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SUPPLEMENT TO THEORY OF NEUTRON CHAIN REACTIONS

By
Alvin M. Weinberg
L. C. Noderer

May 26, 1952

Oak Ridge National Laboratory
Oak Ridge, Tennessee

Technical Information Extension, Oak Ridge, Tennessee

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OAK RIDGE NATIONAL LABORATORY
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The Exponential Experiment

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The multiplying properties of a subcritical chain reacting assembly can be determined, as just indicated, by measuring the multiplication rate, R. This, close to critical, is

$$R = \frac{-1}{k_{ex}(B_o^2)}$$

which, according to (5.28), gives

$$-\frac{1}{R} = k_{ex}(B_o^2) = \frac{k}{p}\,\frac{\bar{P}_\infty(E_s,B_o^2)}{1 + L^2B_o^2} - 1 \qquad (5.96)$$

The quantity B_o^2 is of course known if the size and shape of the reactor are known; for a sphere of radius R, $B_o^2 = \frac{\pi^2}{R^2}$. Often $\frac{1}{p}\bar{P}_\infty(E_s,B_o^2)$ and L^2 are known from diffusion type experiments; in this event the multiplication constant, k, can be found from the measurement of R.

Of particular interest is the situation where the subcritical assembly is so large that the simplified equation (5.22) holds. Then the excess multiplication constant becomes

$$k_{ex}(B_o^2) = k - M^2B_o^2 - 1;$$

hence if M^2 is known, and if R, i.e., k_{ex}, is measured, then

$$k = 1 + M^2B_o^2 - \frac{1}{R}.$$

The measurement of multiplication rate is a practical way to determine the multiplying properties of the assembly only if the system is so close to

critical that R is large; otherwise, the two quantities $M^2B_o^2$ and $\frac{1}{R}$ will be very large, and their difference, which determines k, will have little accuracy. The multiplication rate method is therefore best suited to enriched systems whose critical size is small and which can be built large enough to give multiplication rates of 10 or 15 rather readily.

Since the size of a critical ordinary U chain reactor is very large, the multiplication rate method is applicable only with difficulty to such systems. While the German project nevertheless made most determinations of the multiplying properties of graphite-uranium, and D_2O-uranium systems by measuring multiplication rates, the American and Canadian projects obtained information on such systems by actually measuring the details of the neutron distribution in subcritical assemblies in which the multiplication rate was as little as 1.5 to 2. The advantage of such experiments, called exponential or "intermediate" experiments, was that they could give information on the multiplying system with relatively little material. Since the high quality graphite and uranium needed for the assemblies was very scarce, the possibility afforded by these experiments of predicting the critical size or optimizing uranium-graphite configurations was of the highest importance for the Hanford project.

The exponential experiment ideally consists of a long parallelepiped or cylinder of the multiplying configuration whose properties are to be determined. A neutron source is placed in the plane $z = 0$. The experiment consists in measuring the slow neutron distribution in the assembly; the details of this distribution determine the neutron <u>multiplying</u> properties of the system just as the details of the neutron distribution in the rather similar diffusion length experiment (Chapter I, P. 77) serve to determine

the neutron absorbing properties of the material out of which the diffusion length experiment is constructed.

The analysis of the neutron distribution in the exponential experiment is best done by an application of the Fourier transform. This procedure, first used by G. Placzek and G. Volkoff, Canadian Journal of Research, A, 25, 276 (1947) is of such wide application in all of nuclear reactor theory that it will be described in detail.

Consider a uniform multiplying system in the shape of an infinitely long parallelepiped of square cross section whose extrapolated width, assumed to be the same for neutrons of all energy is a. In the $z = 0$ plane there is placed a source of fast neutrons distributed like

$$Q(x,y,z) = Q_0 \delta(z) \cos B_x x \cos B_y y \qquad (5.97)$$

where

$$B_x^2 = B_y^2 = \frac{\pi^2}{a^2}$$

are the geometric bucklings in the x and y directions. It is required to find $\Phi_s(x, y, z)$, the thermal flux throughout the system.

The problem as set forth is somewhat specialized since the source distribution is given simply by the fundamental mode cox $B_x x$ cos $B_y y$. However since any source distribution f(x, y) can be expressed as a superposition of Fourier components

$$f(x,y) = \sum \sum f_{i_j} \text{ cox } B_{i_x} x \cos B_{j_y} y$$

the restriction to only the fundamental is not serious: the neutron distribution for an arbitrary source disposition is a superposition of the distribution for each harmonic.

In much the same way, the restriction to an infinitely long system is not essential since the distribution in a block of length H is found by placing + sources of equal strength at the points $z = \pm 4\,n\,H$ and - image sources at $z = \pm 4\,(n + \frac{1}{2})H$ in an infinitely long block and superposing.

The equation for the asymptotic slow neutron density is

$$D_o \Delta \Phi_s - N\sigma_a \Phi_s + \frac{k\,N\sigma_a}{p} \int \Phi_s(\underline{r}')P(E_s,\underline{r},\underline{r}')d\underline{r}' + \int Q(\underline{r}')\,P^*(E_s,\underline{r},\underline{r}')d\underline{r}' = 0 \qquad (5.98)$$

The slowing down kernels $P^*(E_s,r)$ and $P(E_s,r)$ are in general different since the energy of the source neutrons need not be the same as the energy of the fission neutrons. To solve (5.98) we put

$$\Phi_s(r) = f(z)\cos B_x x \cos B_y y \qquad (5.99)$$

and we introduce the Fourier transforms

$$f(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(w)e^{iwz}\,dw$$
$$\delta(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iwz}\,dw \qquad (5.100)$$

If (5.97), (5.99), and (5.100) are substituted into (5.98), and if use is made of the fact that

$$\int_{\text{Pile}} \cos B_x x \cos B_y y\, e^{iwz} P(E_s,\underline{r},\underline{r}')d\underline{r}' = \int_{\substack{\text{all}\\ \text{space}}} \cos B_x x \cos B_y y\, e^{iwz} P_\infty(E_s,|\underline{r}-\underline{r}'|)d\underline{r}'$$

$$= \cos B_x x \cos B_y y\, e^{iwz}\, \bar{P}_\infty\,(E_s,B^2)$$

where

$$B^2 = B_x^2 + B_y^2 + w^2,$$

then $\bar{f}(w)$ is found to satisfy

$$\bar{f}(w) = \frac{Q_o \, \bar{P}_\infty{}^*(E_s, B^2)}{N\sigma_a \left[1 + L^2B^2 - \frac{k}{p} \bar{P}_\infty(E_s, B^2) \right]} \tag{5.101}$$

Hence the required distribution is

$$f(z) = \frac{Q_o}{2\pi} \int_{-\infty}^{\infty} \frac{\bar{P}_\infty{}^*(E_s, B^2)^{iwz}}{N\sigma_a \left[1 + L^2B^2 - \frac{k}{p} \bar{P}_\infty(E_s, B^2) \right]} \, dw. \tag{5.102}$$

To compute this integral we follow a semicircular contour which embraces the entire positive half-plane. Then if $B_\nu{}^2 = B_x{}^2 + B_y{}^2 + w_\nu{}^2$ are the zeros (assumed simple) with positive imaginary part of the denominator, i.e.,

$$\frac{k \, \bar{P}_\infty(E_s, B_\nu{}^2)}{p(1 + L^2B_\nu{}^2)} = 1 \tag{5.103}$$

then integral (5.102) has the value

$$f(z) = \frac{Q_o}{2} \sum_\nu \frac{\bar{P}_\infty{}^*(E_s, B_\nu{}^2) e^{iw_\nu |z|}}{B_\nu \, N\sigma_a \left[\frac{k}{p} \bar{P}'_\infty(E_s, B_\nu{}^2) - L^2 \right]} \tag{5.104}$$

where

$$\bar{P}'_\infty(E_s, B_\nu{}^2) = \frac{d}{dB^2} \bar{P}_\infty(E_s, B^2) \Big|_{B^2 = B_\nu{}^2}$$

In general the roots B^2 of (5.103) will form a denumerable infinity. The

lowest root, B_o^2, is by definition, the material buckling of the system, and is real. Moreover, since the infinitely long parallelepiped is subcritical,

$$B_x^2 + B_y^2 > B_o^2;$$

i.e., $w_o = \left[B_o^2 - (B_x^2 + B_y^2)\right]^{1/2}$ is a pure imaginary, say $w_o = \frac{i}{\ell}$. The higher roots w_ν are in general complex, and their imaginary parts are all less than $\frac{1}{\ell}$ in absolute value. Far from the source, therefore, the thermal neutron distribution must be given by

$$\Phi_s(\underline{r}) = f(z) \cos B_x x \cos B_y y \sim \frac{Q_o}{2} \frac{\bar{P}_\infty(E_s, B_o^2)\, e^{-\frac{|z|}{\ell}}}{B_\nu N\sigma_a \left[\frac{k}{p} \bar{P}'_\infty(E_s, B_o^2) - L^2\right]} \cos B_x x \cos B_y y \qquad (5.105)$$

where

$$B_x^2 + B_y^2 - \frac{1}{\ell^2} = B_o^2 \qquad (5.106)$$

The asymptotic neutron distribution (5.105) is exponential with attenuation length ℓ. This gives the "exponential" experiment its name. The asymptotic attenuation length is related to the material buckling by the simple relation (5.106). Thus by measuring the attenuation length of the neutron distribution in an exponential experiment, and by measuring the geometric quantities B_x^2 and B_y^2, the <u>material</u> buckling of the multiplying system can be obtained. It is to be noted that the exponential experiment affords a measurement of the material buckling directly; an evaluation of k from the experiment is possible only if (5.102) can be solved, and this in general requires detailed knowledge of L^2, p, and $\bar{P}_\infty$. However if the multiplication constant is close to unity (5.103) becomes

$$k = 1 + M^2 B_o^2$$

and k can therefore be estimated from a measurement of B_o^2 and an estimate of M^2. This is essentially the procedure which was used to estimate k in the wartime Chicago experiments.

In carrying out the exponential experiment, care must be taken that it is indeed the asymptotic attenuation length which is measured. Close to the source, the distribution (5.104) consists of a superposition of many exponentials, some with complex attenuation lengths. These non-asymptotic terms can be shown to arise from the spatial variation of the neutron energy spectrum: close to the source the spectrum is much influenced by the energy of the source neutrons, while farther away the spectrum takes on its asymptotic form determined mainly by the slowing down properties of the medium. The slowing down density at energy E is

$$q(\underline{r},u) = \frac{kN\sigma_a}{p} \int_{\text{Pile}} \Phi_s(\underline{r}')\, P(E,\underline{r},\underline{r}')d\underline{r}' + \int Q(\underline{r}')P^*(E,\underline{r},\underline{r}')d\underline{r}' \tag{5.105}$$

which, on substitution of (5.97), (5.99), (5.100), and (5.104), becomes

$$q(\underline{r},u) = \left[\frac{k}{p}\frac{Q_o}{2}\sum_\nu \frac{\bar{P}_\infty^*(E_s,B_\nu^2)\bar{P}(E,B_\nu^2)e^{iw_\nu|z|}}{\frac{k}{p}\bar{P}'_\infty(E_s,B_\nu^2) - L^2} + \int_{-\infty}^{\infty} e^{iwz}\,\bar{P}^*(E,B_x^2,B_y^2,w^2)dw\right]\begin{matrix}\cos B_x x\\ \cos B_y y\end{matrix} \tag{5.106}$$

The second term of course represents neutrons which reach $(\underline{r},E)$ without having undergone multiplication; the first term represents those which have gone through multiplication cycles. At great distances from the source only the $w_o = i/\ell$ term persists; thus

$$q(\underline{r},u) \sim \left[\frac{k}{p} \frac{Q_o}{2} \frac{\bar{P}_\infty^*(E_s,B_o^2)\bar{P}(E,B_o^2)}{\frac{k}{p}\bar{P}'_\infty(E_s,B_o^2)-L^2} e^{-|z|/\ell} + \int_{-\infty}^{\infty} e^{iwz} \bar{P}^*(E,B_x^2,B_y^2,w^2)dw \right] \begin{matrix} \cos B_x x \\ \cos B_y y \end{matrix} \qquad (5.107)$$

In addition to these "spectral" corrections the neutron flux will deviate from a pure exponential close to the source if the source distribution contains more than a single harmonic - e.g., if the source is concentrated in a point. Such "harmonic" corrections are in practice usually more important than the "spectral" corrections, and during the original exponential experiments much attention was paid to evaluating these harmonic corrections. To avoid both the spectral and the harmonic corrections it is necessary in determining $1/\ell$ from the observed neutron distribution to start measurements two or three attenuation lengths from the source.

General Reactor with Resonance Fission

It is possible to write down the characteristic equation and the asymptotic solution for a reactor system in which the slowing down model is continuous and fissions are assumed to occur at arbitrary energies. Such systems are often called "resonance reactors": an example is the SIR.

If the validity of age theory is assumed even with absorption, then the equation for the slowing down, absorption, and multiplication of neutrons in a uniform reactor can be written (cf. Eq. 3.59)

$$\frac{D_o}{N\sigma_{s_o}} \Delta N\sigma_{s_o}F_o(\underline{r},u) - N\sigma_a F_o(\underline{r},u) + f(u)\left[\int_o^{u_o} k(u')N\sigma_a F_o(\underline{r},u')du' + k(u_o)N\sigma_a(u_o)\,\Phi_s\right] =$$

$$\frac{\partial}{\partial u}\,\xi\, N\sigma_{s_o}F_o(\underline{r},u) \tag{6.43}$$

or, in terms of the slowing down density $q(\underline{r},u) = \xi\, N_{\sigma s_o}(F_o(r,u)$,

$$\frac{D_o}{N\sigma_{s_o}\,\xi} \Delta\, q(\underline{r},u) - \frac{N\sigma_a}{\xi\, N\sigma s_o}\, q(\underline{r},u) + f(u)\left[\int_o^{u_o} k(u')\,\frac{N\sigma_a}{N\sigma_{s_o}\,\xi}\, q(\underline{r},u')du' = \frac{\partial q(\underline{r},u)}{\partial u} + k(u_o)N\sigma_a(u_o)\Phi_s\right] \tag{6.44}$$

In these equations, $f(u)$ is the fission neutron spectrum in lethargy, $u = \ln E_h/E$, E_h being the high energy cut-off of the fission spectrum, $k(u) = \frac{\nu\, N\sigma_f}{N\sigma_a}$ is the number of fast neutrons produced per neutron absorbed at lethargy u, u_o is the logarithmic energy loss corresponding to thermal energy, $k(u_o)$ is the number of fast neutrons produced per thermal neutron absorbed, and Φ_s is the thermal neutron flux. It is assumed that u_1, the lower limit of the fission spectrum, is above thermal lethargy.

For the thermal neutrons,

$$D_o(u_o)\,\Delta\,\Phi_s - N\sigma_a(u_o)\Phi_s + q(u_o) = 0 \tag{6.45}$$

It will now be shown that asymptotic solutions of (6.44) and (6.45) in which space and lethargy are separable can be constructed provided the neutron fluxes at all energies are distributed in space as solutions of

$$\Delta\,\psi\,(\underline{r}) + B_g^{\,2}\,\psi\,(\underline{r}) = 0, \tag{6.46}$$

and $B_g^{\,2}$ satisfies a characteristic equation which will be described below.

In other words, the most general reactor equation (6.43), in which resonance fission occurs, has an asymptotic solution, and a characteristic equation, just as in the case of thermal fission reactors. Thus even when resonance fissions occur, the power distribution in a bare, uniform reactor with energy independent extrapolation length, is given by a solution of the wave equation (6.46).

To prove this "generalized" fundamental theorem of pile theory, set

$$\Delta\,\Phi_s + B_g^2\,\Phi_s = 0,\quad q(\underline{r},u) = X(u)\psi(\underline{r}).$$

Then from (6.45),

$$\Phi_s = \frac{q(u_o)}{N\sigma_a(u_o) + B_g^2 D_o(u_o)}\,. \tag{6.47}$$

If this expression for Φ_s is substituted into (6.44), and if q is replaced by $X(u)\psi(\underline{r})$, $\Delta\,q$ by $-\,B_g^{\,2}\,X(u)\psi(\underline{r})$, there results

$$-\left(\frac{D_o B_g^{\,2}}{N\sigma_{s_o}\,\xi} + \frac{N\sigma_a}{\xi\,N\sigma_{s_o}}\right)X(u) + f(u)\left[\int_0^{u_o} k(u')\,\frac{N\sigma_a}{N\sigma_{s_o}\,\xi}\,X(u')du' + \frac{k(u_o)\,X(u_o)}{1 + L^2 B_g^{\,2}}\right] = \frac{\partial X(u)}{\partial u} \tag{6.48}$$

Equation (6.48) may be thrown into integral form by solving it as a first order differential equation in X(u). The result is

$$X(u) = e^{-\int_0^u \frac{B_g^2 D_o(y) + N\sigma_a(y)}{\xi\, N\sigma_{s_o}(y)} dy} \left[\int_0^u f(z)\, e^{\int_0^z \frac{B_g^2 D_o(x) + N\sigma_a(x)}{\xi\, N\sigma_{s_o}(x)} dx} \left(\int_0^{u_o} \frac{N\sigma_a(w)}{\xi\, N\sigma_{s_o}(w)} k(\underline{w})\, X(w) dw + \frac{k(u_o)\, X(u_o)}{1 + L^2 B_g^2} \right) dz \right] \tag{6.49}$$

Now the assumption

$$X(u) = e^{-\int_0^u \frac{N\sigma_a(y) + B_g^2 D_o(y)}{\xi\, N\sigma_{s_o}(y)} dy} \left[\int_0^u f(z)\, e^{+\int_0^z \frac{B_g^2 D_o(x) + N\sigma_a}{\xi\, N\sigma_{s_o}} dx} dz \right] \tag{6.50}$$

vanishes, as it should at u = o, and solves (6.48) provided B_g^2 satisfies the following equation

$$\int_0^{u_o} \int_0^{w} k(w) \frac{N\sigma_a}{N\sigma_{s_o}\xi} p(z,w)\, e^{-B_g^2 \tau(z,w)} f(z) dz dw + \frac{k(u_o) \int_0^{u_o} p(z,u_o)\, e^{-B_g^2 \tau(z,u_o)} f(z) dz}{1 + L^2 B_g^2} = 1 \tag{6.51}$$

In this equation

$$p(z,u) = e^{-\int_z^u \frac{N\sigma_a(u)}{\xi\, N\sigma_{s_o}(u)} du} \tag{6.52}$$

is the resonance escape probability from lethargy z to u as computed from simple age theory, while

$$\tau(z,u) = \int_z^u \frac{D_o(u)}{\xi N\sigma_{s_o}} du \tag{6.53}$$

is the neutron age from lethargy z to u, again according to simple age theory.

The sense of this generalized resonance pile equation is the following: the second term, which is practically the same as (6.14) is the product of (1)$k(u_o)$, the number of neutrons produced per thermal absorbed; (2), the probability of escaping leakage while slow, $\frac{1}{1 + L^2B_g^2}$; and (3), the probability of escaping resonance capture or leakage while slowing down, viz.,

$$\int_o^{u_o} p(z,u_o)\, e^{-B_g^2\left[\tau(z,u_o)\right]} f(z)dz.$$

The product of these three factors may be called the slow effective multiplication constant, $k_{t_{eff}}$. Since ordinarily resonance capture does not set in until well below the lower limit of the fission spectrum, $p(z,u_o)$ is replaceable by p, the total resonance escape probability. Also u_o may be replaced by u_1 in the limit of the integral since f(z) is zero below lethargy u_1. Then, since $k(u_o)$ was defined as $\frac{\nu N\sigma_f}{N\sigma_a}$, $k(u_o)p$ is the same as the usual multiplication constant provided ν includes ε. The thermal effective multiplication constant, with these simplications, is therefore

$$k_{t_{eff}}(B_g^2) = \frac{k \int_{E_1}^{E_o} f(E')\, e^{-B_g^2 \tau(E_s,E')} dE'}{1 + L^2B_g^2} \tag{6.54}$$

which is essentially (6.13) since f(E) is assumed to be zero outside the energy range E_1 to E_o.

The first term may be called the effective resonance multiplication constant, $k_{f_{eff}}(B_g^2) = \int_0^{u_0}\int_0^{w} k(w)\,\frac{N\sigma_a}{N\sigma_{s_0}\,\xi}\,p(z,w)\,e^{-B_g^2\,\tau(z,w)}\,f(z)\,dz\,dw$. The integrand is the product of: (1) $k(w)$, the number of neutrons produced per neutron absorbed at lethargy w; (2) $\frac{N\sigma_a}{\xi\,N\sigma_{s_0}}$, the probability that a neutron will be absorbed rather than scattered in unit lethargy interval; and (3), $\int_0^{w} f(z)\,p(z,w)\,e^{-B_g^2\,\tau(z,w)}\,dz$, the probability that a fission neutron will escape resonance capture or leakage while slowing down to lethargy, w, where further fission may take place.

If the fission spectrum were a δ-function, $f(z) = \delta(u_1 - z)$, the generalized characteristic equation (6.51) becomes

$$\int_0^{u_0} k(w)\left(\frac{-dp}{dw}\right) e^{-B_g^2\,\tau(u_1,w)}\,dw + \frac{k_t\,e^{-B_g^2\,\tau(u_1,u_0)}}{1 + L^2B^2} = 1, \qquad (6.55)$$

where $\frac{-dp}{dw} = \frac{N\sigma_a}{\xi\,N\sigma_{s_0}}\,e^{-\int_{u_1}^{w}\frac{N\sigma_a}{\xi\,N\sigma_{s_0}}\,dw}$, being the derivative of the resonance escape probability, is the probability of resonance <u>absorption</u> per unit lethargy interval.

If the reactor system were infinitely large, $B_g^2 = 0$, and the effective multiplication constants become the infinite multiplication constants. This suggests that in any chain reactor in which non-thermal as well as thermal fission takes place, two multiplication constants, k_f for non-thermal and k_t for thermal fissions, ought to be defined. Putting $B_g^2 = 0$ in the formulas for the effective multiplication constants and equating the results to k_f and k_t,

$$k_f \equiv \int_0^{u_0}\int_0^{w} \frac{\nu\,N\sigma_f(w)}{N\sigma_{s_0}\,\xi\,(w)}\,p\,(z,w)\,f\,(z)\,dz\,dw \qquad (6.56)$$

$$k_t \equiv \frac{\nu N\sigma_f(u_o)}{N\sigma_a(u_o)} \int_o^{u_1} p\,(z,u_o)\, f\,(z)\, dz \tag{6.57}$$

or, if an average resonance escape probability is defined by

$$\bar{p}(w) = \int_o^{w} p\,(z,w)\, f\,(z)\, dz, \tag{6.57a}$$

this being the same as the usual resonance escape probability, then

$$k_f \equiv \int_o^{u_o} \frac{\nu N\sigma_f(w)}{N\sigma_{s_o}\,\xi\,(w)}\, \bar{p}\,(w)\, dw \tag{6.58}$$

$$k_t \equiv \frac{\nu N\sigma_f(u_o)}{N\sigma_a(u_o)}\, \bar{p}\,(u_o) \equiv \eta\, f\, \bar{p} \tag{6.59}$$

since $\eta\, f = \dfrac{\nu N\sigma_f}{N\sigma_a}$.

The quantity k_f is simply the number of neutrons produced by non-thermal fission per neutron produced by all fissions. The total number of neutrons produced by thermal fissions that become thermal per thermal neutron absorbed is k_t. The fast effect, $\mathcal{E}$, is by definition the number of neutrons produced by <u>all</u> fissions per neutron produced by thermal fissions. Hence

$$\mathcal{E} = \frac{1}{1 - k_f} = \frac{1}{1 - \int_o^{u_o} \frac{\nu N\sigma_f(w)}{N\sigma_{s_o}\,\xi\,(w)}\, \bar{p}\,(w)\, dw} \tag{6.60}$$

and the multiplication constant

$$k = \frac{\text{number produced by all fissions}}{\text{number absorbed}} = k_t + k_f \mathcal{E} k_t = k_t \mathcal{E} = \eta \bar{p} f \mathcal{E}$$

It is in this way that the continuous slowing down theory for a resonance reactor can be related to the usual four-factor equation for k which has validity in a predominantly thermal reactor.

Historically, formulas such as (6.57) for $\mathcal{E}$ were not introduced into the literature mainly because (6.57) assumes that the slowing down is given by elastic scattering age theory whereas in the Hanford reactors, for which the original theory was formulated, the slowing down below the fast fission threshold of U^{238} is by inelastic scattering with U nuclei. This degradation of the energy below the fast fission threshold is most conveniently handled by tracing the first few collision histories of a neutron as it is born inside a uranium lump until the neutron energy is degraded by inelastic scattering below the U^{238} fission threshold or the neutron escapes from the lump. On the other hand, in a homogeneous reactor system where the energy loss of the neutron is by elastic scattering the theory of the fast effect described here is appropriate.

A further simplification in the characteristic equation can be effected by defining average ages for neutrons inducing resonance fissions and for neutrons inducing thermal fission. Thus if $\bar{\tau}_f$ and $\bar{\tau}_t$ are defined by the equations

$$e^{-B_g^2 \bar{\tau}_f} = \frac{\int_0^{u_0} \int_0^{w} k(w) \frac{N\sigma_a}{N\sigma_{s_0} \xi} p(z,w) e^{-B_g^2 \tau(z,w)} f(z)dzdw}{k_f} \tag{6.61}$$

$$e^{-B_g^2 \bar{\tau}_t} = \frac{k(u_0) \int_0^{u_1} p(z,u_0) e^{-B_g^2 \tau(z,u_0)} f(z)dz}{k_t} \tag{6.62}$$

then the most general reactor equation based on simple age theory is

$$k_f\, e^{-B_g^2 \overline{\tau}_f} + \frac{k_t\, e^{-B_g^2 \overline{\tau}_t}}{1 + L^2 B_g^2} = 1 \tag{6.63}$$

which is a very reasonable generalization of the usual Fermi equation. Since $\overline{\tau}_f$ is always smaller than $\overline{\tau}_t$, the non-leakage probability for the resonance fission cycle is greater than for the thermal neutron cycle. Hence resonance fissions are more effective in maintaining the chain reaction than are thermal fissions.

The essence of the thermal reactor model is that all non-resonance fissions are assumed to be induced by neutrons which do not diffuse; i.e., $\overline{\tau}_f = 0$. In other words, ν is increased by the factor ε. Now if $\overline{\tau}_f = 0$, and if the multiplication constant is put equal to $k = k_t(1 + k_f\varepsilon) = k_t\varepsilon$, then (6.63) becomes

$$\frac{k\, e^{-B_g^2 \overline{\tau}_t}}{1 + L^2 B^2} = 1 \tag{6.64}$$

which is the usual characteristic equation for the continuous slowing down model.

The foregoing is the most general asymptotic theory for reactors in which non-thermal fission occurs and in which the age theory slowing down model is applicable. The major weakness of the theory is of course the assumption that age theory is valid even though absorption is taking place. This weakness manifests itself in that the correct formula for the resonance escape probability involves $\sigma_{a_{eff}}$ while (6.52) involves σ_a. If the absorption is sufficiently weak - as it often is in heavily moderated systems - $\sigma_{a_{eff}}$ and σ_a are nearly the same and the errors are small. On the other hand, if the resonance absorption is heavy - so heavy that q attenuates drastically in one slowing down interval - more nearly accurate results will be obtained if σ_a is everywhere replaced by $\sigma_{a_{eff}}$.

An attempt to derive the first order corrections to the age theory when absorption is present has been made by Goertzel and Greuling and by Edlund[1]. The result is that the usual equation relating q and F, viz.,

$$q(u) = N\sigma_{s_0} \xi F(u)$$

should be replaced by the equation

$$q(u) = \left[\xi N\sigma_{s_0} + \gamma \Sigma_s \right] F(u)$$

where

$$\gamma = \frac{1 - \alpha - \alpha \mathcal{E} - \alpha \mathcal{E}\, 2/2}{1 - \alpha - \alpha}$$

$$\gamma = \frac{M - 1}{M + 1}^{2}, \quad \mathcal{E} = - \ln \alpha.$$

For further details the reader is referred to the original paper, CF No. 51-5-98, Volume II, Chapter VI.

(1) ORNL-1154.

VOLUME III, PART I

MICROSCOPIC PILE THEORY

The previous part of this book has been concerned primarily with macroscopic pile theory, i.e., the calculation of the extensive properties of a chain reactor. These properties include the over-all neutron distribution, the time behavior of the neutron flux when the reactor is non-critical, and the critical size of the system. In all these considerations, the intensive properties of the chain reactor, namely, the multiplication constant, the infinite slowing down kernel, the fine structure of the neutron distribution in a heterogeneous system, or the material buckling were given as preassigned quantities. The calculation of the intensive properties of a chain reactor comprises microscopic pile theory, and the remainder of the book is devoted to an exposition of microscopic pile theory.

Actually the division of chain reactor theory into microscopic pile theory and macroscopic pile theory depends on the possibility of defining a clear cut "generation" or neutron cycle. In a chain reactor in which fissions occur only at thermal energy, a generation is well defined: one starts with, say, an average fast neutron, which has a certain probability, p, of escaping resonance capture, has a certain probability, f, of being absorbed by uranium, and, once absorbed, yields η fast neutrons on the average. This complete cycle constitutes a generation, and the multiplication factor is

$$k = \eta pf .$$

In a natural uranium chain reactor, some non-thermal fission occurs. Since the number of fast fission-produced neutrons is small compared to the number of thermal fission-produced neutrons, the few fast fissions hardly alter the generation. Thus the generation is a relatively valid notion in a predominantly thermal neutron chain reactor such as a large natural uranium pile.

The procedure for determining extensive properties of such a thermal reactor consists of computing k, M^2, and B_g^2, and then applying the ideas of macroscopic pile theory to an equivalent uniform system having the same k, M^2, and B_g^2 as the actual system.

If appreciable fissions occur at non-thermal energies, the definition of a generation, and therefore the division into macroscopic and microscopic theory loses much of its validity. Since neutrons go through complete cycles of absorption and reproduction at several energies, there is no unique way to specify the beginning and the end of the neutron cycle. There are, so to speak, "multiplication constants" for each energy at which fission occurs, and the true multiplication constant is the sum of the separate multiplication constants in each energy range cycle. It is customary in dealing with such non-thermal reactors to avoid reference to a neutron cycle or even to a multiplication constant; instead multi-group equations of the reactor are written down which involve only the absorption, scattering, and fission cross sections and number of neutrons per fission, rather than such quantities as multiplication constant, resonance escape probability, or migration length. The solution of these multi-group equations gives the relevant macroscopic information - critical size or neutron distribution - without recourse to any of the notions of microscopic theory.

So long as a chain reactor is homogeneous, the derivation of the appropriate multi-group equations is straightforward, and there is little incentive to separate microscopic reactor theory from macroscopic theory. However, when the reactor configuration is heterogeneous, as in a natural uranium lattice, the fine structure fluctuations of the neutron density in effect modify the cross sections which are to enter the macroscopic equations. It is the purpose of microscopic, or lattice, theory to compute the details of the neutron fine structure and to thereby obtain the modified values of the neutron cross sections. The calculations of this fine structure and the multiplication constant, migration area, and material buckling in such a system is a complicated matter which will be dealt with at length in Part III of this book.

CHAPTER XI

THEORY OF HOMOGENEOUS SLOW NEUTRON REACTORS

As an introduction to Chapter XII on the microscopic theory of heterogeneous reactors the microscopic theory of a homogeneous thermal neutron reactor will be developed in this chapter. Many of the results have already been given in Chapter V, and so much of the discussion will be a review.

While the microscopic theory of homogeneous thermal reactors is presented mainly because it points up the modifications required for heterogeneous theory, the homogeneous slow neutron theory is nevertheless of considerable practical importance. An example of a homogeneous slow neutron reactor is the Los Alamos water boiler which consists of a solution of enriched uranyl nitrate and H_2O. Chain reacting solutions or dispersions of natural or very slightly enriched uranium in heavy water have been suggested since 1943 as a very attractive means for producing plutonium and power on a large scale. The inherent simplicity of such a homogeneous reactor makes it likely that such systems may eventually become very important in the large scale release of nuclear energy. The homogeneous reactor theory discussed here is particularly relevant to such slightly enriched, heavy water moderated chain reactors.

The critical equation

$$\frac{k}{p} \quad \frac{\bar{P}_{\infty}(E_s, B_g^2)}{1 + L^2 B_g^2} = 1 \tag{11.1}$$

for a thermal chain reactor involves the quantities k, p, L^2, and $\bar{P}_{\infty}(E_s, B_g^2)$.

As was pointed out in Chapter V, the multiplication constant k is the product

$$k = \eta \varepsilon p f \tag{11.2}$$

where

η = number of neutrons produced per neutron absorbed in uranium.

ε = total number of neutrons produced by all fissions/total number of neutrons produced by thermal fission.

p = resonance escape probability.

f = thermal utilization.

Each quantity in this "four factor" formula will be computed separately below(1).

In addition, it will be necessary to know the diffusion area, L^2, and the slowing down kernel, $P_{\infty}(E_s,r)$, in order to determine the critical size or the material buckling from (11.1).

Determination of η

The number of neutrons produced per neutron absorbed in fissionable material was directly given, in terms of cross sections, in Chapter V. (Eq. 5.2). Thus, if the mole ratio of U^{235} to U^{238} is denoted by R, then

$$\eta(R) = \frac{R\nu\,\sigma_f(U^{235})}{R\,\sigma_f(U^{235}) + R\,\sigma_r(U^{235}) + \sigma_r(U^{238})} \tag{11.3}$$

where ν is the number of neutrons produced per fission, $\sigma_f(U^{235})$ is the fission cross section of U^{235}, σ_r is the radiative capture cross section

(1) The "fissionable material" in the system will be considered to include all isotopes of uranium or plutonium. Thus natural uranium as well as U^{235} will be called fissionable material.

in each of the uranium isotopes.

As was mentioned earlier, η in general is a function of energy. This is so mainly because the cross sections, rather than ν, are energy sensitive, and further, the fission cross sections depend on energy in a manner different from the way the capture cross sections depend on energy. The details of the energy dependence of σ_r and σ_f in the fissionable isotopes are given in Chapter IV.

For natural uranium, R = .0071. If the following thermal neutron cross sections and ν, given by the A.E.C. Neutron Cross-Sections Advisory Group (BNL-136), are assumed:

$\nu(U^{235})$	2.51
$\sigma_f(U^{235})$	549 b
$\sigma_r(U^{235})$	101 b
$\sigma_r(U^{238})$	2.8 b

then η for natural uranium is 1.32. There is considerable uncertainty even now as to the thermal capture cross section of U^{238}, and on this account the above quoted value of η cannot be considered reliable to better than perhaps three percent. The values of η for thermal neutrons in pure fissionable isotopes, as given by BNL-136, are as follows:

Thermal Neutron η for U^{233}, U^{235}, Pu^{239}

Isotope	η
U^{233}	2.37
U^{235}	2.12
Pu^{239}	2.00

The thermal η for natural uranium was of course a number of extreme importance in the early days of the Manhattan Project since, unless it was appreciably larger than unity, it would not have been possible to establish a chain reaction in normal uranium. Several experiments were therefore performed to try to measure η directly. One such experiment by Fermi and Anderson consisted of a long block of graphite in which was placed a Ra-Be neutron source. The total number of neutrons appearing as thermals throughout this block was measured both before and after a layer of U_3O_8 had been placed in front of the Ra-Be source. With the U_3O_8 in place the total number of fast neutrons should be increased because of fissions. The experiment was never entirely clear-cut, however, because many of the neutrons leak out of the sides of the block, and in addition, some of the neutrons produced by the U fission are captured by the resonances in U^{238}. Experiments of this type were performed on two different occasions by Fermi and Anderson and the results were the following.

Thermal η for Natural Uranium as Measured by Fermi and Anderson

η	Date Measured
1.73	January 1941
1.29	July 1942

Historically the very high value $\eta = 1.73$ was significant since it made the outlook for a chain reaction with unenriched uranium very optimistic at a time when no chain reaction had even been established.

The directly determined η disagrees with the value obtained from ν and the cross sections. It also disagrees with the value of η deduced from

measurements of the multiplication in chain reacting systems containing natural uranium. The most directly interpretable experiments of this character were done by Wattenberg, et al* on solutions of natural UO_2F_2 in heavy water. The experimentally determined buckling as a function of UO_2F_2 concentration seems to be best fitted if η for natural U is assumed to be 1.35. This is in rather good agreement with the value found directly from the cross sections but is in serious disagreement with the value found by direct measurements. It is generally believed that the directly measured η is the least reliable value.

Recently it has been possible to measure η for U^{235} and other fissionable materials very directly by observing the effect that a small amount of U^{235} has on the reactivity of a chain reactor. To increase accuracy, the sample is oscillated in and out of the reactor, and the resulting modulation of the reactor power is recorded. Such "pile oscillator" measurements have been performed by Cruikshank, Littler and Ward at Chalk River and by Muelhause at Argonne. These measurements are now believed to give the most reliable values for the thermal η of the fissionable materials, and they are tabulated below.

Pile Oscillator Values of η

	Argonne	Chalk River
$\eta(U^{233})$ =	2.31	2.38
$\eta(U^{235})$ =	2.10	---

* CP-3364

The Thermal Utilization, f

The thermal utilization, f, is the number of thermal neutrons absorbed in the fissionable material divided by the total number of thermal neutrons absorbed in the whole system. In a homogeneous system this must simply be the ratio of the absorption cross sections of uranium to the absorption cross section of the whole system:

$$f = \frac{N_o\sigma_{a_o}}{N_o\sigma_{a_o} + N_1\sigma_{a_1}} \tag{11.4}$$

where N_o, N_1 are the number of atoms of fissionable and non-fissionable species present per cubic centimeter, and σ_{a_o}, σ_{a_1} are their respective thermal absorption cross sections. In a natural uranium-graphite chain reactor, f is about 0.9. For natural U, σ_{a_o} is of course the sum of fission, 235 capture, and 238 capture cross sections. It is often more convenient to consider the reciprocal of f:

$$\frac{1}{f} = 1 + \frac{N_1\sigma_{a_1}}{N_o\sigma_{a_o}} \; ; \tag{11.5}$$

the ratio $N_1\sigma_{a_1}/N_o\sigma_{a_o}$ is called the relative absorption since it is the ratio of absorption in non-fissionable to absorption in fissionable species.

The thermal utilization, f, is the number of neutrons absorbed in fissionable material divided by the number absorbed in the whole system; this is the same as the number absorbed divided by the number of thermal neutrons produced only if the system is infinitely large, since otherwise some of the thermal neutrons will leak out rather than be absorbed. This latter ratio,

(number absorbed/number produced), is called the effective thermal utilization and depends on the size of the reactor. It is therefore not an intensive property of the reactor. Since the ratio

$$\frac{\text{thermal neutrons produced}}{\text{thermal neutrons absorbed}} = 1 + \frac{\text{thermal neutrons leaking}}{\text{thermal neutrons absorbed}}$$

is, according to (5.24), just $(1 + L^2B^2)$ so long as the reactor is bare, the effective thermal utilization, f_{eff} in a bare reactor is

$$f_{eff} = \frac{f}{1 + L^2B^2} \quad . \tag{11.6}$$

The name, effective thermal utilization, of course derives from the fact that the effective multiplication constant is

$$k_{eff} = \frac{k}{p} \frac{\bar{P}_{\infty}(E,sB^2)}{1 + L^2B^2} = \frac{\eta \varepsilon p f}{p} \frac{\bar{P}_{\infty}(Es,B^2)}{1 + L^2B^2}$$

or

$$k_{eff} = \eta \varepsilon p_{eff} f_{eff} \tag{11.7}$$

if f_{eff} is defined as $\frac{f}{1 + L^2B^2}$, and p_{eff} is defined as

$$p_{eff} = \bar{P}_{\infty}(E_s,B^2) \quad . \tag{11.8}$$

The effective resonance escape probability is the ratio of the number of neutrons escaping resonance capture to the number of resonance neutrons produced; it, like f_{eff}, depends on the pile size and is smaller than p by

the fraction of neutrons which escape while slowing down - i.e., by $\frac{1}{p}\ \bar{P}_{\infty}(E_s,B^2)$, since by convention $P_{\infty}(E_s,\underline{r})$ has been already normalized to p rather than to 1 (cf. eq. 5.14).

The Resonance Escape Probability and the Fast Effect

Since in a homogeneous slow neutron reactor the resonance escape, the resonance fission, and the fast fission all involve the probability that a neutron will lose energy by elastic collision with the moderator, it will be convenient to treat them all together. In a heterogeneous, natural U arrangement, on the contrary, it is inelastic collision with U^{238} atoms or escape from the uranium lump and subsequent elastic slowing down which are the major mechanisms for making a neutron unavailable for inducing fission in U^{238}. Thus the fast effect in a heterogeneous arrangement does not have the same direct relation to the resonance fission that the phenomenon has in a homogeneous, moderated reactor.

The relevant theory for the resonance escape and the fast effect in a homogeneous system was already worked out in Chapter IV and Chapter VI. In Chapter VI it was shown that the critical equation for a homogeneous system in which resonance capture and non-thermal fission could occur, and in which elementary age theory is applicable, was

$$k_f e^{-B_g^2 \bar{\tau}_f} + \frac{\eta f \bar{p} e^{-B_g^2 \bar{\tau}_t}}{1 + L^2 B_g^2} = 1 \qquad (11.9)$$

where $\bar{\tau}_f$ and $\bar{\tau}_t$ are average ages for fast and slow fission neutrons, k_f is the non-thermal multiplication constant, and $\bar{p}$ was the average resonance escape probability. For $\bar{p}$ there is the formula (6.57a).

$$\bar{p} = \int_0^{u_1} p(z,u_o)\, f(z)\, dz \qquad (11.10)$$

$f(z)$ being the fission spectrum, u_o the thermal lethargy, u_1 the low energy lethargy cutoff of the fission spectrum. Since there is ordinarily practically no resonance capture at fission spectrum lethargies, (11.10) is just the usual

$$\bar{p} = p = \exp. - \frac{N_o}{\xi(N_1\sigma_{s_1} + N_o\sigma_{s_o})} \int_0^{u_o} \sigma_{a_o\ eff}\, du \qquad (11.11)$$

where σ_{a_o} has been replaced by $\sigma_{a_o\ eff}$, as in Chapter IV. In (11.11) N_1 is the number of moderator atoms per cc, ξ is the average slowing down parameter

$$\xi = \frac{N_1\sigma_{s_1}\xi_1 + N_o\sigma_{s_o}\xi_o}{N_1\sigma_{s_1} + N_o\sigma_{s_o}} \qquad (11.12)$$

and σ_{s_1} and σ_{s_o} are the scattering cross sections of moderator and absorber respectively. The resonance integral $\int\sigma_{a_o\ eff}\, du$ depends on the scattering cross section per cc and is in principle calculable according to (4.30) if all cross sections are known. However, it is much more practical to use the measured variation of the integral with cross section. This dependence is given in Fig. 4-VI.

In an actual homogeneous slow neutron chain reactor using normal uranium, there is always some - though usually very little - non-thermal neutron fission. Below the U^{238} threshold this comes from U^{235} fissions; above the U^{238} threshold (0.9 Mev) it comes predominantly from U^{238}. However, the scheme for computing these effects based on (11.9) is entirely

adequate in a homogeneous, moderated reactor. According to (6.53) k_f is to be computed from

$$k_f = \int_0^{u_o} \int_0^{w} \frac{\nu N \sigma_f (w)}{\xi(N_o\sigma_{s_o} + N_1\sigma_{s_1})} p(z,w) f(z) dz dw \tag{11.13}$$

where $N\sigma_f(w)$ is the fission cross section per cc.

At energies above the U^{238} fission threshold, contribution to (11.13) comes practically entirely from U^{238} fission. Below this threshold, which is shown in the accompanying graph of $\sigma_f(U^{238})$ to be about 0.9 Mev, the resonance fission comes only from U^{235}. The expression for k_f can therefore be broken into two parts

$$k_f = k_{f28} + k_{f25} \tag{11.14}$$

the first term coming from U^{238} fissions occurring above the U^{238} threshold, u_{28}, the second from U^{235} fissions. Thus

$$k_{f28} = \int_0^{u_{28}} \int_0^{w} \frac{\nu_{28} N \sigma_{f28}}{\xi(N_o\sigma_{s_o} + N_1\sigma_{s_1})} p(z,w) f(z) dz dw \tag{11.15}$$

and

$$k_{f25} = \int_{u_{28}}^{u_o} \int_0^{w} \frac{\nu_{25} N \sigma_{f25}}{\xi(N_o\sigma_{s_o} + N_1\sigma_{s_1})} p(z,w) f(z) dz dw \tag{11.16}$$

Now in the lethargy interval $0 - u_{28}$, there is practically no resonance capture, $p(z,w) \approx 1$. If $\int_0^w f(z)dz = F(w)$, then

$$k_{f28} = \int_{o}^{u_{28}} \frac{\nu_{28} N \sigma_{f28}(w)}{\xi(N_o\sigma_{s_o} + N_1\sigma_{s1})} F(w)dw \tag{11.17}$$

i.e., the fast fission probability per lethargy interval, $\frac{N \sigma_{f28} dw}{\xi(N_o\sigma_{s_o} + N_1\sigma_{s1})}$, must be weighted with that part of the fission spectrum which, in the course of slowing, will enter dw, and this is F(w).

For the true resonance fission contribution, k_{f25}, the fission spectrum can be replaced by a δ function, $\delta(\mathfrak{z} - u_{28})$, since the actual spread of $f(\mathfrak{z})$ in $\mathfrak{z}$ is relatively very small. Hence

$$k_{f25} = \int_{u_{28}}^{u_o} \frac{\nu_{25} N \sigma_{f25}}{\xi(N_o\sigma_{s_o} + N_1\sigma_{s1})} p(o,w)dw \tag{11.18}$$

and

$$p(o,w) = \exp. - \frac{N_o}{\xi(N_1\sigma_{s1} + N_o\sigma_{s_o})} \int_{o}^{w} \sigma_{a_{oeff}} du \quad ; \tag{11.19}$$

p(o,w) really should be written $p(u_{28},w)$, but the two are the same since $\sigma_{a_{oeff}} = 0$ above the 28 fission cutoff. Actually (11.17) and (11.18) have very much the same form: in the one case the number of neutrons crossing lethargy w is F(w), in the other case, p(o,w).

The fast effect, $\mathcal{E}$, is given, as in Chapter VI, by

$$\mathcal{E} = \frac{1}{1 - k_f} \approx 1 + k_f \tag{11.20}$$

since k_f is always much less than unity in a thermal reactor.

The relative importance of the U^{238} fast fission and the U^{235} resonance fission - i.e., the relative importance of k_{f28} and k_{f25} - is strongly dependent on the enrichment and on the uranium to moderator ratio. In an unenriched heavy water moderated system, k_{f28} will run around 1/2% or less,

k_{f25} will be of the same order. As the enrichment increases, the relative importance of k_{f25} grows until in a completely enriched system $k_{f28} = 0$. Further increase in the U^{235} to moderator ratio will make k_{f25} grow at the expense of k_t until finally, when $k_t \ll k_{f25}$, most of the fissions are in the non-thermal region and the reactor becomes a resonance reactor.

The Migration Area

The critical condition (11.1) for a predominantly thermal reactor may now be written

$$\eta \, \varepsilon \, p \, f \, \frac{\bar{P}_{\infty}(E_s, B_g^2)}{p\left[1 + L^2 B_g^2\right]} = 1 \tag{11.21}$$

which reduces to

$$\frac{\eta \, \varepsilon \, p \, f - 1}{L^2 + \eta \, \varepsilon \, p \, f \, \tau} = B_g^2 \tag{11.22}$$

if the reactor is large. The slowing down kernel is primarily an experimentally measured function; for materials such as graphite, it is usual to consider it to be Gaussian. As usual, $P_{\infty}(E_s, r)$ is normalized to p.

The buckling given by (11.22) clearly goes through a maximum as the ratio of fissionable to moderator atoms is changed. At high N_0/N_1 (relatively large amounts of U), the thermal utilization f approaches unity, while the resonance escape probability, p, approaches zero. At small N_0/N_1 (relatively large amounts of moderator), the resonance escape approaches unity while the thermal utilization approaches zero. Since ε is practically constant, the maximum k, and therefore the maximum B^2, must be somewhere between these extremes.

The diffusion length, L, has already been computed in Chapter I. If the thermal transport cross sections for moderator (including impurities or poisons) and fissionable material are σ_{tr_1} and σ_{tr_o}, and the thermal absorption cross sections at the average Maxwellian velocity are σ_{a_1} and σ_{a_o}, then

$$L^2 = \frac{1}{3(N_o\ \sigma_{tr_o} + N_1\ \sigma_{tr_1})(N_o\ \sigma_{a_o} + N_1\ \sigma_{a_1})} \tag{11.23}$$

provided the absorption is weak compared to the total cross section. This must be the case in a slow neutron chain reaction for otherwise a large fraction of the neutrons would be caught before they become thermal.

The number of fissionable atoms is always small compared to the number of moderator atoms. Since the transport cross sections of practically all elements are comparable (except for H all are within a factor of 3 of each other) $N_1\ \sigma_{tr_1} \gg N_o\ \sigma_{tr_o}$. Furthermore, since

$$\frac{1}{f} = 1 + \frac{N_1\ \sigma_{a_1}}{N_o\ \sigma_{a_o}}\ , \tag{11.24}$$

$$\frac{1}{1 - f} = 1 + \frac{N_o\ \sigma_{a_o}}{N_1\ \sigma_{a_1}}\ , \tag{11.25}$$

the expression for L^2 is often written

$$L^2 = (1 - f)\ L_m^2 \tag{11.26}$$

where L_m is the diffusion length of the moderator alone.

Physically, the reduction in diffusion length implied by (11.26) which results when a heavy absorber like U is dissolved in the system is the result

of the increase in macroscopic absorption from $N_1 \sigma_{a_1}$ to $N_1 \sigma_{a_1} + N_o \sigma_{a_o}$; i.e., from

$$N_1 \sigma_{a_1} \qquad\qquad No\ U$$

to

$$\frac{N_1 \sigma_{a_1}}{1 - f} \qquad\qquad U$$

Now the thermal neutron lifetime in pure moderator is

$$\ell = \frac{1}{N_1 \sigma_{a_1} v} ,$$

v being the average neutron velocity. Hence with U dissolved in the moderator, the lifetime is reduced to

$$\frac{\ell}{1 - f} , \qquad (11.27)$$

provided the effect of the additional absorption on the neutron temperature is ignored.

Since the thermal utilization is usually about 0.9, L^2 in, say, a graphite chain reactor will be 1/10 the L^2 in pure C; similarly the neutron lifetime is shortened by a factor of ten. This is a fairly general rule - that in most slow neutron chain reactors using unenriched uranium the generation time and the L^2 are about ten times lower than their values in the moderator alone.

Temperature Coefficient of Reactivity

The temperature behavior of a chain reactor is of the utmost importance in devising an adequate control system, and for this reason the calculation

of temperature coefficients of reactivity is always a central part of any reactor design problem. Of the quantities η, ε, p, f, L^2, and $\bar{P}$ which enter into the expression for k_{eff}, only the k_{f28} part of ε (which involves very fast neutron cross sections) is independent of the temperature. All the others may vary with the temperature of either the moderator, the fissionable material, or the neutrons.

In general, three temperatures are relevant in a slow neutron chain reactor:

1) Moderator temperature: the moderator temperature determines the neutron temperature (though the two are usually not quite the same); in addition it determines the moderator density, and therefore the macroscopic cross sections of the moderator.

2) Neutron temperature: this is in simplest approximation the same as the moderator temperature. More exactly, because of the U absorption, the neutrons never quite come into thermal equilibrium with the moderator, and their temperature is somewhat higher than the moderator temperature.

Experiments by L. Woods (CP-475) have shown that the neutron temperature in a graphite natural U reactor is perhaps 60 - 85°C higher than the graphite temperature. It is the neutron temperature which enters in calculations of microscopic thermal neutron cross sections. In lieu of a better theory, it will be assumed that a change in moderator temperature causes the same change in neutron temperature.

3) Uranium temperature: if the U and the moderator are not in intimate contact, the uranium and the moderator temperatures can vary independently. This is particularly the case in a heterogeneous reactor where the heat transfer from U to moderator may be very poor. In such a system, a sudden rise in neutron level will cause the fission rate, and therefore the temperature, of the U to increase before the heat gets a chance to enter the moderator. The uranium temperature affects the thermal cross sections of U only insofar as the temperature-induced density change affects N_o, and therefore the macroscopic cross sections. The more important effect of changes in U temperature is to change the resonance absorption because of Doppler broadening.

We turn now to a consideration of each of the temperature dependent quantities, η, p, k_{f25}, f, L^2, and $\bar{P}$.

Temperature Coefficient of η

Since η involves the ratio of U^{235} to U^{238} cross sections, namely

$$\eta = \frac{R\,\nu\,\sigma_{f25}}{R\sigma_{f25} + R\sigma_{r25} + \sigma_{r28}} \tag{11.28}$$

(where subscripts refer to the nuclear species), η will vary with temperature only if the absorption or fission cross sections of U^{235}, and the absorption cross section of U^{238} deviate in different ways from 1/v behavior. Data on this point are not entirely conclusive since $\sigma_r(U^{238})$ has not been measured in the region just beyond .025 volts with any precision.

If the cross sections are considered to deviate from 1/v, then η of course must be averaged over the neutron spectrum which is assumed to

be Maxwellian, $M(E) = \sqrt{E}\, e^{-E/kt}$ Thus (11.28) should be replaced by

$$\eta(T) = \frac{R\,\nu \int \sigma_{f_{25}}(E)\, \sqrt{E}\, M(E)\, dE}{\int \left[R\, \sigma_{f_{25}}(E) + R\, \sigma_{r_{25}}(E) + \sigma_{r_{28}}(E)\right] \sqrt{E}\, M(E)\, dE} \tag{11.29}$$

where T is the absolute neutron temperature. The extra factor $\sqrt{E}$ in the average comes from the fact that the cross sections must be weighted with the flux, nv, rather than with the neutron density, n.

To evaluate $\eta(T)$ it is convenient to make some analytic approximations to the experimental cross section. Wigner, from analysis of all available data has suggested the following analytic approximations (all σ's in barns):

$$\left.\begin{aligned}
\sigma_{r_{28}}(E) &= \frac{.44}{\sqrt{E}} \left[1 + 0.50\, E + 0.11\, E^2\right] \\
\sigma_{a_{25}}(E) &= \sigma_{f_{25}} + \sigma_{r_{25}} = \frac{113}{\sqrt{E}} \left[1 - 3.24\, E + 12.8\, E^2\right] \\
\sigma_{f_{25}}(E) &= \frac{113}{1.18\, \sqrt{E}} \left[1 - \left(3.24 + \frac{1}{1.18}\, \frac{d\alpha}{dE}\right) E + 12.8\, E^2\right]
\end{aligned}\right\} \tag{11.30}$$

where $\alpha = \sigma_{f_{25}}/\sigma_{r_{25}} \simeq 0.18$. Considerable uncertainty in the value of $\frac{d\alpha}{dE}$ exists. Recently (KAPL-511) J. Sampson and H. Hurwitz, Jr. have analyzed the neutron spectrometer data on U^{235} and experiments by Bragdon, Hughes, and Marshall (CP-1381) in which the change in U^{235} fissions to U^{238} captures was measured with and without a silver filter around the sample: the average temperature of the neutrons was higher with the silver filter than without. Their analysis gives the following:

	$d\alpha/dE$	$\frac{d\alpha(kT)}{dT}$
Bragdon et al	- .46/ev to - .82/ev	- $7\times10^{-5}/^{o}C$ to - $11\times10^{-5}/^{o}C$
Neutron Spectrometer	+ 0.6/ev	+ $10\times10^{-5}/^{o}C$

The true value of $d\alpha/dT$ is thus seen to be in grave doubt. The range of values for $d\eta/dT$ of natural U found by substituting (11.31) into (11.30) are given by Sampson and Hurwitz thus:

	$d\alpha/dE$	$\frac{d\alpha}{dT}$	$\frac{d\eta}{dT}$
Bragdon Experiment	- 0.74/ev	- $10\times10^{-5}/^{o}C$	- $6\times10^{-5}/^{o}C$
Neutron Spectrometer	+ 0.6/ev	+ $10\times10^{-5}/^{o}C$	- $29\times10^{-5}/^{o}C$
Constant α	0	0	- $18\times10^{-5}/^{o}C$

All that can be concluded from this is that $d\eta/dT$ is very probably negative, and around $10 \times 10^{-5}/^{o}C$.

<u>Temperature Coefficient of p</u>

Since

$$p = \exp. - \frac{N_o}{\xi(N_1\sigma_{s_1} + N_o\sigma_{s_o})} \int \sigma_{a_{o_{eff}}} \, dE/E \qquad (11.31)$$

and since $\int \sigma_{a_{o_{eff}}} dE/E$ increases with uranium temperature in the manner discussed in Chapter IV, the resonance escape probability <u>decreases</u> as a natural uranium chain reactor heats up. The change in $\int \sigma_{a_{o_{eff}}} dE/E$ with

temperature becomes small as the scattering to absorption ratio increases; in the limit of ∞ scattering cross section, $\int \sigma_{a_{0eff}}\, dE/E$ becomes temperature independent. On this account the change in p with temperature in a homogeneous reactor - in which the scattering per uranium atom is evidently very large - is of less consequence than in a heterogeneous system.

There are no data available on the temperature coefficient of $\int \sigma_{25_{eff}}\, dE/E$; nor are there data on temperature variation of $\int \sigma_{a_{0eff}}\, dE/E$ except for natural U metal and U oxide. For these systems, the temperature coefficient of the resonance integral is about $-1.5 \times 10^{-4}/^{o}C$. Since the fraction of neutrons "trapped" by resonance absorption is about ten percent in a natural uranium chain reactor, the temperature coefficient of p in a reactor such as Hanford is about

$$\frac{1}{p}\,\frac{dp}{dt} \approx -1.5 \times 10^{-5}/^{o}C \qquad (11.32)$$

In a highly enriched reactor such as the EBR or the SIR in which a considerable fraction of the fissions occur at resonance energies, the Doppler broadening can of course increase the fission rate, i.e., k_{f25}, with increased temperature. At the time of this writing not enough is known about the details of the U^{235} resonance structure to predict the sign of the Doppler-induced temperature coefficient in an intermediate energy U^{235} reactor such as SIR.

Temperature Coefficient of f

The thermal utilization involves the ratio of absorption cross sections of uranium and moderator, etc. So long as all cross sections are 1/v, and the system is truly homogeneous, f will be independent of neutron temperature. If some of the cross sections deviate from 1/v, then f will be neutron temperature dependent, just as η is, viz,

$$\frac{1}{f}(T) = 1 + \frac{N_1 \int \sigma_{a_1}(E)\, E\, e^{-E/kT}}{N_o \int \sigma_{a_o}(E)\, E\, e^{-E/kT}} \frac{dE}{dE} \tag{11.33}$$

In (11.33), the assumption is made that the thermal neutron spectrum is Maxwellian. No general conclusion regarding the variation of f with T is possible. In a homogeneous system it is customary to assume f is temperature independent, while in a heterogeneous system in which the details of the thermal neutron distribution will be sensitive to neutron temperature, the thermal utilization will be dependent on temperature even though all cross sections are 1/v. This effect will be discussed in the following chapter.

Temperature Coefficient of L^2

The diffusion area, L^2, in the reactor is given by

$$L^2 = (1 - f)\, L_m^2$$

Hence

$$\frac{1}{L^2}\frac{dL^2}{dT} = \frac{1}{L_m^2}\frac{dL_m^2}{dT} - \frac{1}{1-f}\frac{df}{dT}$$

$$\approx \frac{1}{L_m^2}\frac{dL_m^2}{dT} \tag{11.34}$$

in a homogeneous system in which $\frac{df}{dT} \approx 0$. The diffusion length of the moderator is

$$L_m^2 = \frac{1}{3\, N_1^2\, \sigma_{a1}\, \sigma_{tr}}, \tag{11.35}$$

and therefore

$$\frac{1}{L_m^2}\,\frac{dL_m^2}{dT} = -\frac{2}{N_1}\,\frac{dN}{dT} - \frac{1}{\sigma_{tr}}\,\frac{d\,\sigma_{tr}}{dT} - \frac{1}{\sigma_a}\,\frac{d\sigma_a}{dT} \tag{11.36}$$

Now $\frac{1}{N_1}\frac{dN_1}{dT}$ is simply the negative of the reciprocal expansion coefficient of the moderator, β; and $\frac{1}{\sigma_a}\,\frac{d\sigma_a}{dT} = -\frac{1}{2T}$ if σ_a is $1/v$ and T is the absolute temperature of the neutrons. Thus

$$\frac{1}{L_m^2}\,\frac{dL_m^2}{dT} = 2\beta + \frac{1}{2T} - \frac{1}{\sigma_{tr}}\,\frac{d\,\sigma_{tr}}{dT} \tag{11.37}$$

since $\lambda_{tr} \sim (\sigma_{tr})^{-1}$.

If the transport cross section, σ_{tr}, is known as a function of neutron energy, then the proper average transport cross section, which is to be used for computing the diffusion length, is, according to (1.91),

$$\frac{1}{\sigma_{tr}} = \frac{\int_0^\infty \frac{v}{\sigma_{tr}(v)}\, M(v)\, dv}{\int_0^\infty v\, M(v)\, dv} \tag{11.38}$$

provided the neutron distribution is nearly Maxwellian, and the scattering

atoms are so light that the "amnesia" approximation discussed in I-38 is justified. Although the complete justification for (11.38) has never been given, it has been customary in project literature to use it in computing the average transport mean free path. The important point about (11.38) is that it is the reciprocal cross section which is averaged.

Two mechanisms are of importance in imposing a temperature dependence on the transport cross section. These are

a) Crystalline effects

b) Chemical binding effects.

Although the theory of both effects in application to the problem of neutron diffusion in a chain reactor is not complete, it will be instructive to describe what has been done.

Crystalline Effects

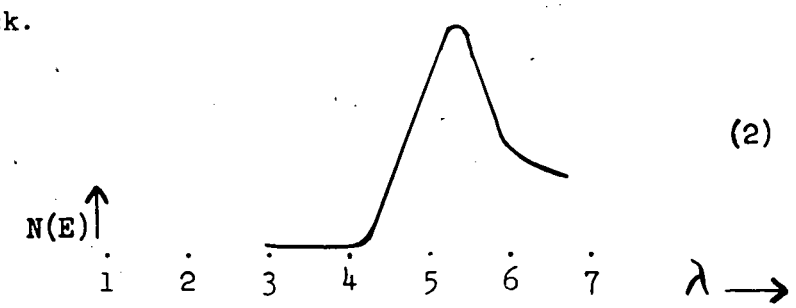
The scattering of slow neutrons in crystalline material is of two kinds: coherent scattering, in which the neutron waves form diffraction patterns as the wavelets from the regularly spaced scattering centers interfere or reinforce; and incoherent scattering, in which, for various reasons such as spin dependence of the scattering, presence of different isotopes, or inelastic scattering, there is no regular phase relation between the neutron wavelets scattered from different scattering centers, and interference is therefore impossible. Coherent scattering will occur only if the scattering angle θ and the neutron wave length λ satisfy the Bragg relation

$$n\lambda = 2d \sin\theta ,$$

n being the order of the reflection, and d the distance between lattice

planes.

If λ exceeds 2d, coherent scattering is impossible; only incoherent processes can lead to scattering. The total incoherent scattering cross section is often very small ($\sim$ 0.7 b in BeO); thus, very slow neutrons will hardly be scattered at all in polycrystalline media. A long thin block of BeO can therefore be used to filter out neutrons of very low energy since neutrons of more than about 4.5 Å wave length will be scattered only incoherently (with cross section of 0.7 b) while neutrons of shorter wave length will be scattered with a cross section perhaps ten times higher. A striking demonstration of this is seen in the accompanying graph of Fermi and Marshall[1] showing the spectrum of neutrons filtered through a 40 cm x 10 cm x 10 cm prism of BeO when a Maxwellian spectrum was incident on the block.



(2)

If λ is short enough for the Bragg relation to be satisfied, coherent scattering can take place. Fermi, Sturm, and Sachs[3] have shown that the scattering cross section per nucleus corresponding to

(1) Physical Review, Vol. 71, p. 666, 1947.

(2) Ibid. p. 672 (Fig. 3).

(3) Ibid. p. 592.

scattering of a particular order from a particular set of lattice planes in a randomly oriented polycrystalline aggregate is

$$\sigma_b = \frac{N F \lambda^2 e^{-2wb^2}}{8 \pi b}$$

where N is the number of nuclei per unit volume, F is the crystal form factor (i.e., the scattering cross section of the crystalline unit cell), $b = \frac{n}{d}$, n = order of the reflection, w = Debye-Waller temperature factor. The formula for w is

$$w = \frac{3}{2} \frac{h^2}{m k \theta} \left(\frac{\Phi (\theta/T)}{\theta/T} + 1/4 \right)$$

where h = Planck's constant, m = the atom's mass, k = Boltzmann's constant, θ = characteristic temperature, T = absolute temperature, and Φ = Debye's function.

The total coherent scattering is the sum of the scattering from all lattice planes for which diffraction can take place. Thus

$$\sigma = \Sigma_{b \leq 2\lambda} F N \lambda^2 \exp(-2wb^2)/8 \pi b ; \qquad (11.39)$$

the cross section given by (11.39) is zero until the energy increases enough for the Bragg condition for the most widely separated crystal planes to be satisfied. The cross section then decreases essentially proportionally to the square of the wave length until the Bragg condition is satisfied for the next most widely separated planes at which energy there is another sudden increase in scattering cross section. These discontinuous jumps, followed by monotonic decreases, are repeated until the wave length is so short that many planes contribute to the scattering, and the Bragg scattering

becomes indistinguishable from the ordinary free atom scattering. This "jagged-tooth" variation of the cross section with energy in the case of BeO is shown in the accompanying Figure 11 - I. At very low energies where (11.39) gives no scattering, there is of course some residual incoherent scattering.

To show how the crystalline scattering affects the average mean free path, the cross sections of microcrystalline BeO have been averaged over a Maxwellian neutron flux according to (11.38). The resulting averages are compared with the averages obtained from the high energy BeO scattering cross sections in the accompanying Figure 11 - II. It is seen that in BeO the crystalline effects increase the free atom mean free path by about 17%.

Chemical Binding Effects

It can be shown on rather general grounds that the cross section for the potential scattering of a neutron is proportional to the square of the reduced mass of the system neutron + scatterer. [See, e.g., Bethe, Elementary Nuclear Theory, (Wiley, 1947), p. 47] When a neutron of high energy strikes an atom in a molecule, the struck atom acts as though it were free. However, when the neutron energy is comparable to the energy of chemical binding, the molecule will tend to recoil as a whole. Thus the effective reduced mass of the scattering center + neutron rises as the neutron energy decreases; since the scattering cross section also increases with the reduced mass, the thermal scattering cross section tends to increase as the neutron energy decreases.

Since the reduced mass of neutron-proton system changes from $\frac{1}{2}$ m, the nuclear mass when the proton is free, to m when the proton is tightly

attached to an infinitely heavy molecule, the thermal neutron-proton scattering cross section increases by about a factor of four as the neutron energy decreases. The change in reduced mass with binding is greater for a proton than for any other nucleus. For deuterons, the reduced mass of the system neutron + deuteron changes from

$$\frac{2}{3}\,m \qquad \text{free deutron}$$

to

$$m \qquad \text{completely bound deuteron}$$

so that the deuteron scattering cross section ought to change by a factor of 9/4 as the energy of the neutron approaches zero. With heavier nuclei the effect is less important.

Experimental evidence for the "reduced mass" effect in neutron-proton and neutron-deuteron scattering is abundant. The accompanying graph Figure 11 - III gives the n-d scattering cross section as measured in D_2O at several neutron energies.

Since the strength of the binding depends on the particular compound containing the proton, it is not surprising to find the details of the energy dependence of the n-p scattering cross section are sensitive to the composition of the proton containing molecules. This is shown in Figure 11 - IV where the n-p cross sections as measured in H_2 gas, methane, propane and cetane are given.

It is seen in the graphs that the n-p cross section (as measured in cetane) falls from about 71 barns to 21 barns, while the n-d cross section falls from 6.6 to 3.3 barns in going from .01 e.v. to 1 e.v. These values are close to the theoretical values of 4 x 21 = 84 barns for H and 9/4 x 3.3 = 7.4 barns for D to which the cross sections should extrapolate at zero energy.

The chemical binding variation of cross section must of course be taken into account when estimating the dependence of L^2 on temperature. In H_2O moderated systems, where the effect is by far the largest, the variation in scattering cross section accounts for about 40% of the temperature coefficient of L^2. The remainder comes from the 1/v absorption cross section.

Temperature Variation of $\frac{1}{p}\bar{P}$

Little need be said about the variation of $\frac{1}{p}\bar{P}$ with temperature. Since $\frac{1}{p}\bar{P}$ (1) is determined by the fast neutron scattering properties of the moderator, and since these depend on temperature only insofar as thermal expansion changes the number of nuclei per c.c., $\frac{1}{p}\bar{P}$ is affected also only insofar as the density of the moderator changes with temperature.

Examples of Homogeneous Reactor Systems

The nuclear energy literature abounds with suggestions for slow neutron chain reacting systems in which fissionable material and moderator are intimately mixed. In addition, there are the fast and intermediate systems in all of which heterogeneities of fuel distributions are rather unimportant because the mean free path of the system is large compared to either the fuel element thickness or the space between the fuel elements.

Of the true homogeneous chain reacting systems only two, the Los Alamos water boiler and the HRE, have actually been built and are operating. Both of these, from a nuclear standpoint, are H_2O moderated and use U^{235} as fuel. Nuclear characteristics of the HRE are given in ORNL-1121 and ORNL CF-51-11-94.

(1) The factor $\frac{1}{p}$ normalizes the slowing down kernel to unity, rather than to p.

There have been numerous critical experiments performed on H_2O solutions of various concentrations with both U^{235} and Pu^{239} as fuel. The U^{235}-H_2O solution experiments were performed by Callihan, et al. A comparison of their observations with two-group calculations and with the variational method of Greuling is given in Fig. 11-V.

Pu-H_2O experiments have been done at Los Alamos during the war by P. Morrison, B. T. Feld, L. Slotin, et al. An estimated minimum critical mass of 606 $\pm$ 50 grams of Pu^{239} in a nitrate solution with an infinite H_2O reflector was obtained by extrapolating these experiments.

Finally, in the H_2O moderated systems, there have been numerous experiments in which the U is borne in plates which are so close to each other that in effect the system is practically homogeneous. The MTR (and its children, the LITR and BSF) are all reactors of this type, the plates being aluminum.

One characteristic of H_2O moderated systems is that because of the very large thermal neutron scattering by protons, the thermal diffusion length in H_2O systems is always extremely small. For this reason the fast leakage is always much greater than the slow leakage. In the MTR, for example, the fast leakage is eighteen times the slow. Thus the characteristics equation in a water moderated system can with good approximation be replaced by

$$\frac{k}{p}\,\overline{P}_{\infty}(E_s,B^2) = 1 \qquad (11.40)$$

since the slow leakage, L^2B^2, is small compared to unity.

Greuling has worked out a complete variational treatment for water moderated, water reflected systems which has been used extensively in calculations on such systems, and the reader is referred to this work

(LA-393) for details.

Heavy water enriched fuel systems have been studied by Snell. He measured critical masses of D_2O - U^{235} solutions which were essentially homogeneous. The results of his measurements are summarized in MonP-454 and are analyzed in some detail by Garabedian[1] and Nordheim[2]. Snell's experiments were heavy water reflected, and therefore bare reactor theory is inapplicable. The minimum critical mass he found was 859 gm of U^{235}.

The critical mass in these H_2O and D_2O moderated systems as a function of fuel to moderator ratio shows a minimum. That the critical mass should be very large at very low concentrations follows because the more dilute the solution, the more neutrons are absorbed by the moderator. In fact, since $k = f\eta$ (ε and p are practically 1 at low concentrations) must exceed unity in order for the system to chain react at all, the concentration at which the system is infinite is given by

$$\frac{1}{f} = \eta \tag{11.41}$$

Since $\frac{1}{f}$ is given by (11.24),

$$\frac{1}{f} = 1 + \frac{N_1\sigma_{a_1}}{N_o\sigma_{a_o}} , \tag{11.24}$$

the minimum chain reacting concentration is

$$\frac{N_1}{N_o} = (\eta - 1)\frac{\sigma_{a_o}}{\sigma_{a_1}} \tag{11.42}$$

(1) CNL-36.

(2) MonP-314.

In H_2O and U^{235} this is 1200 or 10.9 gm U^{235}/liter; in D_2O it is 9.6×10^5 or .0135 gm U^{235}/liter.

At the high concentration end the critical mass also goes up, although the thermal reactor theory breaks down as soon as the concentration is so high that an appreciable fraction of the fissions are epithermal. At the high end of the concentration scale, the multiplication constant becomes practically constant and equal to η since eventually all the neutrons are absorbed by the fuel. The L^2 approaches zero as U concentration, and therefore absorption, increase, but, until resonance fission becomes important, the τ remains practically constant. Consequently the buckling, which roughly approaches

$$B^2 \approx \frac{\eta - 1}{\tau} \tag{11.43}$$

remains unchanged as the concentration increases, and therefore the total mass grows larger and larger. Actually the maximum critical mass on the high concentration side is just the mass computed for a fast neutron reaction in a solid chunk of fuel. Since in a fast system almost every collision leads to a fission, the buckling is much more nearly

$$B^2 \approx \frac{\eta - 1}{\frac{1}{6}\lambda_{tr}^2} \tag{11.44}$$

where λ_{tr} is of the order of the fast transport m.f.p. Since $\frac{1}{6}\lambda_{tr}^2 \ll \tau$ (for U^{235} density 18.6, $\frac{1}{6}\lambda_{tr}^2 \approx 5.4$, $\tau_{H_2O} \approx 33$) the actually maximum

critical mass is far smaller than that obtained from (11.43).

The thermal neutron homogeneous systems of perhaps greatest interest are those which involve solutions of natural or slightly enriched U in D_2O. Such systems can in principle be used to convert uranium into plutonium, and they have therefore been studied rather extensively. Critical mass as a function of U^{235} to U^{238} ratio and of concentration for UO_2SO_4, UO_2, and UO_2NO_3 in D_2O is given in Fig. 11-VI. The method of calculation is simply to use (11.28), (11.11), (11.4), (11.26), and (11.22) for η, p, f, L^2 and B^2 and assume that $\varepsilon = 1$.

From the calculation it appears that any of the natural uranium systems, except the nitrate, can be made chain reacting at the appropriate U to D_2O ratio. The optimum concentration, i.e., the concentration leading to minimum U investment, is about 60 - 70 gm U/liter. Below this concentration the parasitic thermal absorption by the D_2O tends to reduce k and therefore the critical mass is large; above this concentration the U^{238} resonance absorption begins to become important, k decreases, and again the critical mass grows.

No slightly enriched homogeneous D_2O system has been made critical as yet. However, exponential experiments on sub-critical assemblies of natural UO_2F_2 in D_2O were made by Wattenberg in 1945, and the results of these compared with Dancoff's calculations are given in CP-3364. In such systems which contain natural U the resonance absorption integral, and its variation with concentration of moderator, is extemely important for determining the multiplication constant. The values of $\int \sigma_{a_{oeff}} dE/E$

which Dancoff had used to achieve agreement with experiment are the same as given in Fig. VI of Chapter IV. The comparison with experiment is given in the accompanying graph, Fig. 11-VII.

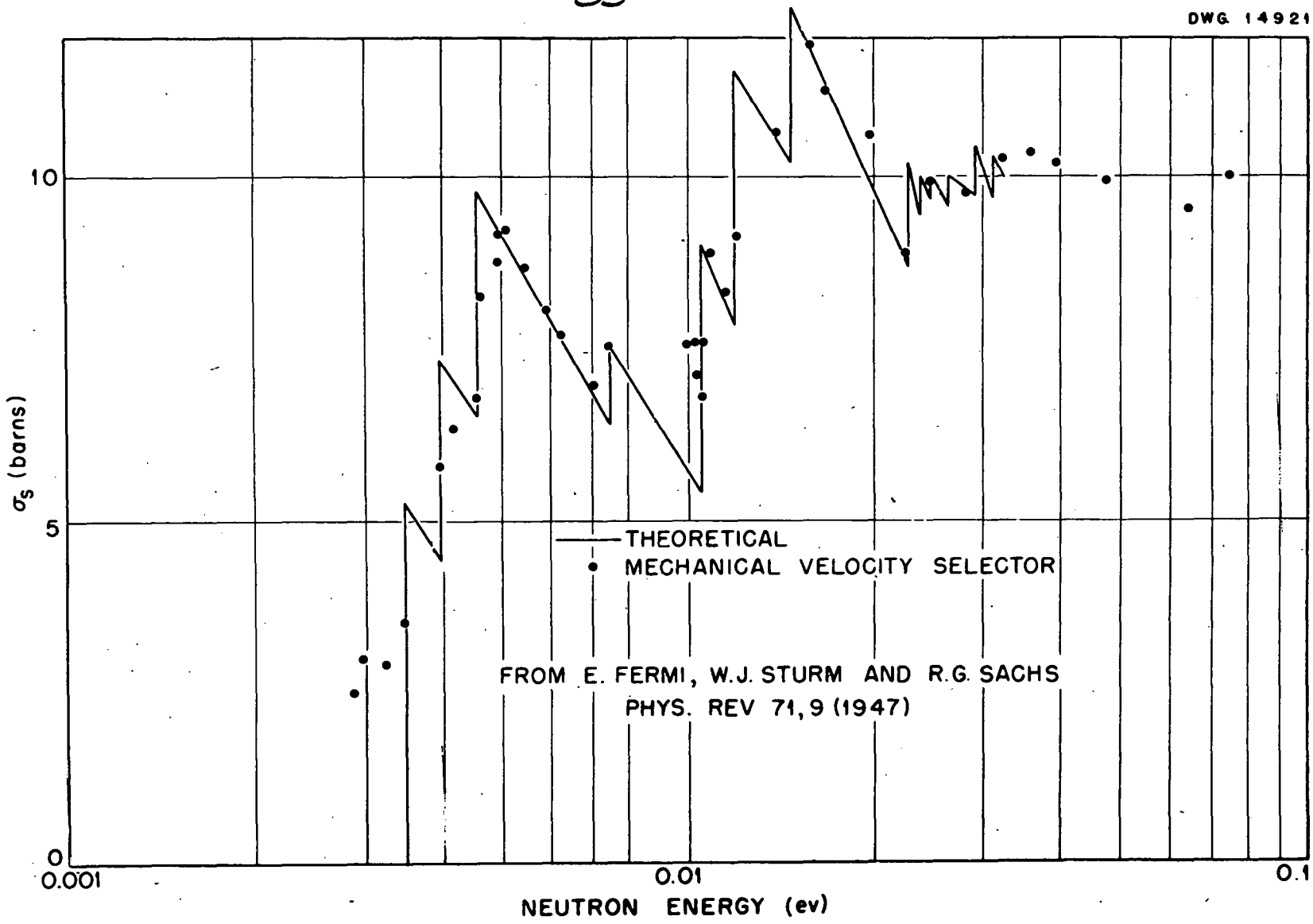


Fig. 1-I Scattering Cross Section of BeO

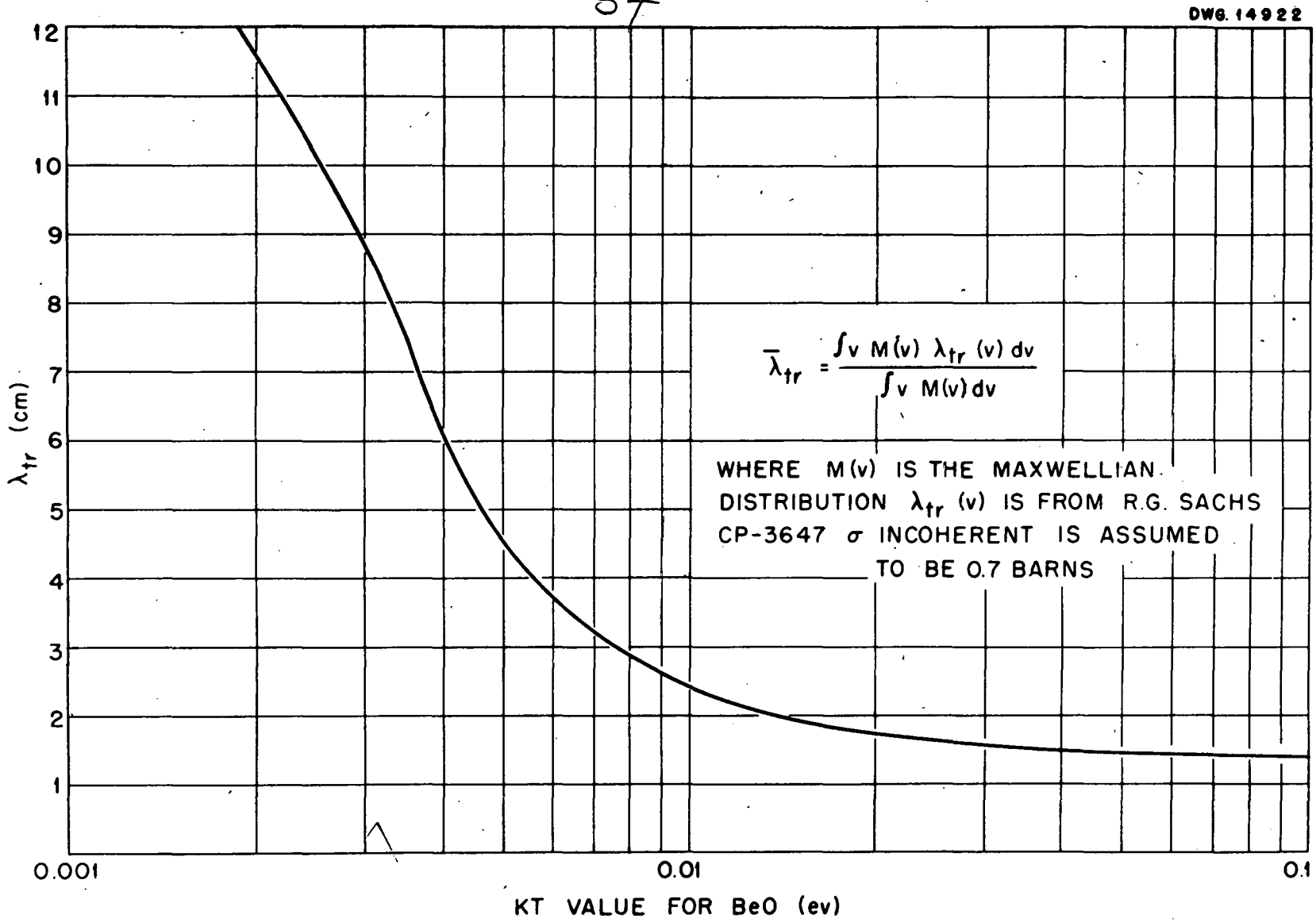


Fig. 11-II Transport Mean Free Path for BeO

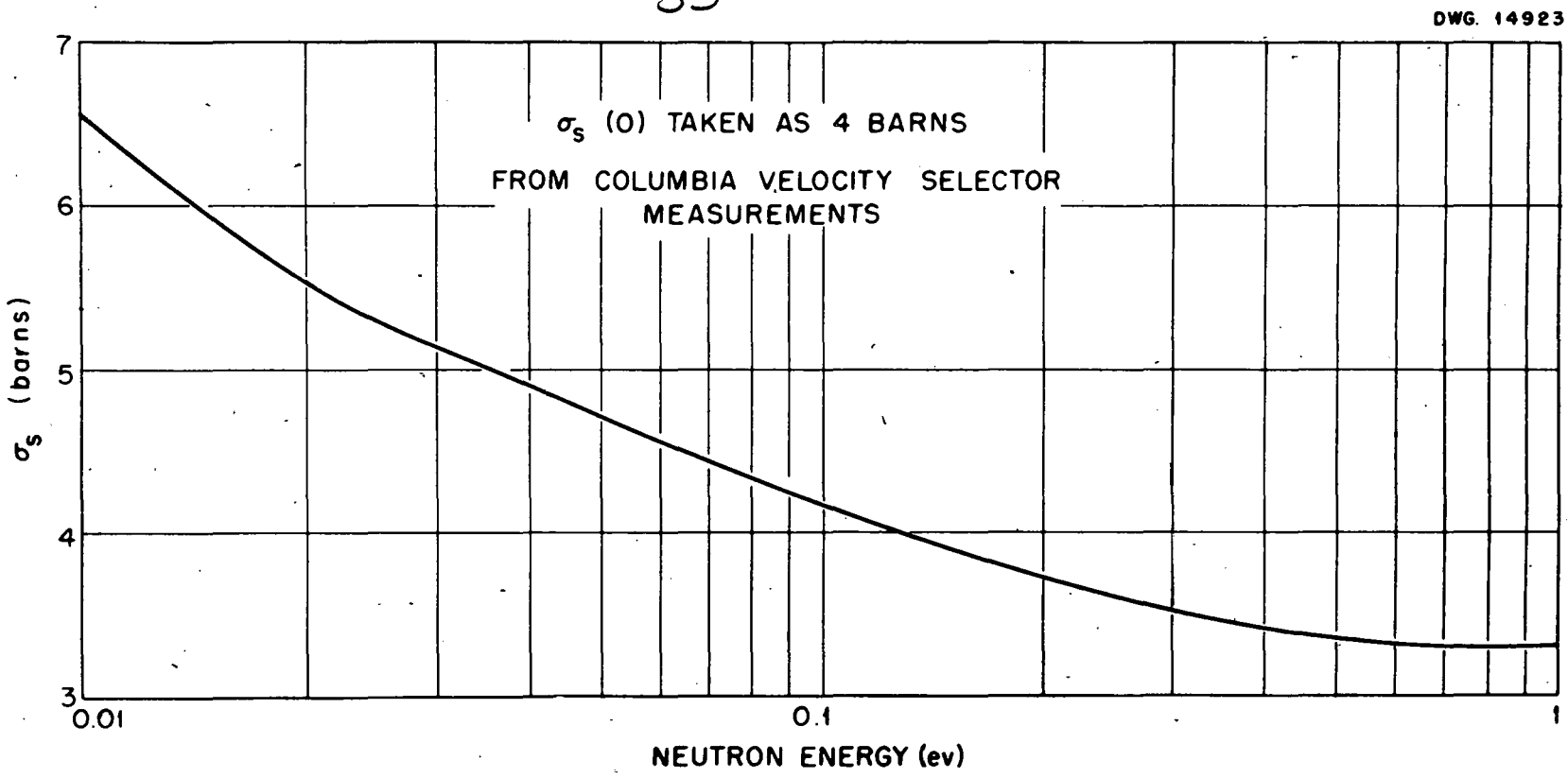


Fig. 11-III σ_s for D in D_2O

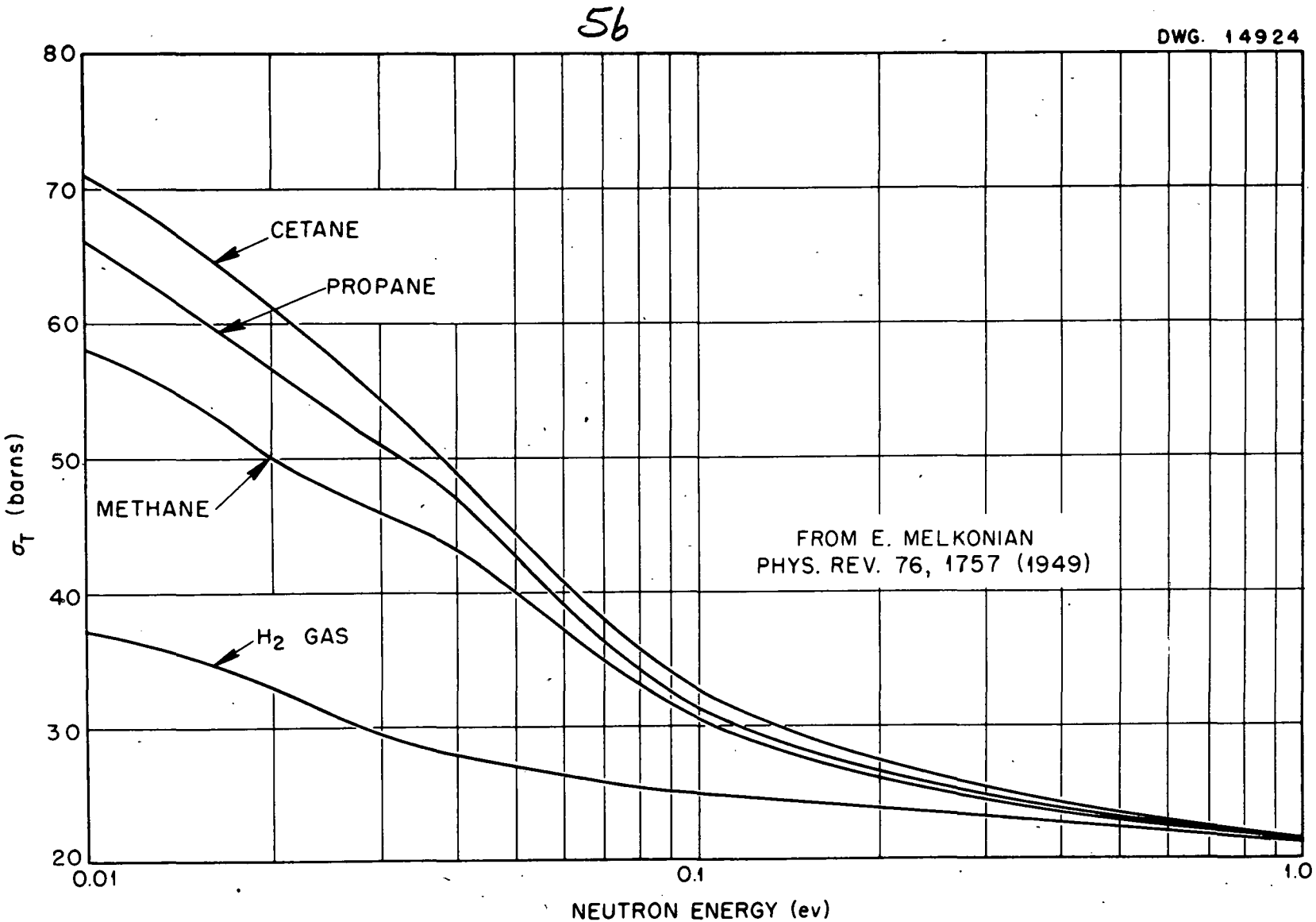


Fig. 11-IV Slow Neutron Cross Section of H in Various Hydrocarbons and H_2 Gas

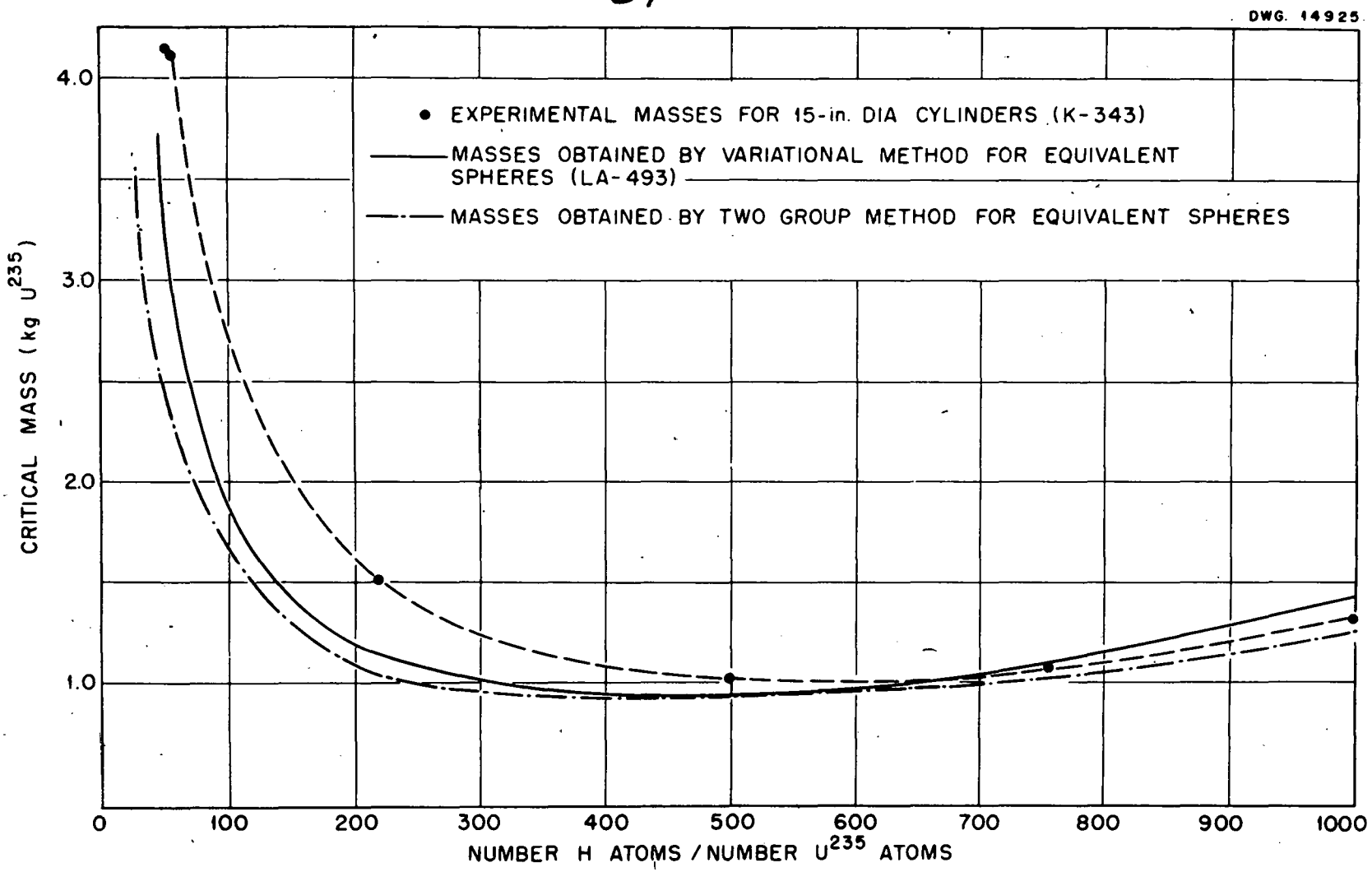


Fig. 11-V Comparison of Critical Masses for H_2O Moderated and Reflected U^{235} Reactors

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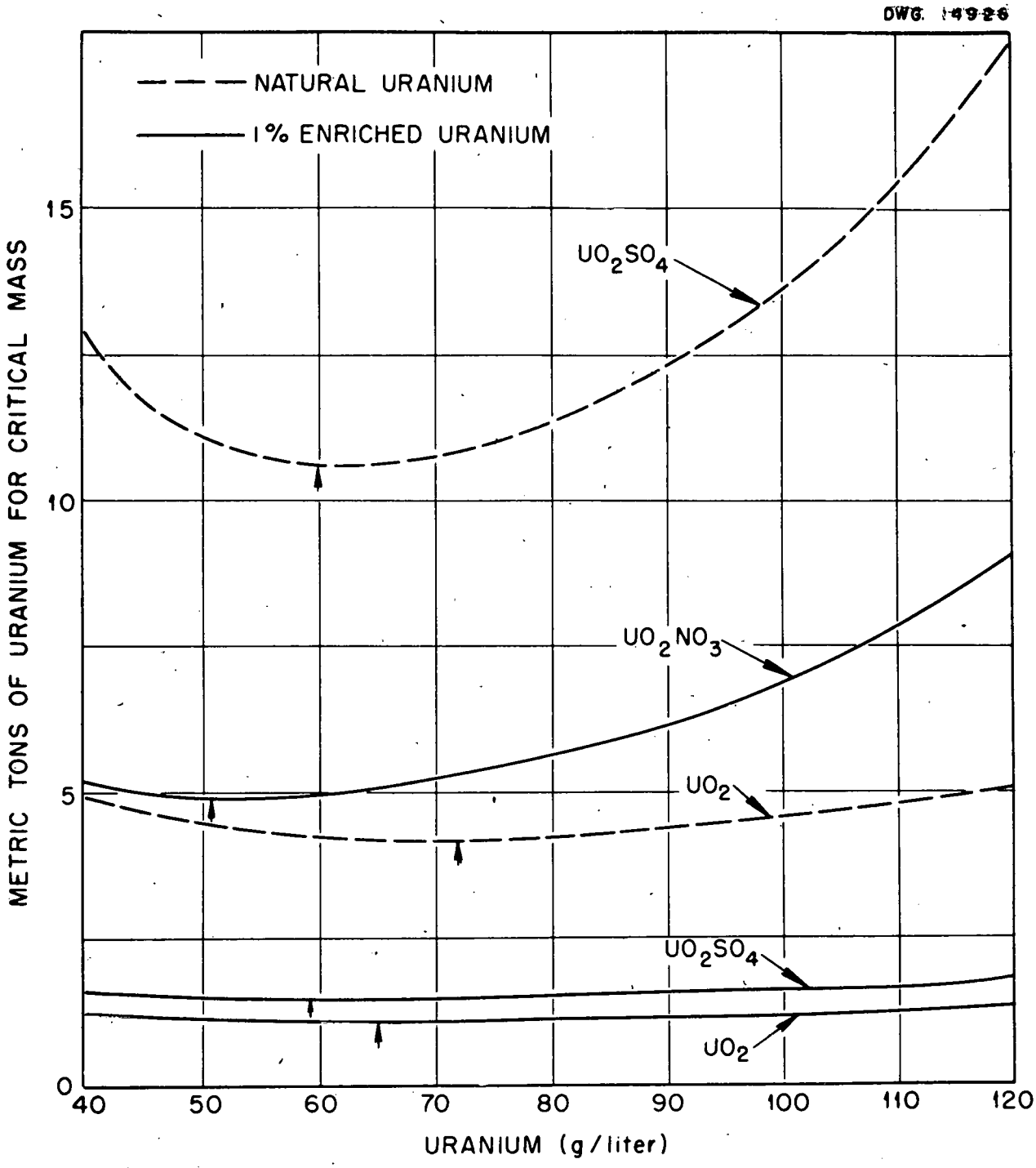

Fig. 11-VI Critical Masses of Natural and 1% Enriched Uranium for Bare Spheres Containing Various Uranium Salts in D_2O at 20°C

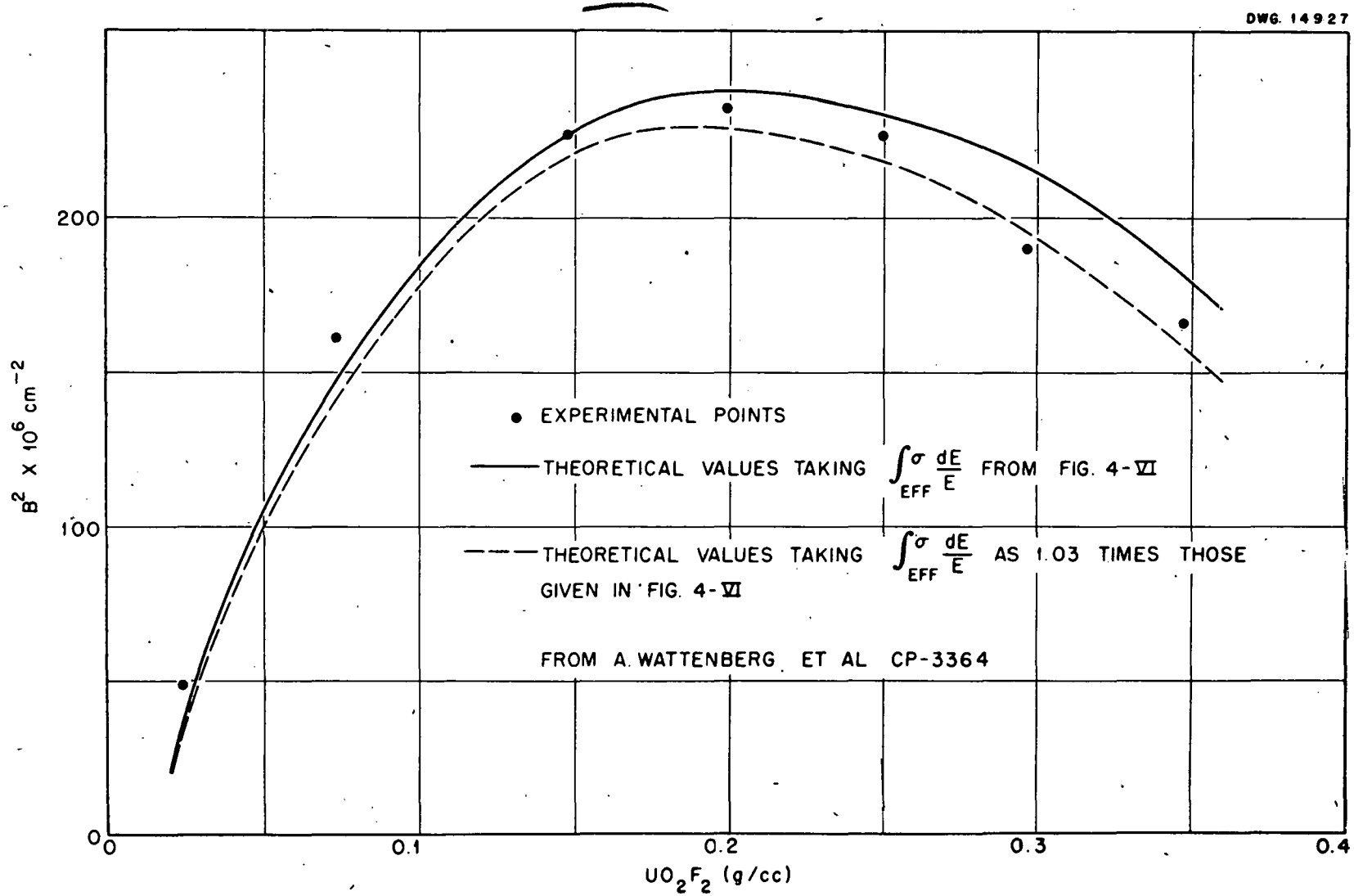

Fig. 11 - VII Variation of LaPlacian Values With Concentration of UO_2F_2 in D_2O Solutions, for $\eta = 1.350$

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