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Generalized Miura Transformations, Two-boson KP Hierarchies and Their Reduction to KdV Hierarchies

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ABSTRACT

Bracket preserving gauge equivalence is established between several two-boson generated KP type of hierarchies. These KP hierarchies reduce under symplectic reduction (via Dirac constraints) to KdV, mKdV and Schwarzian KdV hierarchies. Under this reduction the gauge equivalence is taking form of the conventional Miura maps between the above KdV type of hierarchies.

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1. Introduction

The main aim of this paper is to provide arguments for the existence and usefulness of the linking thread in form of a symplectic gauge transformation connecting various integrable systems. It seems to be quite relevant to introduce such an equivalence principle in view of the growing number of integrable models which recently entered high energy physics (e.g. matrix models). The linkage is discussed here in a simple context of two-boson KP hierarchies. One of advantages of using two-boson systems is that the gauge map connecting them reduces in the special limit to the well-known Miura transformation; another is that their provide a common origin for various KdV type of hierarchies.

We argue that the right setting to study gauge equivalence between integrable systems is provided by Adler-Kostant-Symes (AKS) [1] theory with a Poisson bracket structure defined in terms of the R-matrix Lie-Poisson (LP) bracket. There are several reasons behind this statement. The theory of classical R-matrices provides a unified approach to dealing with most, if not all, existing integrable systems. One practical advantage of using the classical R-matrix theory augmented by the AKS scheme is that it provides a simple method to construct commuting integrals for a wide class of integrable models. In the recent paper [2] classical R-matrix theory based on AKS approach was applied to the algebras of pseudo-differential symbols. This paper has utilized another feature of AKS approach, namely the possibility of establishing symplectic gauge invariance between various related integrable systems (see also [3] for similar approach). Especially, three integrable models [4, 5, 2], called in [2] as KP_ℓ with ℓ taking values 0, 1 and 2 were shown to be connected by the symplectic gauge transformation, acting on Lax operators parametrizing the coadjoint orbits. Symplectic character of the gauge mapping ensured that the R-matrix LP bracket was preserved. Note, that for $\ell = 0$ the AKS system KP_0 is nothing but a standard KP hierarchy.

In Section 2 we briefly recapitulate the AKS method as applied to the algebras of pseudo-differential symbols and state the results of [2] concerning the gauge equivalence of relevant integrable models. In Section 3 we explain the status of the two-boson KP hierarchy, which appears in this setting as an invariant subspace of the coadjoint orbit within the $KP_{\ell=1}$ hierarchy. We will work with two main cases of two-boson KP hierarchies, one defined within $KP_{\ell=1}$ hierarchy will be called Faá di Bruno KP hierarchy, while the second defined within KP hierarchy for a quadratic two-boson KP hierarchy. We will establish for them the gauge invariance playing the role of generalized Miura transformations. We emphasize the symplectic character of equivalence of $KP_{\ell=1}$ and KP and show how this feature explains the 2-boson representation of $W_{1+\infty}$ and $\hat{W}_\infty$ in terms of the Faá di Bruno polynomials. We also made a point that the gauge equivalence established for two-boson systems is valid for an arbitrary n-th Poisson bracket structure and not only the first Poisson bracket structure. In Section 4 we apply Dirac reduction scheme to both two-boson KP hierarchies. We obtain in the process of reduction the standard "one-boson" KdV and mKdV hierachies. On reduced manifold the gauge transformation connecting the two models takes the form of the Miura transformation. We also present some comments on generalizing Schwarzian KdV (SKdV) hierarchy to the two-boson system.

The generalized Miura maps appeared before in literature (see [5, 6, 7]) without however being studied in terms of the gauge transformations between Lax operators and connected

with conventional Miura maps via Dirac reduction.

2. AKS Construction of Generalized KP Hierarchies

Here we will apply the AKS construction [1] on Lie algebra $\mathcal{G}$ of pseudo-differential operators on a circle. An element of $\mathcal{G}$ is an arbitrary pseudo-differential operator $X = \sum_{k\geq -\infty} D^k X_k(x)$. An infinitesimal version of adjoint transformation is given by $ad(Y)X = [Y, X]$. In this setting, an identification of the dual space $\mathcal{G}^*$ with $\mathcal{G}$ and of the coadjoint action with the adjoint action is allowed by the Adler trace; an invariant, non-degenerate bilinear form given by $\langle L | X\rangle \equiv \mathrm{Tr}_A(LX) = \int \mathrm{Res} LX$.

There exist three decompositions of $\mathcal{G}$ into a linear sum of two subalgebras [4, 5, 2] (i.e. $\mathcal{G} = \mathcal{G}^\ell_+ \oplus \mathcal{G}^\ell_-$, with index ℓ taking values $\ell = 0, 1, 2$):

$$\mathcal{G}^\ell_+ = \{ X_{\geq \ell} = \sum_{i=\ell}^{\infty} D^i X_i(x) \} \quad ; \quad \mathcal{G}^\ell_- = \{ X_{<\ell} = \sum_{i=-\ell+1}^{\infty} D^{-i} X_{-i}(x) \} \tag{1}$$

The dual spaces to subalgebras $\mathcal{G}^\ell_\pm$ are given by:

$$\mathcal{G}^{\ell\,*}_+ = \{ L_{<-\ell} = \sum_{i=\ell+1}^{\infty} u_{-i}(x) D^{-i} \} \; ; \; \mathcal{G}^{\ell\,*}_- = \{ L_{\geq -\ell} = \sum_{i=-\ell}^{\infty} u_i(x) D^i \} \tag{2}$$

All three decompositions give rise to integrable models via the AKS construction. Define the R matrix for all the above cases as $R_\ell \equiv P^\ell_+ - P^\ell_-$, where $P^\ell_\pm$ are projections on $\mathcal{G}^\ell_\pm$. It follows from the general formalism that $[X, Y]_{R_\ell} \equiv [RX, Y]/2 + [X, RY]/2 = [X_{\geq \ell}, Y_{\geq \ell}] - [X_{<\ell}, Y_{<\ell}]$ defines an additional (with respect to usual commutator) Lie structure on $\mathcal{G}$ (see [2] and references therein). This additional structure gives rise to the LP R-bracket for $F, H \in C^\infty(\mathcal{G}^*, \mathbb{R})$:

$$\{F, H\}_R(L) = \langle L | [\nabla F(L), \nabla H(L)]_R \rangle \tag{3}$$

where the gradient $\nabla F : \mathcal{G}^* \to \mathcal{G}$ is defined by the standard formula given in [1, 2]. The AKS scheme defines the Hamiltonian equations of motion to be $dF/dt = \{H, F\}_R$ for $F \in C^\infty(\mathcal{G}^*, \mathbb{R})$. The basic result of AKS formalism states that the Ad^*-invariant functions (*Casimirs*) Poisson commute on $(\mathcal{G}^*, \{\cdot,\cdot\}_R)$ establishing integrability of the system and providing quantities in involution.

From the general relation for the R-coadjoint action of $\mathcal{G}$ on its dual space we find that the infinitesimal shift along an R-coadjoint orbit $\mathcal{O}(R_\ell)$ has the form: $\delta_{R_\ell} L \equiv ad^*_{R_\ell}(X)L = [X_{\geq \ell}, L_{<-\ell}]_{<-\ell} - [X_{<\ell}, L_{\geq -\ell}]_{\geq -\ell}$.

We will now focus on the Hamiltonian structure of the integrable systems given by decompositions labelled by $\ell = 0, 1$ (as in [2] we call them here as KP_ℓ hierarchies). For the remaining $\ell = 2$ model we refer the reader to [2].

KP_ℓ Hierarchies. The $\mathrm{KP}_{\ell=0}$ model is defined on the manifold being the R-coadjoint orbit of the form $\mathcal{O}(R_0) = \left\{ L = D + \sum_{k=1}^{\infty} u_k(x) D^{-k} \right\}$. The functions $H_{r+1} = \frac{1}{r+1} \int \mathrm{Res} L^{r+1}$ are Casimir functions on $\mathcal{G}^*$, which in R-matrix approach [1] produce commuting integrals of motion. In fact we find for this model

$$\partial L/\partial t_r = \frac{1}{2} ad^* \left((\nabla H_{r+1})_+ - (\nabla H_{r+1})_- \right) L = [((L^r)_+), L] \tag{4}$$

The subscript (+) means taking the purely differential part of L^r and $t = \{t_r\}$ are the evolution parameters (infinitely many time coordinates). We recognize in (4) the standard KP flow equation. The flows (4) are bi-Hamiltonian [8], i.e. there exist two Poisson bracket structures $\{\cdot,\cdot\}_{1,2}$, such that we can rewrite (4) as a Lenard recursion relation $\partial L/\partial t_r = \{H_r\,,\,L\}_2 = \{H_{r+1}\,,\,L\}_1$ for the hierarchy of Poisson bracket structures with $r = 1, 2, \ldots$. The first Hamiltonian structure is found to be induced by the LP structure: $\{u_i(x)\,,\,u_j(y)\}_{R_0} = \Omega^{(\ell=0)}_{i-1,j-1}(u(x))\,\delta(x-y)$, with [9, 5]:

$$\Omega^{(\ell)}_{i,j}(u(x)) = -\sum_{k=0}^{i+\ell}\binom{i+\ell}{k}u_{i+j+\ell-k+1}(x)D_x^k + \sum_{k=0}^{j+\ell}(-1)^k\binom{j+\ell}{k}D_x^k u_{i+j+\ell-k+1}(x) \quad (5)$$

This LP bracket algebra is isomorphic to the centreless $\mathbf{W}_{1+\infty}$ algebra [10]. All this just classifies $\mathrm{KP}_{\ell=0}$ as the standard KP hierarchy.

We now turn our attention to $\mathrm{KP}_{\ell=1}$ hierarchy. Here the Lax operator takes the form $L^{(1)} = D + u_0 + u_1 D^{-1} + \sum_{i\geq 2} u_{i-2}D^{-i}$. Application of (3) gives a Hamiltonian structure that is a direct sum of the 2×2 matrix $P^{(1)}$ with non-zero matrix elements $P^{(1)}_{12} = P^{(1)}_{21} = D$ associated with the modes $\{u_0, u_1\}$ and the Hamiltonian structure $\Omega^{(1)}$ from (5) associated with $\{v_i | i \geq 0\}$ [5]. Note that $\Omega^{(1)}$ corresponds to the centreless $\mathbf{W}_\infty$ algebra.

"Gauge" Equivalence of $\mathrm{KP}_{\ell=1}$ *Hierarchy to Ordinary* KP. The fundamental result of [2] was the proof that all three hierarchies are "gauge" equivalent via generalized Miura transformations. Here we focus on the link between two KP_ℓ systems discussed above. Reference [2] presented a symplectic (Hamiltonian) map between the orbits such that $G : \mathcal{O}(R_1) \longrightarrow \mathcal{O}(R_0)$. The term "symplectic" ("Hamiltonian") means that under the map G, the LP bracket structure on $\mathcal{O}(R_1)$ is mapped into the LP bracket structure on $\mathcal{O}(R_0)$.

$$\{F_1, F_2\}_{R_0}(G(L)) = \{F_1(G(L))\,,\,F_2(G(L))\}_{R_1} \quad (6)$$

where $F_{1,2}$ are arbitrary functions on $\mathcal{O}(R_0)$ and L and $G(L)$ denote coordinates on the orbits $\mathcal{O}(R_1)$ and $\mathcal{O}(R_0)$, respectively. As a consequence of (6), the infinite set of involutive integrals of motion $\{\widetilde{H}_N[G(L)]\}$ of the integrable system on $\mathcal{O}(R_0)$ are transformed into those of the integrable system on $\mathcal{O}(R_1)$: $H_N[L] = \widetilde{H}_N[G(L)]$.

As shown in [2] the right choice of the map $G : \mathcal{O}(R_1) \longrightarrow \mathcal{O}(R_0)$ is given by

$$G(L^{(1)}) = D + \sum_{k=1}^{\infty}\tilde{u}_k(x)\,D^{-k} = Ad^*(g(L^{(1)}))(D + u_0(x) + u_1(x)D^{-1} + \sum_{k=2}^{\infty}v_{k-2}(x)D^{-k}) \quad (7)$$

where the group element $g(L^{(1)})$ depends on $L^{(1)}$ in $\mathcal{O}(R_1) \subset \mathcal{G}^*$.

It is in the sense of eqs. (6) and (7) that the integrable systems on the orbits $\mathcal{O}(R)$ for different R-matrices are called "gauge" equivalent. We will provide further arguments in Section 4 for that the mapping of one Poisson bracket structure of an integrable model into another one by the group coadjoint action (7) deserves a name of the generalized Miura transformation.

Due to the simple formula $\exp(\phi_0(x))D\exp(-\phi_0(x)) = D - \partial_x\phi_0(x)$, it is easy to see that a "gauge" generator in (7) must be given by

$$g(L) = \exp\phi_0(x) \quad , \quad \partial_x\phi_0(x) = u_0(x) \quad (8)$$

The proof for gauge equivalence in [2] amounted to verifying (7) with (8). This established the "gauge" equivalence of $KP_{t=1}$ and KP by explicitly constructing the generalized Miura-like transformation (7)-(8), which maps the Poisson bracket structure of KP into that of $KP_{t=1}$ and vice versa.

3. Main Two Two-Boson KP Systems

Faá di Bruno Hierarchy. Here we go back to $KP_{t=1}$ and make the following crucial observation. Consider truncated elements of $\mathcal{G}_-^1$ of the type $L_J^{(1)} = D + u_0 + u_1 D^{-1} = D - J + \bar{J}D^{-1}$, where we have introduced two Bose currents $(J, \bar{J})$ to create fit with notation used in [11]. One easily verifies that under the coadjoint action $\delta_{R_1} L_J^{(1)} = \mathrm{ad}^*_{R_1}(X) L_J^{(1)}$ this finite Lax operator maintains its form, i.e. the two-boson Lax operators span an R_1-orbit of finite functional dimension 2. This observation, already present in [4] clarifies status of two-boson $(J, \bar{J})$ system as a consistent restriction of the full $KP_{t=1}$ hierarchy understood as an orbit model. Note that there are only two possibilities for the invariant R_1-orbit; the two-boson system and the full $KP_{t=1}$ system (in quasiclassical limit situation is much richer). A calculation of the Poisson bracket according to (3): $\{\langle L_J^{(1)}|X\rangle, \langle L_J^{(1)}|Y\rangle\}_{R_1} = \langle L_J^{(1)}|[X, Y]_{R_1}\rangle$ yields the first bracket structure of two-boson $(J, \bar{J})$ system given as LP R-bracket: $\{J(x), J(y)\} = -\delta'(x-y)$ and zero otherwise.

As we show now the two-boson KP hierarchy is associated with so called Faá di Bruno polynomials and we will call it therefore Faá di Bruno hierarchy. Consider namely the gauge transformation between $KP_{t=1}$ and $KP_{t=0}$ generated by Φ such that $\Phi' = J$:

$$L_J = e^{-\Phi} L_J^{(1)} e^{\Phi} = D + J(D+J)^{-1} = D + \sum_{n=0}^{\infty} (-1)^n J P_n(J) D^{-1-n} \tag{9}$$

where $P_n(J) = (D+J)^{n-1}\cdot 1$ are the Faá di Bruno polynomials. As a corollary of the symplectic character of the "gauge" transformation used in (9), we conclude that $u_n = (-1)^n J P_n(J)$ satisfy the Poisson-bracket $\mathbf{W}_{1+\infty}$ algebra described by the form $\Omega^{(0)}$ from (5) [5, 11]. It is possible to introduce a deformation parameter into the Faá di Bruno representation of $\mathbf{W}_{1+\infty}$ algebra by redefining u_n to $u_n(h) = (-1)^n J(hD+J)^n \cdot 1$ [11]. Now the semiclassical limit is simply obtained by taking $h \to 0$ in $u_n(h)$ and yields the generators of $\mathbf{w}_{1+\infty}$ algebra.

The higher bracket structures have been investigated in [5, 11] and the result can be summarized as follows. The three lowest Hamiltonian functions are:

$$H_{J1} = \int J \quad ; \quad H_{J2} = \int -JJ \quad ; \quad H_{J3} = \int \left(JJ^2 + JJ' + J^2\right) \tag{10}$$

For the general Hamiltonian matrix structure P_r we have

$$\frac{\partial}{\partial t_r}\begin{pmatrix} J \\ \bar{J} \end{pmatrix} = P_r \begin{pmatrix} \delta H_{r+2-i}/\delta J \\ \delta H_{r+2-i}/\delta \bar{J} \end{pmatrix} = P_{J1}\begin{pmatrix} \delta H_{r+1}/\delta J \\ \delta H_{r+1}/\delta \bar{J} \end{pmatrix} = P_{J2}\begin{pmatrix} \delta H_r/\delta J \\ \delta H_r/\delta \bar{J} \end{pmatrix} \tag{11}$$

Among the multi-Hamiltonian structures only P_{J1} and P_{J2} are independent. All other matrices P_i, $i = 3, 4, \ldots$ are related to P_{J2} through $P_{Ji} = (P_{J2}(P_{J1})^{-1})^{i-2} P_{J2}$ involving the so-called recurrence matrix $P_{J2}(P_{J1})^{-1}$ [6, 11]. The explicit form of first and second

local Hamiltonian structures is:

$$P_{J1} = \begin{pmatrix} 0 & -D \\ -D & 0 \end{pmatrix} \ , \ P_{J2} = \begin{pmatrix} 2D & D^2 + DJ \\ -D^2 + JD & D\bar{J} + \bar{J}D \end{pmatrix} \tag{12}$$

Taking $r = 2$ in (11) we especially get the Boussinesq equation:

$$J_{t_2} = \{J, H_{J3}\}_1 = -\hbar J''' - \left(J^2\right)' - 2\bar{J}' \ ; \ \bar{J}_{t_2} = \{\bar{J}, H_{J3}\}_1 = \hbar \bar{J}''' - 2\left(\bar{J}J\right)' \tag{13}$$

where we re-introduced $\hbar$ as a deformation parameter. In the dispersiveless limit $\hbar \to 0$ taken in (13) we obtain the classical dispersiveless long wave equations (Benney equations) [12, 5].

Quadratic Two-Boson KP Hierarchy. Here we call quadratic two-boson KP hierarchy the construction presented by Wu and Yu [13] in order to realize $\hat{W}_\infty$ as a hidden current algebra in the 2d $SL(2,\mathbb{R})/U(1)$ coset model. Construction is based on the pseudo-differential operator:

$$L_J = D + \bar{J}\,(D - J - \bar{J})^{-1} J \tag{14}$$

Let us discuss the Hamiltonian structure first. The three lowest Hamiltonian functions are:

$$H_{J1} = \int J\bar{J} \ ; \ H_{J2} = \int -J'\bar{J} + J^2\bar{J} + J\bar{J}^2 \ ; \ H_{J3} = \int J''\bar{J} - 3JJ'\bar{J} - 2J'\bar{J}^2 - J\bar{J}\bar{J}' + J^3\bar{J} + 3J^2\bar{J}^2 + J\bar{J}^3 \tag{15}$$

Among the Hamiltonian structures only second and third are local and are given by

$$P_{J2} = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \ , \ P_{J3} = \begin{pmatrix} DJ + JD & -D^2 + DJ + \bar{J}D \\ D^2 + JD + D\bar{J} & D\bar{J} + JD \end{pmatrix} \tag{16}$$

Proposition. The Hamiltonian structure corresponding to the Lax operator L_J in (14) is invariant under the following two transformations:

$$J \ \to \ \bar{J} - \frac{J'}{J} \qquad \text{and} \qquad \bar{J} \to J \tag{17}$$

$$\bar{J} \ \to \ J + \frac{\bar{J}'}{\bar{J}} \qquad \text{and} \qquad J \to \bar{J} \tag{18}$$

Proof. One verifies relatively easily that the bracket structures induced by both P_{J2} and P_{J3} are invariant under the transformations (17) and (18). Since $P_1 = P_2 P_3^{-1} P_2$, a recurrence matrix $P_2(P_1)^{-1}$ and all remaining higher hamiltonian structures must therefore remain invariant under (17) and (18). This completes the proof. One can also directly verify that all three Hamiltonians (15) are invariant under (17) and (18). Hence we conclude that the Lax operators given by

$$L_J = D + J\left(D - J - \bar{J} + \frac{J'}{J}\right)^{-1}\left(\bar{J} - \frac{J'}{J}\right) \tag{19}$$

$$L_J = D + \left(J + \frac{\bar{J}'}{\bar{J}}\right)\left(D - J - \bar{J} - \frac{\bar{J}'}{\bar{J}}\right)^{-1}\bar{J} \tag{20}$$

lead to the same Hamiltonian functions as (14).

Gauge Equivalence between Faá di Bruno and Quadratic Two-Boson Hierarchies. Generalized Miura Map. We apply on L_j from (14) the gauge transformation generated by $\xi = \left(\phi + \bar{\phi} - \ln j\right)$ with result:

$$L_j \to \exp(-\xi)\, L_j \exp(\xi) = D + j + \bar{j} + j(j^{-1})' + \bar{j}j D^{-1} = D - J + \bar{J} D^{-1} \qquad (21)$$

where we have introduced

$$J = -j - \bar{j} + \frac{j'}{j} \quad ; \quad \bar{J} = \bar{j}j \qquad (22)$$

One can now verify explicitly that with the bracket structure given by P_{j2} in (16) variables defined in (22) satisfy the second bracket structure P_{J2} (12) of Faá di Bruno hierarchy. As a corollary we obtained therefore a short proof for the quadratic two-boson KP hierarchy [13] system realizing $\hat{W}_\infty$. We also obtained a Miura transform for two-Bose hierarchies in form of (22) which generalizes the usual Miura transformation between one-bose KdV and mKdV structures (as given below).

It is intriguing that the higher hamiltonian structures of quadratic two-boson hierarchy are being mapped by (22) to their counterparts in Faá di Bruno hierarchy while the gauge equivalence established in Section 1 is limited to the first bracket structure. Explanation for the equivalence of higher structures follows however easily from two additional features. First, it is true that the Hamiltonian functions are invariant under the gauge equivalence. Second, we note that the Lenard recursion relations extend to two-boson system in $\mathrm{KP}_{\ell=1}$ as observed in [2]. One can now use the above two facts to extend the gauge equivalence (in the symplectic sense) to the arbitrary order of bracket structure for the two-boson systems.

Let us go back to the alternative expression (19) for the quadratic two-boson hierarchy. It can be rewritten under multiplication by $1 = jj^{-1}$ from the right and left as $L_j = 1\, L_j 1 = jj^{-1} L_j jj^{-1} = D + (D - j - \bar{j})^{-1}(\bar{j}j - j')$. Next step is to gauge transform L_j from KP to KP_1 hierarchy by acting with gauge transformation generated by $\exp(\phi + \bar{\phi})$ obtaining

$$L_j \sim \exp\left(-\phi - \bar{\phi}\right) L_j \exp\left(\phi + \bar{\phi}\right) = D + j + \bar{j} + D^{-1}(\bar{j}j - j') \qquad (23)$$

which is of the form of the Faá di Bruno hierarchy (up to conjugation) with $J = -j - \bar{j}$ and $\bar{J} = \bar{j}j - j'$. This reproduces construction given in [11] to get the second bracket structure from the first. Note that under (18) this is transformed into $J = -j - \bar{j} - \frac{j'}{j}$ and $\bar{J} = \bar{j}j$ differing from (22) by a conjugation $j \leftrightarrow \bar{j}$.

Similarly for (20) we find $L_j = \bar{j}^{-1}\bar{j} L_j \bar{j}^{-1}\bar{j} = D + (\bar{j}j + \bar{j}')(D - j - \bar{j})^{-1}$. The same transformation as in (23) gives

$$L_j \sim \exp\left(-\phi - \bar{\phi}\right) L_j \exp\left(\phi + \bar{\phi}\right) = D + j + \bar{j} + (\bar{j}j + \bar{j}')\, D^{-1} \qquad (24)$$

producing KP_1 object with $J = -j - \bar{j}$ and $\bar{J} = \bar{j}j + \bar{j}'$. This time under (17) these variables are transformed into $J = -j - \bar{j} + \frac{\bar{J}'}{\bar{J}}$ and $\bar{J} = \bar{j}j$ identical to (22).

We see that because of (17) and (18) there is an ambiguity in the possible form of generalized Miura transformation and (22) can appear also in other forms. All of them are connecting the Poisson bracket structure of Faá di Bruno hierarchy with the corresponding Poisson bracket structure of the quadratic two-boson hierarchy.

AKNS/NLS Hierarchy. AKNS or NLS system is a constrained KP system described by:

$$L_{AKNS} = D + \bar{\psi} D^{-1} \psi \tag{25}$$

with the first two bracket structures given by (see for instance [14]):

$$P_{AKNS1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \ P_{AKNS2} = \begin{pmatrix} -2\bar{\psi} D^{-1} \bar{\psi} & D + 2\bar{\psi} D^{-1} \psi \\ D + 2\psi D^{-1} \bar{\psi} & -2\psi D^{-1} \psi \end{pmatrix} \tag{26}$$

One can easily show that AKNS hierarchy is equivalent to Faá di Bruno two-boson KP. The proof is based, in the spirit of [2], on establishing gauge transformation between two hierarchies. We show now the argument to illustrate the power of gauge transformation argument in the KP setting. Consider $L_{AKNS} \to G^{-1} L_{AKNS} G = G^{-1} D G + G^{-1} \psi D^{-1} \psi G$. Choose $G^{-1} = \bar{\psi}$, which leads to

$$G^{-1} L_{AKNS} G = \bar{\psi} D \bar{\psi}^{-1} + \bar{\psi} \psi D^{-1} = D + \bar{\psi} (\bar{\psi}^{-1})' + \bar{\psi} \psi D^{-1} \tag{27}$$

Clearly the gauge transformed L_{AKNS} is an element of KP_1 hierarchy and it is therefore natural to introduce new variables such that $J = -\bar{\psi} (\bar{\psi}^{-1})'$ and $j = \bar{\psi} \psi$ and the inverse relation being $\bar{\psi} = \exp(\int J)$ and $\psi = j \exp - (\int J)$. Since we now have established a gauge equivalence between two hierarchies it is clear that the first bracket structure in (26) leads to $\{ J(x), J(y) \}_1 = -\delta'(x-y)$ and therefore a linear $W_{1+\infty}$ algebra. The second bracket structure in (26) leads to the second structure in (12) and correspondingly non-linear $\hat{W}_\infty$. If we only took the linear structure in P_{AKNS2} (i.e. $\{ \bar{\psi}(x), \psi(y) \} = \delta'(x-y)$) we would have induced (12) in its "un-deformed" form with upper left corner of $P_{J,j}$ in (12) being zero, corresponding to $\Omega^{(1)}$ or W_∞.

The AKNS system is also gauge equivalent to quadratic KP hierarchy if we make in (27) a substitution $\bar{\psi} = j \exp(\phi + \bar{\phi})$ and $\psi = \exp(-\phi - \bar{\phi}) j$ or inversely $\psi'/\psi = -j - j + j'/j$ and $\bar{\psi}\psi = j j$.

4. Reduction to "One-boson" KdV Systems

We apply here the Dirac reduction scheme to obtain one-boson hierarchies from two-boson hierarchies. The general feature will be a transformation of some two-boson Hamiltonian equations of motion expressed by 2-nd bracket structure $\delta \Gamma / \delta t_r = \{ \Gamma, H_r \}_2$ (where Γ denote original degrees of freedom) to one-boson Hamiltonian system according to the Dirac scheme:

$$\frac{\partial X}{\partial t_r} = \{ X, H_r^D \}_{Dirac} \tag{28}$$

with X denoting a surviving one-boson degree of freedom. Another point is that it is a presence of symmetry in (17) and (18) that enables reduction to be made.

KdV Hierarchy. Consider the Dirac constraint: $\Theta = J = 0$ for system in (9). First let us discuss the resulting Dirac bracket structure. We find for the surviving variable J:

$$\{J(x), J(y)\}_2^D = \{J(x), J(y)\}_2 - \int dzdz' \{J(x), \Theta(z)\}_2 \{\Theta(z), \Theta(z')\}_2^{-1} \{\Theta(z'), J(y)\}_2$$

$$= 2J(x)\delta'(x-y) + J'(x)\delta(x-y) + \frac{1}{2}\delta'''(x-y) \tag{29}$$

The reduced Lax operator looks now as:

$$l_J = D + JD^{-1} \tag{30}$$

and the corresponding (non-zero) lowest Hamiltonian functions $H_r^{KdV} \equiv \mathrm{Tr}\, l_J^r/r$ are:

$$H_1^{KdV} = \int J \quad ; \quad H_3^{KdV} = \int J^2 \quad ; \quad H_5^{KdV} = \int \left(2J^3 + JJ''\right) \tag{31}$$

Moreover one checks that the flow equation:

$$\delta l_J/\delta t_r = [(l_J^r)_+, l_J] \tag{32}$$

gives on the lowest level $\delta J/\delta t_1 = J'$ and $\delta J/\delta t_3 = J''' + 6JJ'$, with the second equation reproducing the well-known KdV equation. This equation can also be obtained by inserting $X = J$ and H_3^{KdV} into (28).

mKdV Hierarchy. Now consider the quadratic two-boson hierarchy with Lax given in (14), (19) or (20). We choose as a Dirac constraint: $\theta = j + \bar{j} = 0$. The resulting Dirac bracket structure is:

$$\{j(x), j(y)\}_2^D = -\int dzdz' \{j(x), \theta(z)\}_2 \{\theta(z), \theta(z')\}_2^{-1} \{\theta(z'), j(y)\}_2 = -\frac{1}{2}\delta'(x-y) \tag{33}$$

and the reduced Lax operator is:

$$l_j = D - jD^{-1}j = D + \sum_{n=0}^{\infty} (-1)^{n+1} j j^{(n)} D^{-1-n} \tag{34}$$

Note that imposing the constraint $\theta = 0$ on the equivalent Lax operators from (19) and (20) respectively, we get:

$$l_j = L_j|_{\theta=0} = D + j\left(D + \frac{j'}{j}\right)^{-1}\left(-j - \frac{j'}{j}\right) = D + D^{-1}\left(-j^2 - j'\right) \tag{35}$$

$$l_j = L_j|_{\theta=0} = D - \left(j + \frac{j'}{j}\right)\left(D - \frac{j'}{j}\right)^{-1} j = D + \left(-j^2 - j'\right) D^{-1} \tag{36}$$

Obviously we could have expressed everywhere j by $-j$ hence the one-boson system must be invariant under transformation $j \leftrightarrow -j$. The flow equations calculated as in (32) are

$$\frac{dj}{dt_1} = j' \quad ; \quad \frac{dj}{dt_2} = 0 \quad ; \quad \frac{dj}{dt_3} = j''' + 6j^2(j)' \tag{37}$$

Hence the flow equation for dj/dt_3 in the mKdV equation. Furthermore the mKdV equation could also be obtained from Hamiltonian H_3^{mKdV} defined in a standard way:

$$H_1^{mKdV} = -\int j^2 \; ; \; H_3^{mKdV} = \int \left(j^4 - jj''\right) \; ; \; H_5^{mKdV} = -\int \left(2j^6 + 10j^2(j')^2 + jj^{(IV)}\right) \tag{38}$$

(and zero for even indices). Because of existence of symmetry described in (17) (and (18)) we could equivalently impose the constraints $\theta_1 = j + \bar{j} - j'/j = 0$ or alternatively $\theta_2 = j + \bar{j} + j'/\bar{j} = 0$ without changing the Dirac bracket structure and the constraint manifold. Imposing $\theta_1 = 0$ on the Lax operator in (14) we get

$$l_j = D + \left(-j + \frac{j'}{j}\right)\left(D - \frac{j'}{j}\right)^{-1} j = D + \left(-j^2 + j'\right) D^{-1} \tag{39}$$

Taking however the equivalent Lax operator as given in (19) we get $l_j = L_j|_{\bar{j}=0} = D - jD^{-1}j$. Hence the mKdV hierarchy is given in terms of three alternative and equivalent Lax operators given in (34), (35) and (39). Especially the mKdV Hamiltonians (including those in (38)) are invariant under transformation $j \to -j$.

Miura Map. Let us now impose the Dirac constraint $J = -j - \bar{j} + j'/j = 0$ on the generalized Miura transformation (22). As a result we get the conventional Miura map:

$$\bar{J}|_{J=0} = j\left(-j + \frac{j'}{j}\right) = -j^2 + j' \tag{40}$$

It is easy to find via Dirac procedure that j satisfies the bracket

$$\{j(x), j(y)\}_2^D = -\int dx dx' \{j(x), J(z)\}_2 \{J(z), J(z')\}_2^{-1} \{J(z'), j(y)\}_2 = -\frac{1}{2}\delta'(x-y) \tag{41}$$

which is perfectly consistent with $\bar{J} = -j^2 + j'$ satisfying the bracket (29).

Especially we see that all Hamiltonians from (31) go to Hamiltonians in (38) under $J \to -j^2 \pm j'$.

Bi-Hamiltonian Structure of KdV Hierarchy. The evolution equation (28) specified to the constrained manifold $J = 0$ results in

$$\frac{\partial \bar{J}}{\partial t_r}|_{J=0} = \{\bar{J}, H_r^{KdV}\}_2^D = \left(D\bar{J} + \bar{J}D + \frac{1}{2}D^3\right)\frac{\delta H_r}{\delta J}|_{J=0} \tag{42}$$

in which one recognizes the second Hamiltonian structure of KdV hierarchy. We now show how to recover the first Hamiltonian structure of the KdV hierarchy (our discussion is here parallel to that given in [5]). Recall now (11) and take r odd so $H_{r\pm1} \to 0$ for $J \to 0$. We find from (11) using P_{j_1} that $\partial\bar{J}/\partial t_r|_{J=0} = -D\delta H_{r+1}/\delta J|_{J=0}$. On the other hand calculating $\partial J/\partial t_{r+1}$ using both P_{j_1} and P_{j_2} we find the following consistency relation in the two-boson case:

$$2D\frac{\delta H_{r+1}}{\delta J} + \left(D^2 + D\bar{J}\right)\frac{\delta H_{r+1}}{\delta \bar{J}} = -D\frac{\delta H_{r+2}}{\delta \bar{J}} \tag{43}$$

However in the limit $J \to 0$ since $H_{r+1} \to 0$ we have $\delta H_{r+1}/\delta \tilde{J} \to 0$. Therefore summarizing we find

$$\frac{\partial \tilde{J}}{\partial t_r}\Big|_{J=0} = -D\frac{\delta H_{r+1}}{\delta J}\Big|_{J=0} = \frac{1}{2}D\frac{\delta H_{r+2}^{KdV}}{\delta \tilde{J}} = \left(\frac{1}{2}D^3 + \tilde{J}D + D\tilde{J}\right)\frac{\delta H_r^{KdV}}{\delta \tilde{J}} \tag{44}$$

which reproduces well-known result about bi-Hamiltonian structure of KdV (see also [5]). Equation (44) can also be treated as a recurrence relation which proves that the system defined by Lax given in (30) is indeed KdV system to all orders of the Hamiltonian function.

One can now find the bi-Hamiltonian structure for the case of mKdV. First we recall a formula [15]:

$$(D \mp 2j)\,D\,(D \pm 2j) = 2\left(\frac{1}{2}D^3 + (-j^2 \pm j')D + D(-j^2 \pm j')\right) \tag{45}$$

Next from Miura transformation we find [15] $\delta H_r^{mKdV}/\delta j = (-D - 2j)\,\delta H_r^{KdV}/\delta \tilde{J}$. We therefore have

$$\frac{\delta H_{r+2}^{mKdV}}{\delta j} = (-D - 2j)\frac{\delta H_{r+2}^{KdV}}{\delta \tilde{J}} = (D + 2j)\,D^{-1}\,(D - 2j)\,D\frac{\delta H_r^{mKdV}}{\delta j} \tag{46}$$

where we used both (44) and (45). Relation (46) reveals a bi-Hamiltonian (but non-local) structure of mKdV hierarchy and can be rewritten in a more simple way as the Lenard recursion relation:

$$D\frac{\delta H_{r+2}^{mKdV}}{\delta j} = \left(D^3 - 4D\,j\,D^{-1}\,j\,D\right)\frac{\delta H_r^{mKdV}}{\delta j} \tag{47}$$

Schwarzian KdV Hierarchy. Here few remarks are given about Schwarzian KdV (SKdV) hierarchy, e.g. [16]. We start by discussing invariance of the Miura map $\tilde{J} = -j^2 + j' = -(\phi')^2 + \phi''$ where as before $\phi' = j$. Let δ be some transformation which leaves $\tilde{J}$ invariant, then $\delta(-(\phi')^2 + \phi'') = 0$ or $\delta\phi'' = 2\phi'\delta\phi'$. Solution to this takes a simple form

$$\delta\phi' = \delta j = \epsilon^{-1}\exp(2\phi) \qquad \text{or} \qquad \delta\phi = \frac{\epsilon^0}{2} + \epsilon^{-1}\exp(2\phi) \tag{48}$$

where ϵ^0 and ϵ^{-1} are some arbitrary constants. Introduce now the function f connected to j through the Cole-Hopf type of transformation $j = \phi' = f''/2f'$ or $f' = \exp(2\phi)$. We find that (48) corresponds to sl_2 transformation $\delta f = \epsilon^1 + \epsilon^0 f + \epsilon^{-1}f^2$ and leaves $\tilde{J} = S(f)/2$ invariant, where $S(f)$ is a Schwarzian.

It is known that the Cole-Hopf transformation relates the mKdV hierarchy to the SKdV hierarchy with equation $f_t/f' = S(f)$. Hence we will be interested in one-boson Lax operator of the form

$$L = D - \frac{1}{2}\frac{f''}{f'}D^{-1}\frac{1}{2}\frac{f''}{f'} \tag{49}$$

There are many ways of promoting this operator to two-boson system. If we consider a very simple choice

$$L = D + \frac{1}{2}\frac{f''}{f'} + \frac{f''}{f'}D^{-1}j \tag{50}$$

the second bracket structure is $\{ j(x), f(y) \} = -2f'(x)D_x^{-1}\delta(x-y)$. Another choice could be $L = D + (f''/f') + 2\rho + ((f''/f') + \rho) D^{-1}\rho$ leading to (49) under constraint $(f''/f') + 2\rho = 0$. Defining $\rho = v'$ we can now make contact with quadratic KP hierarchy by defining a map: $\bar{j} = v' + (f''/f')$ and $j = v'$. Of course ambiguity of (17) allows equally well a map: $\bar{j} = v' + (f''/f') - (v''/v')$ and $j = v'$. The corresponding bracket equivalent to P_{j2} in (16) is non-local and we find easily e.g. $\{ v(x), f(y) \} = D_x^{-1} f'(x) D_x^{-1} \delta(x-y)$.

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