$

As the constants c_m are universal, we may very well assume that $X=\mathbb{P}^2$ and that C is a line.

Lemma 7.6. *Let C and C' be two curves in X intersecting transversally in one point; e.g., two different lines in $\mathbb{P}^2$. Then*

$$\int_{X^{[n]}} \sigma_n \cdot \sigma_n' = \begin{cases} 1 & \text{if } n \leq 1 \\ 0 & \text{else} \end{cases}$$

Proof. If $t = 0$ and $t' = 0$ are local equations for C and C' at the common point, a subscheme in $C^{(n)}$ supported at this point is necessarily of the form $\mathbb{C}[t, t']/(t, t'^n)$ and one in $C'^{(n)}$ must be of the form $\mathbb{C}[t, t']/(t^n, t')$. If a subscheme W simultaneously is of these two forms, necessarily $n \leq 1$. □

Finally we prove

Theorem 7.7.

$$c_i = i.$$

Proof. The idea is to intersect (13) with σ_n. For $n = 1$ we get

$$\begin{aligned} 1 = \int_X \sigma\sigma &= \frac{1}{c_1} \int_X \sigma_1 \cdot q_1[\sigma]\sigma_0 \\ &= \frac{1}{c_1} \int_X (-q_{-1}[\sigma]\sigma_1) \cdot \sigma_0 \\ &= \frac{1}{c_1} \int_X \sigma_0 \cdot \sigma_0 = \frac{1}{c_1}. \end{aligned}$$

This gives $c_1 = 1$. Assume now that $n \geq 2$. Then we obtain

$$\begin{aligned} 0 = \int_{X^{[n]}} \sigma_n \cdot \sigma_n &= \sum_{m=1}^{n} \frac{(-1)^{m-1}m}{c_m} \int_{X^{[n]}} \sigma_n \cdot q_m[\sigma]\sigma_{n-m} \\ &= \sum_{m=1}^{n} \frac{(-1)^{m-1}m}{c_m} (-1)^m \int_{X^{[n-m]}} q_{-m}[\sigma]\sigma_n \cdot \sigma_{n-m} \\ &= \sum_{m=1}^{n} \frac{(-1)^{m-1}m}{c_m} \int_{X^{[n-m]}} \sigma^{n-m} \cdot \sigma^{n-m} \\ &= \frac{(-1)^{n-1}n}{c_n} + \frac{(-1)^{n-2}(n-1)}{c_{n-1}}. \end{aligned}$$

Hence

$$\frac{c_n}{n} = \frac{c_{n-1}}{n-1}$$

from which we get $c_n = n$. □

8. Computation of the Betti numbers of $X^{[n]}$

As before, let X be a smooth projective surface over $\mathbb{C}$. We will now show formula (1) for the Betti numbers of the Hilbert scheme $X^{[n]}$ of points. We needed it in the first part to show that

$$\mathbb{H}(X) := \bigoplus_{n\geq 0} H^\bullet(X^{[n]})$$

is an irreducible representation of the Heisenberg algebra. There are at least three possible different approaches which have been used to prove this result; using the Weil conjectures [8], using perverse sheaves and intersection cohomology [9], or finally one can use the so-called virtual Hodge polynomials [3]. The last two approaches will in addition give the Hodge numbers of the Hilbert schemes. In these notes we will use the second approach. It has the advantage of leading to the shortest and most elegant proof, and to almost completely avoid any computations. The disadvantage is that it requires very deep results about intersection cohomology and perverse sheaves. We will first briefly describe these results and then show how one can use them as a black box, which with rather little effort gives the desired result.

Let Y be an algebraic variety over $\mathbb{C}$. In this section we only use the complex (strong) topology on Y. We want to stress again that all the cohomology that we consider is with $\mathbb{Q}$-coefficients. In particular $H^i(Y)$ stands for $H^i(Y, \mathbb{Q})$. There exists a complex IC_Y of sheaves on Y (for the strong topology), such that

$$\mathrm{IH}^\bullet(Y) := H^\bullet(Y, IC_Y)$$

is the *intersection homology* of Y (strictly speaking IC_Y is an element in the derived category of Y). Recall that the intersection cohomology groups $\mathrm{IH}^i(Y)$ are defined for any algebraic variety and fulfill Poincaré duality (between $\mathrm{IH}^{-i}(Y)$ and $\mathrm{IH}^i(Y)$). IC_Y is called the *intersection cohomology complex* of Y. If Y is smooth and projective of dimension n, then

$$IC_Y = \mathbb{Q}_Y[n],$$

is just the constant sheaf $\mathbb{Q}$ on Y put in degree n. Therefore $\mathrm{IH}^{i-n}(Y) = H^i(X, \mathbb{Q})$. More generally, if $Y = X/G$ is a quotient of a smooth variety of dimension n by a finite group, then $IC_Y = \mathbb{Q}_Y[n]$, and thus again $\mathrm{IH}^{i-n}(Y) = H^i(X, \mathbb{Q})$.

Let now $f : X \to Y$ be a projective morphism of varieties over $\mathbb{C}$. Suppose that Y has a stratification

$$Y = \coprod_\alpha Y_\alpha$$

into locally closed strata. Let $X_\alpha := f^{-1}(Y_\alpha)$. Assume that $f : X_\alpha \to Y_\alpha$ is a locally trivial bundle with fiber F_α (in the strong topology).

Definition 8.1. f is called *strictly semismall* (with respect to the stratification), if, for all α,

$$2dim(F_\alpha) = codim(Y_\alpha).$$

We will use the following facts:

: Fact 1. Assume that $f : X \to Y$ is strictly semismall, and that the F_α are irreducible, then

$$Rf_*(IC_X) = \sum_\alpha IC_{\overline{Y}_\alpha}.$$

(see [9]). Here Rf_* is the push-forward in the derived category, and $\overline{Y}_\alpha$ is the closure of Y_α. This is a consequence of the *Decomposition Theorem* of Beilinson-Bernstein-Deligne [1].

: Fact 2. Let $\kappa : X \to Y$ be a finite birational map of irreducible algebraic varieties, then

$$R\kappa_*(IC_X) = IC_Y$$

(see [9]).

Now we want to see how these facts about the intersection cohomology complex can be applied to compute the Betti numbers of the Hilbert schemes of points.

Let $\rho : X^{[n]} \to X^{(n)}$ be the Hilbert-Chow morphism. The symmetric power $X^{(n)}$ is stratified as follows: Let $\nu = (n_1, \ldots, n_r)$ be a partition of n. We also write $\nu = (1^{\alpha_1}, 2^{\alpha_2}, \ldots n^{\alpha_n})$, where α_i is the number of l such that $n_l = i$. We put

$$X^{(n)}_\nu := \left\{\sum n_i x_i \in X^{(n)} \;\middle|\; \text{the } x_i \text{ are distinct}\right\},$$

and $X^{[n]}_\nu := \rho^{-1}(X^{(n)}_\nu)$. The $X^{(n)}_\nu$ form a stratification of $X^{(n)}$ and similarly the $X^{[n]}_\nu$ form a stratification of $X^{[n]}$. The smallest stratum

$$X^{[n]}_{(n)} := \left\{W \in X^{[n]} \;\middle|\; \mathrm{Supp}(W) \text{ is a point}\right\}$$

is just the variety M_n. It is a locally trivial fiber bundle (in the strong topology) over $X^{(n)} \simeq X$, with fiber

$$F_{(n)} := M_n(P).$$

In particular the fiber is independent of X. This is because finite length subschemes concentrated in a point depend only on an analytic neighborhood of the point. It follows that each stratum $X^{[n]}_{(n_1,\dots,n_r)}$ is a locally trivial fiber bundle over the corresponding stratum $X^{(n)}_{(n_1,\dots,n_r)}$, with fiber $F_{(n_1)} \times \dots \times F_{(n_r)}$.

By Theorem 1.3 $M_n(P)$ is irreducible of dimension $(n-1)$, which is half the codimension of $X^{(n)}_{(n)}$ in $X^{(n)}$. It follows that $\rho : X^{[n]} \to X^{(n)}$ is strictly semismall with respect to the stratification by partitions. Therefore we obtain by Fact 1. above

$$R\rho_*(\mathbb{Q}_{X^{[n]}}[2n]) = R\rho_*(IC_{X^{[n]}}) = \bigoplus_\nu IC_{\overline{X}^{(n)}_\nu}.$$

We write

$$\nu = (n_1, \dots, n_r) = (1^{\alpha_1}, 2^{\alpha_2}, \dots, n^{\alpha_n}),$$

and denote $(\alpha) := (\alpha_1, \alpha_2, \dots, \alpha_n)$. Then there is a morphism

$$\kappa_\alpha : X^{(\alpha)} := X^{(\alpha_1)} \times \dots \times X^{(\alpha_n)} \to \overline{X}^{(n)}_\nu$$
$$(\xi_1, \dots, \xi_n) \mapsto \sum_{i=1}^n i \cdot \xi_i.$$

It is easy to see that κ_α is the normalization of $\overline{X}^{(n)}_\nu$. Therefore Fact 2. above implies

$$IC_{\overline{X}^{(n)}_\nu} = R(\kappa_\alpha)_*(IC_{X^{(\alpha)}}) = R(\kappa_\alpha)_* \mathbb{Q}_{X^{(\alpha)}}\left[2|\alpha|\right],$$

where $|\alpha| = \sum_i \alpha_i$. Putting this together, we get that

$$R\rho_*(\mathbb{Q}_{X^{[n]}}[2n]) = \bigoplus_\alpha R(\kappa_\alpha)_*\big(\mathbb{Q}_{X^{(\alpha)}}\left[2|\alpha|\right]\big). \tag{14}$$

Here the sum runs through all $(\alpha) = (\alpha_1, \dots, \alpha_n)$ with $\sum_i i\alpha_i = n$. Finally we take the cohomology of relation (14). We recall that taking the cohomology of a complex of sheaves commutes with push-forward. Therefore we obtain

$$H^{i+2n}(X^{[n]}) = \bigoplus_\alpha H^{i+2|\alpha|}(X^{(\alpha)}).$$

So with this we have completely determined the additive structure of the cohomology of the Hilbert schemes $X^{[n]}$ in terms of that of the symmetric powers $X^{(k)}$. The cohomology of the symmetric powers is well known. As $X^{(n)}$ is the quotient of X^n by the action of the symmetric group $\mathfrak{S}_n$ by permuting the factors, we see that $H^i(X^{(n)}) = H^i(X^n)^{\mathfrak{S}_n}$ is the invariant part of the cohomology of X^n under the action of $\mathfrak{S}_n$.

Now we want to turn this into a generating function for the Betti numbers of the Hilbert schemes $X^{[n]}$.

Let $p(Y) := \sum_i dim(H^{i+dim(Y)}(Y))z^i$ be the (shifted) Poincaré polynomial of a variety Y. The description above of the cohomology of the symmetric powers leads, by Macdonald's formula [12], to a generating function for their Poincaré polynomials.

$$\sum_{n=0}^{\infty} p(X^{(n)})t^n = \frac{(1+z^{-1}t)^{b_1(X)}(1+zt)^{b_3(X)}}{(1-z^{-2}t)^{b_0(X)}(1-t)^{b_2(X)}(1-z^2t)^{b_4(X)}}.$$

Here the $b_i(X) = dim(H^i(X))$ are the Betti numbers of X. We are now able to put all the ingredients together to get our desired generating function for the Betti numbers of the Hilbert schemes.

$$\begin{aligned}\sum_{n=0}^{\infty} p(X^{[n]})t^n &= \sum_{n=0}^{\infty} \sum_{\alpha_1+2\alpha_2+\ldots n\alpha_n=n} p(X^{(\alpha_1)})p(X^{(\alpha_2)})\ldots p(X^{(\alpha_n)})t^{\alpha_1+2\alpha_2+\ldots n\alpha_n} \\ &= \prod_{k=1}^{\infty}\left(\sum_l p(X^{(l)})t^{kl}\right) \\ &= \prod_{k=1}^{\infty} \frac{(1+z^{-1}t^k)^{b_1(X)}(1+zt^k)^{b_3(X)}}{(1-z^{-2}t^k)^{b_0(X)}(1-t^k)^{b_2(X)}(1-z^2t^k)^{b_4(X)}}.\end{aligned}$$

This (keeping track of the shift in the Poincaré polynomial) is the formula of Theorem 2.1.

9. The Virasoro algebra

The rest of these lectures is mostly based on the paper [11] of Lehn. Before we got a nice description of the additive structure (+ the intersection pairing) of the Hilbert schemes, which put all the Hilbert schemes together into one structure. Our aim now is to get some insight into the ring structure of the cohomology rings of the Hilbert schemes of points $X^{[n]}$. We want to see how the ring structure is related to the action of the Heisenberg algebra. That is; for any cohomology class $\alpha \in H^{\bullet}(X^{[n]})$ we can look at the operator of multiplying by α. We want to try to express these operators in terms of the Heisenberg operators. In particular we will be interested in the Chern classes of tautological sheaves on the Hilbert schemes, which are useful in many applications of Hilbert schemes.

As a first step we will construct an action of a Virasoro algebra on the cohomologies of the Hilbert schemes. This is not such a surprising result: There is a standard construction, which associates to a Heisenberg algebra a

Virasoro algebra. This construction is essentially translated into geometric terms. One of the main technical results will be a geometric interpretation of the Virasoro generators.

We will, in the future, ignore all signs coming from odd-degree cohomology classes.

Definition 9.1. Let $\delta : H^\bullet(X) \to H^\bullet(X \times X) = H^\bullet(X) \otimes H^\bullet(X)$ be the push-forward via the diagonal embedding $\delta : X \to X \times X$. If $\delta(\alpha) = \sum_i \beta_i \otimes \gamma_i$, we write

$$q_n q_m \delta(\alpha) := \sum_i q_n[\beta_i] q_m[\gamma_i].$$

We define operators $L_n : H^\bullet(X) \to End(\mathbb{H}(X))$ by

$$L_n := \frac{1}{2} \sum_{\nu \in \mathbb{Z}} q_\nu q_{n-\nu} \delta \quad \text{, if } n \neq 0$$

$$L_0 := \sum_{\nu > 0} q_\nu q_{-\nu} \delta.$$

The sums appear to be infinite, but, for fixed $y \in \mathbb{H}(X)$ and $\alpha \in H^\bullet(X)$, only finitely many terms contribute to $L_n[\alpha]y$.

Theorem 9.2. 1. $\big[L_n[u], q_m[w]\big] = -m q_{m+n}[uw]$.

2.

$$\big[L_n[u], L_m[w]\big] = (n-m) L_{n+m}[uw] - \frac{n^3 - n}{12} \left(\int_X c_2(X) uw \right) \mathbf{1}.$$

Part 2. can be viewed as saying that the Virasoro algebra given by the $L_n[X]$ acts on $\mathbb{H}(X)$ with central charge $c_2(X)$.

The proof of the theorem is mostly formal. We will show part 1. in case $n \neq 0$. Writing

$$\delta(u) = \sum_i s_i \otimes t_i,$$

we get

$$\begin{aligned} \big[q_\nu[s_i] q_{n-\nu}[t_i], q_m[w]\big] &= q_\nu[s_i] \big[q_{n-\nu}[t_i], q_m[w]\big] + \big[q_\nu[s_i], q_m[w]\big] q_{n-\nu}[t_i] \\ &= (-m) \delta_{n+m-\nu} q_{n+m}[s_i] \left(\int_X t_i w \right) + (-m) \delta_{m+\nu} \left(\int_X w s_i \right) q_{n+m}[t_i]. \end{aligned}$$

We sum this up over all ν and i, to obtain

$$2\big[L_n[u], q_m[w]\big] = (-m) q_{n+m}[Z],$$

with

$$Z = \sum_i s_i \int_X t_i w + \sum_i t_i \int_X w u_i.$$

Each of the sums on the right-hand side equals uw. This shows part 1. Part 2. can easily be reduced to part 1.

10. Tautological sheaves

We can, in a natural way, associate a tautological sheaf $F^{[n]}$ on $X^{[n]}$ to a vector bundle F on X. These sheaves are very important in geometric applications of the Hilbert scheme $X^{[n]}$. Let again

$$Z_n := \{(W, x) \in X^{[n]} \times X \mid x \in W\}$$

be the universal family with the projections $p : Z_n \to X^{[n]}$, $q : Z_n \to X$. Then the *tautological sheaf*

$$F^{[n]} := p_* q^*(F)$$

is a locally free sheaf of rank rn on $X^{[n]}$, where r is the rank of F. (This is because $p : Z_n \to X^{[n]}$ is flat of degree n.) In particular $F^{[1]} = F$. By definition the fiber $F^{[n]}(W)$ of $F^{[n]}$ over a point $W \in X^{[n]}$ is naturally identified with $H^0(W, F|_W)$.

If $0 \to F \to E \to G \to 0$ is an exact sequence of locally free sheaves, then so is

$$0 \to F^{[n]} \to E^{[n]} \to G^{[n]} \to 0.$$

Therefore $(\ )^{[n]} : F \mapsto F^{[n]}$ defines a homomorphism from the Grothendieck group $K(X)$ of locally free sheaves on X to $K(X^{[n]})$.

The Chern classes of the tautological sheaves have interesting geometric interpretations.

1. Let L be a line bundle on X. Then $c_n(L^{[n]}) \in H^n(X^{[n]})$ is the Poincaré dual of the class of $C^{[n]} = C^{(n)}$, where $C \in |L|$ is a smooth curve.
2. More generally $c_{n-l}(L^{[n]})$ is the Poincaré dual of the class of all $W \in X^{[n]}$ with $W \subset C_t$ for C_t a curve in a general l-dimensional linear subsystem of $|L|$.
3. The top Segre class $s_{2n}(L^{[n]})$ is by definition just the top Chern class $c_{2n}(-L^{[n]})$ (here $(-L^{[n]})$ is the negative of $L^{[n]}$ in the $K(X^{[n]})$). In other words that means that $s_{2n}(L^{[n]})$ is the part of degree $2n$ of

$$1/(1 + c_1(L^{[n]}) + c_2(L^{[n]}) + \ldots).$$

> The degree of $s_{2n}(L^{[n]})$ is the number of all $W \in X^{[n]}$, which do not impose independent conditions on curves in a general $(3n-2)$-dimensional sub-linear system of $|L|$.

All these identifications are under the assumption that L is sufficiently ample. It is e.g. sufficient, but not necessary that L is the n-th tensor power of a very ample line bundle on X. The identifications are proven by using the Thom-Porteous formula ([7] Theorem 4.4), which gives the class of the degeneracy locus of a map of vector bundles in terms of their Chern classes. There is a natural evaluation map

$$ev_n : H^0(X,L) \otimes \mathcal{O}_{X^{[n]}} \to L^{[n]}, (s,W) \mapsto s|_W \in H^0(Z, L|_W) = L^{[n]}(W)$$

from the trivial bundle with fiber $H^0(X,L)$ to $L^{[n]}$. The assumption that L is sufficiently ample ensures that ev_n is surjective. In this situation the Thom-Porteous formula says that $c_{n-l}(L^{[n]})$ is the class of the locus where the restriction of ev_n to a trivial vector subbundle of rank $l+1$ is not injective. Such a vector subbundle corresponds to an l-dimensional linear subsystem M of $|L|$, and the locus where the map is not injective is easily seen to be the locus of $W \in X^{[n]}$ with $W \in C$ for a curve $C \in M$. This shows parts 1. and 2.

Part 3 is similar. In this case we look at the dual map

$$(ev_n)^\vee : (L^{[n]})^\vee \to H^0(X,L) \otimes \mathcal{O}_{X^{[n]}},$$

and the locus we are looking for is the locus where $(ev_n)^\vee$ is not injective.

So we get in particular

$$\int_X s_2(L) = \#\{\text{base points in a pencil of } |L|\} = c_1(L)^2.$$

$\int_{X^{[2]}} s_4(L^{[2]})$ is the number of double points of the map $X \to \mathbb{P}_4$ given by a general 4-dimensional linear subsystem of $|L|$. The numbers $s_{2n}(L^{[n]})$ are, for instance, interesting from the point of view of Donaldson invariants.

11. Geometric interpretation of the Virasoro operators

Our aim is to give a more geometric interpretation of the action of the Virasoro algebra, which was defined in section 9. We shall see that they are related to the "boundary" of $X^{[n]}$, i.e. the locus of subschemes of X with support less then n points. If we write ∂ for the operation of multiplying by the cohomology class of the boundary, then L_n will turn out to be essentially the commutator $q_n\partial - \partial q_n$. In order to be able to prove this result we have to give another description of the class ∂, which relates it to tautological sheaves. This is done by looking at the incidence scheme

$$X^{[n,n+1]} := \left\{ (Z,W) \in X^{[n]} \times X^{[n+1]} \;\middle|\; Z \subset W \right\}.$$

As a special case of diagram (5) we have the diagram

$$\begin{array}{ccccc} X & \xleftarrow{\rho} & X^{[n,n+1]} & \xrightarrow{p_{n+1}} & X^{[n+1]} \\ & & {\scriptstyle p_n}\downarrow & & \\ & & X^{[n]}. & & \end{array} \tag{15}$$

In particular there is a morphism $\pi := (p_n, \rho) : X^{[n,n+1]} \to X^{[n]} \times X$, which sends a pair (Z,W) of subschemes of X to Z and the residual point. It is evident that π is an isomorphism over the open subset of all $(Z,x) \in X^{[n]} \times X$ with $x \neq Z$, i.e. over the complement of the universal family

$$Z_n := \left\{ (Z,x) \in X^{[n]} \times X \;\middle|\; x \in Z \right\}.$$

More precisely we have the following theorem:

Theorem 11.1. [5] *$X^{[n,n+1]}$ is the blowup of $X^{[n]} \times X$ along the universal family Z_n.*

Proof. Let $\pi : Y \to X^{[n]} \times X$ be the blowup along Z_n with exceptional divisor E. On $X^{[n]} \times X \times X$, let W_n be the pull-back of Z_n from the first and third factor. On $Y \times X$, let

$$\widetilde{\Delta} := (\pi \times 1_X)^{-1} \Delta, \quad \widetilde{W_n} := (\pi \times 1_X)^{-1} W_n.$$

Then the projection $p_Y|_{\widetilde{\Delta}} : \widetilde{\Delta} \to Y$ is an isomorphism, which maps $\widetilde{\Delta} \cap \widetilde{W}_n$ isomorphically onto the exceptional divisor E. Therefore $\widetilde{Z}_{n+1} := \widetilde{\Delta} \cup \widetilde{W}_n$ is a flat family of degree $(n+1)$ over Y, and on $Y \times X$ we have a sequence

$$0 \to \mathcal{O}_{\widetilde{\Delta}}(-E) \to \mathcal{O}_{\widetilde{Z}_{n+1}} \to \mathcal{O}_{\widetilde{W}_n} \to 0. \tag{16}$$

The flat family $\widetilde{Z}_{n+1}$ induces a morphism $Y \to X^{[n+1]}$, which together with the projection $Y \to X^{[n]}$ gives a morphism $Y \to X^{[n]} \times X^{[n+1]}$ with image $X^{[n,n+1]}$. One checks that the induced morphism $Y \to X^{[n,n+1]}$ is an isomorphism. ☐

Let E be the exceptional divisor of the blowup $X^{[n,n+1]} \to X^{[n]} \times X$. Then E can be described as

$$E = \left\{ (Z,W) \in X^{[n,n+1]} \;\middle|\; supp(Z) = supp(W) \right\}.$$

Let F be a vector bundle on X. Then tensoring the sequence (16) with $p_X^* F$ and pushing down to $Y = X^{[n,n+1]}$ gives the exact sequence

$$0 \to \rho^* F(-E) \to p_{n+1}^* F^{[n+1]} \to p_n^* F^{[n]} \to 0 \tag{17}$$

which relates the tautological bundles $F^{[n]}$ and $F^{[n+1]}$. This makes it possible to try to treat the tautological bundles via an inductive argument. In particular we get

$$\mathcal{O}_{X^{[n,n+1]}}(-E) = p_{n+1}^* \mathcal{O}^{[n+1]} - p_n^* \mathcal{O}^{[n]}$$

in the Grothendieck group $K(X^{[n,n+1]})$.

Let $\partial X^{[n]}$ be the closure of the stratum $X^{[n]}_{(2,1,\dots,1)}$, i.e. the locus in $X^{[n]}$, where the subscheme does not consist of n distinct points. The class of $[\partial X^{[n]}]$ (i.e. the class Poincaré dual to it) is related to the first Chern class of the tautological sheaves.

Lemma 11.2. $[\partial X^{[n]}] = -2c_1(\mathcal{O}_X^{[n]})$.

Proof. $\partial X^{[n]}$ is the branch divisor of the projection $p : Z_n \to X^{[n]}$, therefore $-2c_1(p_* \mathcal{O}_{Z_n}) = [\partial X^{[n]}]$. □

Definition 11.3. Let $d : \mathbb{H}(X) \to \mathbb{H}(X)$ be the operator of multiplying by $c_1(\mathcal{O}_{X^{[n]}})$, i.e. for $y \in H^\bullet(X^{[n]})$ we have $dy = c_1(\mathcal{O}_{X^{[n]}}) \cdot y$. For $f \in End(\mathbb{H}(X))$ the *derivative* f' of f is defined to be

$$f' := [d, f].$$

It is easy to check that

$$(fg)' = f'g + fg', \quad [f,g]' = [f',g] + [f,g'],$$

which gives some justification for calling it derivative.

We have the following geometric interpretation of the derivative in terms of tautological sheaves. Let $X^{[n,m]} \subset X^{[n]} \times X^{[m]}$ be the incidence variety of pairs of subschemes (Z, W) with $Z \subset W$ (in particular $n < m$). Let p_n and be p_m be the projections of $X^{[n,m]}$ to $X^{[n]}$ and $X^{[m]}$.

Then taking the derivative of $f \in End(\mathbb{H}(X))$ amounts to multiplying with $c_1(p_m^* \mathcal{O}_X^{[m]}) - c_1(p_n^* \mathcal{O}_X^{[n]})$.

Proposition 11.4. *Let $f : H^\bullet(X^{[n]}) \to H^\bullet(X^{[m]})$ be a homomorphism which is given by $f(\alpha) := p_{m*}(p_n^* \alpha \cap u)$, for a suitable $u \in H_\bullet(X^{[n,m]})$. Then*

$$f'(\alpha) = p_{m*}\Big(p_n^*(\alpha) \cdot \big(c_1(p_m^* \mathcal{O}_X^{[m]}) - c_1(p_n^* \mathcal{O}_X^{[n]})\big) \cap u\Big).$$

In particular, in case $m = n+1$, we get

$$f'(\alpha) = p_{n+1\,*}\big(p_n^*(\alpha)\cdot(-E)\cap u\big).$$

Proof.

$$\begin{aligned} f'(\alpha) &= df(\alpha) - f d\alpha \\ &= c_1(\mathcal{O}_X^{[m]})\cdot p_{m\,*}(p_n^*(\alpha)\cap u) - p_{m\,*}\Big(p_n^*\big(\alpha\cdot c_1(\mathcal{O}_X^{[n]})\big)\cap u\Big) \end{aligned}$$

Now we apply the projection formula. □

$X^{[n,n+m]}\times X$ carries two universal families $Z_n \subset Z_{m+n}$. The above result can also be reinterpreted as saying that we multiply by the first Chern class of the push-forward to $X^{[n,n+m]}$ of the ideal sheaf $\mathcal{I}_{Z_n/Z_{n+m}}$.

Now we come to the most important technical result of Lehns paper. It gives a geometric interpretation of the Virasoro operators L_n.

Theorem 11.5. 1.

$$\big[q_n'[u], q_m[w]\big] = -nm\left(q_{n+m}[uw] + \frac{|n|-1}{2}\delta_{n+m}\left(\int_X K_X uw\right)\mathbf{1}\right).$$

2.

$$q_n'[u] = nL_n[u] + \frac{n(|n|-1)}{2}q_n[K_X u].$$

Part 2. Says that the Virasoro generators $L_n[u]$ are essentially the derivatives of the $q_n[u]$.

Proof. We show that 1. implies 2. By the Heisenberg relations for the q_n and from the formula $\big[L_n[u], q_m[w]\big] = -mq_{n+m}[uw]$ from Theorem 9.2, we get

$$\begin{aligned} &\left[nL_n[u] + \frac{n(|n|-1)}{2}q_n[K_X u], q_m[w]\right] \\ &\qquad = -nmq_{n+m}[uw] + \frac{n^2(|n|-1)}{2}\delta_{n+m}\left(\int_X K_X uw\right)\mathbf{1}. \end{aligned}$$

Therefore the difference between the right-hand side and the left-hand side in 2. commutes with all the $q_m[u]$. Since $\mathbb{H}(X)$ is an irreducible Heisenberg module, it follows by Schurs lemma that the difference is the multiplication by a scalar. This scalar must be zero, because the difference has weight n (i.e. sends $H^\bullet(X^{[l]})$ to $H^\bullet(X^{[l+n]})$).

The proof of part 1. requires a complicated geometric argument, and it is also difficult to keep track of the indices. The most difficult part is the case $n = -m$ (when the Theorem also has an extra term). We will sketch the proof of

$$\left[q'_1[X], q_n[u]\right]\mathbf{1} = -nq_{n+1}[u]\mathbf{1},$$

which illustrates some of the geometric ideas, without running into any of the technicalities. In the application to Chern classes of tautological sheaves, we mostly use $q'_1[X]$.

Let $U \subset X$ be the submanifold represented by u. Let

$$M_n(U) := \left\{ Z \in X^{[n]} \;\middle|\; supp(Z) \text{ is one point of } U \right\},$$

$$M_{n,n+1}(U) := \left\{ (Z, W) \in X^{[n,n+1]} \;\middle|\; supp(Z) = supp(W) \text{ is one point of } U \right\}.$$

By definition and by Proposition 11.4 we obtain

$$q'_1[X]q_n[u]\mathbf{1} = q'_1[X][M_n(U)] = p_{n+1\,*}((-E) \cap p_n^*[M_n(U)]).$$

We recall that

$$E = \left\{ (Z, W) \in X^{[n,n+1]} \;\middle|\; supp(Z) = supp(W) \right\}.$$

Therefore, set-theoretically $M_{n,n+1} = M_n \times_{X^{[n]}} E$, but the map $E \to X^{[n]}$ has degree n, and the map $M_{n,n+1} \to M_n$ has degree 1. Therefore

$$\begin{aligned} p_{n+1\,*}((-E) \cap p_n^*[M_n(U)]) &= -p_{n+1\,*}(n[M_{n,n+1}(U)]) \\ &= -np_{n+1\,*}[M_{n+1}(U)] \\ &= -nq_{n+1}[U]\mathbf{1}. \end{aligned}$$

On the other hand $q'_1[X]\mathbf{1} = 0$. □

Corollary 11.6. *d and the $q_1[u]$ for $u \in H^\bullet(X)$ suffice to generate $\mathbb{H}(X)$ from $\mathbf{1}$.*

12. Chern classes of tautological sheaves

We define operators on $\mathbb{H}(X)$ of multiplying by the Chern classes of the tautological sheaves $F^{[n]}$ on $X^{[n]}$. If we can understand how these commute with the q_n, this allows us to compute the Chern numbers of all tautological sheaves, and to partially understand the ring structure of the $H^\bullet(X^{[n]})$.

Definition 12.1. Let $u \in K(X)$. We define operators $\bar{c}[u] \in End(\mathbb{H}(X))$ by

$$\bar{c}[u]y = c(u^{[n]}) \cdot y \quad \text{for } y \in H^\bullet(X^{[n]}).$$

So if u is the class of a vector bundle on X, then $\bar{c}[u]$ just multiplies for each n a class on $X^{[n]}$ with the total Chern class of the corresponding tautological sheaf $F^{[n]}$. We also write $\bar{c}_k[u]y$ for $c_k(u^{[n]}) \cdot y$. Note that by definition $d = \bar{c}_1[\mathcal{O}_X]$. Obviously the $\bar{c}[u]$ commute among each other (and therefore they also commute with d). We put

$$\bar{C}[u] := \bar{c}[u] q_1[X] \bar{c}[u]^{-1}.$$

We can use the operator $\bar{C}[u]$ to write down the total Chern classes of the tautological sheaves in a compact way.

Proposition 12.2.

$$\sum_{n \geq 0} c(u^{[n]}) = \exp(\bar{C}[u]) \mathbf{1}.$$

Proof. We note that

$$\frac{q_1[X]^n}{n!} \mathbf{1} = 1_{X^{[n]}}.$$

Therefore

$$\begin{aligned} \sum_{n \geq 0} c(u^{[n]}) &= \bar{c}[u] \exp(q_1[X]) \mathbf{1} \\ &= \bar{c}[u] \exp(q_1[X]) \bar{c}[u]^{-1} \mathbf{1} \\ &= \exp(\bar{c}[u] q_1[X] \bar{c}[u]^{-1}) \mathbf{1}. \end{aligned}$$

□

Now we express $\bar{C}[u]$ in terms of the derivatives of the Heisenberg operator q_1 applied to the Chern classes of u. This establishes a relation between the Chern classes of the tautological sheaves and the Heisenberg generators.

Theorem 12.3.

$$\bar{C}[u] = \sum_{\nu, k \geq 0} \binom{r-k}{\nu} q_1^{(\nu)}[c_k(u)],$$

(here $q_1^{(\nu)}[c_k(u)]$ is the ν-th derivative of $q_1[c_k(u)]$).

Proof. Let F be a locally free sheaf on X. Recall the incidence variety

$$\begin{array}{ccccc} X & \xleftarrow{\rho} & X^{[n,n+1]} & \xrightarrow{p_{n+1}} & X^{[n+1]} \\ & & \downarrow{\scriptstyle p_n} & & \\ & & X^{[n]} & & \end{array}$$

and the exact sequence

$$0 \to \rho^* F(-E) \to p_{n+1}^* F^{[n+1]} \to p_n^* F^{[n]} \to 0.$$

This gives

$$p_{n+1}^* c(F^{[n+1]}) = p_n^* F^{[n]} \cdot \sum_{\nu,k\geq 0} \binom{r-k}{\nu} (-E)^\nu \rho^* c_k(F). \tag{18}$$

So, for $y \in H^\bullet(X^{[n]})$, we get

$$\begin{aligned}\bar{C}[F]y &= c(F^{[n+1]}) \cdot p_{n+1\,*}\big(p_n^*(y \cdot c(F^{[n]})^{-1})\big) \\ &= p_{n+1\,*}\big(p_{n+1}^* c(F^{[n+1]}) \cdot p_n^* c(F^{[n]})^{-1} \cdot p_n^* y\big).\end{aligned}$$

We insert (18) into this formula and apply Proposition 11.4, which says that multiplying by $(-E)$ corresponds to taking derivatives. □

At least in the case of a line bundle L on X, the results obtained so far are enough for finding an elegant formula for the Chern classes of $L^{[n]}$.

Theorem 12.4.

$$\sum_{n\geq 0} c(L^{[n]}) = \exp\left(\sum_{m\geq 1} \frac{(-1)^{m-1}}{m} q_m[c(L)]\right)\mathbf{1}.$$

Remark 12.5. Note that for the top Chern classes this gives the following. Let $D \in |L|$ be a smooth curve, then $c_n(L^{[n]}) = [D^{[n]}] = [D^{(n)}]$. Then the theorem gives

$$\sum_{n\geq 0} [D^{(n)}] = \exp\left(\sum_{m\geq 1} \frac{(-1)^{m-1}}{m} q_m[c_1(L)]\right)\mathbf{1}.$$

This is Theorem 7.5, which was used to determine the constant in the Heisenberg relations.

Proof. Let

$$U(t) := \sum_{n\geq 0} c(L^{[n]}) t^n = \exp(\bar{C}[L]t)\mathbf{1}.$$

The second equality is by Proposition 12.2. Therefore U satisfies the differential equation

$$\frac{d}{dt}U(t) = \bar{C}[L]U(t), \qquad U(0) = \mathbf{1}.$$

Now let

$$S(t) := \exp\left(\sum_{m\geq 1} \frac{(-1)^{m-1}}{m} q_m[c(L)] t^m\right);$$

we want to show that $S(t)\mathbf{1}$ satisfies the same differential equation. By definition

$$\frac{d}{dt}S(t) = S(t)\cdot\sum_{m\geq 0}(-1)^m q_{m+1}[c(L)]t^m.$$

By the Lehns Main Theorem 11.5, we have

$$\big[q_1'[X], q_m[c(L)]\big] = -mq_{m+1}[c(L)].$$

As this commutes with $q_m[c(L)]$, we get

$$\Big[q_1'[X], \frac{q_m[c(L)]^n}{n!}\Big] = \frac{q_m[c(L)]^{n-1}}{(n-1)!}(-m)q_{m+1}[c(L)].$$

Therefore we obtain

$$\big[q_1'[X], S(t)\big] = S(t)\cdot\sum_{m\geq 1}(-1)^m q_{m+1}[c(L)]t^m.$$

We recall from Theorem 12.3 that

$$\bar{C}(L) = q_1[c(L)] + q_1'[X].$$

So we finally get by putting everything together

$$\begin{aligned}\bar{C}(L)S(t)\mathbf{1} &= \big[q_1'[X], S(t)\big]\mathbf{1} + q_1[c(L)]S(t)\\ &= S(t)\cdot\sum_{m\geq 0}(-1)^m q_{m+1}[c(L)]t^m.\end{aligned}$$

□

Let L again be a line bundle on X. We want to compute the top Segre classes

$$N_n := \int_{X^{[n]}} s_{2n}(L^{[n]})$$

as polynomials in the intersection numbers L^2, LK_X, K_X^2, $c_2(X)$ on X. A priory it is not clear that this should be possible. We rewrite

$$N_n = \int_{X^{[n]}} c_{2n}((-L)^{[n]}) = \int_{X^{[n]}} \frac{\bar{C}[-L]^n}{n!}\cdot 1.$$

By Theorem 12.3 we get

$$\bar{C}[-L] = \sum_{\nu\geq 0}(-1)^\nu q_1^{(\nu)}[c(-L)^{\nu+1}].$$

By the main theorem 11.5 we can express the derivatives of q_1 in terms of the Virasoro generators L_n and the Heisenberg generators q_n. Applying the definitions 9.1 of the Virasoro generators, we can express this in terms of the Heisenberg generators. We can do all these computations explicitly to compute the N_n for sufficiently small n. The calculation shows that the following conjecture is true until $n = 7$.

Conjecture 12.6. *(Lehn) Let k be the inverse power series to*

$$t = \frac{k(1-k)(1-2k)^4}{(1-6k+6k^2)}.$$

Then

$$\sum_{n\geq 0} N_n t^n = \frac{(1-k)^{LK_X - 2K_X^2}(1-2k)^{(L-K_X)^2+3\chi(\mathcal{O}_X)}}{(1-6k+6k^2)^{\chi(L)}}.$$

(Here $\chi(L) = L(L-K_X)/2 + (K_X^2 + c_2(X))/12$ is the holomorphic Euler characteristic of L.)

References

[1] A. Beilinson, J.N. Bernstein, P. Deligne, *Faisceaux pervers.* Astérisque, vol 100.

[2] J. Briançon, *Description de* $Hilb^n \mathbb{C}\{x, y\}$. Inventiones Math. 41, 45-89 (1977).

[3] J. Cheah, *Cellular decompositions for various nested Hilbert schemes of points.* Pac. J. Math., 183 (1998), 39–90.

[4] G. Ellingsrud and S. A. Strømme, *On the homology of the Hilbert scheme of points in the plane.* Invent. Math. 87 (1987), 343–352.

[5] G. Ellingsrud and S. A. Strømme, *An intersection number for the punctual Hilbert scheme of a surface.* Trans. Amer. Math. Soc. 350 (1998), 2547–2552.

[6] J. Fogarty, *Algebraic Families on an Algebraic Surface.* Am. J. Math. 10 (1968), 511–521.

[7] W. Fulton, *Intersection Theory.* Ergebnisse 3. Folge, Band 2, Springer 1984.

[8] L. Göttsche, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface.* Math. Ann. 286 (1990), 193–207.

[9] L. Göttsche and W. Soergel, *Perverse sheaves and the cohomology of the Hilbert schemes of smooth algebraic surfaces.* Math. Ann. 296 (1993), 235–245.

[10] I. Grojnowski, *Instantons and affine algebras. I. The Hilbert scheme and vertex operators.* Math. Res. Letters 3 (1996) 275-291.

[11] M. Lehn, *Chern classes of tautological sheaves on Hilbert schemes of points on surfaces.* Inventiones Math. 136, 157-207 (1999).

[12] I. G. Macdonald, *The Poincaré Polynomial of a Symmetric Product.* Proc. Cambridge Phil. Soc. 58 (1962), 563-568.

[13] H. Nakajima, *Heisenberg algebra and Hilbert schemes of points on projective surfaces.* Ann. Math. 145 (1997), 379-388.

[14] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces.* University Lecture Series, 18. American Mathematical Society, Providence, RI, 1999.