See The Physical Theory of Neutron Chain Reactors, by Weinberg & Wigner. Chicago. Univ. of Chicago Press, 1958. Price $13.50

I

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THEORY OF NEUTRON CHAIN REACTIONS

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II

Extracts From

Volume I

DIFFUSION AND SLOWING DOWN OF NEUTRONS

Chapter I. ELEMENTARY THEORY OF NEUTRON DIFFUSION
Chapter II. SECOND ORDER DIFFUSION THEORY
Chapter III. SLOWING DOWN OF NEUTRONS

PHYSICS

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III

VOLUME I

DIFFUSION AND SLOWING DOWN OF NEUTRONS

INTRODUCTION

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The large scale release of nuclear energy in a uranium fission chain reaction involves two essentially distinct physical phenomena. On the one hand there are the individual nuclear processes such as fission, neutron capture, and neutron scattering. These are essentially quantum mechanical in character, and their theory is non-classical. On the other hand, there is the process of diffusion -- in particular, diffusion of neutrons, which is of fundamental importance in a nuclear chain reaction. This process is classical; insofar as the theory of the nuclear chain reaction depends on the theory of neutron diffusion, the mathematical study of chain reactions is an application of classical, not quantum mechanical, techniques.

In a uranium chain reaction, neutrons are produced at very high energy ($\sim$ several Mev) from the fission of uranium nuclei. The neutrons then diffuse through the mass of chain reacting material and are ultimately either captured (and produce further fission), or are lost through the boundaries of the system. Most of the questions of importance in studying chain reactions are concerned with the diffusion of the neutrons rather than with the details of the nuclear processes by which the neutrons are captured or scattered. The probabilities for these nuclear processes enter the theory, for the most part, as empirically given properties of the medium in which the neutrons diffuse. The theory of neutron chain reactions is thus to a very large extent an application of the classical theory of diffusion in a capturing medium rather than an application of the theory of nuclear reactions.

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IV

Because diffusion of neutrons plays such a large role, it will be necessary to develop the mathematical theory of neutron diffusion before embarking on a study of chain reactions. The first section of this volume will therefore be devoted to an exposition of the mathematical theory of neutron diffusion.

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CHAPTER I AECD-3471

ELEMENTARY THEORY OF NEUTRON DIFFUSION

The gross phenomenon of neutron diffusion is in some respects comparable to other diffusion phenomena, such as diffusion of heat or of gas molecules. It is found that whenever there is a spatial gradient in the density of mono-energetic neutrons in a medium, the neutrons tend to move from the region of high density to the region of low density. This behavior is understandable if it is supposed that the neutrons suffer random collisions with the atoms of the matter in which they are diffusing. The number of collisions in a given volume element will be relatively greater where the density of neutrons is high. Since a neutron comes off each collision, the number of neutrons traveling away from the region of high density should be greater than the number traveling toward the region of high density. This gives rise to a net current of neutrons in a sense opposite to the neutron density gradient.

Neutron diffusion is in principle a simpler process than gaseous diffusion. The reason is that, on account of the extraordinarily small density of neutrons in almost all physical systems, neutron-neutron collisions are exceedingly rare. Practically all collisions that take place when a neutron diffuses through a system are collisions between the neutron and the nuclei of the molecules composing the system. This is very different from gaseous diffusion, where intra-molecular collisions are the rule. The phenomenon of neutron diffusion is in this one respect more like diffusion of electrons in a metal than like gaseous diffusion.

The fact that neutron-neutron collisions can be neglected implies that the fundamental equations describing the diffusion of neutrons are linear. These equations can therefore be integrated in many more cases than can the

non-linear equations of gaseous diffusion, and in general, the theory of neutron diffusion is simpler than the diffusion theory encountered in the dynamical theory of gases.

In all diffusion phenomena the net diffusion flow of a substance -- that is, the difference between the number of particles crossing, from each side, a unit area per second -- is taken to be proportional to the negative gradient of a quantity Φ_o which is simply related to the density of the substance diffusing. In the case of heat diffusion, Φ_o is just the temperature, and the constant of proportionality is called the thermal conductivity. In the case of neutron diffusion, the quantity Φ_o is the sum of the speeds of all the neutrons in a unit volume. Thus, if n_o is the total number of neutrons in a cubic centimeter, and $\bar{v}$ is their average speed, while n_{o_i} is the number of neutrons per c.c. having speed v_i, then we define

$$\Phi_o = \sum_i n_{o_i} v_i = \sum_i F_o(v_i) = n_o \bar{v}. \tag{1.1}$$

We shall call Φ_o the <u>total neutron flux</u>, and $F_o(v_i) = n_{o_i} v_i$ the total <u>flux of neutrons of speed v_i</u>.

The difference between the number of neutrons striking from each side a square centimeter per second is called the <u>net diffusion current</u>. It is evidently a vector quantity, and we shall denote it by $\underline{j}$. The assumption that the neutron diffusion current is proportional to the negative gradient Φ_o

$$\underline{j} = - D_o(\bar{v}) \text{ grad } \Phi_o \tag{1.2}$$

becomes in the case of monoenergetic neutrons, or if $\bar{v}$ is independent of position,

$$\underline{j} = - D_o\bar{v} \text{ grad } n_o = - D \text{ grad } n_o. \tag{1.3}$$

Equation (1.3) is a statement of what is sometimes called Fick's Law; we shall call equation (1.2) the generalized Fick's Law. The proportionality factors D_o, or $D_o\bar{v} = D$, are called the diffusion coefficients. The quantity D_o may depend on the average velocity, $\bar{v}$, of the neutrons, but according to Fick's Law it has the same dependence on neutron energy everywhere in a uniform medium.

Fick's law is the fundamental assumption of elementary diffusion theory. We shall see presently that it is a valid assumption in general only at large distances from sources, sinks, or boundaries.* Since there are many instances in a chain reacting system where neutron diffusion takes place near sources, or boundaries*, it is not always correct to apply Fick;s Law. It is therefore necessary to derive Fick's Law for neutrons from first principles, and to examine in detail its limitations.

Diffusion Theory and Transport Theory

Ordinary diffusion theory, based on the generalized Fick's Law (1.2), concerns itself only with the neutron flux Φ_o, and with the net diffusion current $\underline{j}$. Evidently, there is more to be known about the distribution of neutrons than simply the flux, Φ_o, or, in the case of monoenergetic neutrons, the density, n_o. In a volume element at a given point in space neutrons travel in all directions. The instantaneous velocity vectors of all the neutrons which are contained in that volume element form a pattern which characterizes the neutron distribution much more fully than does the total flux of neutrons in the volume element. The sum of all the velocity vectors emanating from a unit volume in a particular direction is just equal to the number of neutrons per unit volume which cross perpendicularly, in one second, a unit area normal to the direction of the velocity vectors. The scalar sum of all these velocity vectors is evidently Φ_o.

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* Or, in general, if the neutrons are not monoenergetic.

The calculation of the angular distribution of neutron velocity vectors in a medium is the central problem of what is called "transport" theory. The calculation of Φ_o, the flux, based on the assumption of Fick's Law, is the central problem of elementary diffusion theory. Clearly elementary diffusion theory is less general than transport theory since a knowledge of the angular distribution of the velocity vectors implies a knowledge of Φ_o, but not conversely. It will be our task in this chapter to trace the relation between the two. In this we shall adopt essentially the methods described by W. Bothe, Zeit. für Physik, 401, 118, (1942).

The Boltzmann Equation without Energy Loss

Let $n(\underline{r},v,\theta,\varphi)dvdxdydzd\underline{\Omega}$ be the number of neutrons in the volume element dxdydz at $\underline{r}$, whose speed lies between v and v+dv, and whose directions of motion lie in the element of solid angle $\sin\theta d\theta d\varphi$ around the direction $\underline{\Omega}$. The vector $\underline{\Omega}$ is a unit vector whose components on the x,y,z axes are (Figure 1)

$$\Omega_x = \cos\theta$$

$$\Omega_y = \sin\varphi \sin\theta$$

$$\Omega_z = \cos\varphi \sin\theta.$$

The element of solid angle around $\underline{\Omega}$, that is, the area of an element on a unit sphere of which $\underline{\Omega}$ is a radius, is $\sin\theta d\theta d\varphi$. We denote this element by $d\underline{\Omega}$.

The quantity $n(\underline{r},v,\underline{\Omega})dvd\underline{\Omega}$ is the number of neutrons per unit volume whose directions of motion lie in a solid angle $d\underline{\Omega}$ around $\underline{\Omega}$ and whose speed lies in a range $d\underline{v}$ around v; or more briefly, whose velocity lies in a range $d\underline{v}$ around the vector velocity $\underline{v} = v\underline{\Omega}$. For brevity we shall call $n(\underline{r},v,\underline{\Omega})$ the number of neutrons per unit volume with speed v and direction $\underline{\Omega}$. The total number of neutrons of speed v per unit volume is

$$n_o(\underline{r},v) = \int n(\underline{r},v,\underline{\Omega})d\underline{\Omega} \qquad (1.4)$$

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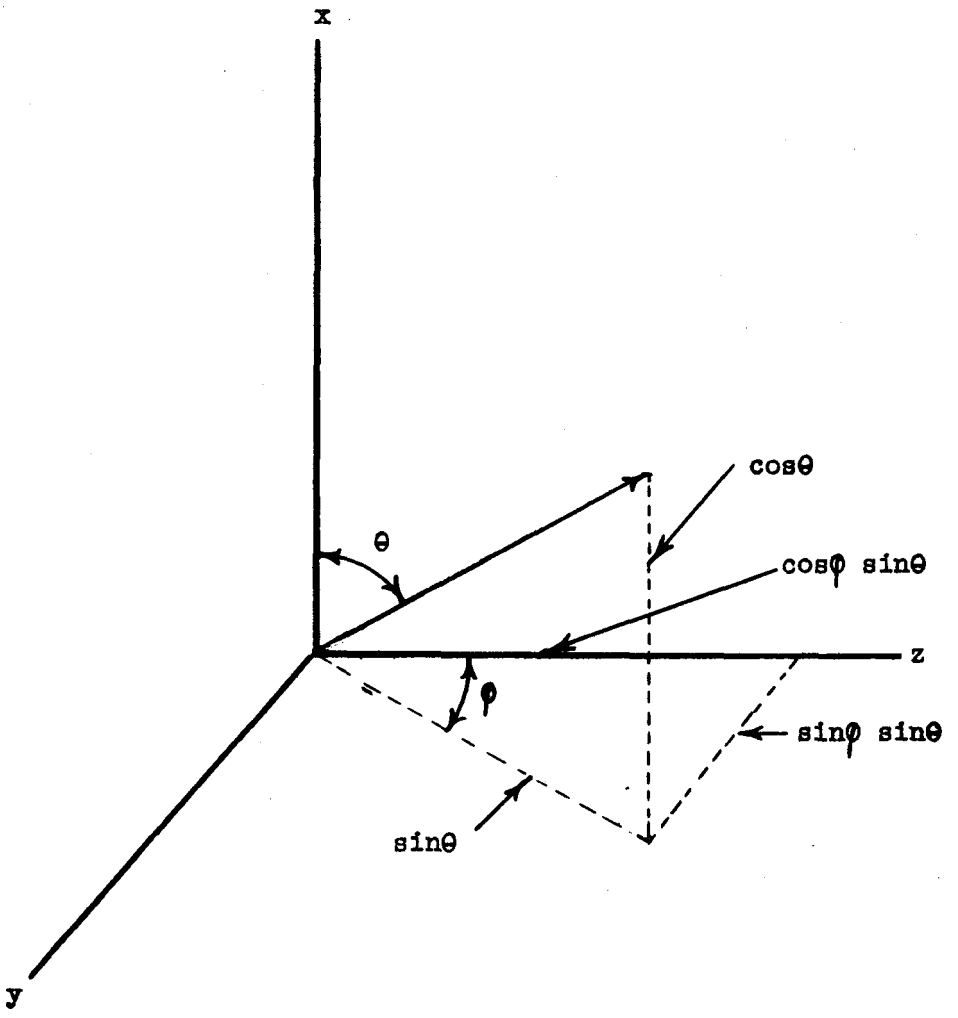

Figure 1.

and the total number of neutrons, per unit volume, of all speeds is

$$n_o(\underline{r}) = \int n_o(\underline{r},v)dv. \tag{1.5}$$

We shall distinguish between $n_o(\underline{r})$ and $n_o(\underline{r},v)$ by the argument following n_o.

In an analogous manner, we define the vector flux, $\underline{F}(\underline{r},v,\underline{\Omega})$, of neutrons of speed v and direction $\underline{\Omega}$ to be

$$\underline{F}(\underline{r},v,\underline{\Omega}) = F(\underline{r},v,\underline{\Omega})\underline{\Omega} = n(\underline{r},v,\underline{\Omega})v\underline{\Omega}. \tag{1.6}$$

The vector $\underline{F}$ is a vector whose magnitude F is the number of neutrons of speed v which in one second cross perpendicularly a unit area normal to the direction $\underline{\Omega}$. The total flux of neutrons of speed v -- that is, the density of neutrons of speed v multiplied by v -- is

$$F_o(\underline{r},v) = \int n(\underline{r},v,\underline{\Omega})vd\underline{\Omega} = \int F(\underline{r},v,\underline{\Omega})d\underline{\Omega}. \tag{1.7}$$

The total number of neutrons crossing perpendicularly a unit area per second in the direction $\underline{\Omega}$, $\underline{\Phi}(\underline{r},\underline{\Omega})$, is a vector which is the sum of the vectors $\underline{F}$ for all speeds but fixed $\underline{\Omega}$:

$$\underline{\Phi}(\underline{r},\underline{\Omega}) = \Phi\underline{\Omega} = \underline{\Omega}\int F(\underline{r},v,\underline{\Omega})dv. \tag{1.8}$$

We call $\underline{\Phi}(\underline{r},\underline{\Omega})$ the vector flux of neutrons in the direction $\underline{\Omega}$. Finally, the total flux of neutrons in all directions and having all speeds -- that is, the sum of all the speeds of neutrons in a unit volume -- is

$$\Phi_o(\underline{r}) = \int n_o(\underline{r},v)vdv = \int \Phi(\underline{r},\underline{\Omega})d\underline{\Omega}. \tag{1.9}$$

With these definitions stated explicitly, we can now proceed to set up an equation which will describe the variation of $F(\underline{r},v,\underline{\Omega})$ in the steady state.

We therefore consider a material balance of neutrons in a cubical volume element of sides dx,dy,dz, at $\underline{r}$. The number of neutrons whose directions of motion lie in $d\underline{\Omega}$ around $\underline{\Omega}$ and whose speeds lie in dv around v which enter the volume element through the face dydz in one second is the x-component of the vector flux:

$$F_x(\underline{r},v,\underline{\Omega})dydzdvd\underline{\Omega},$$

and the number which leave through the opposite face is

$$F_x(x+dx,y,z,v,\underline{\Omega})dydzd\underline{\Omega}dv = F_x(x,y,z,v,\underline{\Omega})dydzd\underline{\Omega}dv + \frac{\partial F_x}{\partial x}(x,y,z,v,\underline{\Omega})dydzd\underline{\Omega}dvdx.$$

The net number of neutrons travelling in this direction element which are gained by the volume element through the yz face is the difference of these two expressions, namely

$$-\frac{\partial F_x}{\partial x}(x,y,z,v,\underline{\Omega})dxdydzdvd\underline{\Omega}.$$

A similar expression holds for the number of neutrons gained through the other two sets of faces. Hence the total number of neutrons of speed between v and v+dv and direction between $\underline{\Omega}$ and $\underline{\Omega}+d\underline{\Omega}$ which are gained by the volume element per second is

$$-\nabla\cdot\ \underline{F}(\underline{r},v,\underline{\Omega})dxdydzdvd\underline{\Omega}. \qquad (1.10)$$

The second way in which the number of neutrons with vector velocity $\underline{v} = v\underline{\Omega}$ in a volume element can be increased is by scattering collisions which change a neutron's velocity from $\underline{v}'$ to $\underline{v}$ -- that is, its speed from v' to v and its direction from $\underline{\Omega}'$ to $\underline{\Omega}$. Let $\sigma_s(\underline{v},\underline{v}')d\underline{v}d\underline{v}' = \sigma_s(v,\underline{\Omega},v',\underline{\Omega}')dvdv'd\underline{\Omega}d\underline{\Omega}'$ be the differential scattering cross-section per atom of the medium for the process of scattering a neutron from its original velocity between $\underline{v}'$ and $\underline{v}'+d\underline{v}'$ into a final vector velocity between $\underline{v}$ and $\underline{v}+d\underline{v}$. Then the total number of scattering

collisions per second in dxdydz which lead to neutrons having speeds between v and v+dv and directions between $\underline{\Omega}$ and $\underline{\Omega}+d\underline{\Omega}$ is

$$dxdydzdvd\underline{\Omega}\int_{v'}\int_{\underline{\Omega}'} n(\underline{r},v',\underline{\Omega}')v'N\sigma_s(v,v',\underline{\Omega},\underline{\Omega}')dv'd\underline{\Omega}' = \tag{1.11}$$

$$= dxdydzdvd\underline{\Omega}\int_{v'}\int_{\underline{\Omega}'} F(\underline{r},v',\underline{\Omega}')N\sigma_s(v,v',\underline{\Omega},\underline{\Omega}')dv'd\underline{\Omega}' \ .$$

The meaning of $\sigma_s(\underline{v},\underline{v}')d\underline{v}d\underline{v}'$ is, more precisely, the following. Suppose a well collimated beam of neutrons of velocity between $\underline{v}'$ and $\underline{v}'+d\underline{v}'$ strikes a square centimeter of a thin foil of scattering material placed normal to the beam. Let the number of atoms per cubic centimeter of the material be N; the number of atoms per square centimeter will then be Nt, where t is the thickness of the scatterer. We suppose t is so small that a single neutron suffers at most one collision in traversing the scatterer. As a result of scattering, some neutrons will be deflected out of their original speed and direction element (between $\underline{v}'$ and $\underline{v}'+d\underline{v}'$) into a new velocity element between $\underline{v}$ and $\underline{v}+d\underline{v}$. The number of such scattered neutrons per second we call $P(\underline{v},\underline{v}')d\underline{v}d\underline{v}'$. Evidently $P(\underline{v},\underline{v}')d\underline{v}d\underline{v}'$ must be the product of $F(\underline{v}')d\underline{v}'$, Nt, and a proportionality factor $\sigma_s(\underline{v},\underline{v}')d\underline{v}$:

$$P(\underline{v},\underline{v}') = N\sigma_s(\underline{v},\underline{v}')t\,|\underline{F}(\underline{v}')| \ . \tag{1.12}$$

The quantity $\sigma_s(\underline{v},\underline{v}')$ is called the microscopic differential scattering cross-section. "Microscopic" indicates that it refers to a single atom, and "differential", that it gives the probability of scattering from one velocity element to another. The quantity $N\sigma_s(\underline{v},\underline{v}')$ is called the macroscopic differential

cross-section; it is the sum of the cross-sections for all the atoms in a cubic centimeter. The microscopic cross-section has dimensions cm^2; the macroscopic cross-section has dimensions cm^{-1}. The reciprocal of a macroscopic cross-section has the dimensions of cm and will be denoted by λ with a subscript which refers to the type of collision process. Thus λ_s, the scattering mean free path, is

$$\lambda_s = (N\sigma_s)^{-1} . \tag{1.13}$$

A third way in which neutrons of a particular velocity can be added to dxdydz is by means of an extraneous source, such as, e.g. a Ra-Be mixture. We denote the number of neutrons produced with velocity $\underline{v}$ in the volume element dxdydz by such a source by

$$S(\underline{r},v,\underline{\Omega})dxdydz\ dvd\underline{\Omega} . \tag{1.14}$$

The quantity $S(\underline{r},v,\underline{\Omega})$ is the source strength.

The total number of neutrons which are removed from the velocity interval around $\underline{v}$ consists of those which are removed by scattering and those which are removed by absorption. Let $\sigma_a(v)$ be the microscopic absorption cross-section of the medium for neutrons of speed v. The total cross-section for removal of neutrons from the velocity $v\underline{\Omega}$ is

$$\sigma(v,\underline{\Omega}) = \sigma_a(v) + \sigma_s(v,\underline{\Omega}) \tag{1.15}$$

where

$$\sigma_s(v,\underline{\Omega}) = \iint_{v'\underline{\Omega}'} \sigma_s(v',\underline{\Omega}',v,\underline{\Omega})dv'd\underline{\Omega}' ,$$

and the total number of neutrons in the element dxdydz which are removed from $d\underline{v}$ per second (either because of deflection or by absorption) is

$$N\sigma\ F(\underline{r},v,\underline{\Omega})dxdydzdvd\underline{\Omega} . \tag{1.16}$$

We assume the medium is isotropic; hence $N\sigma(v,\underline{\Omega})$ is independent of the direction $\underline{\Omega}$. In practice this means that we ignore "crystal" effects.

In a steady state we can equate the number of neutrons removed from a given direction in the volume element (1.16) to the number entering that direction (1.10, 1.11, 1.14). The result is the steady state Boltzmann equation:

$$\nabla \cdot \underline{F}(\underline{r},v,\underline{\Omega}) + No\, F(\underline{r},v,\underline{\Omega}) = \iint_{v'\underline{\Omega}'} F(r,v',\underline{\Omega}')\, N\sigma_s(v,v',\underline{\Omega},\underline{\Omega}')dv'd\underline{\Omega}' + S(\underline{r},v,\underline{\Omega}). \tag{1.17}$$

This is an integro-differential equation since the variable $\underline{F}$ is operated on both by a differential and an integral operator.

We now restrict our considerations to systems in which the neutron distribution depends only on x. Furthermore we assume the angular distribution of the neutrons at a given x depends only on μ, the cosine of the angle between $\underline{\Omega}$ and the x direction. This is the case if the angular distribution of neutrons emitted from any source is radially symmetric about the x axis. With this assumption the term $\nabla \cdot \underline{F}(\underline{r},v,\underline{\Omega})$ in (1.17) becomes $\mu \frac{dF}{dx}(x,v,\underline{\Omega})$.

It is convenient to integrate the Boltzmann equation (1.17) with respect to φ from 0 to 2π; this amounts to lumping all neutrons of a given velocity together whose directions make an angle θ with the x axis, regardless of their azimuth. Since $F(x,v,\underline{\Omega})$ is independent of φ, we obtain

$$\mu \frac{dF}{dx}(x,v,\mu) + N\sigma F(x,v,\mu) = \int_0^{2\pi}\int_0^{\infty}\int_0^{2\pi}\int_{-1}^{1} F(x,v',\underline{\Omega}') \times \tag{1.18}$$

$$N\sigma_s(v,v',\underline{\Omega},\underline{\Omega}')d\mu'd\varphi'dv'd\varphi + S(x,v,\mu)$$

where

$$F(x,v,\mu)d\mu = \int_0^{2\pi} F(x,v,\underline{\Omega})d\mu d\varphi = 2\pi F(x,v,\underline{\Omega})d\mu \tag{1.19}$$

is the number of neutrons of velocity v with direction cosine between μ and $\mu+d\mu$ perpendicularly crossing in one second a ring of area 2π on the unit sphere around the point x, and $S(x,v,\mu) = \int_0^{2\pi} S(x,v,\underline{\Omega})d\varphi = 2\pi S(x,v,\underline{\Omega})$ is the number of neutrons of velocity v emitted with direction cosine μ per second by the source. In the integral term of (1.18) it is not possible in general to integrate out the φ variable because, while $F(x,v,\underline{\Omega}')$ is independent of φ, $\sigma_s(\underline{\Omega},\underline{\Omega}')$ is not, necessarily.

In order to proceed further, it will be necessary to express the scattering cross-section in terms of the direction coeines μ and μ'. The average scattering cross-section, in an isotropic medium, can depend only on the angle between the direction of the incident and scattered neutrons and must be independent of the azimuth of the scattering, i.e.,

$$\sigma_s(v,v',\underline{\Omega},\underline{\Omega}') = \sigma_s(v,v',\underline{\Omega}\cdot\underline{\Omega}') = \frac{1}{2\pi}\sigma_s(v,v',\mu_o) \tag{1.20}$$

where $\mu_o = \underline{\Omega}\cdot\underline{\Omega}'$ is the cosine of the angle between incident (direction $\underline{\Omega}'$) and scattered ($\underline{\Omega}$) neutrons, and $\sigma_s(v,v',\mu_o)d\mu_o$ is the v' into v cross-section for scattering neutrons from their original direction into an elementary ring on the unit sphere which subtends an angle between $\cos^{-1}\mu_o$ and $\cos^{-1}(\mu_o+d\mu_o)$ with the initial direction. From the components of $\underline{\Omega}$ and $\underline{\Omega}'$ we have that

$$\mu_o = \underline{\Omega}\cdot\underline{\Omega}' = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi'). \tag{1.21}$$

The scattering cross-section $\sigma_s(v,v',\mu_o)$ is a known function (experimentally determined usually) of the scattering angle $\cos^{-1}\mu_o$ and of course, of v and v'. Hence we suppose that it can be represented as a sum of Legendre Polynomials

$$\sigma_s(v,v',\mu_o) = \sum_{\ell} \frac{2\ell+1}{2} \sigma_{s\ell}(v,v')P_\ell(\mu_o) \tag{1.22}$$

where

$$\sigma_{s\ell}(v,v') = \int_{-1}^{1} \sigma_s(v,v',\mu_o)P_\ell(\mu_o)d\mu_o. \tag{1.23}$$

The coefficient $\sigma_{s\ell}(v,v')$ of the ℓ-th harmonic gives the contribution to the total scattering from velocity v' to v of that particular harmonic. It is customary to use spectroscopic terminology and refer to σ_{s_o} as the cross-section for s (spherically symmetric) scattering, σ_{s_1} the cross-section for p-scattering, and so on.

The first two terms in the expansion of the scattering cross-section have particularly simple physical interpretations. Thus

$$\sigma_{s_o}(v,v') = \int_{-1}^{1} \sigma_s(v,v',\mu_o)d\mu_o \tag{1.24}$$

is the total scattering cross-section from v' to v, while

$$\sigma_{s_1}(v,v') = \int_{-1}^{1} \mu_o\sigma_s(v,v',\mu_o)d\mu_o = \bar{\mu}_o\sigma_{s_o}(v,v') \tag{1.25}$$

is the total scattering cross-section times the average cosine of the scattering angle.

It is still necessary to express $\sigma_s(v,v',\mu_o)$ in terms of μ and μ'. To do this we make use of the addition theorem for Legendre functions (Jahnke-Emde, p. 115):

$$P_\ell(\mu_o) = P_\ell(\mu)\, P_\ell(\mu') + 2 \sum_{m=1}^{\ell} \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\mu)\, P_\ell^m(\mu') \cos m(\varphi-\varphi') \tag{1.26}$$

where $P_\ell^m(\mu_o)$ is an associated Legendre function. Substituting (1.26) into (1.22) we have

$$\sigma_s(v,v',\underline{\Omega},\underline{\Omega}') = \frac{1}{2\pi}\sigma_s(v,v',\mu_o) = \frac{1}{2\pi}\sum \frac{2\ell+1}{2}\sigma_{s_\ell}(v,v')P_\ell(\mu)P_\ell(\mu') +$$

$$+ \frac{1}{2\pi}\sum_{\ell}\sum_{m=1}^{\ell}\frac{(\ell-m)!}{(\ell+m)!}(2\ell+1)\sigma_{s_\ell}(v,v')P_\ell^m(\mu)P_\ell^m(\mu')\cos m(\varphi-\varphi') \,. \tag{1.27}$$

If we now substitute this into the integrand of (1.18), the double sum drops out after integration over φ, as does the factor $\frac{1}{2\pi}$ in the first term. Since $F(x,v',\underline{\Omega}')$ is independent of φ', the integration over φ' just introduces a factor 2π which is absorbed by (1.20) in transforming from $F(x,v',\underline{\Omega}')$ to $F(x,v,\mu')$. The final result is therefore

$$\mu\frac{dF}{dx}(x,v,\mu) + N\sigma F(x,v,\mu) = \frac{1}{2}\sum_{\ell}(2\ell+1)\int_0^\infty N\sigma_{s_\ell}(v,v')P_\ell(\mu)\int_{-1}^{1}P_\ell(\mu')F(x,v'\mu')d\mu'dv' +$$

$$+ S(x,v,\mu). \tag{1.28}$$

In this chapter we shall be interested primarily in diffusion without energy loss. The scattering cross-sections $\sigma_{s_\ell}(v,v')$ will therefore be taken to be zero unless $v = v'$; in that case they will be δ-functions such that

$$\int_0^\infty N\sigma_{s_\ell}(v,v')\, F(x,v'\mu)dv' = N\sigma_{s_\ell}(v)\, F(x,v,\mu).$$

The transport equation without energy loss therefore is

$$\mu \frac{dF}{dx}(x,v,\mu) + N\sigma F(x,v,\mu) = \frac{1}{2}\sum_{\ell}(2\ell+1)N\sigma_{s_\ell}(v)P_\ell(\mu)\int_{-1}^{1}P_\ell(\mu')F(x,v,\mu')d\mu' + S(x,v,\mu). \tag{1.28a}$$

Usually the scattering is sufficiently isotropic so that only the first two terms in the spherical harmonics expansion of σ_s are needed. With this restriction (1.28a) becomes

$$\mu \frac{dF}{dx}(x,v,\mu) + N\sigma F(x,v,\mu) = \frac{N\sigma_{s_0}}{2}(v)\int_{-1}^{1}F(x,v,\mu')d\mu' + \frac{3}{2}\mu N\sigma_{s_1}(v)\int_{-1}^{1}F(x,v,\mu')\mu'd\mu' + S(x,v,\mu). \tag{1.29}$$

Boltzmann's equation in the forms (1.28) and (1.29) will be used as the basis of most of our remaining discussion.

Derivation of Fick's Law

We proceed to derive a generalization of Fick's law from the transport equation (1.28a). In the first place we observe that the net number of neutrons of speed v which cross a square centimeter of surface in the y-z plane per second is

$$j(x,v) = \int_{-1}^{1} nv(x,v,\mu)\mu d\mu = \int_{-1}^{1} F(x,v,\mu)\mu d\mu , \tag{1.30}$$

while the total current of all velocities is

$$j(x) = \int_0^{\infty} j(x,v)\,dv = \int_{-1}^{1} \Phi(x,\mu)\mu d\mu \ . \tag{1.31}$$

We shall call $j(x,v)$ the net diffusion current of neutrons of velocity v; and $j(x)$ the total net diffusion current. The net current $j(x)$ is the difference between the current of neutrons going to the right, $j_+(x,v)$, and the current of neutrons going to the left, $j_-(x,v)$. Since neutrons travelling toward the right must have direction cosines between 0 and 1, while those travelling to the left must have direction cosines between 0 and -1, we write

$$j_+(x,v) = \int_0^{1} F(x,v,\mu)\mu d\mu \tag{1.32}$$

$$j_-(x,v) = \int_0^{-1} F(x,v,\mu)\mu d\mu \tag{1.33}$$

and

$$j(x,v) = j_+(x,v) - j_-(x,v) = \int_{-1}^{1} F(x,v,\mu)\mu d\mu \ . \tag{1.34}$$

The formulas for $j(x)$, $j_+(x)$, and $j_-(x)$ are the same except that we must substitute the total flux, Φ, for F, the flux of speed v.

In order to establish Fick's law we must find a relation between $j(x,v)$ and the derivative of the flux, $\frac{dF(x,v)}{dx}$, and also between $j(x)$ and $\frac{d\Phi_o(x)}{dx}$. We start with the source free transport equation (Eq. 1.28a with S = 0),

multiply by μ and integrate with respect to μ from -1 to +1. On the right hand side all terms except the one involving $P_1(\mu)$ drop out. After a little algebra we obtain

$$j(x,v) = -\frac{\int_{-1}^{1} \mu^2 \frac{dF(x,v,\mu)d\mu}{dx}}{N\sigma(v)\left[1 - \frac{\sigma_{s_o}}{\sigma}\bar{\mu}_o\right]} \tag{1.38}$$

Since μ and x are independent variables, it is permissible to interchange the order of differentiation and integration. Then introducing the quantity

$$\overline{\mu^2} = \frac{\int_{-1}^{1} \mu^2 F(x,v,\mu)d\mu}{\int_{-1}^{1} F(x,v,\mu)d\mu} = \frac{\int_{-1}^{1} \mu^2 F(x,v,\mu)d\mu}{F_o(x,v)} \tag{1.39}$$

so that

$$\int_{-1}^{1} \mu^2 \frac{dF(x,v,\mu)d\mu}{dx} = \frac{d}{dx}\int_{-1}^{1} \mu^2 F(x,v,\mu)d\mu = \frac{d\left[\overline{\mu^2}\, F_o(x,v)\right]}{dx}, \tag{1.40}$$

we can write (1.38) in the form

$$j(x,v) = \frac{-\frac{d}{dx}\overline{\mu^2}F_o}{N\sigma(1 - \frac{\sigma_{s_o}}{\sigma}\bar{\mu}_o)} \tag{1.41}$$

which is the generalized Fick's Law for neutrons of speed v.

It is important not to confuse $\overline{\mu^2}$ and μ_o. The quantity $\overline{\mu^2}$ is the mean square cosine of the neutron angular distribution at speed v; in general it

will vary from point to point if the neutron angular distribution varies with x. The quantity $\bar{\mu}_o$ is the average cosine in an individual scattering process at speed v. It depends only on the interaction between neutron and scattering atom and is the same throughout a medium having the same nuclear composition everywhere, and through which monoenergetic neutrons diffuse.

The coefficient of $\frac{-d(\overline{\mu^2 F_o})}{dx}$ is the reciprocal of a macroscopic cross-section. It therefore has the dimensions of a mean free path, and in fact is called the "diffusion" mean free path, $\lambda_d(v)$:

$$\lambda_d(v) = \frac{1}{N\sigma(v)\left[1 - \frac{\sigma_{s_o}(v)}{\sigma(v)}\bar{\mu}_o(v)\right]} = \frac{1}{N\sigma_{s_o}(1 - \bar{\mu}_o) + N\sigma_a} \quad . \tag{1.42}$$

With this notation, (1.41) may be written

$$j(x,v) = -\lambda_d(v)\frac{d}{dx}\left[\overline{\mu^2}\, F_o(x,v)\right] . \tag{1.41a}$$

If the absorption cross-section is zero then the diffusion mean free path becomes equal to the so-called "transport mean free path", λ_t, which is defined as

$$\lambda_t(v) = \frac{1}{N\sigma_{s_o}(v)\left[1 - \bar{\mu}_o(v)\right]} \quad . \tag{1.43}$$

It is worthwhile to note that the transport mean free path is larger than the scattering mean free path, $\lambda_s = \frac{1}{N\sigma_s}$, by the factor $\frac{1}{1 - \bar{\mu}_o}$. This factor measures how strong is the forward bias in a scattering collision. If the scattering is completely forward, $\bar{\mu}_o = 1$, $\lambda_t = \infty$; while if the scattering is isotropic, $\bar{\mu}_o = 0$ and $\lambda_t = \lambda_s$. For forward biased scattering, the

coefficient of $-\frac{d}{dx}(\overline{\mu^2 F_o})$, and therefore $j(x,v)$ is greater than for isotropic scattering. Hence a given gradient of $\overline{\mu^2 F_o}$ will maintain a larger net flow of neutrons the greater is the "forwardness" of the scattering. This result is very reasonable physically.

In order to find a relation between the total net diffusion current, $j(x)$, and the total flux, $\Phi_o(x)$, we integrate (1.141) over all velocities. Then

$$j(x) = \int_0^\infty j(x,v)dv = -\int_0^\infty \lambda_d(v) \frac{d\left[\overline{\mu^2 F_o(x,v)}\right]}{dx} dv \tag{1.44}$$

is the generalized Fick's law when the neutrons are not monoenergetic, although the scattering is assumed to occur without energy loss.

Another way of writing the generalized Fick's law, for polyenergetic neutrons, -- and this holds whether or not the scattering occurs with energy loss -- can be found by multiplying Eq. (1.128) by μ and then integrating with respect to both μ and v. The result is

$$\frac{d\overline{\overline{\mu^2}}\Phi_o(x)}{dx} = -\int_0^\infty N\sigma(1 - \frac{\sigma_{s_o}}{\sigma}\overline{\mu}_o)\ j(x,v')dv' \tag{1.45}$$

where $\overline{\overline{\mu^2}}$ is the mean square cosine averaged over both angle and velocity, and where σ_{s_o} and $\overline{\mu}_o$ are functions of v' only, since their dependence on v has been integrated out. Evidently neither (1.41a), (1.44), nor (1.45) are in the form of Fick's law; for example, in (1.45) $\overline{\overline{\mu^2}}\Phi_o$ and not Φ_o, appears under the derivative sign.

We now discuss the relation between the generalized Fick's law for monoenergetic neutrons (1.41), and the Fick's law of elementary diffusion theory.

Elementary diffusion theory, which is based on the proportionality between $j(x,v)$ and dF_o/dx holds rigorously for monoenergetic neutrons only if $\overline{\mu^2}$ is independent of position; for in that case (1.41) can be written

$$j(x,v) = -\lambda_d \overline{\mu^2} \frac{dF_o}{dx} \tag{1.41b}$$

and the diffusion coefficient is

$$D_o(v) = \lambda_d \overline{\mu^2} . \tag{1.41c}$$

If the neutrons are not monoenergetic, then the conditions for the validity of elementary diffusion theory, i.e., that $j(x)$ be proportional to $\frac{d\Phi_o(x)}{dx}$ are considerably more complicated. Let us suppose that the condition for elementary diffusion theory of monoenergetic neutrons is fulfilled, namely $\overline{\mu^2}(v)$ is independent of x. Then $\overline{\overline{\mu^2}}$ will also be independent of x, and we can write (1.45) as

$$\overline{\overline{\mu^2}} \frac{d\Phi_o(x)}{dx} = -\int_0^\infty N\sigma(1 - \frac{\sigma_{s_o}}{\sigma} \bar{\mu}_o)\ j(x,v')dv'. \tag{1.46}$$

In general, the energy spectrum of the neutrons will change from place to place in the medium. Consequently the average of $N\sigma(1 - \frac{\sigma_{s_o}}{\sigma} \bar{\mu}_o)$ over $j(x,v')$ which enters in (1.46) cannot be evaluated, and so it is impossible to write a proportionality between $j(x)$ and $\frac{d\Phi_o(x)}{dx}$ without first solving the transport equation. This we interpret as a deviation from elementary theory. We therefore state that deviations from elementary diffusion theory are associated in general with non-constancy of the neutron energy spectrum as well as with non-constancy of the $\overline{\mu^2}$ for neutrons of each energy.

There is a rather trivial case in which the conditions for applicability of elementary theory is the same for both polyenergetic and monoenergetic neutrons, namely, when all cross-sections are energy independent. In this case we can integrate the Boltzmann equation (1.28) over v. The resulting equation is identical with (1.28) except that the total flux, $\Phi(x,\mu)$ replaces $F(x,v,\mu)$ everywhere. Hence all statements about $F(x,v,\mu)$ apply in toto for $\Phi(x,\mu)$; in particular, if the cross-sections are energy independent then it is sufficient for $\overline{\overline{\mu^2}}$ to be independent of x in order for elementary theory to be applicable.

The assumption of constant cross-sections does not correspond very well to any physical situation. The nearest approach would be the diffusion of thermal neutrons in a non-capturing, non-crystalline, non-hydrogenous medium -- diffusion in He gas is the only strictly applicable case.

If there is no energy loss on scattering in a uniform medium, the condition for elementary theory can be stated fairly simply. We return to (1.44) where the total current j(x) is expressed as the sum of the individual currents of each speed. Since in a uniform medium $\lambda_d(v)$ is independent of x, we can write (1.44) as

$$j(x) = -\frac{d}{dx}\left[\overline{\lambda_d\overline{\mu^2}}\,\Phi_o(x)\right] \tag{1.47}$$

where

$$\overline{\lambda_d\overline{\mu^2}} = \frac{\int_0^\infty \lambda_d(v)\,\overline{\mu^2}(v)\,F_o(x,v)dv}{\int_0^\infty F_o(x,v)dv} \quad . \tag{1.48}$$

In general $\overline{\lambda_d\overline{\mu^2}}(v)$ may be a function of x, even if $\overline{\mu^2}(v)$ is not. This arises from the fact that the energy spectrum can change from place to place because of energy dependent absorption, even if a neutron cannot change its

energy in a collision. The condition for applicability of elementary theory can evidently be stated as

$$\overline{\lambda_d \overline{\mu^2}} \quad \text{be independent of } x \,. \tag{1.49}$$

If $\overline{\mu^2}(v)$ is independent of x, then $\overline{\lambda_d \overline{\mu^2}}$ will also be independent provided the energy spectrum is the same from point to point.

The case of diffusion without energy loss is really just a superposition of diffusion of neutrons each with a fixed energy. It is therefore a fairly easy problem to solve, even though it is not strictly reducible to elementary theory, and we shall return to it later. Thermal neutron diffusion in a very heavy medium is a good approximation to this situation.

There is one case in which there is energy exchange between medium and neutron and which can be reduced approximately to elementary theory even if the cross-sections are energy dependent. This is the case of thermal neutrons diffusing in a very weak absorber. We shall deal with this problem also in a later paragraph.

Aside from these two cases, all of our discussion of transport theory in this chapter will be confined to monoenergetic neutrons, or the trivial case of constant mean free path for polyenergetic neutrons. Thus all deviations from elementary theory which we encounter will arise from variation in $\overline{\mu^2}$, not from variation in the neutron energy spectrum. The reason we confine attention to variation in $\overline{\mu^2}$ alone is not that the corrections due to the polyenergetic character are unimportant; rather it is because there is no general theory of the effect of energy distribution on the diffusion of neutrons. This fact must always be kept in mind when assessing the value of results obtained from monoenergetic transport theory.

The Angular Distribution in Elementary Diffusion Theory, and the Value of $\overline{\mu^2}$

Thus far we have said nothing about the specific form of the angular distribution of the neutrons diffusing in a medium. We have simply contented ourselves with the observation that Fick's law, with diffusion coefficient which depends only on the energy distribution of the neutrons, is applicable whenever $\overline{\mu^2}$ is independent of x, if the neutrons are monoenergetic, or whenever $\overline{\lambda_d \mu^2}$ is independent of x if the neutrons are polyenergetic.* We shall now inquire more closely into the form of the neutron angular distribution, and shall compute the diffusion coefficient in some simple cases. The argument which follows is stated for the case of monoenergetic neutrons; if the cross-sections are independent of v then it holds with equal weight for polyenergetic neutrons with $\Phi(x,\mu)$ replacing $F(x,v,\mu)$.

It will first be useful to examine how the angular distribution of neutrons in a given volume element arises. The number of neutrons going in the direction μ in a volume element around a given point is made up of all those neutrons which, as a result of their last collision, were re-directed toward the volume element in the direction μ and have reached it without intervening collisions. It must therefore be possible to express the angular distribution at a point in terms of the flux at surrounding points, at least in the case where the scattering is isotropic.

To show this we start with the transport equation for isotropic scattering of monoenergetic neutrons in a uniform medium:

$$\mu \frac{dF}{dx}(x,v,\mu) + N\sigma F(x,v,\mu) = \frac{N\sigma_{s_o}}{2}\int_{-1}^{1} F(x,v,\mu)\,d\mu + S(x,v,\mu) = \frac{N\sigma_{s_o}}{2} F_o(x,v) + S(x,v,\mu). \tag{1.49a}$$

* And the scattering does not change the neutron energy.

Since we wish to express $F(x,v,\mu)$ in terms of $F_o(x,v)$, we shall assume $F_o(x,v)$ as known, and shall therefore view (1.49a) as an inhomogeneous, first order differential equation with constant coefficients. The solution which is finite at $x = \pm\infty$ is

$$F(x,v,\mu) = \frac{N\sigma_{s_o}}{2}\int_{-\infty}^{x} F_o(x',v)\, e^{-\frac{N\sigma}{\mu}(x-x')}\frac{dx'}{\mu} + \int_{-\infty}^{x} S(x',v,\mu)d^{-\frac{N\sigma}{\mu}(x-x')}\frac{dx'}{\mu} \qquad \mu > 0$$

$$= \frac{N\sigma_{s_o}}{2}\int_{\infty}^{x} F_o(x',v)e^{-\frac{N\sigma}{\mu}(x-x')}\frac{dx'}{\mu} + \int_{\infty}^{x} S(x',v,\mu)e^{-\frac{N\sigma}{\mu}(x-x')}\frac{dx'}{\mu}\,. \qquad \mu < 0 \tag{1.50}$$

The pair of equations is the required relation between the angular distribution $F(x,v,\mu)$ and the total flux of velocity v, $F_o(x,v)$. Physically the first integral has the following simple interpretation: $\frac{N\sigma_{s_o}}{2} F_o(x',v)$ is the number of particles per unit volume at x' which are scattered into unit solid angle per second. The probability that these neutrons reach a volume element at x is $e^{-\frac{N\sigma}{\mu}(x-x')}$. The total number reaching x in the direction μ is the sum of all those from the elements of volume lying along the radius vector connecting the elements at x and x': the length of each such element is dx'/μ. Hence the flux $F(x,v,\mu)$, which is the total reaching unit volume per second from all elements along the radius whose direction cosine is μ, is

$$\frac{N\sigma_{s_o}}{2}\int_{-\infty}^{x} F_o(x',v)\, e^{-\frac{N\sigma}{\mu}(x-x')}\frac{dx'}{\mu}\,. \qquad \mu > 0 \tag{1.51}$$

The second integral in (1.50) evidently represents those neutrons which reach the volume element directly from the source.

We now suppose that $F_o(x',v)$ can be expanded in Taylor's series around x:

$$F_o(x',v) = \sum_{\nu=o}^{\infty} \frac{(x'-x)^\nu}{\nu!} \frac{d^\nu F_o(x,v)}{dx^\nu} \tag{1.52}$$

and that the point x is so far from the source that the source integral contribution to (1.50) is negligible.

If we substitute the series (1.52) into (1.50), and integrate term by term we obtain a series in ascending powers of μ. If we now recall that any power of μ is expressible as a linear combination of Legendre polynomials, e.g.,

$$\mu^0 = P_o(\mu)$$

$$\mu^1 = P_1(\mu)$$

$$\mu^2 = \frac{1}{3} P_o(\mu) + \frac{2}{3} P_2(\mu);$$

then, upon collecting coefficients of Legendre polynomials of the same order, we can write the result of this term by term integration as

$$F(x,v,\mu) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} F_\ell(x,v)\, P_\ell(\mu). \tag{1.53}$$

The general problem of transport theory is to determine the functions $F_\ell(x,v)$; this can be done, in principal, by substituting (1.53) into the transport equation [even the non-isotropic equation (1.28)] and equating coefficients of spherical harmonics of the same order. This procedure leads to an infinite set of linear equations, the solutions of which are $F_\ell(x,v)$.

We defer consideration of these equations until Chapter III and consider instead the relation between $\overline{\mu^2}$ and the $F_\ell(x,v)$. Since $\mu^2 = \frac{1}{3} P_o(\mu) + \frac{2}{3} P_2(\mu)$, we find from (1.53)

$$\overline{\mu^2}(x,v) = \frac{\int_{-1}^{1} \mu^2 F(x,v,\mu)\,d\mu}{\int_{-1}^{1} F(x,v,\mu)\,d\mu} = \frac{1}{3}\left[1 + 2\,\frac{F_2(x,v)}{F_o(x,v)} \right]. \tag{1.54}$$

Since the requirement for the applicability of simple diffusion theory is that $\overline{\mu^2}$ be independent of x, from (1.54) we can say that simple diffusion theory in these cases is applicable if the ratio of $F_2(x,v)/F_o(x,v)$ is independent of x.

There are two important cases in which $F_2(x,v)$ and $F_o(x,v)$, that is the strength of the second and zero-th spherical harmonics, are in constant ratio. These are

$$\text{(I)} \qquad F(x,v,\mu) = \frac{1}{2} F_o(x,v) + \frac{3}{2} F_1(x,v)\, P_1(\mu). \tag{1.55}$$

(Asymptotic distribution in non-capturing medium.)

In this case $F_2(x) = 0$ and $\overline{\mu^2} = 1/3$. Under what physical conditions can we expect (1.55) to hold? The answer is that the very simple angular distribution (I) holds, strictly, only in a non-capturing medium far from sources and far from boundaries.

That the medium must be non-capturing in order for (1.55) to hold rigorously can be shown by returning to (1.50), where, with S = 0, we express the angular distribution in terms of the density. Evidently, if the series (1.53) is to break off after the second term, it is necessary that the Taylor expansion (1.52) break off with the second term also:

$$F_o(x',v) = F_o(x,v) + (x'-x)\, F_o'(x,v). \tag{1.56}$$

Substituting this into (1.50) we obtain as the angular distribution corresponding to a linear density distribution,

$$F(x,v,\mu) = \frac{\sigma_{s_o}}{2\sigma} F_o(x,v) - \frac{\sigma_{s_o}}{2\sigma} \frac{F_o'(x,v)}{N\sigma} P_1(\mu). \tag{1.57}$$

Now by definition,

$$\int_{-1}^{1} F(x,v,\mu)d\mu \equiv F_o(x,v) \tag{1.58}$$

and this condition can be fulfilled by (1.57) only if

$$\sigma_{s_o} = \sigma \ ;$$

i.e., if there is no capture in the medium. In other words, if the angular distribution at every point is strictly representable by a sum of only the first two Legendre Polynomials, then there can be no capture in the medium.

Close to a localized source the neutron angular distribution will generally require many spherical harmonics for its representation. Thus if the source is an isotropically emitting plane sheet at x_o, i.e., $S(x',v,\mu) = \delta(x_o-x')$, then the contribution to the angular distribution of those neutrons which have made no collisions is, according to (1.50),

$$\frac{1}{\mu} e^{-\frac{N\sigma}{\mu}|x_o-x|} ,$$

and this evidently requires an infinite number of harmonics for its representation. A few mean free paths away the effect of the source angular distribution will be "forgotten" by the neutrons, the distribution characteristic of the medium will take over, and $\overline{\mu^2}$ will become constant.

Near a boundary between medium and vacuum, or between non-capturing and capturing media, the two harmonic representation which leads to $\overline{\mu^2}$ = constant = $\frac{1}{3}$ will also be invalid. For in the case where the medium is bounded by a vacuum, no neutrons can be reflected back. Hence the angular distribution at this interface (x = 0) must satisfy

$$F(x,v,\mu) \equiv 0 \qquad -1 \leqslant \mu \leqslant 0. \tag{1.59}$$

Since this distribution is very different from that which characterized the interior of the medium, there must be a transition region in which the $P_2(\mu)$ harmonic is involved in the angular distribution. In this region, which is also of the order of a few mean free paths, simple diffusion theory is inapplicable. A vacuum can be considered as the extreme case of a medium which has infinitely large capture. If the capture is finite, the angular distribution at the interface will not be as drastic as (1.59) -- i.e., source neutrons can return to the medium but it will in general still require more than two harmonics for its representation, and $\overline{\mu^2}$ will not be constant near the interface.

It is easy to show that the angular distribution of the general form (1.55), which we have shown corresponds to the linear total flux distribution (1.52), does indeed satisfy the Boltzmann equation if there is no capture. We start with the general equation with no capture and no source,

$$\mu \frac{dF}{dx} + N\sigma_{s_o}F = \frac{1}{2} \sum (2\ell+1)\, P_\ell(\mu)\, N\sigma_{s_\ell} \int_{-1}^{1} P_\ell(\mu) F d\mu \tag{1.60}$$

and we try to determine a solution

$$F(x,v,\mu) = A + Bx + CP_1(\mu). \tag{1.61}$$

where A, B, C, for a given v, are constants which are to be determined. The total flux $F_o(x,v)$ is linear,

$$F_o(x,v) = 2A + 2Bx \tag{1.61a}$$

as can be verified by integrating F over μ. Substituting (1.61) into (1.60) we observe that all terms in the infinite sum except the first two fall out; what is left is

$$B\mu + (C\mu + Bx + A)N\sigma_{s_o} = N\sigma_{s_o}(Bx + A) + C\mu N\sigma_{s_1} \cdot$$

This is an identity in μ which is satisfied if

$$C = \frac{-B}{N(\sigma_{s_o} - \sigma_{s_1})},$$

or, by the definition of σ_{s_1} and $\lambda_{tr}(v)$,

$$C = \frac{-B}{N\sigma_{s_o}(1 - \bar{\mu}_o)} = -\lambda_{tr}B. \tag{1.62}$$

Now B is related to the space derivative of the total flux F'_o; for, from (1.61),

$$F'_o(x,v) = \int_{-1}^{1} F'(x,v,\mu)d\mu = 2B,$$

so that (1.61) can, according to (1.62) be written either as

$$F(x,v,\mu) = A + B\left[x - \lambda_{tr} P_1(\mu)\right] \tag{1.63}$$

or as

$$F(x,v,\mu) = \frac{1}{2} F_o(x,v) - \frac{1}{2} \lambda_{tr} F'_o(x,v) P_1(\mu), \tag{1.64}$$

where $F_o(x) = 2A+2Bx$, A and B being arbitrary. Since, by comparing (1.61) with (1.53) we can identify C with $\frac{3}{2} F_1(x,v)$, we see that (1.63) -- which we have shown is a solution of the Boltzmann equation -- is just the same as the originally assumed angular distribution, $\frac{1}{2} F_o(x,v) + \frac{3}{2} F_1(x,v) P_1(\mu)$, with the understanding that $F_o(x,v)$ is linear in x, $F_1(x,v)$ is constant in x.

The value of the diffusion coefficient corresponding to (1.63) is found, according to the formula for D_o (Eq. 1.41c), by evaluating $\overline{\mu^2}$ for the angular distribution (1.63). The result is

$$D_o = \frac{1}{3}\left[N\sigma_{s_o}(1 - \bar{\mu}_o)\right]^{-1} = \frac{1}{3}\lambda_t(v). \tag{1.65}$$

Formula (1.65) is the one ordinarily derived in kinetic theory. As we see, it is rigorously correct only in a non-capturing medium far from sources or boundaries.

The angular distribution (1.63) is the one which is usually understood as the angular distribution corresponding to elementary diffusion theory. It applies strictly to media which do not capture neutrons; it is customary to assume this distribution in weakly capturing media, but this is correct only to the extent that the capture is small. Actually there is another angular distribution which is somewhat more general than (1.63) and which applies rigorously to media with finite capture which also leads to elementary diffusion theory, and this is described in the following paragraph.

II. <u>If $F(x,v,\mu)$ is a product of function of x and function of μ</u> (asymptotic distribution in capturing medium).

In this case the coefficients of all the harmonics bear a constant ratio to $F_o(x,v)$ for all x; in particular F_2/F_o = constant, and so $\overline{\mu^2}$ will be independent of position. However, the formula for D will not be (1.65).

To calculate $\overline{\mu^2}$ in this case we start with the no-source slightly non-isotropic transport equation with capture in which the scattering cross-section is representable by just two harmonics. Then we have

$$\mu \frac{dF}{dx}(x,v,\mu) + N\sigma F(x,v,\mu) = \frac{N\sigma_{s_o}}{2}\int_{-1}^{1} F(x,v,\mu')d\mu' + \frac{3}{2} N\sigma_{s_o}\overline{\mu}_o\mu\int_{-1}^{1} \mu' F(x,v,\mu')d\mu' . \tag{1.29}$$

We try a separable solution

$$F(x,v,\mu) = e^{\pm \varkappa x} f(\mu), \tag{1.66}$$

and seek to determine $\mathcal{K}$ and $f(\mu)$. Substituting into (1.29) we obtain

$$(N\sigma \pm \mathcal{K}\mu)f(\mu) = \frac{N\sigma_{s_o}}{2}\int_{-1}^{1} f(\mu')d\mu' + \frac{3N\sigma_{s_o}}{2}\mu\bar{\mu}_o\int_{-1}^{1}\mu' f(\mu')d\mu' . \tag{1.67}$$

Our separable solution (1.66) will indeed satisfy (1.29) if we can find an $f(\mu)$ which satisfies the integral equation (1.67). We therefore try a solution of (1.67) of the form

$$f(\mu) = \text{constant} \times \frac{A + \bar{\mu}_o\mu}{N\sigma \pm \mathcal{K}\mu} ; \tag{1.68}$$

in general, this assumed solution will not satisfy (1.67). Only for particular values of A and $\mathcal{K}$ will $f(\mu)$ (1.68) solve equation (1.67), and to find these characteristic values will be our task. We substitute (1.68) into (1.67), and, making use of the integrals

$$\int_{-1}^{1} \frac{d\mu}{N\sigma \pm \mathcal{K}\mu} = \frac{1}{\mathcal{K}} \ln \frac{N\sigma + \mathcal{K}}{N\sigma - \mathcal{K}}$$

$$\int_{-1}^{1} \frac{\mu d\mu}{N\sigma \pm \mathcal{K}\mu} = \pm\left(\frac{2}{\mathcal{K}} - \frac{N\sigma}{\mathcal{K}^2} \ln \frac{N\sigma + \mathcal{K}}{N\sigma - \mathcal{K}}\right)$$

$$\int_{-1}^{1} \frac{\mu^2 d\mu}{N\sigma \pm \mathcal{K}\mu} = -\frac{2N\sigma}{\mathcal{K}^2} + \frac{N^2\sigma^2}{\mathcal{K}^3} \ln \frac{N\sigma + \mathcal{K}}{N\sigma - \mathcal{K}}$$

we obtain

$$A + \bar{\mu}_o\mu = \frac{N\sigma_{s_o}A}{2\mathcal{K}} \ln \frac{N\sigma + \mathcal{K}}{N\sigma - \mathcal{K}} \pm \frac{N\sigma_{s_o}\bar{\mu}_o}{2}\left(\frac{2}{\mathcal{K}} - \frac{N\sigma}{\mathcal{K}^2}\ln \frac{N\sigma + \mathcal{K}}{N\sigma - \mathcal{K}}\right) +$$

$$\pm \frac{3N\sigma_{s_o}\mu\mu_o A}{2}\left(\frac{2A}{\mathcal{K}} - \frac{N\sigma}{\mathcal{K}^2}\ln \frac{N\sigma + \mathcal{K}}{N\sigma - \mathcal{K}}\right) + \frac{3N\sigma_{s_o}}{2}\mu\bar{\mu}_o^2\left(-\frac{2N\sigma}{\mathcal{K}^2} + \frac{N^2\sigma^2}{\mathcal{K}^3}\ln \frac{N\sigma + \mathcal{K}}{N\sigma - \mathcal{K}}\right) .$$

This is an identity in μ; hence we can equate coefficients of powers of μ, and we obtain

$$A = \pm \frac{\frac{N\sigma_{s_o}\bar{\mu}_o}{2}\left(\frac{2}{\mathcal{H}} - \frac{N\sigma}{\mathcal{H}^2}\ln\frac{N\sigma + \mathcal{H}}{N\sigma - \mathcal{H}}\right)}{1 - \frac{N\sigma_{s_o}}{2\mathcal{H}}\ln\frac{N\sigma + \mathcal{H}}{N\sigma - \mathcal{H}}} \tag{1.69}$$

and

$$\frac{N\sigma_{s_o}}{2\mathcal{H}}\ln\frac{N\sigma + \mathcal{H}}{N\sigma - \mathcal{H}} = \frac{1 + \frac{3N^2\sigma_{s_o}\sigma_a}{\mathcal{H}^2}\bar{\mu}_o}{1 + \frac{3N^2\sigma_{s_o}\sigma_a}{\mathcal{H}^2}\bar{\mu}_o} \tag{1.70}$$

When $\bar{\mu}_o = 0$ (isotropic scattering), $\mathcal{H}$ is determined as the root of

$$\frac{N\sigma_{s_o}}{2\mathcal{H}}\ln\frac{N\sigma + \mathcal{H}}{N\sigma - \mathcal{H}} = 1, \tag{1.71a}$$

or

$$\frac{N\sigma_{s_o}}{\mathcal{H}}\tanh^{-1}\frac{\mathcal{H}}{N\sigma} = 1. \tag{1.71b}$$

Equation (1.71a) has one pair of real roots, $\pm\mathcal{H}$, which are less than $N\sigma$ in absolute value, and an infinite set of imaginary ones. We will not be concerned here with the imaginary roots. We note in passing that in order for our derivation of the form of $f(\mu)$ to be valid it was necessary for $(\mathcal{H})$ to be less than $N\sigma$; otherwise $f(\mu)$ would have singularities for those values of μ at which the denominator of (1.68) vanishes.

The quantity A has a finite value, in the isotropic case, which we need not compute. The isotropic scattering angular distribution $F(x,v,\mu)$ is therefore

$$F(x,v,\mu) = \text{constant}\ \frac{e^{\pm\mathcal{H}x}}{N\sigma \pm \mathcal{H}\mu} \tag{1.72}$$

provided $\mathcal{H}$ is a root of (1.71).

To calculate the diffusion coefficient, D_o, we must first compute $\overline{\mu^2}$. We find, for the value of $\overline{\mu^2}$ in the isotropic case (1.72),

$$\overline{\mu^2} = \frac{N^2\sigma\sigma_a}{\mathcal{H}^2} \tag{1.73}$$

and

$$D_o = \frac{\overline{\mu^2}}{N\sigma} = \frac{N\sigma_a}{\mathcal{H}^2} . \tag{1.74}$$

We shall see in the next section that (1.74) reduces to the usual $D_o = \frac{1}{3}\lambda t$ if the absorption is very weak.

When the absorption is very small compared to the scattering, then as will be shown later, $\mathcal{H}$ is small compared to $N\sigma$. In this case it is permissible to expand the denominator of (1.72) and keep only the first two terms:

$$F(x,v,\mu) \approx \text{constant } x\, e^{\pm \mathcal{H} x}(1 \mp \frac{\mathcal{H}}{N\sigma}\mu).$$

The angular distribution contains only the first two Legendre functions and is therefore the angular distribution of elementary diffusion theory. Since

$$F_o(x,v) \sim e^{\pm \mathcal{H} x}, \quad \lambda_{tr} F_o'(x,v) \sim \frac{\pm\mathcal{H}}{N\sigma} e^{\pm \mathcal{H} x} ,$$

this form is the same as (1.64). It is clear then that in a weak absorber, the angular distribution of elementary diffusion theory is a good approximation to the asymptotic distribution (1.72).

The angular distributions computed in this and in the previous paragraph are very special ones. They involve only a single multiplicative constant and are not sufficiently general to enable one to satisfy arbitrary boundary conditions, such as would be imposed on the neutron angular distribution near a source or a boundary. Thus, as we have already stressed, the two harmonic solutions in the case of the no capturing medium, and the separable solution in the capturing

medium are solutions which apply only at great distances from sources, or boundaries between two media at least one of which has absorption. In other words, these solutions are "asymptotic" solutions of the Boltzmann equation, the first one being the asymptotic solution in a non-capturing medium, the second the solution in a capturing medium. We can therefore summarize the situation by the statement: simple diffusion theory, based on Fick's Law, holds strictly only for the asymptotic solutions of the Boltzmann equation. Only insofar as the neutron distribution approaches the asymptotic distribution, as it does far from sources and boundaries, can simple diffusion theory be applied.

The Diffusion Length

In both the isotropic and the slightly anisotropic scattering cases, the asymptotic flux distribution in a capturing medium is

$$F_o(x,v) = \text{constant } e^{\pm \mathcal{H} x} . \tag{1.75}$$

The quantity $1/\mathcal{H}$ has the dimensions of length; it is called the diffusion length of the medium and is denoted by L.

The asymptotic flux, since it is exponential, satisfies the following differential equation:

$$\frac{d^2}{dx^2} F_o(x,v) - \mathcal{H}^2 F_o(x,v) = 0. \tag{1.76}$$

This equation can be written, by virtue of the relation between D_o, $N\sigma_a$, and $\mathcal{H}^2$, as

$$D_o \frac{d^2}{dx^2} F_o(x,v) - N\sigma_a F_o(x,v) = 0. \tag{1.77}$$

In this form it is easily recognizable as the elementary diffusion equation without sources. It could have been derived from a consideration of the material balance in an element of volume $\Delta x \Delta y \Delta z$: the net flow per second into the volume minus the net flow per second out of the volume is just $D_o \frac{d^2}{dx^2} F_o \Delta x \Delta y \Delta z$, which, in the steady state, is equal to $N\sigma_a F_o \Delta x \Delta y \Delta z$, the number captured per second.

If the capture cross-section is small compared to the scattering cross-section, the transcendental equation, (1.70) which defines the diffusion length, can be solved by series expansion. The result is, correct to terms of the first degree in σ_a/σ,

$$\varkappa^2 = \frac{1}{L^2} = 3N\sigma(1 - \bar{\mu}_o)\, N\sigma_a\left(1 - \frac{4}{5}\frac{\sigma_a}{\sigma} + \frac{\sigma_a\, \bar{\mu}_o}{\sigma(1 - \bar{\mu}_o)} \cdots\right) \tag{1.78}$$

For isotropic scattering ($\bar{\mu}_o = 0$) this becomes

$$\varkappa^2 = \frac{1}{L^2} = 3N\sigma N\sigma_a\left(1 - \frac{4}{5}\frac{\sigma_a}{\sigma}\right), \tag{1.79}$$

while, if the capture is so weak that σ_a/σ is negligible,

$$\varkappa^2 = \frac{1}{L^2} = 3N\sigma(1 - \bar{\mu}_o)N\sigma_a\ . \tag{1.80}$$

The relation (1.80) between diffusion length and cross-sections is an immediate consequence of the elementary diffusion equation and follows from the assumption $D_o = \dfrac{1}{3N\sigma(1 - \bar{\mu}_o)}$. However, the derivation given here, besides being more fundamental has the advantage of yielding the exact formula for $\varkappa^2$.

The diffusion length of a medium is important practically because it is a measure of the distance that neutrons travel before being captured. From a

plane source the asymptotic flux is of the form $e^{-\varkappa x}$; in spherical coordinates the asymptotic distribution from a point source is $\frac{e^{-\varkappa r}}{r}$. The mean square distance, $\overline{r^2}$, that a slow neutron travels from a point before it is captured is therefore

$$\overline{r^2} = \frac{\int_0^\infty r^3 e^{-\varkappa r} dr}{\int_0^\infty r e^{-\varkappa r} dr} = \frac{6}{\varkappa^2} = 6L^2; \tag{1.81}$$

that is, the square of the diffusion length is 1/6 times the mean square distance that a slow neutron travels from birth until death, assuming the asymptotic neutron distribution holds up to the point source. Actually the asymptotic distribution does not hold very close to the source, but the error caused by this is small if the capture is not very great compared to the scattering.

The diffusion length in a medium will be large if the transport mean free path is long, or if the capture cross-section is very small. Since the diffusion length is directly proportional to the capture mean free path, it is a convenient measure of the relative neutron absorption of two media which are weak neutron absorbers, as for example, two different samples of graphite.

end M-3336 pt 1
begin M-3336 pt 2

Diffusion of Thermal Neutrons

The formulas for the diffusion length and for the diffusion coefficient which we obtained in the preceding section were derived for

a) monoenergetic neutrons

b) polyenergetic neutrons scattered in a medium in which all cross-sections are independent of speed.

Neither of these assumptions is very realistic for the diffusion problem of most relevance to slow neutron chain reactions -- viz., the diffusion of thermal neutrons. Thermal neutrons are not monoenergetic -- for example, their energy distribution is Maxwellian in a non-capturing medium. Furthermore, most slow neutron capture cross-sections are 1/v, and in crystalline media, the scattering cross-section is strongly affected by interference phenomena which depend on the wave length, i.e., on the velocity, of the neutrons. For both of these reasons it is not justified to regard thermal neutron cross-sections as velocity independent.

An accurate calculation of the energy distribution of thermal neutrons has been performed by Wigner and Wilkins for the case of neutrons slowing down in atomic hydrogen gas whose atoms are in motion with a Maxwellian distribution of velocities. In this calculation the details of the energy exchange between neutron and proton at each collision were taken into account rigorously. The details of the transition from the energy distribution of neutrons which suffer elastic collisions at high energy, where temperature motions are unimportant, to the thermal neutron distribution, which is predominantly Maxwellian, were traced.

We shall concern ourselves here with two very much simpler, and over-idealized problems, which nevertheless approximate certain important cases of thermal neutron diffusion. The two cases are

I. The neutron energy is unaffected by scattering collisions with the medium. This corresponds to diffusion in a medium of infinite atomic weight; it is a fairly good approximation for thermal neutrons diffusing in U.

II. The neutron suffers complete "amnesia" after each scattering collision; that is, its energy after collision is uncorrelated with its energy

before a collision, and it is thrown into a Maxwellian energy distribution, regardless of its initial energy. This is more nearly correct for collisions with light nuclei than with heavy ones, but is of course not strictly correct for either.

Case I. In this case the neutrons of each speed, can be treated independently of all others. The flux, $F_o(x,v)$ of neutrons of a particular speed will fall off exponentially far from a source, and the diffusion length for each speed will be different:

$$F_o(x,v) = \text{constant } e^{-\mathcal{H}(v)x} , \tag{1.82}$$

where $\mathcal{H}(v)$ is the root of the usual characteristic equation (1.70) with all cross-sections evaluated at speed v. If the neutrons start at x = 0 with a Maxwellian energy distribution,

$$F_o(o,v) = M(v) = v^2 \exp - \frac{v^2}{\alpha^2} , \tag{1.83}$$

where α is the most probable speed of the neutron; then in a 1/v absorbing medium the Maxwellian distribution will not be maintained as the neutrons move out from the source. The low energy neutrons, being absorbed more strongly, have a shorter diffusion length than do the more energetic ones. As a result the average energy of the neutrons increases with the distance from the source; at very large distances only the most energetic, which have a very small cross-section, survive. This phenomenon is called "hardening". It occurs in any medium in which the absorption cross-section increases as neutron energy decreases.

The total flux $\Phi_o(x)$ can be computed by summing $F_o(x,v)$ over all v. For neutrons which were Maxwellian at x = 0, the total flux at x is

$$\Phi_o(x) = \int_0^\infty F_o(x,v)dv = \int_0^\infty v^2 \exp - \left[\frac{v^2}{\alpha^2} + \mathcal{H}(v)x \right] dv. \tag{1.84}$$

A comparison of $\Phi_o(x)$ and $F_o(x,\bar{v})$, the flux of neutrons of average speed $\bar{v}$, is given for neutrons diffusing in uranium metal in the accompanying Figure 2.

Case II. In a medium which does not absorb thermal neutrons at all, temperature equilibrium will ultimately be established between the neutrons and the medium. When this state of affairs has been achieved, the total number of neutrons thrown into a given speed range in one second is the same as the number removed. Now if the neutrons are in a Maxwellian distribution, $M(v)$, then the number per c.c. removed from unit speed range in one second is

$$N\sigma_{s_o}(v)\ v\ M(v);$$

since the total number of collisions in one second is

$$\int_0^\infty N\sigma_{s_o}(v)\ v\ M(v)dv$$

the probability $P(v)$ that on the average a collision will result in neutrons of unit speed range around v is

$$P(v) = \frac{N\sigma_{s_o}(v)\ v\ M(v)}{\int_0^\infty N\sigma_{s_o}(v)\ v\ M(v)dv}\ . \tag{1.85}$$

In any individual collision the initial and final neutron speeds are, of course, correlated; the heavier the scattering material, the greater the correlation. Thus $P(v)$ is not the probability that each collision will give rise to neutrons of unit speed range around v; rather it is the probability, averaged over a large number of collisions (in each of which there is correlation before and after), that a neutron will come off a collision in a particular speed range. The assumption of complete "amnesia" is that the probability of a given neutron speed following each collision is independent of the initial speed and is the

same as P(v), the probability averaged over a large number of collisions.

The form for P(v) assumed here will not be correct, even on the average, if there is velocity dependent absorption in the medium. However, if the absorption is very small, so that many scattering collisions occur for each absorption, we can expect results based on this P(v) to be reasonably good. The neutron distribution which we calculate will therefore be approximately right for diffusion of thermal neutrons in a weakly absorbing, low atomic weight material.

The Boltzmann equation for thermal neutrons which suffer complete amnesia at each collision is, in the case of isotropic scattering,

$$\mu \frac{dF}{dx}(x,v,\mu) + N\sigma F(x,v,\mu) = \frac{P(v)}{2}\int_0^\infty\int_{-1}^{1} N\sigma_{s_0} F(x,v',\mu')d\mu' dv', \tag{1.86}$$

where P(v) is given by (1.85). We wish to find an asymptotic solution of (1.86). We therefore try

$$F(x,v,\mu) = \text{constant } e^{\pm \varkappa x} \frac{P(v)}{N\sigma \pm \varkappa\mu} \tag{1.87}$$

and substitute into (1.86). After cancelling common factors on both sides of the resulting equation, we find that in order for (1.87) to be a solution of (1.86) it is both necessary and sufficient that $\varkappa$ satisfy

$$1 = \frac{1}{2}\int_0^\infty \left[\frac{N\sigma_{s_0}}{\varkappa} \ln \frac{N\sigma + \varkappa}{N\sigma - \varkappa}\right] P(v)dv. \tag{1.88}$$

This is the generalized characteristic equation which determines the diffusion length, $1/\varkappa$, for thermal neutrons in a weakly absorbing medium, if the neutrons

suffer complete "amnesia" at each collision. Since the whole argument is valid only if $\sigma_a/\sigma_{s_o} \ll 1$, we may expand (1.88), keeping only the first two terms, and we may identify σ_{s_o} and σ. After a little algebra we obtain

$$L^2 = \frac{1}{\mathcal{K}^2} = \frac{\bar{\lambda}}{3N\bar{\sigma}_a} \tag{1.89}$$

where

$$\bar{\sigma}_a = \frac{\int_0^\infty \sigma_a \, v \, M(v) \, dv}{\int_0^\infty v \, M(v) \, dv} \tag{1.90}$$

and

$$\bar{\lambda} = \frac{\bar{1}}{N\sigma} = \frac{\int_0^\infty \frac{v}{N\sigma} M(v) \, dv}{\int_0^\infty v \, M(v) \, dv} . \tag{1.91}$$

The total flux of speed v, $F_o(x,v)$, is, according to (1.87)

$$F_o(x,v) = \int_{-1}^{1} F(x,v,\mu) d\mu = e^{\pm \mathcal{K} x} N\sigma_{s_o} v \frac{M(v)}{\mathcal{K}} \ln \frac{N\sigma + \mathcal{K}}{N\sigma - \mathcal{K}} \tag{1.92}$$

which for small σ_a/σ is

$$nv(x,v) = F_o(x,v) \approx e^{\pm \mathcal{K} x} \frac{N\sigma_{s_o} \, v \, M(v)}{N\sigma} . \tag{1.93}$$

Thus the neutron density, n(x,v), is not quite Maxwellian; in a 1/v absorber with constant scattering cross-section, for example, there is a deficiency of neutrons at low energies where σ_a, and hence σ, is large.

The total flux of all neutrons, $\Phi_o(x)$, is

$$\Phi_o(x) = \int_0^\infty F_o(x,v)dv = \frac{e^{\pm\mathcal{H}x}}{\mathcal{H}} \int_0^\infty N\sigma_{s_o} v M(v) \ln \frac{N\sigma + \mathcal{H}}{N\sigma - \mathcal{H}} \, dv; \tag{1.94}$$

it is exponential in space and therefore obeys the differential equation

$$\frac{d^2\Phi_o(x)}{dx} - \mathcal{H}^2 \Phi_o(x) = 0$$

which is the elementary diffusion equation in a capturing medium. Hence elementary diffusion theory holds, to the extent that the complete "amnesia" assumption is good, for the asymptotic distribution of thermal neutrons in a weakly absorbing medium.

What we have shown in the preceding paragraphs is that thermal neutrons in a weakly absorbing medium behave, approximately, like monoenergetic neutrons whose absorption cross-section is given by (1.90) and whose mean free path is (1.91). It is important to note that it is not the cross-section, but rather the mean free path which is averaged in (1.91).

If the absorption cross-section is 1/v, as is usually the case for thermal neutrons, then $\sigma_a v = s$, a constant. Hence

$$\bar{\sigma}_a = \frac{s \int_0^\infty M(v)dv}{\int_0^\infty v\, M(v)\, dv} = \frac{s}{\bar{v}}; \tag{1.95}$$

that is, the appropriate average absorption cross-section for a 1/v absorber is exactly the absorption cross-section at the average speed $\bar{v}$ of the Maxwellian distribution M(v).

If the mean free path $\bar{\lambda}$ is known, then a measurement of the diffusion length yields a value for $\bar{\sigma}_a$. This is a practical way of measuring the absorption cross-section of weak absorbers for thermal neutrons, although it is complicated by the fact that $\bar{\lambda}$ may be hard to evaluate on account of crystal effects. The absorption cross-section which is obtained from these measurements refers according to (1.95) to neutrons at the average speed of the Maxwellian distribution. Since the average speed $\bar{v}$ is $\frac{2}{\sqrt{\pi}} = 1.128$ times higher than the most probable speed, α, the absorption cross-section in a 1/v absorber found from a diffusion length measurement is $\frac{1}{1.128}$ times the absorption cross-section at the most probable velocity. It is customary to refer cross-sections to neutrons of energy kT; this is exactly the energy of a neutron with the most probable speed α, since $\alpha^2 = \frac{2kT}{M}$, where M is the neutron mass. Thus the kT absorption cross-section of a 1/v absorber is 1.128 times larger than the cross-section found directly from a diffusion length measurement.

Boundary Conditions between Two Media

Suppose neutrons diffuse in a composite system consisting of one medium in the left half-space ($x < 0$) and a different medium, with different scattering and absorption properties, in the right half-space ($x > 0$). We denote quantities referring to the left hand medium by a superscript $^{(-)}$, and those referring to the right hand medium by a superscript $^{(+)}$.

At the interface ($x = 0$) between the two media, the flux of speed v, $F(x,v,\mu)$, must be continuous for all μ and v. For if $F(x,v,\mu)$ were discontinuous at $x = 0$ -- for example, if $F^{(-)}(o,v,\frac{1}{2}) \neq F^{(+)}(o,v,\frac{1}{2})$ -- then the number of neutrons of speed v reaching the interface from the left in the direction $\mu = \frac{1}{2}$ would not be the same as the number leaving the interface toward the right.

This would be possible only if there were a source or sink of neutrons at the interface which made up the difference between the two neutron currents.

Now $F(x,v,\mu)$ is expressible as a sum of spherical harmonics in both media:

$$F(x,v,\mu) = \sum_{\ell} \frac{2\ell+1}{2} F_{\ell}(x,v)\, P_{\ell}(\mu) \tag{1.96}$$

and the requirement that $F^{(+)} = F^{(-)}$ at the boundary can be stated

$$F_{\ell}^{(-)}(o,v) = F_{\ell}^{(+)}(o,v) \text{ for all } v, \ell. \tag{1.97}$$

Equation (1.97) is the correct boundary condition at the interface.

If neither of the media captures, and the neutrons are monoenergetic, then the asymptotic solution (far from sources)

$$F(x,v,\mu) = \tfrac{1}{2} F_o(x,v) - \tfrac{1}{2} \lambda_{tr} F_o'(x,v)\mu = \tfrac{1}{2} F_o(x,v) + \tfrac{3}{2} j(x,v)\mu \tag{1.98}$$

where $F_o(x,v) = 2A + 2Bx$, holds even at the interface. For (1.97) is automatically satisfied if we choose the arbitrary constants $A^{(+)}$, $B^{(+)}$, $A^{(-)}$, and $B^{(-)}$ so that

$$F_o^{(-)}(o,v) = F_o^{(+)}(o,v) \tag{1.99}$$

$$\lambda_{tr}^{(-)} \frac{dF_o^{(-)}}{dx}(o,v) = \lambda_{tr}^{(+)} \frac{dF_o^{(+)}}{dx}(o,v). \tag{1.100}$$

The first boundary condition (1.99) states that the total flux of neutrons is continuous; the second condition states that the net diffusion current is also continuous. Since F_o in elementary diffusion theory is the solution of a second order differential equation, it is possible to specify both F_o and F_o' at an interface. This pair of elementary theory boundary conditions leads to a rigorously correct solution when neither medium captures.

If one of the media captures, then continuity of flux and current is not a rigorously correct set of boundary conditions even if the neutrons are monoenergetic; for the asymptotic distribution in a capturing medium involves all the spherical harmonics, whereas continuity of flux and current simply ensures that the coefficients of the first two spherical harmonics are continuous. In order to satisfy the rigorous boundary condition (1.97), a large number of non-asymptotic solutions, which die away within a mean free path or so of the interface, must be added. This is rather complicated; what is usually done is simply to use the elementary theory boundary conditions, F_o and j continuous. By so doing we ensure that the number of neutrons striking the interface is continuous up to terms in $P_1(\mu)$, and that there is no discontinuity in net flow of neutrons. There may be discontinuities in flow in particular directions, but by requiring continuity of net flow we balance discontinuities in different angles against each other so that the average flow is continuous, and neutrons do not build up indefinitely at the interface.

The diffusion coefficient in a medium with capture is $D_o = \frac{N\sigma_a}{\varkappa^2}$. Hence the boundary condition (1.100) becomes, for capturing media,

$$\left[\frac{N\sigma_a}{(\varkappa^2)}\right]^{(-)} \frac{dF_o^{(-)}}{dx}(x,v) = \left[\frac{N\sigma_a}{(\varkappa^2)}\right]^{(+)} \frac{dF_o^{(+)}}{dx}(x,v). \tag{1.101}$$

We shall take this as the second boundary condition of elementary diffusion theory.

Since the asymptotic distribution breaks down near the interface, and the diffusion coefficient $N\sigma_a/\varkappa^2$ is based on the asymptotic distribution, there is some question whether the formula (1.101) is a consistent approximation. An alternative approximation would be to assume in the capturing medium the two

harmonic solution (1.98); although this does not satisfy the Boltzmann equation with capture, it is a good approximation if the capture is weak. The diffusion coefficient would then be given by $1/3\,\lambda_{tr}$ and (1.101) would be replaced by

$$(\lambda_{tr}F_o')^- = (\lambda_{tr}F_o')^+ \quad . \tag{1.102}$$

In general it is impossible to say which approximation is better without solving the transport problem exactly first. Both (1.101) and (1.102) have been used by different authors. We shall return to this question again in Chapter III.

If the neutrons are not monoenergetic and the media are at different temperatures, then the elementary boundary conditions again become not strictly applicable, in general. If all cross-sections are energy independent, this case reduces to the monoenergetic one with Φ replacing F everywhere. But if the cross-sections are energy dependent, then the neutron energy spectrum will change near the interface; in this region elementary theory is inapplicable. The rigorous (1.97) is still correct of course, but (1.99) and (1.100) or (1.101) with Φ_o replacing F_o will only be approximations.

For two adjoining weak absorbers in which thermal neutrons are diffusing, it seems reasonable to assume the validity of elementary diffusion theory with the average cross-sections found in the preceding section (Eq. 1.84, 1.90, 1.91). We therefore take as our boundary conditions

$$\Phi_o^{(-)}(x) = \Phi_o^{(+)}(x) \tag{1.103}$$

$$\left(\frac{\overline{N\sigma_a}}{\varkappa^2}\right)^{(-)} \frac{d\Phi_o^{(-)}(x)}{dx} = \left(\frac{\overline{N\sigma_a}}{\varkappa^2}\right)^{(+)} \frac{d\Phi_o^{(+)}(x)}{dx} \tag{1.104}$$

where the $(\overline{N\sigma_a}/\mathcal{K}^2)$ denote the diffusion coefficients found in (1.89).

It is important to notice that at the interface between two media it is the flux Φ_o (that is, $\int nvdv$) which is continuous -- not the density, n. The distinction between continuity of flux and continuity of density becomes very important when the two media are at different temperatures. The density will be discontinuous at the interface, being lower on the side with the higher termperature. The situation is quite analogous to the phenomenon of thermal transpiration in a two-compartment vessel containing gas at low pressure; a pressure difference can be maintained across an open hole in the membrane subdividing the vessel if the temperature of the gases in the two compartments are different.

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Boundary Conditions at Interface between Medium and Vacuum; the Extrapolation Distance

If the right hand medium is a vacuum, or is a perfect absorber, the boundary conditions of the previous paragraph become meaningless. In a vacuum there are no collisions at all, and so it is impossible to speak of a net diffusion current. In order to find an appropriate boundary condition we return to the fact, already pointed out, that since no neutrons can return to the medium from a perfect absorber, the rigorous boundary condition must be

$$F(o,v,\mu) = 0 \qquad \mu \leqslant 0. \qquad (1.105)$$

Since we are confining ourselves to elementary diffusion theory, it is evident that we cannot hope to satisfy (1.105) rigorously. Instead we shall be content with requiring that there is no net current of neutrons back from the vacuum. This is much less stringent than (1.105), since we allow neutrons to be reflected from the vacuum in certain directions if we compensate by increasing the flow into the vacuum in other directions. Now the net current

reflected from the vacuum is

$$j_-(o,v) = \int_0^{-1} \mu F(o,v,\mu)d\mu \tag{1.106}$$

and we require $j_-(o,v) = 0$ instead of $F(o,v,\mu) \equiv 0$.

In a non-capturing medium the asymptotic, i.e.. elementary diffusion theory, distribution is

$$F(x,v,\mu) = A + B(x - \lambda_{tr}\mu) = \tfrac{1}{2} F_o(x,v) - \tfrac{1}{2} \lambda_{tr}F'_o(x,v)\mu \tag{1.107}$$

and

$$j_-(x,v) = \tfrac{1}{4} F_o(x,v) + \tfrac{1}{6} \lambda_{tr}F'_o(x,v) \quad . \tag{1.08}$$

Equation (1.108) is the familiar formula for the number of neutrons striking a unit area per second from the right. If $j_-(o,v)$ vanishes then

$$\frac{F_o(o,v)}{F'_o(o,v)} = - \tfrac{2}{3} \lambda_{tr}; \tag{1.109}$$

that is, at a plane interface between a non-capturing medium and a vacuum the logarithmic derivative of the total neutron flux must have the value $- \frac{3}{2} \lambda_{tr}^{-1}$. Another way of saying this is that the asymptotic distribution of the neutron flux extrapolates to zero at a distance $\frac{2}{3} \lambda_{tr}$ beyond the edge of the medium.

An accurate calculation in which all the non-asymptotic solutions are used to satisfy (1.105) rigorously gives the value $0.7104\ \lambda_{tr}$ for the extrapolation distance in a non-capturing medium. In the rigorous theory it is the asymptotic flux distribution which extrapolates to zero at the extrapolation distance $.7104\ \lambda_{tr}$; the actual flux falls below the asymptotic distribution (which is linear) close to the boundary. At the boundary its ratio to the asymptotic flux is $\frac{1}{(.7104)(\sqrt{3})} = 0.81$. It is not surprising that the flux falls off

faster near the boundary than in the interior, since close to the boundary neutrons arrive at a point almost entirely from one side while in the interior they come from both right and left sides in comparable amounts.

The extrapolation distance $0.71\,\lambda_{tr}$ is correct for the case of a non-capturing medium at a plane boundary. If the medium captures, then the formula for the extrapolation distance is more complicated; however, if the capture is not very strong and if the scattering is isotropic the only change in the formula is that the mean free path for scattering, rather than the total mean free path, is substituted for λ_{tr}.

If the boundary is curved, then the extrapolation distance is longer than $0.71\,\lambda_{tr}$. In the limit of vanishingly small radius of curvature of the boundary -- this would correspond to an extremely small black sphere imbedded in a scattering material -- the extrapolation distance is $\frac{4}{3}\lambda$. The extrapolation distance for intermediate radii of curvature has been calculated by B. Davison.* We shall refer to his results later in connection with the theory of control rods.

The Albedo

If the angular distribution in a medium is known, then it is a simple matter to compute the albedo, or reflecting power, i.e., the ratio of the number of neutrons reflected from a medium to the number of neutrons incident on the medium. Evidently the reflecting power of a medium depends upon its size; a thin sheet of scatterer will reflect fewer neutrons than a thick one. It also depends on the angular distribution of the incident neutrons. Thus more neutrons will be reflected if they enter the medium at a sharp angle (and therefore do not get very far from the surface) than if they enter with a uniform angular distribution. We shall calculate here the fraction of neutrons which are reflected from the plane surface of an infinite medium assuming that the angular

* MT - 93 - B. Davison - "Influence of a Large Black Sphere upon the Neutron Density in an Infinite Non-Capturing Medium"

distribution of the incident neutrons is the angular distribution of elementary diffusion theory.

We consider an interface between two media and, as before, let - denote the left hand medium, + the right hand medium. The albedo, β, of the right hand medium, is by definition the ratio of the current reflected from the right hand medium, j_-, to the current incident on the right hand medium, j_+,

$$\beta = + \frac{j_-}{j_+} \quad . \tag{1.110}$$

Now since we have assumed elementary diffusion theory,

$$F(o,v,\mu) = \frac{1}{N\sigma - \mathcal{H}\mu} \tag{1.111}$$

and so

$$\beta = \frac{j_-}{j_+} = \frac{\int_0^{-1} \frac{\mu d\mu}{N\sigma - \mathcal{H}\mu}}{\int_0^{1} \frac{\mu d\mu}{N\sigma - \mathcal{H}\mu}} = \frac{1 - \frac{N\sigma}{\mathcal{H}} \ln(1 + \mathcal{H}/N\sigma)}{-1 - \frac{N\sigma}{\mathcal{H}} \ln(1 - \mathcal{H}/N\sigma)} \quad . \tag{1.112}$$

If the absorption is weak, then $\mathcal{H} \approx N\sqrt{3\sigma\sigma_a} \ll N\sigma$, and we can expand the logarithms. The result is

$$\beta \approx 1 - \frac{4}{\sqrt{3}} \sqrt{\frac{\sigma_a}{\sigma}} = 1 - 2.31 \sqrt{\frac{\sigma_a}{\sigma}} \quad . \tag{1.113}$$

The albedo of a medium decreases as the ratio σ_a/σ increases. This is natural since σ/σ_a is the number of collisions before capture, and the smaller the number of collisions before capture, the smaller is the probability that an incident neutron will be reflected before it is captured. If the capture is 0, then $\beta = 1$; this again is a physically obvious consequence of (1.113).

In case the absorption is so slight that the two harmonic angular distribution (1.98) is justified, we can write

$$\beta = \frac{j_-}{j_+} = \frac{\frac{1}{2} F_o(o,v) \int_0^{-1} \mu d\mu + \frac{3}{2} j(o,v) \int_0^{-1} \mu^2 d\mu}{\frac{1}{2} F_o(o,v) \int_0^{1} \mu d\mu + \frac{3}{2} j(o,v) \int_0^{+1} \mu^2 d\mu} = \frac{\frac{F_o}{4} - \frac{j}{2}}{\frac{F_o}{4} + \frac{j}{2}} \tag{1.114}$$

and, since $j = \frac{-1}{3} \lambda_d F_o'$,

$$\beta = \frac{1 + \frac{2}{3} \lambda_d \frac{F_o'}{F_o}}{1 - \frac{2}{3} \lambda_d \frac{F_o'}{F_o}} \approx 1 + \frac{4}{3} \lambda_d \frac{F_o'}{F_o} . \tag{1.115}$$

Since for isotropic scattering and weak absorption,

$$\lambda_d = \frac{3N\sigma_a}{\chi^2} \approx \frac{1}{N\sigma} , \quad F_o'/F_o = -\chi \approx -N\sqrt{3\sigma_a\sigma} ,$$

formula (1.115) is really equivalent to (1.113). We can express the logarithmic derivative F_o'/F_o in terms of the albedo by solving (1.115):

$$\lambda_d \frac{F_o'}{F_o} = \frac{3}{2} \frac{\beta - 1}{\beta + 1} . \tag{1.116}$$

Now the boundary condition of elementary diffusion theory is that $\lambda_d \frac{F_o'}{F_o}$ is continuous across an interface. Hence, if β_+ denotes the albedo of the right hand medium, and $\left(\lambda_d \frac{F_o'}{F_o}\right)^-$ refers to the left hand medium, then the boundary condition of elementary diffusion theory can be written

$$\left(\lambda_d \frac{F_o'}{F_o}\right)^- = \frac{3}{2} \frac{\beta_+ - 1}{\beta_+ + 1} . \tag{1.117}$$

Some Plutonium Project writers, notably Fermi, have used the boundary condition in this form rather than in the form discussed previously (1.99) - (1.100). Since the albedo is computed by elementary theory, the two formulations are equivalent.

The albedo of a perfect absorber or a vacuum is 0. Hence, from (1.117), at an interface with a vacuum we must have

$$\frac{F_o}{F'_o} = -\frac{2}{3}\lambda_d \;;$$

that is, the elementary theory extrapolation distance is $\frac{2}{3}\lambda_d$, as we have already calculated.

A problem which is of considerable importance in pile neutron physics, and which can be solved rather handily by use of the albedo is the following. Suppose the net current, j, of neutrons in a weakly absorbing medium is known. Now suppose a thin, weak absorbing sheet is placed normal to the neutron flux gradient. It is required to calculate what fraction of the net neutron current is absorbed in the sheet.

We assume, as usual, elementary diffusion theory. Then the current j is

$$j = -D_o\Phi'_o = D_o\kappa\Phi_o$$

while the number of neutrons absorbed per cm^2 per sec in the sheet, assuming it is very thin, is

$$(N\sigma_a)_s\,\Phi_o t$$

where $(N\sigma_a)_s$ is the absorption cross-section of the sheet and t is its thickness. Hence we have

$$\frac{\text{absorption}}{\text{current}} = \frac{N\sigma a_s t}{D_o\kappa} \;;$$

and, since by (1.116)

$$D_o \mathcal{L} = \frac{1}{2} \frac{1 - \beta}{1 + \beta},$$

we can write

$$\frac{\text{absorption}}{\text{current}} = 2N\sigma_{a_s} t \frac{1 + \beta}{1 - \beta}, \tag{1.117a}$$

a formula which is applicable if the sheet is thin and absorbs very weakly.

This formula Eq. (1.117a), has a rather interesting physical interpretation. The quantity $\frac{1 + \beta}{1 - \beta}$ is the number of times a neutron crosses the sheet per unit net current, while $N\sigma_{a_s} t$ is the number of absorptions which would occur, per pass, if the neutrons traversed the sheet normally. Actually the neutrons strike with a cosine distribution* (since $j(\mu) = \mu\Phi$); hence the average path length that a neutron travels through the sheet is greater than t, and in fact is 2t. The fact that the average distance traversed by neutrons in a sheet is 2t, if the incident flux has a cosine distribution, is a special case of a famous geometric theorem due to Gauss; namely, that in a solid of volume v and surface s, the average chord length, $\bar{e}$, weighted with the cosine of the angle between chord and normal is

$$\bar{e} = \frac{4v}{s}.$$

Diffusion through Channels

If a scattering medium is traversed by long cylindrical channels or gaps, the net diffusion current for a given flux gradient is greater than it would be in the absence of such gaps. In a general way this can be predicted from

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* In the approximation used here the angular distribution $\Phi_o(x,\mu) = \Phi_o/2 - \frac{\lambda}{2}\Phi_o'\mu$; hence $j_+ = \Phi_o/4 - \lambda/6\ \Phi_o'$ and $j_- = \Phi_o/4 + \lambda/6\ \Phi_o'$. In calculating the net number which strike from both sides of the sheet, only the Φ_o term, which corresponds to the uniform part of the flux, (and therefore a cosine current distribution), need be considered.

the fact that in a vacuum the mean free path of a neutron, and therefore the diffusion coefficient, is infinitely long. The average mean free path is thus increased by the presence of gaps, and this increase is reflected in an increase in the net diffusion current per unit gradient of flux.

The net diffusion current through a channel depends upon (1) the dimensions of the channel; (2) the orientation of the channel with respect to the neutron flux gradient; (3) the scattering properties of the medium in which the channel is placed; (4) the flux distribution. Because the flow per unit gradient depends on the orientation of the channel, if a medium is traversed by a large number of parallel channels, the net diffusion current will be anisotropic; i.e., the diffusion coefficient is a tensor.

The current through an infinitely long channel is not always finite; for example, if the neutron flux increases exponentially, say like $e^{\varkappa x}$, and the channel axis is parallel to x, then the current is infinite.

On the other hand, in a non-capturing medium in which the asymptotic flux distribution is

$$\Phi(x,\mu) = \frac{1}{2}\Phi_0 + \frac{x}{2}\Phi_0' - \frac{1}{2}\Phi_0'\mu \ , \tag{1.118}$$

where Φ_0 and Φ_0' are constants, the total number of neutrons which traverse per second a plane perpendicular to the channel is finite. We proceed to compute this current; in this calculation we follow essentially the method of Nordheim and Soodak.

Consider a cylindrical channel of arbitrary convex shape (Figure 3) whose axis lies along x, in a non-capturing infinite medium which scatters isotropically. The net number of neutrons which cross the plane x = 0 per second consists of

those which cross the plane $x = 0$ inside the cylinder, plus the number which cross $x = 0$ outside the cylinders. We denote the net number which cross inside the cylinder by J_i, and the difference between the net number which cross $x = 0$ outside the cylinder after and before the cylinder was in place by J_{ex}. We assume the validity of elementary diffusion theory; i.e., we assume the asymptotic flux distribution (1.118). This is correct if the total neutron flux is a linear function of x.

To calculate J_i, we first find the net number of neutrons which cross an element of area $d\sigma$ in the $x = 0$ plane inside the cylinder; we then integrate over the interior surface. The number of neutrons which strike the element $d\sigma$ in the $x = 0$ plane (Figure 3) is the sum of all neutrons which strike all elements dA on the wall of the cylinder and are directed toward $d\sigma$. Now the number which strike dA per second and reach $d\sigma$ is

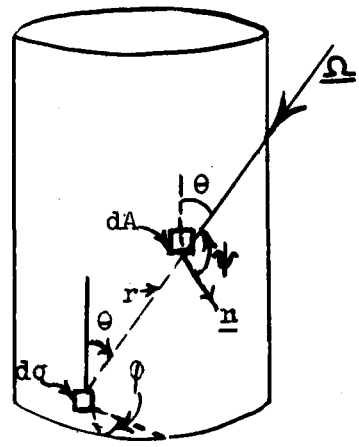

Figure 3

$$\Phi(x,\underline{\Omega})\ \cos\psi\ \cos\theta\ dA\ \frac{d\sigma}{r^2}$$

where $\Phi(x,\underline{\Omega})$ is the total flux per unit solid angle in the direction $\underline{\Omega}$ $\frac{d\sigma}{r^2}\cos\theta$ is the solid angle subtended by $d\sigma$ at dA, and ψ is the angle between the direction of motion of the neutron and the outward normal n to the element dA. The total flux $\Phi(x,\underline{\Omega})$ is $\frac{1}{2\pi}$ times the flux $\Phi(x,\mu)$ of Eq. (1.118):

$$\Phi(x,\underline{\Omega}) = \frac{\Phi_o}{4\pi} + \frac{x\Phi_o'}{4\pi} + \frac{\lambda\Phi_o'\cos\theta}{4\pi} \quad .$$

(The $\cos\theta$ term has a + sign here because θ is the supplement of the angle which the neutron direction makes with the x-axis.) If we denote the element of the bounding contour of the cylinder by dc, then dA = dcdx. Hence the total number of neutrons which cross, from above, the x = 0 plane inside the cylinder is

$$\int_\sigma\int_c\int_0^\infty \Phi(x,\underline{\Omega})\cos\psi\cos\theta\,\frac{d\sigma}{r^2}\,dc\,dx = \frac{1}{4\pi}\int_c\int_0^\infty\left[\Phi_o + x\Phi_o' + \lambda\Phi_o'\cos\theta\right]\cos\psi\cos\theta\,\frac{d\sigma}{r^2}\,dc\,dx, \tag{1.119}$$

and from below it is

$$\frac{1}{4\pi}\int_\sigma\int_c\int_{-\infty}^0\left[\Phi_o + x\Phi_o' + \lambda\Phi_o'\cos\theta\right]\cos\psi\cos\theta\,\frac{d\sigma}{r^2}\,dc\,dx.$$

The total net flow is the difference between the upward and downward flow:

$$J_i = \frac{-\Phi_o'}{4\pi}\int_\sigma\int_c\int_{-\infty}^{\infty}(x + \lambda\cos\theta)\cos\psi\cos\theta\,\frac{d\sigma}{r^2}\,dc\,dx\ . \tag{1.120}$$

In taking the difference, the term involving Φ_o, being odd in x cancels while the terms in Φ_o', being even in x, add. To simplify the integral for J_i, we use the geometric relations

$$\cos\psi = \sin\theta\cos\varphi$$

$$\sin\theta = \frac{\rho}{(x^2+\rho^2)^{1/2}}\ ,\quad \cos\theta = \frac{x}{(x^2+\rho^2)^{1/2}}$$

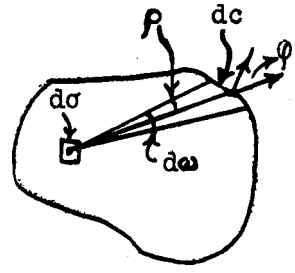

Figure 4

and

$$\cos\varphi\,dc = \rho\,d\omega$$

where $\rho(\varphi)$ is the distance from the element $d\sigma$ to the gap perimeter in the direction which makes angle φ with the normal to dc, and $d\omega$ is the element

of angle subtended by dc at $d\sigma$ (Figure 4). Making these substitutions, and using the integrals

$$\int_{-\infty}^{\infty} \frac{x^2}{(\rho^2 + x^2)^2}\, dx = \frac{\pi}{2\rho}, \quad \int_{-\infty}^{\infty} \frac{x^2}{(\rho^2 + x^2)^{5/2}}\, dx = \frac{2}{3\rho^2},$$

we obtain

$$J_i = -\Phi_0' \left[\frac{\lambda}{3} A + \frac{1}{8}\iint \rho(\varphi)\, d\omega\, d\sigma\right] = -A\Phi_0' \left[\frac{\lambda}{3} + 2\gamma\bar{\ell}\right]$$

where $\bar{\ell}$ is 2x channel area/channel perimeter, ($\bar{\ell}$ is sometimes called the hydraulic radius of the channel), and $\gamma = \frac{1}{16\bar{\ell}A}\iint \rho(\varphi)\, d\omega\, d\sigma$. The quantity γ is a numerical factor which depends on the shape of the channel. From the integral (1.119) it is easy to prove the remark made before -- namely, that if Φ increases exponentially in x, then the integrals will not converge. For, when x is very large, $\cos\psi$ is of order $1/x$, and the whole integrand is of order $\sim \Phi/x^3$. Hence the integral does not converge if $\Phi \sim e^{\lambda x}$. end M-3336 pt 3 begin M-3336 pt 4

In order to calculate J_{ex}, the difference between the number of neutrons which cross the x = 0 plane before and after the channel has been introduced, we consider the number of neutrons which cross an element of cylinder wall, dA at a height x_1 in a direction between $\underline{\Omega}$ and $\underline{\Omega} + d\underline{\Omega}$ (Figure 5). Evidently the number which cross dA must have originated from an element dA' at height x_2 on the opposite wall of the cylinder. The projected area dA' normal to the direction of neutron motion

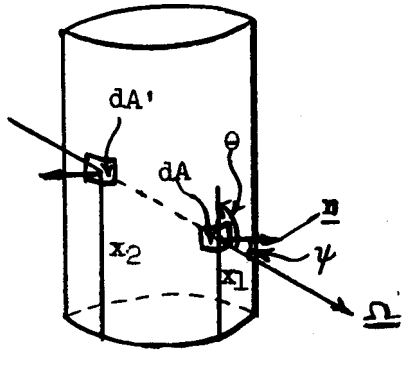

Figure 5

must be the same as the projected area of dA in order for all the neutrons in the proper solid angle to strike dA and dA'; both projected areas must be equal to cos ψ dA where ψ is the angle between $\underline{\Omega}$ and the outward normal to dA. The number which cross dA per second in the direction $\underline{\Omega}$, and survive without further collision until they reach the x = 0 plane is

$$\Phi(x_2,\underline{\Omega})\ \cos\psi e^{-r/\lambda} d\underline{\Omega} dA$$

where $r = |x_1/\cos\theta|$, and θ is the angle between $\underline{\Omega}$ and the x direction. Now in the absence of the channel, the number of neutrons which pass through dA and cross the x = 0 plane per second is

$$\Phi(x_1,\underline{\Omega})\ \cos\psi e^{-r/\lambda} d\underline{\Omega} dA.$$

The extra number of neutrons which cross from above because of the presence of the channels is therefore

$$J^-_{ex} = \int_{\sigma}\int_{\underline{\Omega}} \left[\Phi(x_2,\underline{\Omega}) - \Phi(x_1,\underline{\Omega})\right] \cos\psi e^{-x_1/\lambda \cos\theta} d\underline{\Omega} dA.$$

Since we have assumed the no capture asymptotic distribution for $\Phi(x,\mu)$, we have, according to (1.118),

$$\Phi(x_2,\underline{\Omega}) - \Phi(x_1,\underline{\Omega}) = \frac{\xi\Phi_0'}{4\pi}$$

where $\xi = x_2 - x_1$. The integral for J_{ex}, the difference between what strikes from below and above, can be written

$$J_{ex} = \frac{-2}{4\pi}\int_s^{\infty}\int_0\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\pi}^{-\frac{\pi}{2}} (-\rho)\ \cot\theta\ \cos\varphi\ \sin\theta\ e^{-|x_1/\lambda\cos\theta|} \sin\theta d\theta d\varphi dx_1 dc$$

where $\rho = -\xi \tan\theta$ is the chord length between two points on the circumference of the channel in the $x = 0$ plane, $dA = dc\,dx_1$, $d\underline{\Omega} = \sin\theta\, d\theta\, d\varphi$. The factor -2 comes from taking the difference between the + and - flows. The integration over x_1 and θ can be performed easily; the result is

$$J_{ex} = -\frac{\lambda}{3}\frac{\Phi_o'}{4\pi}\iint \rho\cos\varphi\, d\varphi\, dc = -\frac{\lambda}{3}\Phi_o' A ,$$

where A is the cross-sectional area of the channel.

In the presence of a channel of area A the net longitudinal neutron current per negative unit flux gradient, over and above what the current would be outside the channel (if the channel were absent) is

$$\frac{J_i + J_{ex}}{-\Phi_o'} = A\left[\frac{2\lambda}{3} + 2\gamma\bar{\ell}\right]$$

or the longitudinal diffusion coefficient, D_{oll} per channel, which is the average current per cm^2 of channel per negative unit flux gradient, is

$$D_{oll} = \left[\frac{2\lambda}{3} + 2\gamma\bar{\ell}\right] . \qquad (1.121)$$

If a medium is traversed by many parallel similar channels whose volume is the fraction f* of the total volume, then the total current per unit gradient of flux which crosses a normal plane of area A_s, A_s being so large that many channels traverse it, is

$$\frac{J}{-\Phi_o'} = \frac{\lambda}{3} A_s + fA_s\left(\frac{\lambda}{3} + 2\gamma\bar{\ell}\right)$$

and the average longitudinal diffusion coefficient of the medium with its

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* It is assumed that $f \ll 1$; that is, the channels do not interact.

channels is

$$\bar{D}_{oll} = \frac{J}{-\Phi_o' A_s} = \frac{\lambda}{3}\left[1 + f\left(1 + \frac{6\gamma\bar{\ell}}{\lambda}\right)\right] . \qquad (1.122)$$

The presence of the channels increases the diffusion coefficient in the parallel direction by a fraction

$$f(1 + 6\gamma\bar{\ell}/\lambda).$$

If the radius $\bar{\ell}$ of the channels is very small compared to λ, but f is constant -- this would correspond to uniform decrease in density -- the fractional increase in D is just f, the fractional density decrease.

The quantity $6\gamma\bar{\ell}$ has been calculated for a circular cylinder and for a circular cylindrical annulus. The result for a circular cylinder is $6\gamma\bar{\ell} = 2a$, where a is the radius of the cylinder. For an annulus, the value of $6\gamma\bar{\ell}/b$ is given for annuli with various ratios of a/b, where b is the outer radius and a the inner. These results were obtained by Soodak (CP-2019).

Form factor $6\gamma\bar{\ell}/b$ for annular channels.

a/b	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
$\frac{6\gamma\bar{\ell}}{b}$	2	1.83	1.68	1.52	1.31	1.15	.96	.76	.53	.29	0

Solution of Boltzmann Equation Near a Source

The exact solution of the Boltzmann equation near a source of neutrons can be found rather easily. Since the asymptotic part of this solution is very fundamental for the application of the elementary diffusion theory when sources are present, we shall show here how this solution is obtained. This problem has been treated by Bothe, by Placzek, and by Wigner. We follow the method of Wigner (CP-1120 .

We seek a solution of the Boltzmann equation for a localized, isotropic unit source at the origin. We represent the source by $\delta(x)$; such a source emits $\frac{1}{2}\,d\mu$ neutrons between μ and $\mu + \delta\mu$. The Boltzmann equation is, for isotropic scattering,

$$\mu \frac{dF}{dx}(x,\mu,v) + N\sigma F(x,\mu,v) = \frac{N\sigma_{s_o}}{2}\int_{-1}^{1} F(x,\mu,v)d\mu + \tfrac{1}{2}\,\delta(x). \tag{1.123}$$

To solve this equation we expand $\delta(x)$ and $F(x,\mu,v)$ as Fourier integrals:

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} d\omega$$

$$F(x,\mu,v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{A(\omega)e^{i\omega x}}{N\sigma + i\omega\mu}\, d\omega \ .$$

Substituting these integrals into (1.123) we find that $A(\omega)$ is just

$$A(\omega) = \frac{1}{2}\ \frac{1}{1 - \dfrac{N\sigma_{s_o}}{2i\omega}\,\ell n\,\dfrac{N\sigma + i\omega}{N\sigma - i\omega}} \ .$$

The complete solution of (1.123) can therefore be expressed as the integral

$$F(x,\mu,v) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{1 - \dfrac{N\sigma_{s_o}}{2i\omega}\,\ell n\,\dfrac{N\sigma + i\omega}{N\sigma - i\omega}}\ \frac{d\omega}{N\sigma + i\omega\mu} \ . \tag{1.124}$$

The total flux is

$$F_o(x,v) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\frac{i}{i\omega}\, e^{i\omega x}}{1 - \dfrac{N\sigma_{s_o}}{2i\omega}\,\ell n\,\dfrac{N\sigma + i\omega}{N\sigma - i\omega}}\,\ell n\,\frac{(N\sigma + i\omega)}{N\sigma - i\omega}\, d\omega \ . \tag{1.125}$$

In order to see more clearly the significance of this solution, it is useful to calculate this integral by the method of residues. The integrand of (1.125) has a simple pole where

$$\frac{N\sigma_{s_0}}{2i\omega} \ln \frac{N\sigma + i\omega}{N\sigma - i\omega} = 1 ;$$

i.e., for $i\omega = \pm\varkappa$ where $\varkappa$ is the reciprocal diffusion length, and it has an essential singularity at $i\omega = \pm N\sigma$. The contribution $F_{o_{as}}$ to the integral from the residue at the pole is

$$F_{o_{as}}(x) = \frac{\varkappa}{2N\sigma_a} I e^{-\varkappa|x|}, \quad I = \frac{2\sigma_a}{\sigma_{s_0}} \frac{N^2\sigma^2 - \varkappa^2}{\varkappa^2 - N^2\sigma_a\sigma} \tag{1.126}$$

while the contribution, $F_{o\ell}(x)$, from the essential singularity is expressible as a real integral

$$F_{o\ell}(x) = 2N^2\sigma^2 \int_0^\infty \frac{(1+\eta)\, e^{-(1+\eta)N\sigma x}\, d\eta}{\left[2\sigma(1+\eta) - N\sigma_{s_0} \ln(1 + 2/\eta)\right]^2 + \pi^2 N^2 \sigma_{s_0}^2} . \tag{1.127}$$

The total flux is

$$F_o(x) = F_{o_{as}}(x) + F_{o\ell}(x) . \tag{1.128}$$

It can be shown that $F_{o\ell}(x)$ falls off from the source plane very much quicker than $F_{o_{as}}(x)$; after a few mean free paths it is entirely negligible compared to $F_{o_{as}}$. Thus the asymptotic solution from an isotropic unit source falls off like $e^{-\varkappa|x|}$.

In the next section we show that the solution due to a unit source according to elementary diffusion theory is just

$$\frac{\varkappa}{2N\sigma_a} e^{-\varkappa|x|}$$

Hence the correct asymptotic solution is the same as the elementary theory solution except that the effective source strength is smaller by an amount

$$\frac{2\sigma_a}{\sigma_{s_o}} \frac{N^2\sigma^2 - \varkappa^2}{\varkappa^2 - N^2\sigma_a\sigma_a} \approx 1 - \frac{4}{5}\frac{\sigma_a}{\sigma} \quad \text{if } \sigma_a/\sigma \ll 1.$$

Physically the reason for this reduction in apparent source intensity at distances far from the source is that, because of the extra term $F_{o\ell}$, the neutron density, and therefore the absorption, is greater than it would be if only the asymptotic solution were used. The initial absorption correction is unimportant in any material, such as H_2O or graphite, which is suitable for slowing down neutrons in a chain reaction. In H_2O, for example, the correction is about 0.8%; in graphite it is about 1/10 of this.

Solution of the Diffusion Equation in Various Geometries: The Green's Function or Diffusion Kernel

The steady state elementary diffusion equation for monoenergetic neutrons in a uniform medium, or for polyenergetic neutrons with energy independent cross-section, or finally, for thermal neutrons in a weak absorber, is

$$D_o\Delta\Phi_o - N\sigma_a\Phi_o + S(\underline{r}) = 0 \tag{1.129}$$

where $S(\underline{r})$ is the "effective" number of neutrons produced per second per c.c., and σ_a is the absorption cross-section at the average velocity. The effective source strength S is smaller than the actual source strength by the factor $I \approx 1 - \frac{4}{5}\frac{\sigma_a}{\sigma}$, which arises because of the breakdown of elementary theory near the source.

The solution of (1.129) can be represented most conveniently by means of a Green's function, $G_p(\underline{r},\underline{r}')$. We define the Green's function as the neutron flux Φ_o at $\underline{r}$ due to a point source at $\underline{r}'$, which emits one neutron per second.

The form of the Green's function, of course, depends on the geometry of the medium in which the neutrons diffuse. However, from the Green's function in an infinite medium it is usually possible to construct the Green's function in a finite medium by a judicious superposition of images. We therefore proceed to compute the Green's function for an infinite medium.

The Green's function must be spherically symmetric about the source point $\underline{r}'$. We can therefore, without loss of generality, put the origin at the source point. The Green's function is then the solution of the spherically symmetric diffusion equation

$$D_o \left[\frac{d^2 G_p}{d\rho^2} + \frac{2}{\rho} \frac{dG_p}{d\rho} \right] - N\sigma_a G_p = 0 \ , \tag{1.130}$$

where $\rho = \underline{r} - \underline{r}'$ is the distance from field point (at $\underline{r}$) to source point (at $\underline{r}'$) which is taken as the origin. As boundary condition we require that the number of neutrons leaving the source point per second is just 1. This condition can be written

$$\lim_{\rho \to 0} \ 4\pi\rho^2 D_o \frac{dG_p}{d\rho} = -1 \ . \tag{1.131}$$

The solution of (1.130) which satisfies the boundary condition (1.131) is

$$G_p(\rho) = G_p(\underline{r},\underline{r}') = \frac{e^{-\varkappa|\underline{r}-\underline{r}'|}}{4\pi D_o|\underline{r} - \underline{r}'|} \ ; \tag{1.132}$$

it is important to observe that $G(\underline{r},\underline{r}')$ is a function only of the difference $|\underline{r} - \underline{r}'|$.

The Green's function gives the flux Φ_o at $\underline{r}$ due to a unit source at $\underline{r}'$. Since the diffusion equation is linear, the flux at $\underline{r}$ due to a distributed source $S(\underline{r}')$ is a superposition of fluxes due to point sources,

$$\Phi_o(\underline{r}) = \int S(\underline{r}') \frac{e^{-\mathcal{L}|\underline{r}-\underline{r}'|}}{4\pi D_o|\underline{r}-\underline{r}'|} d\underline{r}' , \tag{1.133}$$

and this is the solution of (1.129) in an infinite medium.

Equation (1.133) expresses the total flux $\Phi_o(\underline{r})$ at a point $\underline{r}$ as the sum of the individual fluxes from a distribution of sources $S(\underline{r})$. Conversely, it can be viewed as an integral equation for the source density $S(\underline{r}')$, if the flux $\Phi_o(\underline{r})$ is known. The kernel of this integral equation,

$$\frac{e^{-\mathcal{L}|\underline{r}-\underline{r}'|}}{4\pi D_o|\underline{r}-\underline{r}'|} ,$$

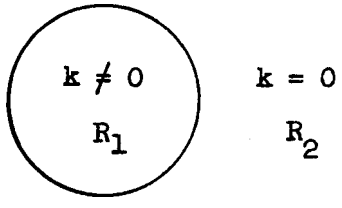
is called the point diffusion kernel, as well as the Green's function for the elementary diffusion equation.

As an example of a physical situation in which an integral equation of the form (1.133) arises, we consider a medium in which neutrons are produced as the result, say, of fission, the number of fissions, and therefore the production rate, being proportional to the neutron flux Φ_o. We suppose that the neutrons, when they are produced, are already thermal; this is not correct in a uranium chain reaction but will serve to illustrate the point. Then, if k is the number of neutrons produced per thermal neutron absorbed,

$$S(\underline{r}) = kN\sigma_a\Phi_o(r),$$

and, if we suppose the system consists of two regions, R_1 and R_2 which have identical nuclear properties except that production occurs only in region R_1, then $k \neq 0$ in the region R_1, $k = 0$ in R_2, (Figure 5a). Substituting for

Figure 5a



$S(\underline{r})$ into (1.133) we find that the integral equation for the neutron flux is (cf. 1.133)

$$\Phi_o(\underline{r}) = \frac{k\mathcal{L}^2}{4\pi}\int_{R_1}\frac{e^{-\mathcal{L}|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|}\Phi_o(\underline{r}')dr' \ . \qquad (1.133a)$$

This is a characteristic value problem which has a non-zero solution $\Phi_o(\underline{r})$ only for particular values of k. Equation (1.133a) is an example of a characteristic equation for a chain reacting system. The quantity k is the "multiplication factor".

Another example of a case in which the source strength is directly proportional to the neutron density is that of a foil in an infinite medium in which q thermal neutrons are produced per second. In the absence of the foil, the neutron distribution is a constant,

$$\Phi_o(\underline{r}) = \frac{q}{N\sigma_a} \ .$$

Suppose the foil has a volume V and an absorption cross-section $(N\sigma_a)_f$. If the foil is very thin, then the number of neutrons absorbed per second per c.c. of foil is $(N\sigma_a)_f\Phi_o$. Hence the neutron distribution when the foil is in place must satisfy the equation

$$\Phi_o(\underline{r}) = \frac{q}{N\sigma_a} - \frac{(N\sigma_a)_f\mathcal{L}^2}{4\pi N\sigma_a}\int_V\frac{e^{-\mathcal{L}|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|}\Phi_o(\underline{r}')dr'$$

where the integration is over V alone. If the foil is small it is satisfactory to solve this equation by replacing $\Phi_o(\underline{r}')$ under the integral with its undisturbed value, $q/N\sigma_a$. The distribution is then

$$\Phi_o(r) = \frac{q}{N\sigma_a}\left[1 - \frac{(N\sigma_a)_f}{N\sigma_a}\frac{\kappa^2}{4\pi}\int_V \frac{e^{-\kappa|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|}\,dr'\right] .$$

This is only an approximate result since we have assumed here the validity of diffusion theory. Also we have ignored the fact that the foil constitutes a second medium and therefore the diffusion kernel for a single medium is not quite appropriate. The error introduced thereby is small if the foil does not absorb too strongly.

If the sources of neutrons are distributed uniformly over infinite planes parallel to the y-z plane, the resulting neutron flux $\Phi_o(x)$ can depend only on the one coordinate x. The flux can then be expressed in terms of the source strength $S(x)$, where $S(x)dx$ is the number of neutrons emitted per cm^2 per second by a thickness dx of source.

Evidently in this case $\Phi_o(x)$ is the solution of the differential equation

$$D_o \frac{d^2\Phi_o}{dx^2}(x) - N\sigma_a\Phi_o(x) + S(x) = 0 . \tag{1.134}$$

The solution of this equation can be expressed, as in the spherical case, in terms of a one-dimensional Green's function, or <u>plane diffusion kernel</u>, $G_{p\ell}(x,x')$, which is the flux at x due to a source of strength one neutron per cm^2 per second at x':

$$\Phi_o(x) = \int S(x')G_{p\ell}(x,x')dx'. \tag{1.135}$$

In an exactly analogous manner, if the sources are distributed over concentric spherical shells of strength $S_s(r)(4\pi r^2 S_s(r)dr$ is the number of

neutrons emitted per second by a shell of radius r and thickness dr), the flux $\Phi_o(r)$ is expressible in terms of the spherical diffusion kernel, $G_s(r,r')$

$$\Phi_o(r) = 4\pi \int S_s(r')\, G_s(r,r') r'^2\, dr' \tag{1.136}$$

where $G_s(\underline{r},\underline{r}')$ is the flux at r due to a shell of radius r and of total strength one neutron per second. Evidently $\Phi_o(r)$ is the solution of

$$\frac{D_o}{r}\frac{d^2[\Phi_o(r)]}{dr^2} - N\sigma_{a_o}\Phi_o(r) + S_s(r) = 0 . \tag{1.137}$$

Finally if the sources are distributed uniformly along infinitely long concentric cylindrical shells, the flux is expressible in terms of the strength of the source and the cylindrical diffusion kernel, $G_o(r,r')$:

$$\Phi_o(r) = 2\pi \int S_c(r')G_c(r,r')r dr' \tag{1.138}$$

where $2\pi S_c(r')r'dr'$ is the number of neutrons emitted per second per cm length of shell of thickness dr, and $G_c(r,r')$ is the flux at r due to a uniform cylindrical source of radius, r', and total strength one neutron per second per cm. The flux $\Phi_o(r)$ is the solution of the diffusion equation in cylindrical coordinates

$$D_o\left[\frac{d^2\Phi_o}{dr^2} + \frac{1}{r}\frac{d\Phi_o}{dr}\right] - N\sigma_a\Phi_o(r) + S_c(r) = 0. \tag{1.139}$$

It is a rather easy matter to compute the diffusion kernels in the various geometries by solving the diffusion equation in the appropriate geometry and applying boundary conditions like (1.131) at the source. However, we will compute the diffusion kernels directly by integrating (1.132) over the proper source distribution. The advantage of this method is that it illustrates

certain general relations between point, plane, sphere, and cylinder kernels which hold even when the kernels are not the solutions of simple second order differential equations, provided only that the point kernel depends on $|\underline{r}-\underline{r}'|$ only. In particular, these relations hold for the so-called "transport" kernels which are the Green's functions for the isotropic transport equation, and are therefore not the solution of any differential equation.

To compute the plane diffusion kernel, we integrate the point kernel over a plane. Thus, if a plane source of unit strength is placed at x', then the flux at x, which is just the plane diffusion kernel, is

$$G_{p\ell}(|x-x'|) = 2\pi \int_0^\infty y' G_p(|\underline{r}-\underline{r}'|)dy' .$$

Now, from Figure 6,

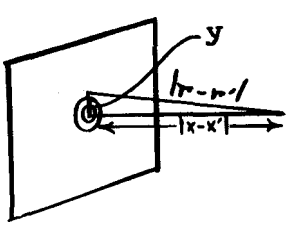


Figure 6

$$y^2 + |x-x'|^2 = |\underline{r}-\underline{r}'|^2$$

and, putting $|x-x'| = \xi$, $|\underline{r}-\underline{r}'| = \rho$, we obtain

$$G_{p\ell}(\xi) = 2\pi \int_\xi^\infty \rho G_p(\rho)d\rho , \tag{1.140}$$

or, conversely,

$$G_p(\rho) = -\frac{1}{2\pi\rho} G'_{p\ell}(\rho). \tag{1.141}$$

The relations (1.140) and (1.141) are general relations between point and plane kernels -- they hold whenever the kernel $G_p(\underline{r},\underline{r}')$ depends only on

the distance between $\underline{r}$ and $\underline{r}'$. This is always true in a homogeneous, isotropic, infinite system regardless of whether diffusion theory is applicable.

The plane <u>diffusion</u> kernel can be evaluated by substituting the formula (1.132) for $G_p(\rho)$ into (1.140) and integrating: the result is

$$G_{p\ell}(\xi) = G_{p\ell}(|x-x'|) = \frac{1}{2\gamma_\ell D_o} e^{-\gamma_\ell |x-x'|} . \tag{1.142}$$

To compute the spherical kernel from the point kernel we integrate (1.142) over a sphere of radius r' (Figure 7). Since the kernel is normalized to one neutron per total shell, we take for the source strength $1/4\pi r'^2$ neutrons per cm^2 per second. The case $r > r'$ (field point outside sphere) and the case $r < r'$ (field point inside sphere) must be treated separately. We consider the case $r > r'$ From the figure

$$\rho^2 = r'^2 + r^2 - 2rr'\cos\theta;$$

$$\rho d\rho = rr'\sin\theta d\theta.$$

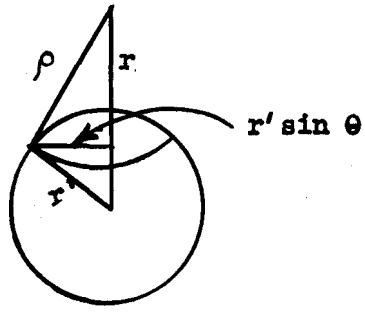


Figure 7

Hence, for $r > r'$,

$$G_s(r,r') = \frac{2\pi}{4\pi r'^2}\int_0^\pi G_p(\rho) r'^2 \sin\theta d\theta = \frac{2\pi}{4\pi r' r}\int_{r-r'}^{r+r'} G_p(\rho)\rho d\rho \quad . \tag{1.143}$$

In case $r < r'$ the formula remains the same except that $r' - r$ replaces $r - r'$. On comparing the right-hand integral in (1.143) with the relation (1.140) between plane and point kernels, we see that

$$G_s(r,r') = \frac{1}{4\pi r' r}\left[G_{p\ell}(|r-r'|) - G_{p\ell}(|r+r'|)\right], \tag{1.144}$$

which is a general relation. For the diffusion kernels, we have

$$G_s(r,r') = \frac{1}{8\pi\mathcal{H}D_o r r'}\left[e^{-\mathcal{H}|r-r'|} - e^{-\mathcal{H}|r+r'|}\right]$$

$$= \frac{\sinh\mathcal{H}r'}{\mathcal{H}r'}\,\frac{e^{-\mathcal{H}r}}{4\pi D_o r} \qquad r > r' \tag{1.145}$$

$$= \frac{e^{-\mathcal{H}r'}}{4\pi D_o r'}\,\frac{\sinh\mathcal{H}r}{\mathcal{H}r} \qquad r < r' \quad .$$

We observe that $G_s(r,r')$ is not a function of $|r-r'|$ alone. It is, however, still symmetric in r and r'.

It is interesting to observe (1.145) that the effect at exterior points of a unit spherical shell of radius r' is the same as though the shell were all concentrated at its center and its strength increased by the factor $\frac{\sinh\mathcal{H}r'}{\mathcal{H}r'}$. In the interior of the shell, $r' > r$, the effect of the shell cannot be replaced by a single point. These results are generalizations of the well known fact in electrostatic potential theory that a uniform sphere attracts like a point charge of the same strength at its center. This is not surprising since the diffusion equation reduces to the potential equation when $\mathcal{H} = 0$; all the diffusion kernels become potential functions when $\mathcal{H} = 0$. In particular the sphere kernels reduce to the familiar potentials in a medium with dielectric constant D_o, $\frac{1}{4\pi D_o r}$ and $\frac{1}{4\pi D_o r'}$ for $r > r'$ and $r < r'$ respectively.

The cylindrical diffusion kernel is obtained by integrating the point kernel, $\frac{e^{-\mathcal{H}|\underline{r}-\underline{r}'|}}{4\pi D_o|\underline{r}-\underline{r}'|}$ over the surface of a circular cylinder of radius r'.

Since the generators of the cylinder are vertical straight lines (Figure 8) it is convenient to integrate first along one generator; this will yield the "line" diffusion kernel, i.e., the flux at (r,φ) due to a line source at (r',φ') of strength one neutron per second per centimeter. The line kernel, $G_\ell(r,\varphi,r',\varphi')$, is

$$G_\ell(r,\varphi,r'\varphi') = \int_{-\infty}^{\infty} \frac{e^{-\mathcal{H}|\underline{r}-\underline{r}''|}}{4\pi D_0|\underline{r}-\underline{r}''|}\,dx \qquad (1.146)$$

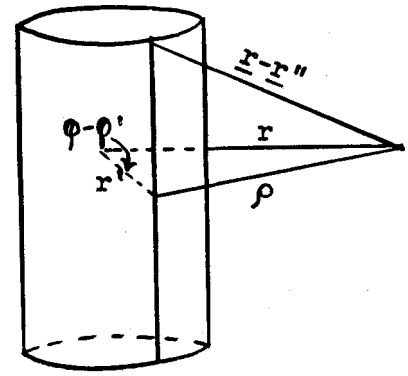


Figure 8

where $|\underline{r}-\underline{r}''|$ is the distance between the element dx and the field point. Now (Figure 8)

$$|\underline{r}-\underline{r}''| = \sqrt{\rho^2 + x^2}\,, \quad \rho = \sqrt{r^2 + r'^2 - 2rr'\cos(\varphi'-\varphi)},$$

and so

$$G_\ell(r,\varphi;r',\varphi') = \int_{-\infty}^{\infty} \frac{e^{-\mathcal{H}\sqrt{\rho^2 + x^2}}}{4\pi D_0\sqrt{\rho^2 + x^2}}\,dx = \frac{1}{2\pi D_0} K_0(\mathcal{H}\rho) \qquad (1.147)$$

where $K_0(\mathcal{H}\rho)$ is the second Bessel function of imaginary argument which has a logarithmic singularity at the origin. The function in (1.146) is the generalization of the logarithmic potential, to which it reduces when $\mathcal{H} = 0$.

In order to obtain $G_c(r,r')$, the cylindrical diffusion kernel, it is necessary to integrate G_ℓ around a circle of radius r'. Since, from symmetry G_c must be independent of φ, we may put $\varphi = 0$ without loss of generality. Also, since G_c is normalized to one neutron per unit length of cylinder, we

must divide G_ℓ by $2\pi r'$ before integrating. Then

$$G_c(r,r') = \frac{1}{2\pi r'} \int_0^{2\pi} G_\ell\,(r,\varphi,r',\varphi')r'd\varphi' \quad . \tag{1.148}$$

This integral can be evaluated by using the addition theorem (Jahnke-Emde, p. 144)

$$\begin{aligned} K_o\left(\varkappa\sqrt{r^2 + r'^2 - 2rr'\cos\varphi'}\right) &= \sum_{-\infty}^{\infty} K_m(\varkappa r)\, I_m(\varkappa r')\cos m\varphi \qquad r > r' \\ &= \sum_{-\infty}^{\infty} K_m(\varkappa r')\, I_m(\varkappa r)\cos m\varphi \qquad r < r' \quad . \end{aligned} \tag{1.149}$$

If (1.149) is substituted into (1.148) then only the m = 0 term survives the integration, and the result is

$$\begin{aligned} G_c(r,r') &= \frac{1}{2\pi D_o} K_o(\varkappa r)\, I_o(\varkappa r') \qquad r > r' \\ &= \frac{1}{2\pi D_o} K_o(\varkappa r')\, I_o(\varkappa r) \qquad r < r' \quad . \end{aligned} \tag{1.150}$$

If $r' = 0$, then $G_c = G_\ell$, as it should. If we compare (1.150) with (1.147) we see that a cylindrical shell can be replaced by a line of strength $I_o(\varkappa r')$ at its center as far as its effect on exterior points is concerned.

We summarize the results in the following table:

(over)

Diffusion Kernels

Geometry	Notation	Source Normalization (per second)	G = flux (nv) at $\underline{r}$
Point	$G_p(\underline{r}-\underline{r}')$	1 neutron	$\dfrac{e^{-\mathcal{H}\lvert\underline{r}-\underline{r}'\rvert}}{4\pi D_o\ r-r'}$
Plane	$G_{p\ell}(x,x')$	1 neut. per cm^2	$\dfrac{e^{-\mathcal{H}\lvert x-x'\rvert}}{2\mathcal{H}D_o}$
Spherical shell	$G_s(r,r')$	1 neut. per shell of radius r'	$\dfrac{1}{8\pi rr'\mathcal{H}D_o}\left[e^{-\mathcal{H}\lvert r-r'\rvert}-e^{-\mathcal{H}\lvert r+r'\rvert}\right]$
Line	$G_\ell(r,\varphi,r'\varphi')$	1 neut. per unit length	$\dfrac{1}{2\pi D_o}K_o(\mathcal{H}\rho)$ $\rho^2=r^2+r'^2-2rr'\cos(\theta'-\theta)$
Cylindrical shell	$G_c(r,r')$	1 neut. per shell of radius r' and unit length	$\dfrac{1}{2\pi D_o}\begin{cases}K_o(\mathcal{H}r)I_o(\mathcal{H}r') & r>r'\\ K_o(\mathcal{H}r')I_o(\mathcal{H}r) & r<r'\end{cases}$

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The Transport Kernels

It is possible to write the Boltzmann equation as an integral equation whenever the scattering and the source are isotropic. To do this we return to Eq. (1.50) in which $F(x,v,\mu)$ is expressed as an integral over F_o and S, the source distribution. Since we assume S is isotropic, we may put $S(x,v,\mu) = \dfrac{S_o(x,v)}{2}$ where S_o is the total number of neutrons produced per c.c. and second. As it stands (1.50) is not quite an integral equation because the total flux $F(x,v,\mu)$ appears outside, and both functions are unknown. If, however, we integrate over μ, then we obtain an integral equation in $F_o(x,v)$:

$$F_o(x,v) = \int_{-1}^{1} F(x,v,\mu)d\mu = \frac{N\sigma_{s_o}}{2}\int_0^1\int_{-\infty}^{x} F_o(x',v)\, e^{-\frac{N\sigma}{\mu}(x-x')}\,\frac{dx'd\mu}{\mu} +$$

$$+ \frac{1}{2}\int_0^1\int_{-\infty}^{x} S_o(x',v)\, e^{-\frac{N\sigma}{\mu}(x-x')}\,\frac{dx'}{\mu}\,d\mu + \frac{N\sigma_{s_o}}{2}\int_{-1}^{0}\int_{\infty}^{x} F_o(x'v)e^{-\frac{N\sigma}{\mu}(x'-x)}\frac{dx'}{\mu}\,d\mu +$$

$$+ \frac{1}{2}\int_{-1}^{0}\int_{\infty}^{x} S_o(x',v)\, e^{\frac{N\sigma}{\mu}(x-x')}\,\frac{dx'}{\mu}\,d\mu,$$

and, by interchanging order of integration,

$$F_o(x,v) = \frac{N\sigma_{s_o}}{2}\int_{-\infty}^{\infty} F_o(x',v)\, E_1(N\sigma|x-x'|)\,dx' + \frac{1}{2}\int_{-\infty}^{\infty} S_o(x',v)\, E_1(N\sigma|x-x'|)dx' \tag{1.151}$$

where

$$E_1(y) = \int_0^1 e^{-y/\mu}\,\frac{d\mu}{\mu} = \int_y^{\infty} e^{-u}\,\frac{du}{u} \tag{1.152}$$

is the exponential integral (denoted by $-E_1(-x)$ in Jahnke-Emde). The total number of neutrons which start fresh flights per second in each cubic centimeter is $N\sigma_{s_o}F_o(x,v) + S_o(x,v)$; this quantity, which we shall call $Q(x,v)$, may be viewed as the source which furnishes neutrons for the remainder of the medium. Hence the integral equation (1.151) may be written:

$$F_o(x,v) = \frac{1}{2}\int_{-\infty}^{\infty} Q(x',v)\ E_1(N\sigma|x-x'|)dx',$$

$$Q(x',v) = N\sigma s_o(x',v) + S_o(x',v). \tag{1.153}$$

In this form the integral equation of isotropic transport theory resembles very much the integral equation (1.135) of diffusion theory. However, the transport kernel [which we denote by $K_p\ (x,x')$] is $\frac{1}{2}\ E_1(N\sigma|x-x'|)$, while the plane diffusion kernel is $\frac{1}{2\mathcal{H}D_o}\ e^{-\mathcal{H}|x-x'|}$. The two are related by the formula

$$K_{p\ell}(x,x') = D_o\int_{N\sigma}^{\infty} G_{p\ell}(\mathcal{H},x,x')d\mathcal{H}\ . \tag{1.154}$$

The transport kernel gives the flux in a unit volume at $\underline{r}$ due to a unit source at $\underline{r}'$. The transport kernels in the other geometries can be obtained similarly from the diffusion kernels by integrating with respect to $\mathcal{H}$ from $N\sigma$ to ∞ and multiplying by D_o. The results are tabulated in the following table:

Transport Kernels

Geometry	Notation	Source Normalization	K
Point	$K_p(\underline{r},\underline{r}')$	1 neut. per sec.	$\frac{1}{4\pi}\ \frac{e^{-N\sigma\|\underline{r}-\underline{r}'\|}}{\|r-r'\|^2}$
Plane	$K_{p\ell}(x,x')$	1 neut. per cm^2 per sec.	$\frac{1}{2}\ E_1(N\sigma\|x-x'\|)$

cont'd . . . . . .

Transport Kernels (cont'd)

Geometry	Notation	Source Normalization	K
Spherical Shell	$K_s(r,r')$	1 neut. per shell per sec.	$\frac{1}{8\pi rr'}\left[E_1(N\sigma\lvert r-r'\rvert) - E_1(N\sigma\lvert r+r'\rvert)\right]$
Line	$K_\ell(r,\varphi,r',\varphi')$	1 neut. per cm. per sec.	$\frac{N\sigma}{2\pi}\int_1^\infty K_o(N\sigma\rho y)dy,$ $\rho = \sqrt{r^2 + r'^2 - 2rr'\cos(\varphi'-\varphi)}$
Cylindrical Shell	$K_c(r,r')$	1 neut. per cm. per sec.	$\frac{N\sigma}{2\pi}\int_1^\infty K_o(N\sigma ry)I_o(N\sigma r'y)dy \quad r > r'$ $\frac{N\sigma}{2\pi}\int_1^\infty K_o(N\sigma r'y)I_o(N\sigma ry)dy \quad r < r'$

The equivalence between a spherical shell and a point, or a cylindrical shell and a line, which holds for the diffusion and the potential kernels does not hold for the transport kernels.

As an example of the use of a transport kernel we calculate, according to transport theory, the depression in neutron density caused by a thin foil which is introduced into an infinite medium in which monoenergetic neutrons are being produced everywhere at the constant rate q. This problem was treated by diffusion theory in a preceding paragraph, and here we use the same notation. The discussion which follows is in part due to Skyrme.

The integral equation for $F_o(\underline{r},v)$ in the absence of the foil, is

$$F_o(\underline{r},v) = \int_{\substack{\text{all}\\\text{space}}} N\sigma_{s_o} F_o(\underline{r}',v)\,\frac{e^{-N\sigma|\underline{r}-\underline{r}'|}}{4\pi|r-r'|^2}\,d\underline{r}' + \int_{\substack{\text{all}\\\text{space}}} q\,\frac{e^{-N\sigma|\underline{r}-\underline{r}'|}}{4\pi|r-r'|^2}\,d\underline{r}' \qquad (1.155)$$

and this has the solution

$$F_o(\underline{r},v) = \frac{q}{N\sigma_a} , \tag{1.156}$$

as can be verified by substitution into (1.155). If the foil is present, then it absorbs $(N\sigma_a)_F F_o$ neutrons per c.c. per second; this acts as a negative source. Further if we assume that the neutron makes no collisions in traversing the foil, except absorptions, so that the one medium transport kernel is applicable, then the integral equation for F_o must be

$$F_o(\underline{r},v) = \int_{\substack{\text{all}\\ \text{space}}} \left[N\sigma_{s_o} F_o(r,v) + q\right] \frac{e^{-N\sigma|\underline{r}-\underline{r}'|}}{4\pi|\underline{r}-\underline{r}'|^2}\, dr' - (N\sigma_a)_F \int_V F_o(\underline{r},v) \frac{e^{-N\sigma|\underline{r}-\underline{r}'|}}{4\pi|\underline{r}-\underline{r}'|^2}\, dr' . \tag{1.157}$$

Again, if the foil is so small that it hardly absorbs any neutrons, we can solve this equation by successive approximations, the first approximation being to put $F_o = \frac{q}{N\sigma_a}$. The result is

$$F_o(\underline{r},v) = \frac{q}{N\sigma_a}\left[1 - \frac{(N\sigma_a)_F}{4\pi}\int_V \frac{e^{-N\sigma|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|^2}\, dr'\right] \tag{1.158}$$

while the diffusion theory result (we replace Φ_o of Eq. 1.133b by F_o) is

$$F_o(\underline{r},v) = \frac{q}{N\sigma_a}\left[1 - \frac{(N\sigma_a)_F}{4\pi D_o}\int_V \frac{e^{-\varkappa|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|}\, dr'\right] .$$

If the foil is a small sphere of radius r_o, center at $r = 0$, these integrals are easily evaluated. The results for the density at the center of the sphere are

$$F_o(o,v) = \frac{q}{N\sigma_a}\left[1 - \frac{(N\sigma_a)_F}{N\sigma}(1 - e^{-N\sigma r_o})\right] \quad \text{by transport theory} \tag{1.159}$$

$$F_o(o,v) = \frac{q}{N\sigma_a}\left[1 - \frac{(N\sigma_a)_F}{N\sigma_a}(1 - e^{-\mathcal{L} r_o}(1 + \mathcal{L} r_o)\right] \quad \text{by diffusion theory.} \tag{1.160}$$

Evidently, since the kernels used in this and in the diffusion calculation applied only to a single medium, the equations (1.159) and (1.160) will not give the depression in the interior of the foil correctly. This correction may be comparable to the depression outside the foil, especially if the foil absorbs neutrons heavily. We will calculate it later.

If the foil is so small that $N\sigma r_o \ll 1$, then the depressions at the center are

$$\frac{F_o}{q/N\sigma_a} = 1 - (N\sigma_a)_F r_o \qquad \text{by transport theory}$$

$$\frac{F_o}{q/N\sigma_a} = 1 - \tfrac{3}{2}(N\sigma_a)_F(N\sigma)r_o^2 \qquad \text{by diffusion theory.}$$

The difference between the two results is by no means negligible.

Solution of the Steady State Diffusion Equation in Various Geometries; Measurement of Diffusion Length

In this section we give a few examples of the calculation of the thermal neutron distribution in systems of particular shape and with certain source distributions.

A. Rectangular parallelopiped of size $x = \bar{a}$, $y = \bar{b}$, $z = \infty$. Source in $z = 0$ plane distributed like $f(x,y)$.

The neutron flux in this case is the solution of

$$\Delta\Phi_o - \varkappa^2\Phi_o = 0 \tag{1.161}$$

with the initial condition

$$D_o \left.\frac{\partial\Phi_o}{\partial z}\right|_{z=0} = -\frac{1}{2} f(x,y), \tag{1.162}$$

and the boundary conditions

$$\Phi_o = 0 \qquad x = \pm\frac{a}{2} \qquad y = \pm\frac{b}{2} \tag{1.163}$$

where a and b are the geometric sides, $\bar{a}$, $\bar{b}$, augmented by twice the extrapolation distance (.71 λ_{tr} in a weakly absorbing medium). The solution of (1.161) which satisfies all boundary and initial conditions is

$$\Phi_o(x,y,z) = \frac{2}{D_o a^2 b^2} \sum_n \sum_m f_{mn} L_{mn} \cos\frac{(2m+1)\pi x}{a} \cos\frac{(2n+1)\pi y}{b}\, e^{-z/L_{mn}} \tag{1.164}$$

where

$$\frac{1}{L_{mn}^2} = \varkappa^2 + \frac{(2m+1)^2\pi^2}{a^2} + \frac{(2n+1)^2\pi^2}{b^2} \tag{1.165}$$

and

$$f_{mn} = ab \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} f(x,y) \cos(2m+1)\frac{\pi x}{a} \cos(2n+1)\frac{\pi y}{b} \cdot dx dy.$$

The neutron flux falls off from the source as a sum of exponentials with relaxation lengths L_{mn} given by (1.165). The relaxation length of each harmonic decreases as the order m, n of the harmonic increases. Far from the source only the (o, o) harmonic remains.

The measurement of the diffusion length of a substance is usually done by measuring the distribution along the z direction of thermal neutrons in a block of the material in which a source of thermal neutrons has been placed. Since the actual distribution in the z direction is a sum of many harmonics, in order to deduce the relaxation length from the observed neutron distribution, it is necessary to take measurements far from the source, if intensity permits, or else to correct the observed relaxation length to the (o,o) relaxation length by subtracting the effect of the higher harmonics. The diffusion length is obtained from the observed L_{oo}^2 by equation (1.165); i.e.,

$$\varkappa^2 = \frac{1}{L^2} = \frac{1}{L_{oo}^2} - \frac{\pi^2}{a^2} - \frac{\pi^2}{b^2} \quad . \tag{1.167}$$

If the absorber is weak the diffusion length is long. Hence, unless the sides of the block are very large $\left(\text{i.e.}\ \frac{\pi^2}{a^2}, \frac{\pi^2}{b^2} \ll \frac{1}{L_{oo}^2}\right)$, the reciprocal diffusion length appears as the small difference of two relatively large numbers. In order to obtain results which are meaningful it is therefore necessary to measure L_{oo}, a, and b with extreme accuracy.

The technique which has generally been used on the Plutonium Project for reducing the data in a diffusion length measurement has been the following:

1) From a knowledge of the source disposition the strength of the higher harmonics is estimated. These are subtracted from the observed neutron distribution to give the (o,o) harmonic. By a judicious choice of x,y coordinates for the neutron detectors, it is possible to eliminate a few of the important harmonics. For example, in a square block, the (1,n) and (m,1) harmonic vanish at x = a/6, y = a/6.

2) Since the block is always of finite length, it is necessary to add an "end correction" to the observed intensities close to the end of the block. Suppose the extrapolated length of the block is z_o. Then the neutron distribution which is zero at $z = z_o$ and satisfies (1.162) in the source plane is

$$\Phi_o(x,y,z) = \frac{2}{D_o a^2 L^2} \sum_m \sum_n \frac{f_{mn} L_{mn}}{1 + e^{-2z_o}} \left[e^{-z/L_{mn}} - e^{(z-2z_o)/L_{mn}} \right] \cos(2m+1)\frac{\pi x}{a} \cos\frac{(2n+1)\pi y}{b} . \tag{1.168}$$

The reflected "wave" $e^{(z-2z_o)/L_{mn}}$, which can be considered to arise from a negative image source in the plane $z = 2z_o$, is negligible compared to the incident wave, $e^{-z/L_{mn}}$, unless z is close to the extrapolated edge of the block. The end correction is made by subtracting the reflected wave, as estimated by the expression (1.168), from the observed distribution near the boundary.

3) After the harmonic and end corrections have been made it is customary to make a least squares fit to the longitudinal (z) distribution. The relaxation length of the best fitting exponential is used as L_{oo} in (1.167), and from this L is determined.

4) Since the results are very sensitive to the values of a and b, transverse (x,y) neutron distributions are usually taken. If only one harmonic is present then the transverse distribution is strictly the product of two cosines, the half wave length of which are the extrapolated dimensions of the block. The strength of higher harmonics can be estimated from a harmonic analysis of the transverse distribution, although usually it is sufficiently accurate to compute these from a knowledge of the source distribution.

The most convenient source for a diffusion length measurement is a large block of graphite set on top of a chain reacting pile. Such a graphite block is called a thermal column, since neutrons from the pile are practically all reduced to thermal energy in the block, provided it is large enough. If the block whose diffusion length is to be measured is placed on top of the thermal column, then the neutrons impinging on it will all be thermal, and the theory outlined is immediately applicable.

Before chain reacting piles were available, thermal columns as neutron sources were impractical because neutron intensities were never high enough. Diffusion length measurements were performed by first measuring the thermal neutron distribution when an uncovered Ra-Be source (of fast neutrons) was in the source position, and then when the Ra-Be source was covered by a Cd sheet which absorbs all thermal neutrons. The difference between the thermal neutron distributions in the two cases is just the thermal neutron distribution due to a source of pure thermal neutrons at the position of the Cd sheet. This can be seen by writing down the equation for the thermal neutron density in the two cases. Without the Cd sheet the flux Φ_0' satisfies

$$\Phi_0' - \mathcal{L}^2\Phi_0' + \frac{q}{D_0} = 0$$

where q(x,y,z) is the number of neutrons which become thermal per second in the block. With the Cd sheet in place the flux Φ_0 satisfies the equation

$$\Phi_0'' - \mathcal{L}^2\Phi_0'' + \frac{q}{D_0} = 0$$

but with the boundary condition $\Phi_0'' = 0$ at $z = 0$, the extrapolated position of the Cd sheet. The difference $\Phi_0 = \Phi_0' - \Phi_0''$ satisfies

$$\Phi_0 - \mathcal{L}^2\Phi_0 = 0,$$

with the boundary condition

$$\Phi_o(x,y,o) = \Phi_o'(x,y,o),$$

where $\Phi_o'(x,y,o)$ is the measured distribution at $z = 0$ with the Cd sheet.

The Time-Dependent Diffusion Equation

The diffusion problems considered in the previous section have all been stationary problems. The neutron density was considered to be independent of time, and only the stationary spatial and angular distributions were sought. In this section we consider the non-stationary neutron diffusion problem from the elementary standpoint.

We consider an infinite plane system in which mono-energetic neutrons diffuse. According to elementary diffusion theory, the equation satisfied by n_o, the neutron density of speed v, is

$$D_o \frac{vd^2n_o}{dx^2}(x,v,t) - N\sigma_a v n_o(x,v,t) + S_o(x,v,t) = \frac{\partial n_o}{\partial t}(x,v,t), \qquad (1.169)$$

where $S_o(x,v,t)$ is the number of neutrons of speed v produced per c.c. and second at (x,t). Equation (1.169) is the same as the heat equation with leakage. Its solution for an instantaneous unit source at the origin, which emits one neutron per cm^2, $S_o(x,v,t) = \delta(x)\,\delta(t)$, is found by the usual Fourier transform method. Thus, putting

$$\delta(x)\,\delta(t) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} e^{i\omega t} e^{i\nu x} d\omega\, d\nu, \quad n_o = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} A(\omega,\nu) e^{i\omega t} e^{i\nu x} d\omega\, d\nu$$

and substituting into (1.169) we obtain

$$A = \frac{1}{D(\nu^2 + \varkappa^2 + \frac{i\omega}{D})} ,$$

where, as usual $D = D_o v$, $\varkappa^2 = \frac{N\sigma_a v}{D}$; hence

$$n_o = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \frac{e^{i\omega t} e^{i\nu x}}{D(\nu^2 + \varkappa^2 + \frac{i\omega}{D})} d\omega d\nu .$$

This integral can be evaluated readily by integrating first over ω and then over ν. The integrand has a simple pole at $\omega = iD(\nu^2 + \varkappa^2)$, and its residue there is $\frac{1}{i} e^{-D(\nu^2 + \varkappa^2)t}$. Hence the integral over ν has the value $2\pi e^{-D(\nu^2 + \varkappa^2)t}$. The integral over ν is just the Fourier transform of a Gaussian function, and this is another Gaussian. Hence

$$n_o(x,v,t) = \frac{e^{-\frac{x^2}{4Dt} - \varkappa^2 Dt}}{(4\pi Dt)^{1/2}} , \qquad (1.170)$$

which is the well known one-dimensional non-steady diffusion kernel. The properties of this function are very familiar since it also represents the temperature distribution from an instantaneous unit heat source. At any given time, the neutron distribution is Gaussian with a range $\sqrt{2Dt}$, and an amplitude $\frac{e^{-D\varkappa^2 t}}{(4\pi Dt)^{1/2}}$. The attenuation factor $e^{-D\varkappa^2 t}$, of course, arises from the absorption by the medium. At any given point the neutron intensity waxes and wanes, reaching a maximum at time $\frac{\sqrt{4\varkappa^2 x^2 + 1} - 1}{4\varkappa^2 D}$. If there is no absorption, the maximum is reached at time $t = x^2/2D$. For thermal neutrons diffusing in

graphite, $\lambda = 2.7$ cm, $v = 2.2 \times 10^5$ cm/sec., $2D = 39.6 \times 10^4$ cm^2/sec., and the time for the neutron intensity to reach its maximum at a distance of 100 cm is about 25 milliseconds. Such a time lag is easily observable with standard electronic equipment.

According to (1.170), the effect of an instantaneous neutron source is felt everywhere immediately, although (except at the source) with small intensity. Evidently this cannot be quite correct since a neutron burst requires at least the time x/v to travel the distance x from the source. During this "retarded time", no neutrons can appear at x. It will be shown later that the correct elementary theory time dependent neutron diffusion equation is really the equation of telegraphy, (whose solutions are in fact retarded) rather than equation (1.169) of heat conduction.

The retardation time is $t_{ret} = x/v$, while the time $t_{diffusion}$ for a neutron burst to reach a maximum at x is $x^2/2D$. The ratio of the two times is

$$\frac{t_{ret}}{t_{diffusion}} = \frac{2D}{xv} = \frac{2}{3}\frac{\lambda_d}{x} .$$

At distances from the source large compared with a diffusion mean free path the retarded time is negligible compared to the diffusion time. Since most experiments involve the neutron distribution far from the source, it is permissible to ignore the retardation and to describe time dependent diffusion by means of the heat equation instead of the telegrapher's equation.

It is convenient at this point to give the non-stationary diffusion kernels, $G(\underline{r},t',\underline{r}',t)$ in various geometries. The method of deriving these kernels from the corresponding plane kernel is exactly the same as that used in a previous section for the steady state diffusion kernels.

Time-Dependent Diffusion Kernels

Geometry	Notation	Source Normalization	
Plane	$G_{p\ell}(x,t,x',t')$	1 neut/cm^2 at (x',t')	$\dfrac{e^{-\frac{\lvert x-x'\rvert^2}{4D(t-t')} - \mathcal{H}^2 D(t-t')}}{\left[4\pi D(t-t')\right]^{1/2}}$
Point	$G_p(\underline{r},t,\underline{r}',t')$	1 neut. at $(\underline{r}',t')$	$\dfrac{e^{-\frac{\lvert \underline{r}-\underline{r}'\rvert^2}{4D(t-t')} - \mathcal{H}^2 D(t-t')}}{\left[4\pi D(t-t')\right]^{3/2}}$
Line	$G_\ell(r,\varphi,t,r',\varphi',t')$	1 neut/cm at (r',φ',t')	$\dfrac{e^{-\frac{\rho^2}{4D(t-t')} - \mathcal{H}^2 D(t-t')}}{\left[4\pi D(t-t')\right]^{3/2}}$ $\rho^2=r^2+r'^2 - 2rr'\cos(\varphi-\varphi')$
Spherical Shell	$G_s(r,t,r',t')$	1 neut. per shell of radius r' at time t'	$\dfrac{e^{-\mathcal{H}^2 D(t-t')}}{4\pi rr'} \times \left[\dfrac{e^{-\lvert r-r'\rvert^2/4D(t-t')}}{\left[4\pi D(t-t')\right]^{1/2}} - \dfrac{e^{-\lvert r+r'\rvert^2/4D(t-t')}}{\left[4\pi D(t-t')\right]^{1/2}}\right]$
Cylindrical Shell	$G_c(r,t,r',t')$	1 neut/cm of shell r' at time t'	$\dfrac{e^{-\frac{\lvert r-r'\rvert^2}{4D(t-t')} - \mathcal{H}^2 D(t-t')}}{4\pi D(t-t')} \times I_o\left(\dfrac{rr'}{2D(t-t')}\right)$

It is useful to observe that the point, line and plane kernels differ only by powers of $\frac{1}{\left[4\pi D(t-t')\right]^{1/2}}$, and that the power of this quantity which appears in the kernel is just equal to the number of dimensions involved; e.g., the point kernel (3 dimensions) has the factor $\left[4\pi D(t-t')\right]^{3/2}$ in the denominator.

Propagation of Neutron Waves

Suppose that instead of an instantaneous source of monoenergetic neutrons at x = 0, there is a localized source whose intensity oscillates with angular frequency ω:

$$S_o(x,v,t) = \delta(x)e^{i\omega t},$$

where it is understood here and in the following that only the real part of a complex function is to be used. The neutron intensity at (x,t) is found, exactly as in the previous paragraph to be

$$n_o(x,v,t) = \frac{e^{i\omega t}}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\nu x}\,d\nu}{D(\nu^2 + \kappa^2 + \frac{i\omega}{D})} .$$

The integrand has a pole above the real axis at $\nu = +i\sqrt{\kappa^2 + i\omega/D}$, and its residue there is just $e^{-x\sqrt{\kappa^2 + i\omega/D}}\Big/\left(2Di\sqrt{\kappa^2 + i\omega/D}\right)$. Hence the value of n_o, which is $2\pi i$ times the residue times $\frac{e^{i\omega t}}{2\pi}$, is

$$n_o(x,v,t) = e^{i\omega t}\,\frac{e^{-x\sqrt{\kappa^2 + i\omega/D}}}{2D\sqrt{\kappa^2 + i\omega/D}} , \qquad (1.171)$$

i.e., the distribution at time t is the same as from a stationary source of strength $e^{i\omega t}$, but the relaxation distance is the complex number $\sqrt{\kappa^2 + i\omega/D}$.

In order to understand the physical significance of the complex relaxation distance we write

$$\sqrt{\varkappa^2 + i\omega/D} = \sqrt{\frac{\varkappa^2 + \sqrt{\varkappa^4 + \omega^2/D^2}}{2}} + i\sqrt{\frac{-\varkappa^2 + \sqrt{\varkappa^4 + \omega^2/D^2}}{2}}$$

Substituting into (1.171),

$$n_o(x,v,t) = \frac{e^{i\left(\omega t - \sqrt{\frac{-\varkappa^2 + \rho^2}{2}}\, x\right)} e^{-\sqrt{\frac{\varkappa^2 + \rho^2}{2}}\, x}}{2D\sqrt{\varkappa^2 + i\omega/D}} \tag{1.172}$$

where

$$\rho^2 = \sqrt{\varkappa^4 + (\omega/D)^2} .$$

According to (1.172) the neutron density from an oscillating source is propagated as a damped wave; the velocity v_w of the wave is

$$v_w = \omega\sqrt{\frac{2}{\rho^2 - \varkappa^2}} \quad ; \tag{1.173}$$

its wave length ℓ_w is

$$\ell_w = \frac{2\pi v_w}{\omega} = 2\pi\sqrt{\frac{2}{\rho^2 - \varkappa^2}} ; \tag{1.174}$$

and its attenuation distance α, which is the distance over which its intensity falls by a factor e, is

$$\alpha = \sqrt{\frac{2}{\varkappa^2 + \rho^2}} . \tag{1.175}$$

The propagation velocity depends on the frequency, becoming larger as the frequency increases. Thus the medium in which the neutron waves travel is dispersive. The amplitude and the wave length of the wave fall off with increasing ω (cf. Eq. 1.172).

In the extreme cases, $\omega/D \gg \varkappa^2$ and $\omega/D \ll \varkappa^2$, the formulas for velocity, etc., become quite simple. We tabulate them below:

	$\omega/D \gg \varkappa^2$	$\omega/D \ll \varkappa^2$
v_w	$\sqrt{2D\omega/(1 - \varkappa^2 D/\omega)}$	$2\varkappa D\left[1 + \frac{1}{8}\left(\frac{\omega}{D\varkappa^2}\right)^2\right]$
ℓ_w	$2\pi\sqrt{\frac{2D}{\omega - \varkappa^2 D}}$	$(4\pi\varkappa D/\omega)\left[1 + \frac{1}{8}\left(\frac{\omega}{D\varkappa^2}\right)^2\right]$
α	$\sqrt{2(D/\omega)/(1 + \varkappa^2 D/\omega)}$	$\frac{1}{\varkappa}\left[1 - \frac{1}{8}\left(\frac{\omega}{D\varkappa^2}\right)^2\right]$.

The complete analogy between the propagation of neutron waves and the propagation of heat waves is evident from the foregoing discussion. The analogy was first pointed out by E. P. Wigner who proposed that such basic constants as the thermal diffusion length, $1/\varkappa$, and diffusion constant, D, could be determined by measuring the wave length and attenuation constant of neutron waves in a weakly absorbing medium. Such experiments would be completely analogous to the famous Ångström method for measuring thermal conductivity. With the high neutron intensities available from a chain reacting pile, such experiments should now be feasible.

We list below representative values of v_w, ℓ_w, and α for thermal neutrons in graphite (D_o = 0.9 cm, v = 2.2 x 10^5 cm/sec, $\mathcal{H}$ = .02 cm^{-1})

	$\omega = 10\ sec^{-1}$	$\omega = 1000\ sec^{-1}$
v_w	7.9 x 10^3 cm/sec	2.1 x 10^4 cm/sec
ℓ_w	5.0 x 10^3 cm	132 cm
α	50 cm	19 cm.

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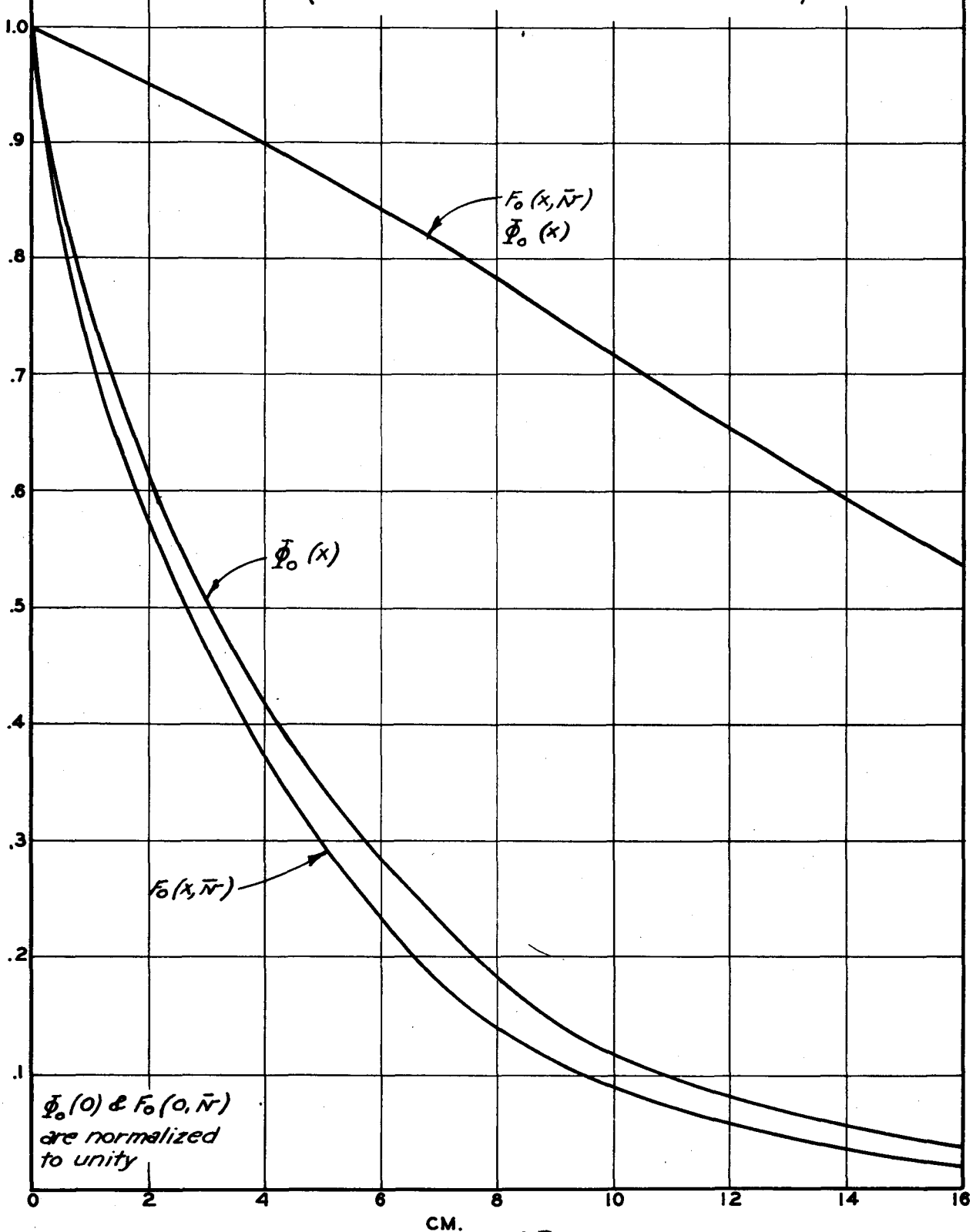
Drawing # 11265

FIG. I-II

MAXWELLIAN NEUTRONS DIFFUSING IN A URANIUM SLAB
(TOTAL FLUX & FLUX FOR AVERAGE VELOCITY)



CHAPTER II

SECOND ORDER DIFFUSION THEORY

The diffusion theory described in the previous chapter is called "first-order" diffusion theory since it is based on the assumption that the angular distribution of the neutron flux could be represented adequately by the simplest possible combination of spherical harmonics, namely $P_0(\mu)$ and $P_1(\mu)$. Under conditions where this assumption is valid -- e.g., in weakly absorbing media far from sources or boundaries -- it was seen that Fick's Law was valid and the diffusion process could be represented adequately by a second-order differential equation.

A very natural extension of first-order diffusion theory is obtained by cutting off the series in spherical harmonics which represents the distribution not at the first two terms, as in first-order theory, but at a higher number of terms. The coefficients in the expansion are then determined so that the Boltzmann equation is satisfied approximately by the assumed distribution. Such "second-order" diffusion theory would be expected to predict neutron distributions which are more accurate than those of simple diffusion theory, the neutron distribution approaching more and more closely the exact distribution as more terms are kept.

The method of "second-order" diffusion theory (also called the spherical harmonics method) was first suggested by Jeans in connection with certain problems of transfer of radiation in stars. It was proposed and used on the Plutonium Project essentially independently by both the Chicago and Montreal groups;

its great power as a computational technique was recognized only after the systematic work of Placzek and his co-workers in Montreal.

The main features of the spherical harmonics method can be illustrated by considering the one-dimensional, monoenergetic transport equation with isotropic scattering:

$$\frac{\mu dF(x,u)}{dx} + N\sigma F(x,\mu) = \frac{N\sigma_{s_0}}{2}\int_{-1}^{1} F(x,\mu)d\mu + S(x,\mu) \tag{2.1}$$

We expand the source distribution and the neutron angular distribution in Legendre polynomials:

$$S(x,\mu) = \sum \frac{2\ell+1}{2} S_\ell(x) P_\ell(\mu) \tag{2.2}$$

$$F(x,\mu) = \sum \frac{2\ell+1}{2} F_\ell(x) P_\ell(\mu) \tag{2.3}$$

where, as usual,

$$S_\ell(x) = \int_{-1}^{1} S(x,\mu) P_\ell(\mu)d\mu, \quad F_\ell = \int_{-1}^{1} F(x,\mu) P_\ell(\mu)d\mu \tag{2.4}$$

The coefficients S_ℓ are known, since the source distribution $S(x,\mu)$ is known; on the other hand, the coefficients F_ℓ are to be determined so that (2.3) satisfies the Boltzmann equation. We now substitute (2.3) and (2.2) into (2.1):

$$\sum \frac{2\ell+1}{2}\mu F_\ell'(x) P_\ell(\mu) + \sum \frac{2\ell+1}{2} N\sigma F_\ell(x)P_\ell(\mu, \quad \frac{N\sigma_{s_0}}{2}\int_{-1}^{1}\sum \frac{2\ell+1}{2} F_\ell(x)P_\ell(\mu)d\mu$$

$$+ \sum \frac{2\ell+1}{2} S_\ell(x) P_\ell(\mu) \tag{2.5}$$

where ' denotes $\frac{d}{dx}$. We multiply (2.5) by $P_\nu(\mu)$ and integrate over μ. Then, upon making use of the identities (Jahnke-Emde, p. 114),

$$\int_{-1}^{1} \mu P_\nu(\mu) P_\ell(\mu)\, d\mu = \frac{(\nu+1)}{(2\nu+1)} \int_{-1}^{1} P_{\nu+1}(\mu) P_\ell(\mu)\, d\mu + \frac{\nu}{2\nu+1} \int_{-1}^{1} P_{\nu-1}(\mu) P_\ell(\mu)\, d\mu$$

$$= \frac{\nu+1}{(2\nu+1)} \cdot \frac{2}{2(\nu+1)+1} \qquad \text{if } \ell = \nu+1$$

$$= \frac{\nu}{2\nu+1} \cdot \frac{2}{2(\nu-1)+1} \qquad \text{if } \ell = \nu-1 \tag{2.6}$$

$$= 0 \qquad \text{if } \ell \neq \nu+1 \text{ or } \nu-1$$

and

$$\int_{-1}^{1} P_\nu^2(\mu)\, d\mu = \frac{2}{2\nu+1}, \quad \int_{-1}^{1} P_\nu(\mu) P_\ell(\mu)\, d\mu = 0, \quad \nu \neq \ell \tag{2.7}$$

we obtain the following:

$$\int_{-1}^{1} P_\nu(\mu) \sum \frac{2\ell+1}{2} \mu F'_\ell P_\ell(\mu)\, d\mu = \frac{\nu+1}{2\nu+1} F'_{\nu+1} + \frac{\nu}{2\nu+1} F'_{\nu-1} \tag{2.8}$$

$$N\sigma \int_{-1}^{1} P_\nu(\mu) \sum \frac{2\ell+1}{2} F_\ell P_\ell(\mu)\, d\mu = N\sigma F_\nu \tag{2.9}$$

$$\frac{N\sigma_{so}}{2} \int_{-1}^{1} P_\nu(\mu')\, d\mu' \int_{-1}^{1} \sum \frac{2\ell+1}{2} F_\ell P_\ell(\mu)\, d\mu = N\sigma_{so} F_0 \qquad \text{if } \nu = 0 \tag{2.10}$$

$$= 0 \qquad \text{if } \nu \neq 0$$

$$\int_{-1}^{1} P_\nu(\mu) \sum \frac{2\ell+1}{2} \; S_\ell \, P_\ell(\mu) \, d\mu = S_\nu \; . \tag{2.11}$$

The Boltzmann equation thus reduces to the following infinite set of linear differential equations:

$$\frac{\nu+1}{2\nu+1} \; F'_{\nu+1} + \frac{\nu}{2\nu+1} \; F'_{\nu-1} + N\sigma F_\nu = N\sigma_{so} \, F_o \, \delta_{\nu o} + S_\nu \quad , \tag{2.12}$$

$$F'_1 + N\sigma F_o = N\sigma_{so} \, F_o + S_o \tag{2.13}$$

$$\frac{2}{3} F'_2 + N\sigma F_1 + \frac{1}{3} F'_o = S_1$$

$$\frac{3}{5} F'_3 + N\sigma F_2 + \frac{2}{5} F'_1 = S_2$$

$$\frac{4}{7} F'_4 + N\sigma F_3 + \frac{3}{7} F'_2 = S_3$$

The essence of the spherical harmonics method is to cut equations (2.13) off at a finite number and to solve the remaining equations for $F_\nu(x)$. The larger the number of equations kept, the more accurate is the resulting distribution; however, the effort required to solve the equations (2.13) increases rapidly as the number of terms goes up.

The great power and versatility of the spherical harmonics method lies in the fact that by means of it an integro-differential equation is replaced by a set of ordinary differential equations. This reduction is possible even if the scattering is non-isotropic or if the cross sections are functions of position.

Hence, in principle, by the spherical harmonics method it is possible to solve any energy independent transport problem to any required degree of accuracy.

Solution of the linear equations

Because there is a source of neutrons, equations (2.13) are non-homogeneous. The general solution is therefore obtained by adding to any particular solution of the inhomogeneous equations, the general solutions of the homogeneous set,

$$
\begin{aligned}
&F_1' + N\sigma_a F_o = 0 \\
&\tfrac{2}{3} F_2' + N\sigma F_1 + \tfrac{1}{3} F_o' = 0 \\
&\tfrac{3}{5} F_3' + N\sigma F_2 + \tfrac{2}{5} F_1' = 0 \\
&\tfrac{4}{7} F_4' + N\sigma F_3 + \tfrac{3}{7} F_2' = 0 \\
&- - - - - - - - - - - - - -
\end{aligned}
\tag{2.14}
$$

The infinite set (2.14) will be solved by successive approximations. The ν-th approximation corresponds to the assumption that the coefficients of all $P_\nu(\mu)$ higher than the ν-th vanish, and will therefore be referred to as the P_ν approximation.

Suppose that the successive approximations are begun with the P_1 approximation. (The P_o approximation, F_o = constant, $F_{1,2,\ldots} \equiv 0$ leads only to the trivial solution $F_o \equiv 0$.) Then $F_{2,3,\ldots} \not\equiv 0$, and the equations (2.14) reduce to

$$
\begin{aligned}
&F_1' + N\sigma_a F_o = 0 \\
&N\sigma F_1 + \tfrac{1}{3} F_o' = 0
\end{aligned}
\tag{2.15}
$$

i.e.,

$$\frac{1}{3N\sigma} F_o'' - N\sigma_a F_o = 0 \tag{2.16}$$

which is just the first order diffusion equation in an absorbing medium whose diffusion length is $\left(\frac{1}{3N\sigma\, N\sigma_a}\right)^{1/2}$.

The next approximation might be taken to include P_2 terms but no $P_{3,4,\ldots}$ terms. The P_2 approximation is

$$F_1' + N\sigma_a F_o = 0 \tag{2.17}$$

$$\frac{2}{3} F_2' + N\sigma F_1 + \frac{1}{3} F_o' = 0$$

$$N\sigma F_2 + \frac{2}{5} F_1' = 0$$

which, upon elimination of F_1 and F_2 leads to the second order differential equation

$$\frac{1}{3N\sigma}\left(1 + \frac{4}{5}\frac{\sigma_a}{\sigma}\right) F_o'' - N\sigma_a F_o = 0 . \tag{2.18}$$

This again has the same form as the first order diffusion equation; however, the reciprocal diffusion length is given this time by

$$\varkappa^2 = \frac{3N\sigma\, N\sigma_a}{1 + \frac{4}{5}\sigma_a/\sigma} . \tag{2.19}$$

If σ_a/σ is small, this can be written

$$\varkappa^2 \approx 3N\sigma\, N\sigma_a \left(1 - \frac{4}{5}\sigma_a/\sigma\right) \tag{2.20}$$

which, on comparison with (1.71b) is seem to be a closer approximation to the asymptotic value for $\mathcal{H}^2$ than is (2.16). Thus the P_2 approximation represents an improvement over the P_1 approximation; however, since both P_1 and P_2 approximations lead to second order equations for F_o, the solutions for F_o are in both cases just the asymptotic solution with $\mathcal{H}^2$ given by $3N\sigma N\sigma_a$ or $3N\sigma N\sigma_a/(1+\frac{4}{5}\frac{\sigma_a}{\sigma})$ respectively. In this sense the P_2 approximation is no more general than the P_1 approximation.

The P_3 approximation leads to a fourth order equation. F_o in this approximation therefore involves, in addition to an asymptotic solution, another solution which, as we shall explain presently, is non-asymptotic. The P_4 approximation again leads to a fourth order equation and therefore is no more general than the P_3 approximation.

More generally, the number of non-asymptotic solutions remains unchanged in going from the P_{2n+1} to the P_{2n+2} approximations; i.e., the successive approximation proceeds in steps of two spherical harmonics. The approximation scheme can therefore proceed according to either of two sequences, the even approximations, $P_1, P_3, P_5 ..$ or the odd approximations P_2, P_4, P_6 ....

The next approximation, after P_1, is P_3; i.e., $F_{0,1,2,3} \neq 0$, $F_{4,5...} \equiv 0$. The equation for F_o in this case is of the fourth order; however, it is more convenient to solve directly the original set of 4 first order equations (2.14). We set

$$F_\nu(x) = \sum_j A_{\nu j}\, e^{\mathcal{H}_j x} \tag{2.21}$$

and substitute into (2.14). The resulting linear algebraic equations for each $A_{\nu j}$ are

$$
\begin{aligned}
N\sigma_a A_{oj} + \mathcal{H}_j A_{1j} &= 0 \\
\frac{\mathcal{H}_j}{3} A_{oj} + N\sigma A_{1j} + \frac{2}{3}\mathcal{H}_j A_{2j} &= 0 \\
\frac{2}{5}\mathcal{H}_j A_{1j} + N\sigma A_{2j} + \frac{3}{5}\mathcal{H}_j A_{3j} &= 0 \\
\frac{3}{7}\mathcal{H}_j A_{2j} + N\sigma A_{3j} &= 0 .
\end{aligned}
\tag{2.22}
$$

The consistency condition for this set is that the determinant of the coefficients vanishes. This leads to the following equation for $\mathcal{H}_j$:

$$
\frac{9}{35(N\sigma)^2}\mathcal{H}_j^4 - \left(1 + \frac{11\sigma_a}{7\sigma}\right)\mathcal{H}_j^2 + 3N\sigma_a N\sigma = 0 \tag{2.23}
$$

of which the four roots are

$$
\mathcal{H}_1 \doteq \pm N\sqrt{3\sigma\,\sigma_a}\left(1 - \frac{2}{5}\frac{\sigma_a}{\sigma}\right) \tag{2.24}
$$

$$
\mathcal{H}_2 \doteq \pm \frac{1}{3}\sqrt{35}\,N\sigma\left(1 + \frac{2}{5}\frac{\sigma_a}{\sigma}\right) . \tag{2.25}
$$

The neutron distribution is therefore

$$
F_o = A_{o1}\, e^{|\mathcal{H}_1|x} + A_{o2}\, e^{-|\mathcal{H}_1|x} + A_{o3}\, e^{|\mathcal{H}_2|x} + A_{o4}\, e^{-|\mathcal{H}_2|x} \tag{2.26}
$$

where the A_{oj} are arbitrary. The expressions for the other F_γ are easily found from the equations (2.22). Thus

$$F_1 = -\frac{N\sigma_a}{|\mathcal{H}_1|}\left[A_{o1}\, e^{|\mathcal{H}_1|x} - A_{o2}\, e^{-|\mathcal{H}_1|x}\right] - \frac{N\sigma_a}{|\mathcal{H}_2|}\left[A_{o3}\, e^{|\mathcal{H}_2|x} - A_{o4}\, e^{-|\mathcal{H}_2|x}\right]$$

$$F_2 = \left(-\frac{1}{2} + \frac{3}{2}\frac{N\sigma_a N\sigma}{|\mathcal{H}_1|^2}\right)\left(A_{o1}\, e^{|\mathcal{H}_1|x} + A_{o2}\, e^{-|\mathcal{H}_1|x}\right) + \tag{2.27}$$

$$+ \left(-\frac{1}{2} + \frac{3}{2}\frac{N\sigma_a N\sigma}{|\mathcal{H}_2|^2}\right)\left(A_{o3}\, e^{|\mathcal{H}_2|x} + A_{o4}\, e^{-|\mathcal{H}_2|x}\right)$$

$$F_3 = \left(\frac{3}{14}\frac{|\mathcal{H}_1|}{N\sigma} - \frac{9}{14}\frac{N\sigma_a}{|\mathcal{H}_1|}\right)\left(A_{o1}\, e^{|\mathcal{H}_1|x} - A_{o2}\, e^{-|\mathcal{H}_1|x}\right) +$$

$$+ \left(\frac{3}{14}\frac{|\mathcal{H}_2|}{N\sigma} - \frac{9}{14}\frac{N\sigma_a}{|\mathcal{H}_2|}\right)\left(A_{o3}\, e^{|\mathcal{H}_2|x} - A_{o4}\, e^{-|\mathcal{H}_2|x}\right).$$

The quantity $\mathcal{H}_1$ is essentially the reciprocal diffusion length while $\mathcal{H}_2$ is of the order of the reciprocal of the total mean free path. Consequently $|\mathcal{H}_1| < |\mathcal{H}_2|$, and therefore the exponential in $-|\mathcal{H}_1|$ falls off slower than the exponential in $-|\mathcal{H}_2|$. In this sense the solution $e^{-|\mathcal{H}_1|x}$ is asymptotic while the solution $e^{-|\mathcal{H}_2|x}$ is non-asymptotic.

The asymptotic solution has already been encountered as the exact solution of the transport equation far from sources or boundaries, provided $\mathcal{H}_1$ is given by

$$\mathcal{H}_1 = N\sigma_{so} \tanh^{-1} \frac{\mathcal{H}_1}{N\sigma} \tag{1.71b}$$

The expression (2.24) for $\mathcal{H}_1$ is of course an approximation to the solution of (1.71b) when the absorption is weak.

Close to sources or boundaries there are excited the non-asymptotic solutions $e^{-|\mathcal{H}_2|x}$. These fall off roughly as the mean free path, and are therefore negligible a few mean free paths away from the boundary.

As the order of the spherical harmonics approximation increases, the number of $\mathcal{H}_j$, and therefore the number of non-asymptotic solutions, increases. The values of $\mathcal{H}_j$ for the first few approximations on the assumption $\sigma_a/\sigma \ll 1$, are given below:

Even approximations:

P_1

$$\mathcal{H}_1 = \pm N\sqrt{3\sigma\sigma_a}$$

P_3

$$\mathcal{H}_1 = \pm N\sqrt{3\sigma\sigma_a}\left(1 - \frac{2}{5}\sigma_a/\sigma\right)$$

$$\mathcal{H}_2 = \pm \frac{1}{3}\sqrt{35}\, N\sigma\left(1 + \frac{2}{5}\sigma_a/\sigma\right)$$

P_5

$$\mathcal{H}_1 = \pm N\sqrt{3\sigma\sigma_a}\left(1 - \frac{2}{5}\sigma_a/\sigma\right) \qquad (2.28)$$

$$\mathcal{H}_2 = \pm 1.225\, N\sigma(1 + .1123\, \sigma_a/\sigma)$$

$$\mathcal{H}_3 = \pm 3.203\, N\sigma(1 + .2877\, \sigma_a/\sigma)$$

P_7

$$\mathcal{H}_1 = \pm N\sqrt{3\sigma\sigma_a}\left(1 - \frac{2}{5}\sigma_a/\sigma\right)$$

$$\mathcal{H}_2 = \pm 1.1032\, N\sigma(1 + .0448\, \sigma_a/\sigma)$$

$$\mathcal{H}_3 = \pm 1.592\, N\sigma(1 + .1400\, \sigma_a/\sigma)$$

$$\mathcal{H}_4 = \pm 4.458\, N\sigma(1 + .2153\, \sigma_a/\sigma)$$

Odd approximations:

P_2

$$\mathcal{H}_1 = \pm \frac{N\sqrt{3\sigma\sigma_a}}{\sqrt{1 + \frac{4}{5}\sigma_a/\sigma}} \tag{2.29}$$

P_4

$$\mathcal{H}_1 = \pm N\sqrt{3\sigma\sigma_a}\left(1 - \frac{2}{5}\sigma_a/\sigma\right)$$

$$\mathcal{H}_2 = \pm \sqrt{\frac{45}{23}}\, N\sigma(1 + .2012\,\sigma_a/\sigma)$$

P_6

$$\mathcal{H}_1 = \pm N\sqrt{3\sigma\sigma_a}\left(1 - \frac{2}{5}\sigma_a/\sigma\right)$$

$$\mathcal{H}_2 = \pm 1.146(1 + .0685\,\sigma_a/\sigma)\, N\sigma$$

$$\mathcal{H}_3 = \pm 1.9825\,(1 + .1994\,\sigma_a/\sigma)\, N\sigma$$

The angular distribution in, say, the P_3 approximation is

$$F(x,\mu) = \frac{1}{2}F_o(x) + \frac{3}{2}F_1(x)\,P_1(\mu) + \frac{5}{2}F_2(x)\,P_2(\mu) + \frac{7}{2}F_3(x)\,P_3(\mu) \tag{2.30}$$

where $F_{o,1,2,3}$ are given by (2.26) and (2.27). The total neutron flux is, according to (2.26),

$$F_o(x) = \sum_j A_{oj}\, e^{\mathcal{H}_j x} \tag{2.31}$$

Now from the neutron flux it is supposedly possible to determine the angular distribution by the "last collision" technique; this leads to Eq. (2.32) :

$$F(x,\mu) = \frac{N\sigma_{so}}{2}\int_{-\infty}^{x} F_o(x')\, e^{-\frac{N\sigma}{\mu}(x-x')}\,\frac{dx'}{\mu} \qquad \mu > 0 \tag{2.32}$$

$$= \frac{N\sigma_{so}}{2}\int_{\infty}^{x} F_o(x')\, e^{-\frac{N\sigma}{\mu}(x-x')}\,\frac{dx'}{\mu} \qquad \mu < 0$$

which, upon substituting (2.31) becomes (for $\mu > 0$)

$$F(x,\mu) = \frac{N\sigma_{so}}{2\mu} \sum A_{oj} \int_{-\infty}^{x} e^{\mathcal{H}_j x'} \, e^{-\frac{N\sigma}{\mu}(x-x')} \, dx' \qquad (2.33)$$

Formally, the integrations can be performed yielding

$$F(x,\mu) = \frac{N\sigma_{so}}{2} \sum_j \frac{A_{oj} \, e^{\mathcal{H}_j x}}{\mu \mathcal{H}_j + N\sigma} \qquad (2.34)$$

The contribution to the angular distribution from the asymptotic solution in case $\sigma_a/\sigma \ll 1$ is

$$\frac{A_{o2} \, \sigma_{so} \, e^{-|\mathcal{H}_1| x}}{2\left[\sigma - \mu\sqrt{3\sigma\sigma_a}\left(1 - \frac{2}{5}\,\sigma_a/\sigma\right)\right]}$$

which has no singularity. However, the non-asymptotic solutions give angular distributions which, according to (2.34), are singular at $\mu = -N\sigma/\mathcal{H}_j$, i.e., at $\mu = \frac{3}{\sqrt{35}}\left(\frac{1}{1 - \frac{2}{5}\frac{\sigma_a}{\sigma}}\right)$ in P_3 approximations. In other words, the angular distribution computed from a P_3 or higher approximation neutron distribution according to the last collision method is singular. This of course is a reflection on the approximate nature of the spherical harmonics solution. Since the spherical harmonics distribution does not really satisfy Boltzmann's equation, there is no justification for computing the angular distribution from the space distribution by the last collision method which, after all, is a consequence of the Boltzmann equation. It is therefore consistent to use the finite expansion in spherical harmonics (2.3) for the angular distribution, but it is incorrect to express the angular distribution by (2.34), at least near $\mu = -\frac{N\sigma}{\mathcal{H}_j}$.

Examples of Problems Solved by the Spherical Harmonics Method

In this section we solve a number of standard diffusion problems in second order approximation. The calculations are straightforward applications of the general theory described in the previous section.

1) Neutron distribution from a unit planeisotropic source at x = 0 in an infinite medium.

This problem was solved rigorously in Chapter I; it will be instructive to compare this rigorous solution with the spherical harmonics solution.

The Boltzmann equation in this case is

$$\mu \frac{dF(x,\mu)}{dx} + N\sigma\, F(x,\mu) = \frac{N\sigma_{so}}{2} \int_{-1}^{1} F(x,\mu)\, d\mu + \frac{1}{2}\,\delta(x). \tag{2.35}$$

The corresponding linear equations for the components of the angular distribution are

$$\begin{aligned}
&F_1' + N\sigma_a F_o = \delta(x) \\
&\tfrac{2}{3} F_2' + N\sigma F_1 + \tfrac{1}{3} F_o' = 0 \\
&\tfrac{3}{5} F_3' + N\sigma F_2 + \tfrac{2}{5} F_1' = 0 \\
&\tfrac{4}{7} F_4' + N\sigma F_3 + \tfrac{3}{7} F_2' = 0 \\
&\text{----------}
\end{aligned} \tag{2.36}$$

In P_1 approximation,

$$\frac{1}{3N\sigma} F_o'' - N\sigma_a F_o + \delta(x) = 0 \tag{2.37}$$

the solution of which is

$$F_o = A_1\, e^{-\varkappa_{o1}|x|} \tag{2.38}$$

where $\mathcal{H}_{o1}^2 = 3N\sigma N\sigma_a$. To determine A_1, we equate the total absorption rate to the total production rate:

$$1 = 2\int_0^\infty N\sigma_a F_o(x)\,dx = 2\frac{N\sigma_a}{\mathcal{H}_{o1}} A_1 ,$$

$$A_1 = \frac{\mathcal{H}_{o1}}{2N\sigma_a} \tag{2.39}$$

Thus

$$F_o = \frac{\mathcal{H}_{o1}}{2N\sigma_a} e^{-\mathcal{H}_{o1}|x|} \tag{2.40}$$

is the required distribution. In terms of the albedo, β, which according to (1.113) is

$$\beta = 1 - \frac{4}{\sqrt{3}}\sqrt{\frac{\sigma_a}{\sigma}} ,$$

$$F_o = \frac{2}{1-\beta} e^{-\mathcal{H}_{o1}|x|} . \tag{2.41}$$

In order to solve this problem in P_3 approximation, we first solve the problem in which the source, of strength S neutrons/cm^2/sec., extends from -b to +b; we then pass to the limit in which b approaches zero but 2Sb remains finite (and equal to unity). We denote quantities in the source region by superscript I, in the outer region by superscript II. In the source region the neutron distribution satifies (2.36) with $\delta(x)$ replaced by S; outside $\pm b$, S = 0. The solutions of (2.36) with $\delta(x)$ replaced by S are, in the source region

$$F_o^I(x) = \frac{S}{N\sigma_a} - A_1 \cosh \mathcal{H}_{11}x - A_2 \cosh \mathcal{H}_{12}x$$

$$F_1^I(x) = A_1 \frac{N\sigma_a}{\mathcal{H}_{11}} \sinh |\mathcal{H}_{11}x| + A_2 \frac{N\sigma_a}{\mathcal{H}_{12}} \sinh |\mathcal{H}_{12}x| \tag{2.42}$$

cont d.....

$$F_2^I(x) = A_1\left(-\frac{3}{2}\frac{N\sigma\, N\sigma_a}{\mathcal{H}_{11}^2} + \frac{1}{2}\right)\cosh \mathcal{H}_{11}x + A_2\left(-\frac{3}{2}\frac{N\sigma\, N\sigma_a}{\mathcal{H}_{12}^2} + \frac{1}{2}\right)\cosh \mathcal{H}_{12}x$$

(2.42 cont'd)

$$F_3^I(x) = A_1\left(\frac{9}{14}\frac{N\sigma_a}{\mathcal{H}_{11}} - \frac{3}{14}\frac{\mathcal{H}_{11}}{N\sigma}\right)\sinh|\mathcal{H}_{11}x| + A_2\left(\frac{9}{14}\frac{N\sigma_a}{\mathcal{H}_{12}} - \frac{3}{14}\frac{\mathcal{H}_{12}}{N\sigma}\right)\sinh|\mathcal{H}_{12}x|$$

Outside the source region,

$$F_o^{II}(x) = B_1 e^{-\mathcal{H}_{11}(|x|-b)} + B_2 e^{-\mathcal{H}_{12}(|x|-b)}$$

$$F_1^{II}(x) = B_1\frac{N\sigma_a}{\mathcal{H}_{11}}e^{-\mathcal{H}_{11}(|x|-b)} + B_2\frac{N\sigma_a}{\mathcal{H}_{12}}e^{-\mathcal{H}_{12}(|x|-b)} \qquad (2.43)$$

$$F_2^{II}(x) = B_1\left(\frac{3}{2}\frac{N\sigma_a\, N\sigma}{\mathcal{H}_{11}^2} - \frac{1}{2}\right)e^{-\mathcal{H}_{11}(|x|-b)} + B_2\left(\frac{3}{2}\frac{N\sigma_a\, N\sigma}{\mathcal{H}_{12}^2} - \frac{1}{2}\right)e^{-\mathcal{H}_{12}(|x|-b)}$$

$$F_3^{II}(x) = B_1\left(\frac{9}{14}\frac{N\sigma_a}{\mathcal{H}_{11}} - \frac{3}{14}\frac{\mathcal{H}_{11}}{N\sigma}\right)e^{-\mathcal{H}_{11}(|x|-b)} + B_2\left(\frac{9}{14}\frac{N\sigma_a}{\mathcal{H}_{12}} - \frac{3}{14}\frac{\mathcal{H}_{12}}{N\sigma}\right)e^{-\mathcal{H}_{12}(|x|-b)}$$

In these equations,

$$\mathcal{H}_{11} = N\sqrt{3\sigma\sigma_a}\left(1 - \frac{2}{5}\frac{\sigma_a}{\sigma}\right)$$

$$\mathcal{H}_{12} = \frac{1}{3}\sqrt{35}\, N\sigma\left(1 + \frac{2}{5}\frac{\sigma_a}{\sigma}\right) \qquad (2.44)$$

At the interfaces $\pm b$, the neutron angular distribution is continuous. The four constants, A_1, A_2, B_1 and B_2, are therefore to be determined from the linear equations

$$F_o^I(\pm b) = F_o^{II}(\pm b),\ F_1^I(\pm b) = F_1^{II}(\pm b),\ F_2^I(\pm b) = F_2^{II}(\pm b),\ F_3^I(\pm b) = F_3^{II}(\pm b). \qquad (2.45)$$

In the limit of small b ($\cosh \mathcal{H} b \to 1$, $\sinh \mathcal{H} b \to \mathcal{H} b$), these linear equations are

$$
\begin{aligned}
&B_1 + B_2 + A_1 + A_2 = \frac{S}{N\sigma_a} \\
&\frac{N\sigma_a}{\mathcal{H}_{11}} B_1 + \frac{N\sigma_a}{\mathcal{H}_{12}} B_2 - bN\sigma_a A_1 - bN\sigma_a A_2 = 0 \\
&\left(\frac{3N^2\sigma_a\sigma}{2\mathcal{H}_{11}^2} - \frac{1}{2}\right)B_1 + \left(\frac{3}{2}\frac{N^2\sigma_a\sigma}{\mathcal{H}_{12}^2} - \frac{1}{2}\right)B_2 + \left(\frac{3}{2}\frac{N^2\sigma\sigma_a}{\mathcal{H}_{11}^2} - \frac{1}{2}\right)A_1 + \left(\frac{3}{2}\frac{N^2\sigma\sigma_a}{\mathcal{H}_{12}^2} - \frac{1}{2}\right)A_2 = 0 \\
&\left(\frac{9}{14}\frac{N\sigma_a}{\mathcal{H}_{11}} - \frac{3}{14}\frac{\mathcal{H}_{11}}{N\sigma}\right)B_1 + \left(\frac{9}{14}\frac{N\sigma_a}{\mathcal{H}_{12}} - \frac{3}{14}\frac{\mathcal{H}_{12}}{N\sigma}\right)B_2 - \left(\frac{9}{14}\frac{N\sigma_a}{\mathcal{H}_{11}} - \frac{3}{14}\frac{\mathcal{H}_{11}}{N\sigma}\right)\mathcal{H}_{11} bA_1 - \\
&\qquad - \left(\frac{9}{14}\frac{N\sigma_a}{\mathcal{H}_{12}} - \frac{3}{14}\frac{\mathcal{H}_{12}}{N\sigma}\right)\mathcal{H}_{12}\, bA_2 = 0 .
\end{aligned}
\tag{2.46}
$$

We shall be interested primarily in the asymptotic neutron density, $F_{oasy}(x)$, far from the source. This is given by

$$F^{II}_{oasy}(x) \approx B_1\, e^{-\mathcal{H}_{11}(|x| - b)}, \tag{2.47}$$

the $\mathcal{H}_{12}$ solution falling off faster than this because $\mathcal{H}_{12} > \mathcal{H}_{11}$. The computation of B_1 from the linear equations (2.46) is straightforward but lengthy; the result is

$$B_1 = \frac{Sb\mathcal{H}_{11}}{N\sigma_a} \frac{1}{1 - \left(\frac{3N^2\sigma\sigma_a}{\mathcal{H}_{11}^2} - 1\right) \Big/ \left(\frac{3N^2\sigma\sigma_a}{\mathcal{H}_{12}^2} - 1\right)}, \tag{2.48}$$

If we substitute the expressions (2.44) for $\mathcal{H}_{11}$ and $\mathcal{H}_{12}$ in the denominator of (2.48) there results, correct to terms of order σ_a/σ,

$$B_1 = \frac{Sb\,\mathcal{H}_{11}}{N\sigma_a}\left(1 - \frac{4}{5}\frac{\sigma_a}{\sigma}\right) \quad . \tag{2.49}$$

The asymptotic neutron density from a unit isotropic plane source is found from (2.49) and (2.47) by putting $\lim_{b\to 0} Sb = \frac{1}{2}$; hence

$$F_{o_{asy}}(x) = \frac{\mathcal{H}_{11}}{2N\sigma_a}\left(1 - \frac{4}{5}\frac{\sigma_a}{\sigma}\right) e^{-\mathcal{H}_{11}|x|} \tag{2.50}$$

provided σ_a/σ is small.

Comparing the P_3 approximation for $F_{o_{asy}}$ with the P_1 approximation, we see that the P_3 approximation and the P_1 approximation differ in two respects:

a) The reciprocal diffusion length $\mathcal{H}_{11}$ is somewhat smaller than $\mathcal{H}_{o1}$.

b) The source strength in P_3 approximation is reduced by the factor $(1 - 4/5\ \sigma_a/\sigma)$. This is exactly the "initial absorption" correction which was found in the rigorous solution, Chap. I, (Eq. 1.26, -7,-8).

The equations (2.46) can also be solved for B_2, the result being, in the limit $b\to 0$,

$$B_2 = \frac{\mathcal{H}_{12}}{2N\sigma_a}\,\frac{1}{1 - \left(\frac{\mathcal{H}_{o1}^2}{\mathcal{H}_{12}^2} - 1\right)\Big/\left(\frac{\mathcal{H}_{o1}^2}{\mathcal{H}_{11}^2} - 1\right)} \tag{2.51}$$

Hence the total neutron flux from a unit source in P_3 approximation is

$$F_o(x) = \frac{\mathcal{H}_{11}}{2N\sigma_a}\,\frac{1}{1 - \left(\frac{\mathcal{H}_{o1}^2}{\mathcal{H}_{11}^2} - 1\right)\Big/\left(\frac{\mathcal{H}_{o1}^2}{\mathcal{H}_{12}^2} - 1\right)}\, e^{-\mathcal{H}_{11}|x|} \tag{2.52}$$

$$+ \frac{\mathcal{H}_{12}}{2N\sigma_a}\,\frac{1}{1 - \left(\frac{\mathcal{H}_{o1}^2}{\mathcal{H}_{12}^2} - 1\right)\Big/\left(\frac{\mathcal{H}_{o1}^2}{\mathcal{H}_{11}^2} - 1\right)}\, e^{-\mathcal{H}_{12}|x|}$$

The exact distribution was found in Chap. I to be

$$F_o(x) = \frac{\mathcal{H}}{N\sigma_s}\left(\frac{N^2\sigma^2 - \mathcal{H}^2}{\mathcal{H}^2 - N^2\sigma_a\sigma}\right) e^{-\mathcal{H}|x|} + 2N^2\sigma^2 \int_0^\infty \frac{(1+\eta)e^{-(1+n)N\sigma x}\, d\eta}{\left[2N\sigma(1+\eta) - N\sigma_s \ln(1+\frac{2}{\eta})\right]^2 + \pi^2 N^2 \sigma_s^2} \tag{2.53}$$

where $\mathcal{H} = \mathcal{H}_{11}$ and $x > 0$.

In Figure (2.I) the exact, P_1 and P_3 distributions are compared.

2) <u>Extrapolation distance in a non-absorbing isotropically scattering medium</u>.

This problem can be restated as the determination of the neutron distribution in a semi-infinite non-absorbing medium, close to the boundary; it is usually called Milne's problem. In P_1 approximation the extrapolation distance was found to be

$$\ell = \text{extrapolation distance} = \frac{2}{3N\sigma},$$

while the exact extrapolation distance can be shown to be

$$\ell = \frac{.7104609}{N\sigma} \; * .$$

It will be instructive to see how successive spherical harmonics approximations give progressively better values for the extrapolation distance.

The plane Boltzmann equation in a non-absorbing medium is

$$\mu \frac{dF(x,\mu)}{dx} + N\sigma F(x,\mu) = \frac{N\sigma}{2} \int_{-1}^{1} F(x,\mu)\, d\mu . \tag{2.54}$$

The neutron distribution in the semi-infinite medium $x > 0$ must satisfy the boundary condition

$$F(0,\mu) \equiv 0 \qquad 0 \leq \mu \leq 1 \tag{2.55}$$

---

* G. Placzek and W. Seidel, Phys. Rev. 72, 550 (1947)

Actually, since the inward angular distribution must vanish over a continuous range while the spherical harmonics approximation involves only a finite number of arbitrary constants, it will be impossible to satisfy (2.55) rigorously.

The harmonics equations equivalent to (2.54) are

$$
\begin{aligned}
F_1' \qquad\qquad\qquad &= 0 \\
\frac{2}{3} F_2' + N\sigma F_1 + \frac{1}{3} F_o' &= 0 \\
\frac{3}{5} F_3' + N\sigma F_2 + \frac{2}{5} F_1' &= 0 \\
\frac{4}{7} F_4' + N\sigma F_3 + \frac{3}{7} F_2' &= 0 \\
\frac{5}{9} F_5' + N\sigma F_4 + \frac{4}{9} F_4' &= 0 \\
\cdots\cdots\cdots\cdots &
\end{aligned}
\tag{2.56}
$$

In P_1 approximation, the solution of (2.56) is

$$F_o = A_o x + B_o$$

$$F_1 = -\frac{1}{3N\sigma} F_o' = \frac{-A_o}{3N\sigma}$$

and

$$F(x,\mu) = \frac{1}{2}\left[A_o x + B_o - \frac{A_o \mu}{N\sigma}\right].$$

To determine the ratio A_o/B_o, it is customary to require, as a simple approximation to the much more stringent (2.55), that the net inward flow at $x = 0$, $j_+(0)$, vanish. Now

$$j_+(0) = \int_0^1 \mu\, F(0,\mu)\, d\mu$$

$$= \frac{1}{2}\left[\frac{B_o}{2} - \frac{A_o}{3N\sigma}\right] = 0\,,$$

///

or

$$\frac{A_o}{B_o} = \frac{3N\sigma}{2} \quad .$$

Thus

$$F_o(x) = A_o(x + \frac{2}{3N\sigma}) \tag{2.57}$$

which extrapolates to zero at $x = -\frac{0.666...}{N\sigma}$.

In P_3 approximation, if we normalize so that $F_1 = -1$, we find

$$\begin{aligned} F_o &= 3N\sigma x + B_1 + C_1 e^{-\mathcal{H}_{12}x} \\ F_1 &= -1 \\ F_2 &= -\frac{1}{2} C_1 e^{-\mathcal{H}_{12}x} \\ F_3 &= -\frac{5}{6} \frac{N\sigma}{\mathcal{H}_{12}} C_1 e^{-\mathcal{H}_{12}x} \end{aligned} \tag{2.58}$$

where $\mathcal{H}_{12} = \frac{1}{3}\sqrt{35}\ N\sigma$. At the boundary, $x = 0$,

$$F(0,\mu) = \frac{1}{2}(B_1 + C_1) - \frac{3}{2} P_1(\mu) - \frac{5}{4} C_1 P_2(\mu) - \frac{\sqrt{35}}{4} C_1 P_3(\mu) \tag{2.59}$$

In order to determine the constants B_1 and C_1 it is necessary to impose conditions on $F(0,\mu)$ which, while not identical with the rigorous (2.55), are at least consistent with it. One such condition was used in P_1 approximation, namely

$$\int_0^1 P_1(\mu)\ F(0,\mu)\ d\mu = 0; \tag{2.60}$$

this condition insures no <u>net</u> inward flow at $x = 0$, although it allows inward currents at certain angles if they are balanced by outward currents at other angles.

A further condition, suggested by Marshak, is that the next odd "moment" vanish, i.e.,

$$\int_0^1 P_3(\mu)\, F(0,\mu)\, d\mu = 0 \tag{2.61}$$

This condition is consistent with the requirement

$$F(0,\mu) \equiv 0 \qquad\qquad 0 \leq \mu \leq 1 \tag{2.55}$$

but of course it does not imply (2.55).

The conditions (2.60) and (2.61) lead to the pair of equations

$$.3750\, C_1 + B_1 = 2$$

$$6.8802\, C_1 + B_1 = 0$$

or

$$C_1 = -.3074\,, \qquad B_1 = 2.1153. \tag{2.62}$$

The total flux distribution, according to (2.62) and (2.58), is therefore

$$F_o(x) = 3N\sigma x + 2.1153 - 0.3074\, e^{-\sqrt{35}/3\, N\sigma x} \tag{2.63}$$

and the angular distribution at $x = 0$ is

$$F(0,\mu) = 0.9039 - 1.5\, P_1(\mu) + 0.3843\, P_2(\mu) + 0.4547\, P_3(\mu)\,. \tag{2.64}$$

The extrapolation distance is the reciprocal slope of the asymptotic part of $F_o(x)$. According to (2.63) this

$$\ell = \frac{2.1153}{3N\sigma} = \frac{0.705}{N\sigma}\,.$$

The method outlined here is readily extended to higher approximations, the boundary conditions at the ν-th stage being the vanishing of the odd moments,

$$\int_0^1 P_j(\mu)\, F(0,\mu)\, d\mu = 0. \qquad j = 1,3,5.....\nu$$

Thus in P_5 approximation, it is found that

$$F_0(x) = 3N\sigma x + 2.1246 - 0.0777\, e^{-1.225N\sigma x} - 0.27145\, e^{-3.2030N\sigma x}$$

$$F(0,\mu) = .8877 - 1.5\, P_1(\mu) + .4365\, P_2(\mu) + .4322\, P_3(\mu) - .07583\, P_4(\mu) - .3641\, P_5(\mu)$$

and the corresponding extrapolation distance is

$$\ell = \frac{2.1246}{3N\sigma} = \frac{.708}{N\sigma} \quad .$$

The angular distributions $F(0,\mu)$ and the spatial distributions $F_0(x)$ near the boundary as given in the various approximations are plotted in Fig. (2.2) and (2.3). The exact distributions as computed by Placzek (Phys. Rev. 72, 556, 1947, and C. Mark, Phys. Rev. 72, 558, 1947), are given there also for comparison.

The odd moment boundary condition is one of many possible conditions consistent with, but not implying, the rigorous boundary condition

$$F(0,\mu) \equiv 0 \qquad\qquad 0 \leq \mu \leq 1.$$

Another approximate boundary condition consistent with (2.55) is to require that, in P_ν approximation (ν odd), $F(0,\mu)$ vanish at the $\frac{\nu+1}{2}$ positive roots of $P_{\nu+1}(\mu) = 0$. This condition is very reasonable on the following grounds: the rigorous boundary condition (2.55) is equivalent to the requirement

$$\int_0^1 F^2(0,\mu)\, d\mu = 0. \tag{2.65}$$

Now according to Gauss's approximate integration formula (Hobson: Spherical and Ellipsoidal Harmonics, p. 77)

$$\int_0^1 F^2(0,\mu)\, d\mu \cong \sum_r H_r F^2(0,\mu_r) \tag{2.66}$$

where μ_r are the positive roots of

$$P_{\nu+1}(\mu) = 0$$

and the H_r are universal constants whose exact values are unimportant. According to (2.66), a choice of arbitrary constants in $F(0,\mu)$ which make $F(0,\mu_r)$ vanish at values of μ_r which are roots of $P_{\nu+1}(\mu_r) = 0$ will imply that the rigorous boundary condition is satisfied to the extent that Gauss's integration formula is exact.

In P_1 approximation, $F(x,\mu) = \frac{1}{2}\left[A_o x + B_o - \frac{A_o\mu}{N\sigma}\right]$, and $F(0,\mu)$ must vanish at the positive root of $P_2(\mu) = 0$. This root lies at $\mu = \frac{1}{\sqrt{3}}$. Hence, to determine A_o/B_o we have

$$F\left(0,\frac{1}{\sqrt{3}}\right) = \frac{1}{2} A_o\left[\frac{B_o}{A_o} - \frac{1}{\sqrt{3}\,N\sigma}\right] = 0,$$

$$\frac{B_o}{A_o} = \frac{1}{\sqrt{3}\,N\sigma}$$

and

$$F(x,\mu) = \frac{A_o}{2}\left[x + \frac{1}{\sqrt{3}\,N\sigma} - \frac{\mu}{N\sigma}\right], \tag{2.67}$$

$$F_o(x) = A_o\left(x + \frac{1}{\sqrt{3}\,N\sigma}\right) \tag{2.68}$$

The extrapolation distance in this approximation is

$$\ell = \frac{1}{\sqrt{3}\,N\sigma} = \frac{.577}{N\sigma}.$$

It is remarkable that the very simple angular distribution given by (2.67) at $x = 0$, never differs from the exact distribution by more than 7.2%.

In P_3 approximation, $F(0,\mu)$ must vanish at the positive roots of $P_4(\mu) = 0$, i.e., at $\mu = 0.861$ and $\mu = 0.340$. According to (2.59),

$$F(0,\mu) = \frac{1}{2}(B_1 + C_1) - \frac{3}{2} P_1(\mu) - \frac{5}{4} C_1 P_2(\mu) - \frac{\sqrt{35}}{4} C_1 P_3(\mu);$$

the requirements

$$F(0,0.340) = F(0,0.861) = 0$$

give

$$B_1 = 2.0848, \quad C_1 = -0.3509.$$

Thus

$$F_0(x) = 3N\sigma x + 2.0848 - 0.3509\, e^{-\sqrt{35}/3\, N\sigma x} \tag{2.69}$$

$$F(0,\mu) = 0.8670 - 1.5\, P_1(\mu) + 0.4386\, P_2(\mu) + 0.5189\, P_3(\mu), \tag{2.70}$$

and the extrapolation distance is

$$\ell = \frac{2.0848}{3N\sigma} = \frac{0.695}{N\sigma}$$

<u>Wick's Method</u>

An ingenious variant of the spherical harmonics method based on the Gauss integration formula has been proposed by G. C. Wick (Zeit. für Physik, <u>121</u>, 702, 1943) and has been applied to many astrophysical transport problems by Chandrasekhar (Astro. Phys. J. 101, 348, 1945). The idea of Wick's Method is to approximate the integral in the transport equation (2.1) by Gauss's integration formula. As will be recalled, in Gauss's formula, the interval $-1 \leq \mu \leq 1$ is divided into $\nu + 1$ (ν odd) intervals spaced at the zeros of $P_{\nu+1}(\mu) = 0$. Thus

$$\int_{-1}^{1} F(x,\mu)\, d\mu \cong \sum_{r=1}^{\nu+1} H_r\, F(x,\mu_r) \tag{2.71}$$

where the universal constants H_r and the roots μ_r have the following values:

$\nu = 1$ approximation, $H_{1,2} = 1$, $\mu_{1,2} = \pm \frac{1}{\sqrt{3}} = \pm .57735$

$\nu = 3$ approximation, $H_{1,2} = 0.6521$, $H_{3,4} = 0.3479$; $\mu_{1,2} = \pm 0.3400$,

$\mu_{3,4} = \pm .8611$

$\nu = 5$ approximation, $H_{1,2} = 0.4679$, $H_{3,4} = 0.3608$, $H_{5,6} = 0.1713$;

$\mu_{1,2} = \pm .2386$, $\mu_{3,4} = \pm 0.6612$, $\mu_{5,6} = \pm 0.9325$

If Gauss's approximation is substituted into the integral equation (2.1), and if the angular distribution is specified at the points μ_i, which are the roots of $P_{\nu+1}(\mu) = 0$, then (2.1) can be replaced, approximately, by the set of equations

$$\mu_i \frac{d}{dx} F(x,\mu_i) + N\sigma F(x,\mu_i) = \frac{N\sigma_{so}}{2} \sum_{r=1}^{\nu+1} H_r F(x,\mu_r). \tag{2.72}$$

The solutions of (2.72) are a linear combination, with arbitrary constants, A_r of the form

$$F(x,\mu_i) = \frac{N\sigma_{so}}{2} \sum_{r=1}^{\nu+1} \frac{A_r}{N\sigma + \varkappa_r \mu_i} e^{\varkappa_r x} \tag{2.73}$$

where each $\varkappa$ satisfies the characteristic equation

$$\frac{N\sigma_{so}}{2} \sum_{i=1}^{\nu+1} \frac{H_i}{N\sigma + \varkappa\mu_i} = 1. \tag{2.74}$$

If the numerical values for H_r and μ_r are substituted into (2.74) and the resulting algebraic equations are solved, it is found that the roots $\varkappa_r$ of (2.74) are identical with the roots listed in (2.28) for the even approximation spherical harmonics scheme. Thus (2.73) is identical in form with (2.34); i.e., the angular distribution predicted in ν-th approximation by Wick's Method is identical, at the points μ_i, with the angular distribution computed by last collisions from the P_ν approximation total flux.

The total flux, $F_o(x)$, is found from (2.73) by integrating over μ. Since

$$F_o(x) = \int_{-1}^{1} F(x,\mu)\, d\mu$$

and, in this approximation

$$\int_{-1}^{1} F(x,\mu)\, d\mu = \sum_{r=1}^{\nu+1} H_r F(x,\mu_r),$$

we have

$$F_o(x) = \int_{-1}^{1} F(x,\mu)\, d\mu = \frac{N\sigma_{so}}{2} \sum_{i=1}^{\nu+1} \sum_{r=1}^{\nu+1} \frac{H_i \cdot A_r\, e^{\varkappa_r x}}{(N\sigma + \varkappa_r \mu_i)} = \sum_{r=1}^{\nu+1} A_r\, e^{\varkappa_r x} . \tag{2.75}$$

by virtue of (2.74). Since the $\varkappa_r$ are the same as were found in the spherical harmonics approximation methods, (2.75), is identical in form with the flux distribution (2.21).

Equation (2.73) defines the angular distribution only for value of μ_i which are roots of $P_{\nu+1}(\mu_i) = 0$. In Wick's Method it is assumed that the actual angular distribution, in directions for which

$$\mu_i \neq - \frac{N\sigma}{\varkappa_r}$$

is the analytic continuation of (2.73), namely

$$F(x,\mu) = \frac{N\sigma_{so}}{2} \sum_{r=1}^{\nu+1} \frac{A_r}{N\sigma + \varkappa_r \mu}\, e^{\varkappa_r x} . \tag{2.76}$$

Evidently at $\mu_i = - \frac{N\sigma}{\varkappa_r}$ the distribution is singular.

The constants A_r are determined from the boundary conditions. For example, in the extrapolation distance problem, the boundary condition is

$$F(0,\mu_i) = 0 \qquad\qquad 0 \leq \mu_i \leq 1$$

and this set of $\frac{\nu+1}{2}$ equations is sufficient to determine, except for a normalization factor, $\left(\frac{\nu+1}{2} + 1\right)$ of the constants A_r; the remaining $(\frac{\nu+1}{2} - 1)$ A_r are zero because only the decreasing exponential non-asymptotic solutions are physically significant.

As an example of the usefulness of Wick's Method, we solve the extrapolation distance problem. Since

$$F(0,\mu) = \frac{N\sigma_{so}}{2} \sum_{r=1}^{\nu+1} \frac{A_r}{N\sigma + \varkappa_r \mu} ,$$

the constants A_r are to be determined so that

$$\sum_{r=1}^{\nu+1} \frac{A_r}{N\sigma + \varkappa_r \mu_i} = 0 \qquad i = 1,2,\ldots \nu+1 \qquad (2.77)$$

it being understood that the A_r corresponding to the non-asymptotic, positive $\varkappa_r$ are zero. The set (2.77) are linear equations and can be solved in a straightforward manner.

If there is no capture in the medium, the angular distribution $F(x,\mu)$ contains a term linear in μ as well as terms of the form (2.76). For, in this case, the equations (2.72) are

$$\mu_i \frac{d}{dx} F(x,\mu_i) + N\sigma F(x,\mu_i) = \frac{N\sigma}{2} \sum_{r=1}^{\nu+1} H_r F(x,\mu_r).$$

We put

$$F(x,\mu) = A + Bx + C\mu \; ; \qquad (2.78)$$

according to Gauss's approximation formula, which is exact if $F(x,\mu)$ is a polynomial in μ of degree $2\nu - 1$

$$\sum_{r=1}^{\nu+1} H_r = \int_{-1}^{1} d\mu = 2$$

$$\sum_{r=1}^{\nu+1} H_r \mu_r = \int_{-1}^{1} \mu \, d\mu = 0.$$

Hence, on substituting (2.78) into (2.72) we find that (2.78) satisfies the Boltzmann equation provided

$$C = -\frac{B}{N\sigma_{so}}.$$

In a non-capturing medium, therefore, the complete solution of (2.72) is

$$F(x,\mu) = A + B\left(x - \frac{\mu}{N\sigma_{so}}\right) + \frac{N\sigma}{2} \sum_{r=3}^{\nu+1} \frac{A_r\, e^{\varkappa_r x}}{(N\sigma + \varkappa_r\, \mu)} \tag{2.79}$$

In $\nu = 1$ approximation, the ratio B/A is determined by requiring $F(0,\frac{1}{\sqrt{3}}) = 0$. This is exactly the condition which led to the P_1 approximation summarized in (2.67) and (2.68).

In $\nu = 3$ approximation, A_3 and B/A are to be determined from the requirements

$$F(0,.861) = F(0,.340) = 0.$$

This leads to

$$F(x,\mu) = A\left[1 + 1.4409\,(N\sigma x - \mu) - \frac{.16808\, e^{-1.9720\, N\sigma x}}{1 - 1.9720\,\mu}\right] \tag{2.80}$$

The value $\frac{.694}{N\sigma}$ for the extrapolation distance is very near the value computed in P_3 approximation with vanishing odd moments.

In $\nu = 5$ approximation

$$F(x,\mu) = A\left[1 + 1.42065\,(N\sigma x - \mu) - \frac{.03596\, e^{-1.2252\, N\sigma x}}{1 - 1.2252\,\mu} - \frac{.14387\, e^{-3.2030\, N\sigma x}}{1 - 3.2030\,\mu}\right] \tag{2.81}$$

here the extrapolation distance is $\frac{.7039}{N\sigma}$.

In Table I the angular distributions for $F(0,\mu)$ approximations are compared with the exact value. It will be noted that these give better values for $F(0,\mu)$ than do the approximations with vanishing odd moments.

In Fig. 2-IV, $F_0(x)$ in P_1, P_3 and P_5 approximations are compared with the exact value.

Table 2-I

$F(0,\mu)$ Approximations (Wick's Method)

μ	Exact	P_1 Approx.	P_3 Approx.	P_5 Approx.	(P_5 - Exact)
0	.5000	.5359	.5111	.5057	.0057
-.1	.6237	.6287	.6167	.6189	-.0048
-.2	.7252	.7215	.7174	.7199	-.0053
-.3	.8213	.8144	.8151	.8180	-.0033
-.4	.9146	.9072	.9108	.9133	-.0013
-.5	1.0064	1.0000	1.0050	1.0068	+.0004
-.6	1.0971	1.0928	1.0983	1.0991	+.0020
-.7	1.1870	1.1856	1.1907	1.1905	+.0035
-.8	1.2764	1.2785	1.2826	1.2813	+.0049
-.9	1.3653	1.3713	1.3739	1.3716	+.0063
-1.0	1.4539	1.4641	1.4649	1.4616	+.0077

3) Neutron distribution near a black sphere.*

A black sphere of radius a is surrounded by a non-absorbing, isotropically scattering infinite homogeneous medium. Because of the spherical symmetry the monoenergetic neutron flux will depend only upon r, the distance from the center of the black sphere and μ, the cosine of the angle between the direction of the neutron's velocity and the radius vector.

Thus if $\underline{\Omega}$ is a unit vector in the direction of the neutron's velocity, the non-absorbing, sourceless Boltzmann transport equation becomes

$$\underline{\Omega} \cdot \nabla F(r,\mu) + N\sigma F(r,\mu) = \frac{N\sigma}{2} \int_{-1}^{1} F(r,\mu)\, d\mu \qquad (2.82)$$

* R. E. Marshak and W. Seidel, MT-49; and G. N. Plass, CP-2453G.

By regarding a given direction $\underline{\Omega}$ in space as the polar axis $\underline{r}$ is expressed in spherical coordinates and Eq. (2.82) will depend on r and θ but not ψ .

$$\underline{\Omega} = \cos\theta\,\hat{\underline{r}} - \sin\theta\,\hat{\underline{\theta}}$$

$$\nabla F(r,\mu) = \frac{\partial F(r,\mu)}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial F}{\partial \theta}\hat{\underline{\theta}} = \frac{\partial F(r,\mu)}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial F}{\partial \mu}\frac{\partial \mu}{\partial \theta}\hat{\underline{\theta}}$$

$$\underline{\Omega}\cdot\nabla F(r,\mu) = \cos\theta\,\frac{\partial F(r,\mu)}{\partial r} - \frac{\sin\theta}{r}\frac{\partial F}{\partial \mu}\frac{\partial \mu}{\partial \theta} = \mu\,\frac{\partial F(r,\mu)}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial F}{\partial \mu}$$

and Eq. (2.82) becomes

$$\mu\,\frac{\partial F(r,\mu)}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial F}{\partial \mu} + N\sigma F(r,\mu) - \frac{N\sigma}{2}\int_{-1}^{1} F(r,\mu)\,d\mu = 0\,. \qquad (2.83)$$

For convenience let the unit of length equal $N\sigma$. Then Eq. (2.83) becomes

$$\mu\,\frac{\partial F(r,\mu)}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial F(r,\mu)}{\partial \mu} + F(r,\mu) - \frac{1}{2}\int_{-1}^{+1} F(r,\mu)\,d\mu = 0. \qquad (2.84)$$

Expand $F(r,\mu)$ in Legendre polynomials as in Eq. (2.3),

$$F(r,\mu) = \sum \frac{2\ell+1}{2} F_\ell(r)\,P_\ell(\mu). \qquad (2.85)$$

Upon substituting (2.85) into (2.84), multiply by $P_\nu(\mu)$ and integrating over μ, in addition to the result given in Eq. (2.12) there will be terms arising from the second term in Eq. (2.84). In this manipulation, use is made of the identity

$$\frac{dP_\ell}{d\mu} = \frac{\ell(\ell+1)}{(2\ell+1)(\mu^2-1)}\left(P_{\ell+1} - P_{\ell-1}\right)$$

$$\int_{-1}^{1} \frac{1-\mu^2}{r}\sum \frac{2\ell+1}{2} F_\ell(r)\,\frac{\partial P_\ell(\mu)}{\partial \mu}\,P_\nu(\mu)\,d\mu$$

$$= \frac{1}{r}\int_{-1}^{1} \frac{\ell(\ell+1)}{2} F_\ell(r)\,P_\nu(\mu)\left[P_{\ell-1} - P_{\ell+1}\right]d\mu$$

$$= \frac{1}{r} \int_{-1}^{1} \left[\frac{(\nu + 1)(\nu + 2)}{2} F_{\nu+1}(r) - \frac{\nu(\nu - 1)}{2} F_{\nu-1}(r) \right] P_\nu^2 \, d\mu$$

$$= \frac{(\nu + 1)(\nu + 2)}{r(2\nu + 1)} F_{\nu+1} - \frac{\nu(\nu - 1)}{r(2\nu + 1)} F_{\nu-1}$$

In obtaining the avove Eq. (2.7) was used. The Boltzmann equation thus reduces to the following infinite set of differential equations

$$\frac{\nu + 1}{2\nu + 1} F'_{\nu+1} + \frac{\nu}{2\nu + 1} F'_{\nu-1} + \frac{(\nu + 1)(\nu + 2)}{r(2\nu + 1)} F_{\nu+1} - \frac{\nu(\nu - 1)}{r(2\nu + 1)} F_{\nu-1} + F_\nu = F_o \, \delta_{o\nu} \tag{2.86}$$

The corresponding harmonics equations are

$$\begin{aligned}
&F'_1 + \frac{2F_1}{r} = 0 \\
&\frac{2}{3} F'_2 + \frac{1}{3} F'_o + \frac{2}{r} F_2 + F_1 = 0 \\
&\frac{3}{5} F'_3 + \frac{2}{5} F'_1 + \frac{12}{5r} F_3 - \frac{2}{5r} F_1 + F_2 = 0 \\
&\frac{4}{7} F'_4 + \frac{3}{7} F'_2 + \frac{20}{7r} F_4 - \frac{6}{7r} F_2 + F_3 = 0 \\
&- - - - - - - - - - - - - - - - - - -
\end{aligned} \tag{2.87}$$

In P_1 approximation the solution of Eq. (2.87) is

$$\begin{aligned}
F_o &= \frac{-3A}{r} + B \\
F_1 &= \frac{-A}{r^2}
\end{aligned} \tag{2.88}$$

and

$$F(r,\mu) = \frac{-3A}{2r} + \frac{B}{2} - \frac{3A\mu}{2r^2} \tag{2.89}$$

By requiring the first odd moments to vanish for $F(a,\mu)$, i.e.

$$\int_0^1 \mu\, F(a,\mu)\, d\mu = 0 \tag{2.90}$$

Eq. (2.89) becomes

$$F(r,\mu) = A\left[\frac{-3}{2r} + \frac{3}{2a}\left(1+\frac{2}{3a}\right) - \frac{3\mu}{2r^2}\right] \tag{2.91}$$

When $F_o(r_o) = 0$ we find from Eq. (2.91)

$$r_o = \frac{a}{1+\frac{2}{3a}}$$

and the extrapolation distance is $a - r_o$.

$$a - r_o = \frac{2}{3+\frac{2}{a}}$$

Note that as $a \longrightarrow \infty$, $a - r_o \longrightarrow \frac{2}{3}$ which is the extrapolation distance obtained in the P_1 approximation for the plane case.

In P_3 approximation the solution of Eq. (2.87) is

$$\begin{aligned}
F_o &= \frac{-3A}{r} + B - \frac{2C}{r}\, e^{-\frac{\sqrt{35}\, r}{3}} \\
F_1 &= -\frac{A}{r^2} \\
F_2 &= -\frac{6A}{5r^3} + \frac{C}{r}\, e^{-\frac{\sqrt{35}\, r}{3}}\left(1 + \frac{9}{\sqrt{35}\, r} + \frac{27}{35r^2}\right) \\
F_3 &= -\frac{18A}{7r^4} + \frac{\sqrt{35}\, C}{7r}\, e^{-\frac{\sqrt{35}\, r}{3}}\left(1 + \frac{18}{\sqrt{35}\, r} + \frac{27}{7r^2} + \frac{81}{7\sqrt{35}\, r^3}\right)
\end{aligned} \tag{2.92}$$

corresponding terms in $e^{+\frac{\sqrt{35}\,r}{3}}$ are excluded because of their behavior as $r \to \infty$.

By requiring the first two odd moments to vanish, i.e., Eq. (2.90) and

$$\int_0^1 P_3(\mu)\, F(a,\mu)\, d\mu = 0$$

the constants B and C can be expressed in terms of A:

$$C = A\,\alpha\, e^{+\frac{\sqrt{35}\,a}{3}}$$

$$B = A\left[\frac{3}{a}\left(1 + \frac{2}{3a} + \frac{1}{2a^2}\right) + \frac{3\alpha}{4a}\left(1 - \frac{15}{\sqrt{35}\,a} - \frac{9}{7a^2}\right)\right]$$

where

$$\alpha = \frac{\dfrac{7}{a4\sqrt{35}}\left(1 + \dfrac{15}{4a} + \dfrac{72}{7a^2}\right)}{\left(1 + \dfrac{175}{32\sqrt{35}}\right) + \dfrac{18}{\sqrt{35}\,a}\left(1 + \dfrac{175}{64\sqrt{35}}\right) + \dfrac{27}{7a^2}\left(1 + \dfrac{35}{32\sqrt{35}}\right) + \dfrac{81}{7\sqrt{35}\,a^3}}$$

the extrapolation distance $a - r_o$ becomes

$$a - r_o = \frac{\left(\dfrac{2}{3a} + \dfrac{1}{2a^2}\right) + \dfrac{1}{4}\alpha\left(1 - \dfrac{15}{\sqrt{35}\,a} - \dfrac{9}{7a^2}\right)}{\dfrac{1}{a}\left(1 + \dfrac{2}{3a} + \dfrac{1}{2a^2}\right) + \dfrac{\alpha}{4a}\left(1 - \dfrac{15}{\sqrt{35}\,a} - \dfrac{9}{7a^2}\right)}$$

and as $a \longrightarrow \infty$; $a - r_o \longrightarrow .7051$.

By using an alternate boundary condition which requires that, in the P_ν approximations, $F(a,\mu)$ vanish at the $\frac{\nu+1}{2}$ positive roots of $P_{\nu+1}(\mu) = 0$, Eq. (2.89) becomes

$$F(r,\mu) = A\left[-\frac{3}{2r} + \frac{1}{2}\left(\frac{\sqrt{3}}{a^2} + \frac{3}{a}\right) - \frac{3\mu}{2r^2}\right]$$

and the extrapolation distance is

$$a - r_o = \frac{\sqrt{3}\,a}{\sqrt{3} + 3a}\,,$$

By applying these conditions to Eq. (2.92) B and C can be solved for in terms of A thusly:

$$C = A\beta e^{+\frac{\sqrt{35}\,a}{3}}$$

$$B = A\left[\frac{3}{a} + \frac{1.01994}{a^2} - \frac{1.95972}{a^3} - \frac{7.41114}{a^4} + \beta\left(\frac{6.06893}{a} + \frac{9.89555}{a^2} + \frac{10.65516}{a^3} + \frac{4.76431}{a^4}\right)\right]$$

where

$$\beta = \frac{1.55555\,(1 + .43683a + .12123a^2)}{1 + 2.40886a + 2.41700a^2 + 1.07753a^3}$$

the extrapolation distance is obtained from

$$a - r_o = \frac{a\,B/A - 3}{B/A} \quad .$$

In Table 2-II the extrapolation distances are given for various black sphere radii in both P_1 and P_3 approximations. For the same approximations the total neutron distribution is given for a black sphere of unit radius ($a = N\sigma$) in Fig. 2-V.

Table 2-II

Extrapolation Distances as a Function of Black Sphere Radius

	$a - r_o$			
	Odd Moments		Gauss Roots	
a	P_1	P_3	P_1	P_3
.1	.08696	.08837	.08524	.08795
.2	.1539	.1609	.1485	.1597
.5	.2857	.3228	.2680	.3184
1	.4000	.4801	.3660	.4707
2	.5000	.6081	.4480	.5937
3	.5556	.6526	.4842	.6370
5	.5882	.6823	.5176	.6668
∞	.6667	.7051	.5774	.6940

The Time-Dependent Boltzmann Equation

The time-dependent Boltzmann equation with plane symmetry is

$$\mu \frac{\partial F(x,\mu,t)}{\partial x} + N\sigma F(x,\mu,t) = \frac{N\sigma_{so}}{2} \int_{-1}^{1} F(x,\mu,t)d\mu - \frac{1}{v} \frac{\partial F(x,\mu,t)}{\partial t} \qquad (2.93)$$

In P_1 approximation,

$$F(x,\mu,t) = \frac{1}{2} F_o(x,t) + \frac{3}{2} F_1 (x,t) \mu \; ; \qquad (2.94)$$

we substitute (2.94) into (2.93) and obtain the following pair of equations:

$$\frac{\partial F_1(x,t)}{\partial x} + N\sigma F_o(x,t) = N\sigma_{so} F_o(x,t) - \frac{1}{v} \frac{\partial F_o(x,t)}{\partial t} \qquad (2.95)$$

cont'd.....

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$$N\sigma F_1(x,t) + \frac{1}{3}\frac{\partial F_0(x,t)}{\partial x} = -\frac{1}{v}\frac{\partial F_1(x,t)}{\partial t} \qquad \text{(2.95 cont')}$$

By differentiating the first of these equations with respect to time and the second with respect to space, and subtracting, we obtain the following second order equation for $F_0(x,t)$:

$$\frac{v}{3N\sigma}\frac{\partial^2 F_0(x,t)}{\partial x^2} - N\sigma_a v F_0(x,t) = (1 + \frac{\sigma_a}{\sigma})\frac{\partial F_0(x,t)}{\partial t} + \frac{1}{N\sigma v}\frac{\partial^2 F_0(x,t)}{\partial t^2} \qquad (2.96)$$

It is a remarkable fact, which is not usually appreciated, that the correct <u>first</u> order approximation to the time-dependent Boltzmann equation is not the simple diffusion equation (which is first order in t), but is the <u>telegrapher's</u> equation which is second order in both t and x. The general solution of the telegrapher's equation shows the phenomenon of retardation -- i.e., the solution has a well-defined wave front, in addition to a residual disturbance which persists, at all points traversed by the wave front. The telegrapher's equation thus lies between the simple wave equation whose solutions have a wave front but no residual disturbance, and the diffusion equation whose solutions have a residual disturbance but no wave front. That a solution of the time dependent Boltzmann equation must have a wave front -- i.e., a region beyond which there are no neutrons -- is obvious physically, since a neutron travelling with velocity v requires at least a "retardation time", x/v to reach a distance x from a source. Beyond the point x there can be no neutrons until time x/v. As the mean free path of the neutron decreases, the probability of the neutron reaching any finite distance without a collision becomes smaller and smaller. Finally in the limit of zero mean free path and zero absorption cross-section but infinite velocity so that the diffusion coefficient

$$D = \frac{1}{3}\frac{v}{N\sigma} \qquad (2.97)$$

and the neutron lifetime, $\tau = 1/N\sigma_a v$ are finite, the telegrapher type equation reduces to the diffusion equation

$$D \frac{\partial^2 F_o(x,t)}{\partial x^2} - \frac{F_o(x,t)}{\tau} = \frac{\partial F_o(x,t)}{\partial t} \quad . \tag{2.98}$$

It is a well-known property of the solution of the diffusion equation that an instantaneous source immediately produces a disturbance at all points in space, -- i.e., there is no retardation in the solution.

Diffusion equations which arise in mathematical physics -- e.g., the heat equation -- usually are approximations to the Boltzmann equation in which the mean free path is assumed vanishingly small but the diffusion coefficient is finite. For heat conduction or molecular diffusion the assumption of zero mean free path is a good approximation and the existence of a retardation time is generally overlooked; in neutron diffusion on the other hand the mean free path is usually so large ($\sim$ several centimeters) that the existence of the wave front in the time-dependent diffusion equation cannot be ignored.

In order to make these qualitative remarks more exact, we write down the solution of (2.96) and of (2.98) from an initial distribution which is zero outside the region $x_1 < x < x_2$; i.e.,

$$\begin{aligned} F(x,0) &= \mathcal{F}(x) && x_1 < x < x_2 \\ &= 0 && \text{elsewhere} \\ \frac{\partial F_o}{\partial t}(x,0) &= 0 && \text{everywhere} \end{aligned}$$

For simplicity, we assume no absorption. Then, for (2.96) the solution is (cf. Webster, Partial Differential Equations of Mathematical Physics [G. E. Stechert, 1933], p. 179)

$$F_o(x,t) = \frac{e^{-\frac{N\sigma v}{2}t}}{2}\left\{\mathcal{F}\left(x + \frac{vt}{\sqrt{3}}\right) + \mathcal{F}\left(x - \frac{vt}{\sqrt{3}}\right)\right. \tag{2.99}$$

$$\left. + \frac{\sqrt{3}}{v}\int_{x-vt/\sqrt{3}}^{x+vt/\sqrt{3}} \mathcal{F}(\alpha)\left(\frac{N\sigma v}{2} + \frac{\partial}{\partial t}\right) I_o\left(\sqrt{\frac{3(N\sigma)^2}{4}\left(\frac{v^2t^2}{3} - (x-\alpha)^2\right)}\right) d\alpha\right\}$$

I_o being the Bessel function of imaginary argument. The first and second terms represent waves travelling to the left and right with velocity $v/\sqrt{3}$. Since $\mathcal{F}(x)$ vanishes outside $x_1 < x < x_2$, the neutron flux is strictly zero, at a point x, until time $t = \sqrt{3}/v\,(x - x_2)$, on the right, or $t = \sqrt{3}/v\,(x_1 - x)$ on the left. Once t exceeds either of these times -- i.e., once the disturbance has reached x, $\mathcal{F}(\alpha)$ in the integrand is non-zero and the integral contributes a "tail" which persists indefinitely. The neutron flux, at such times, is given by

$$F_o(x,t) = \frac{e^{-\frac{N\sigma vt}{2}}}{2}\frac{\sqrt{3}}{v}\int_{x_1}^{x_2} \mathcal{F}(\alpha)\left(\frac{N\sigma v}{2} + \frac{\partial}{\partial t}\right) I_o\left(\sqrt{\frac{3(N\sigma)^2}{4}\left(\frac{v^2t^2}{3} - (x-\alpha)^2\right)}\right) d\alpha .$$

For the diffusion equation, the solution is simply

$$F_o(x,t) = \int_{x_1}^{x_2} \mathcal{F}(x')\,\frac{e^{-\frac{3N\sigma(x-x')^2}{4vt}}}{\left(\frac{4\pi vt}{3N\sigma}\,t\right)^{1/2}}\,dx' ;$$

i.e., the neutron flux appears instantaneously at all points in space -- and persists indefinitely -- there is no retardation and no wave front.

The P_1 approximation time dependent equation is an improvement over the simple diffusion equation since the solution does exhibit the phenomenon of retardation -- however, the propagation velocity is only $v/\sqrt{3}$ instead of v. This short-coming of course arises from the approximate character of the assumed angular distribition.

Evidently those neutrons which reach the point x first must have suffered no collisions -- their angular distribution must be very strongly forward, and could really be represented only by a large number of spherical harmonics. It is therefore not surprising that a P_1 approximation does not treat the first arrival neutrons quite correctly.

The fact that the undeflected neutrons have a strongly forward angular distribution and therefore cannot be represented in P_1 approximation is very important in the discussion of slowing down of neutrons. In computing the spatial distribution of neutrons from a monoenergetic source, the neutrons which have lost little or no energy are analogous to the "first arrivals" in the time dependent problem. The difficulty of representing the angular distribution of these neutrons makes calculation of accurate slowing down distributions at large distances from a high energy source difficult; an asymptotic solution has been given by Wick.*

* G.C.Wick, Phys. Rev. 75, 738, (1949).

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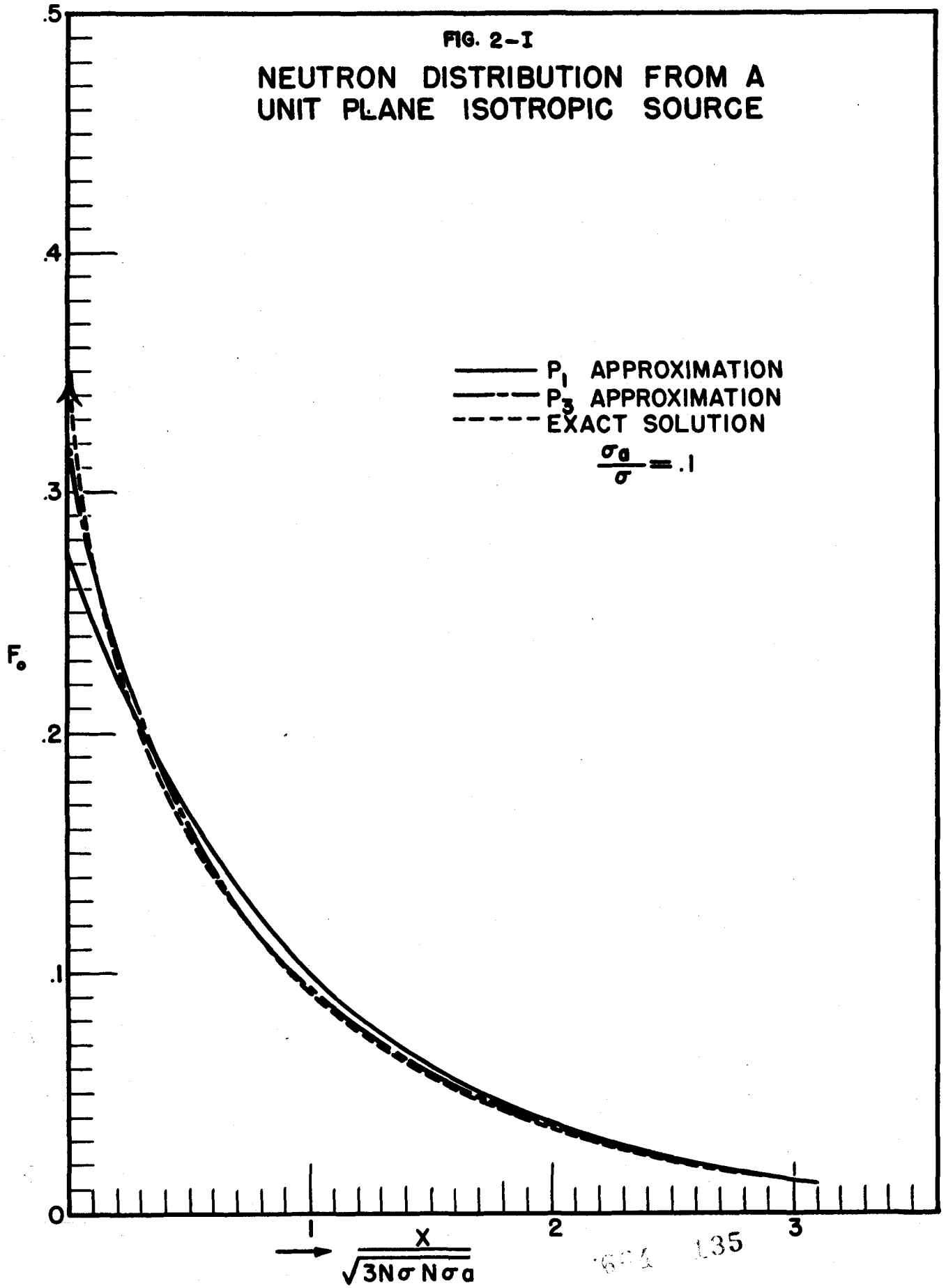
FIG. 2-I

NEUTRON DISTRIBUTION FROM A UNIT PLANE ISOTROPIC SOURCE



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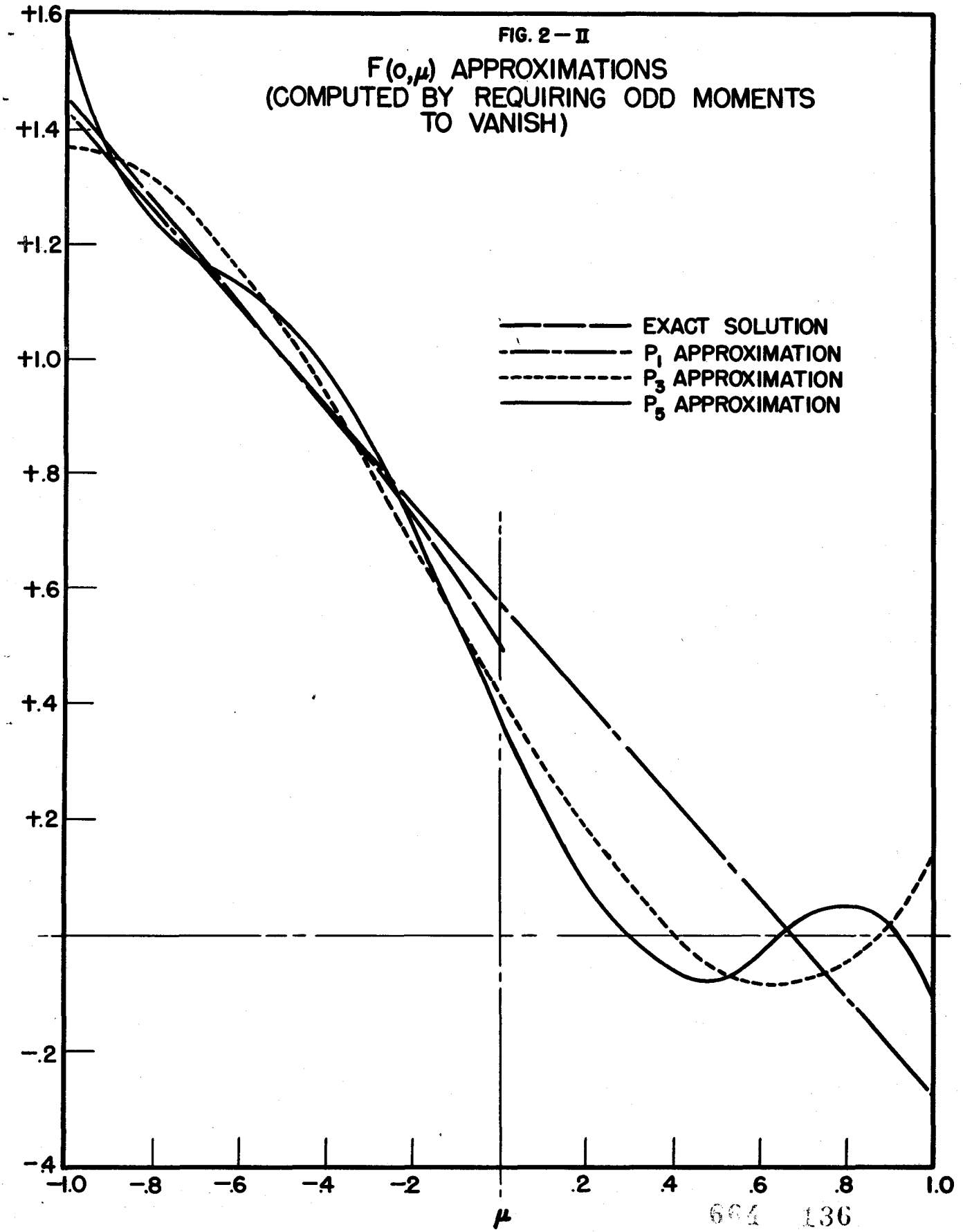


FIG. 2 — II
F(o,μ) APPROXIMATIONS
(COMPUTED BY REQUIRING ODD MOMENTS
TO VANISH)
EXACT SOLUTION
P1 APPROXIMATION
P3 APPROXIMATION
P5 APPROXIMATION
+1.6
+1.4
+1.2
+1.0
+.8
+.6
+.4
+.2
-.2
-.4
-1.0
-.8
-.6
-.4
-.2
.2
.4
.6
.8
1.0
μ

FIG. 2-III

$\frac{1}{3}F_0(x)$ APPROXIMATIONS

(COMPUTED BY REQUIRING ODD MOMENTS TO VANISH)

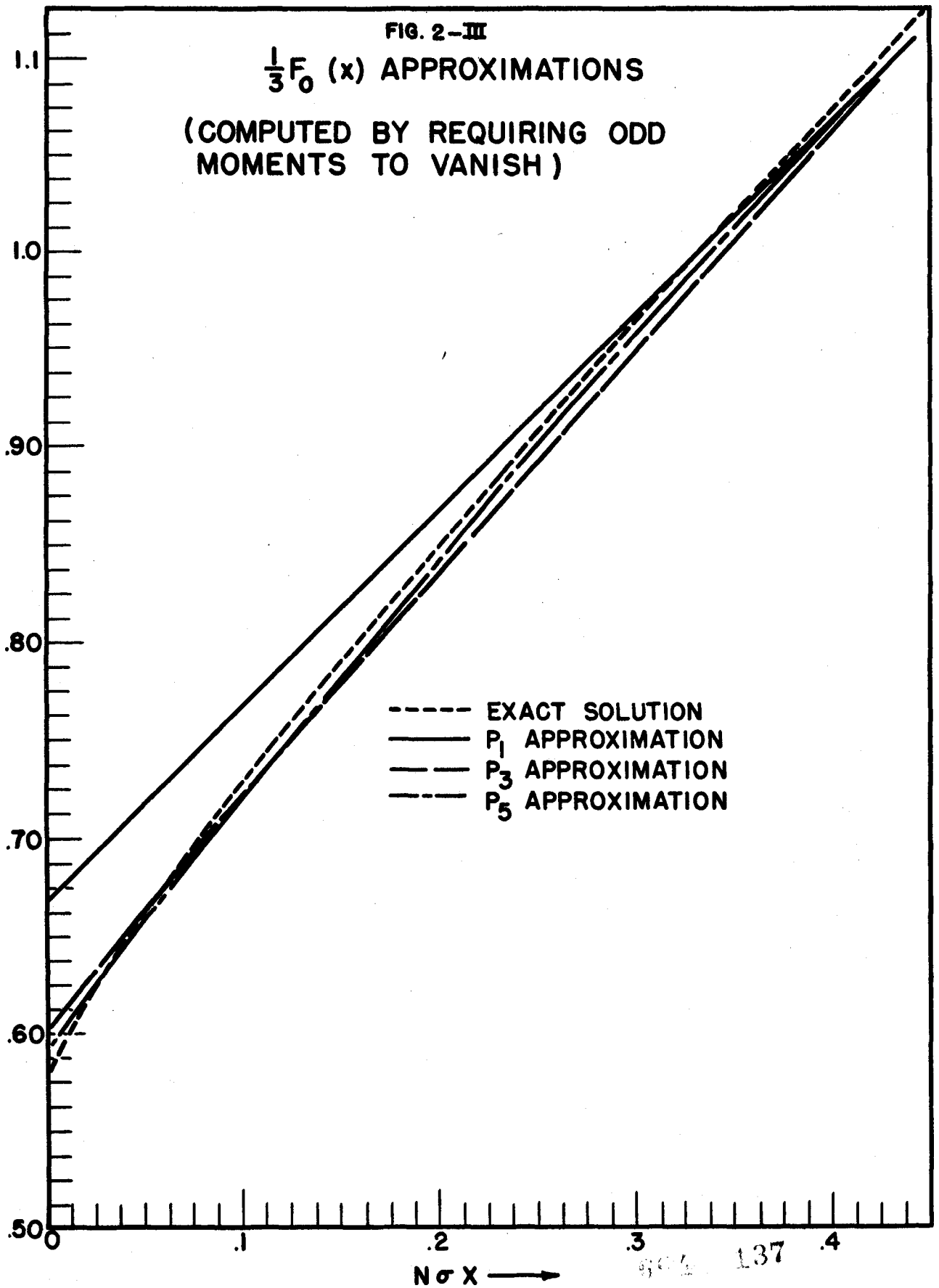


FIG. 2 IV

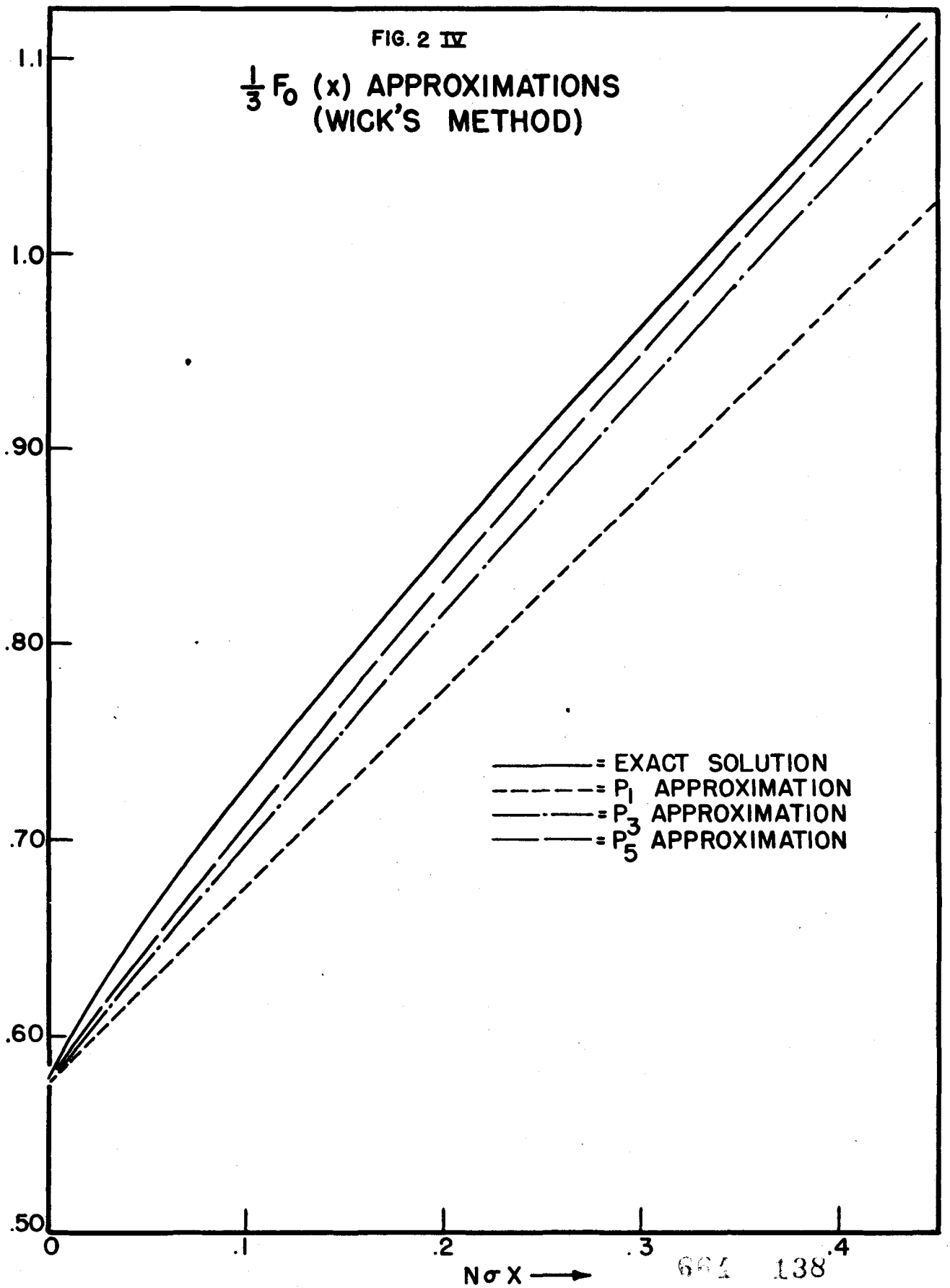
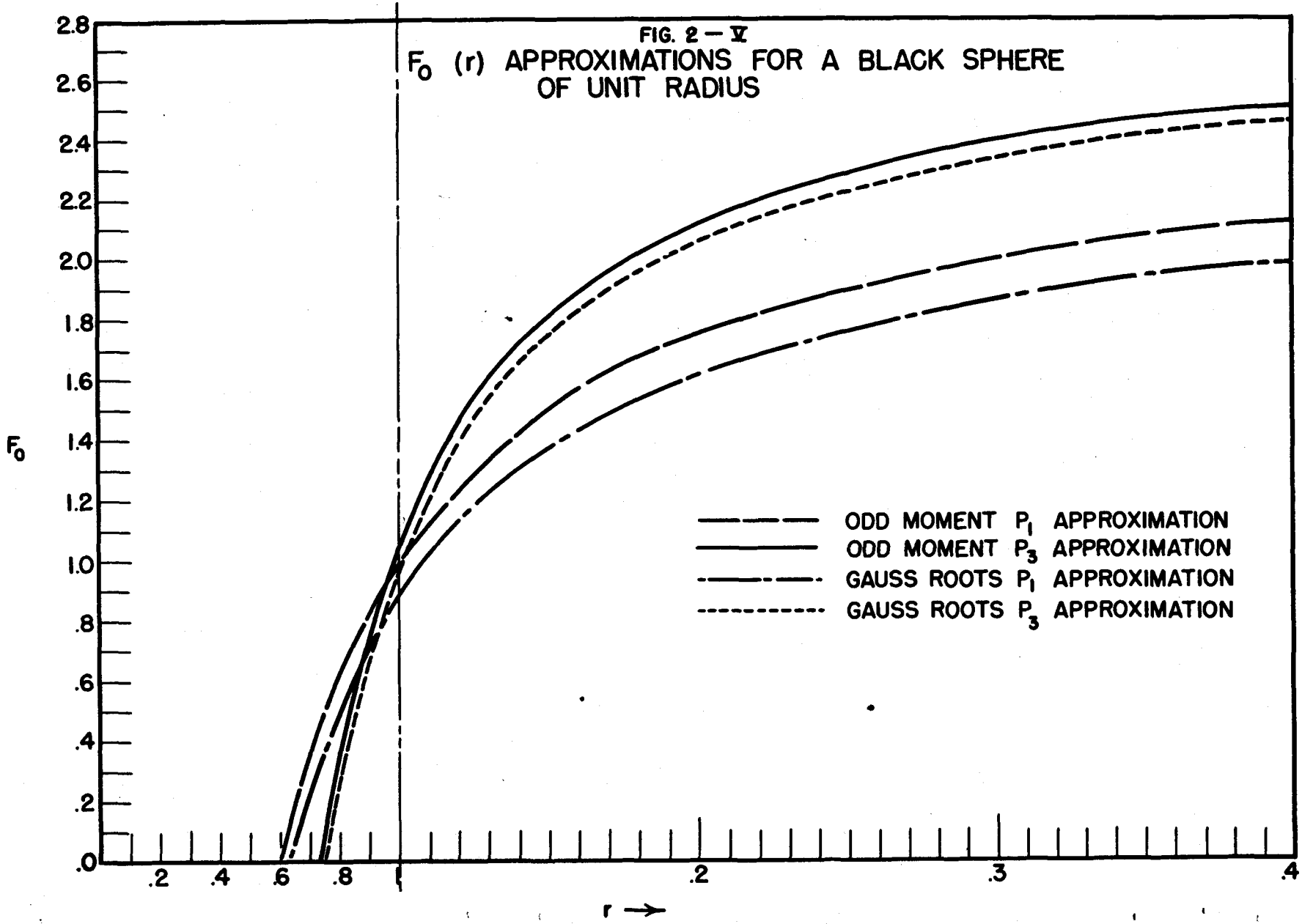

$\frac{1}{3} F_0$ (x) APPROXIMATIONS
(WICK'S METHOD)
1.1
1.0
.90
.80
.70
.60
.50
0
.1
.2
.3
.4
$N\sigma X$
= EXACT SOLUTION
= P_1 APPROXIMATION
= P_3 APPROXIMATION
= P_5 APPROXIMATION

FIG. 2 — V

F_0 (r) APPROXIMATIONS FOR A BLACK SPHERE OF UNIT RADIUS



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CHAPTER III

SLOWING DOWN OF NEUTRONS

Since the average energy of a neutron produced in fission is several Mev, while the neutrons most effective in producing further fission have only thermal velocities, it is necessary in a slow neutron chain reaction to reduce the energy of the fission neutrons. This is accomplished by mixing the uranium with a low atomic weight material, such as graphite. The neutrons suffer elastic collisions with the light nuclei and lose some energy at each encounter. In this way their velocity is "moderated" by elastic collisions -- hence the word "moderator" for the low atomic weight material.

In this chapter we shall study the manner in which neutrons lose their energy by elastic collisions. During the process of slowing down, the neutrons diffuse away from the point at which they originated. Thus after having lost a certain amount of energy the neutrons will be distributed in space, in energy, and in direction of motion. In order to calculate this distribution, it is necessary to solve the Boltzmann equation in which account is taken of the details of the energy and momentum balance at each collision. Unfortunately, the asymptotic solutions for the energy dependent Boltzmann equation are much harder to obtain than were the asymptotic solutions for the energy independent equation. For this reason we shall be obliged to deal largely with approximate solutions of the energy dependent equation.

It has been customary to assume on the Plutonium Project that the scattering of fission neutrons by moderator nuclei is isotropic in the center of gravity system. This assumption is valid only for hydrogen. The maximum value of the isotropic scattering cross section is, on very general grounds, just $4\pi\lambda\!\!\!^{-2}$, where $2\pi\lambda\!\!\!^{-}$ is the deBroglie wave length of the neutron in the center of gravity of

the system neutron+nucleus. In terms* of the neutron energy, E, $\lambda = \frac{M+1}{M} .045/\sqrt{E} \times 10^{-11}$ cm., where E is the neutron energy in Mev. Hence the maximum isotropic scattering cross-section is $\left(\frac{M+1}{M}\right)^2 \times (2.54/E) \times 10^{-24}$ cm^2. A perusal of H. H Goldsmith's and H. I. Ibser's collection of the neutron cross-section shows that, among the light elements, only hydrogen has a cross-section which is less than the isotropic maximum at energy below 1.3 Mev. This means that the occurrence of non-isotropic scattering (corresponding to higher angular momenta) must be a very widespread phenomenon even at neutron energies of only about 1 Mev. Under these circumstances it is rather risky to place too much reliance on any slowing down calculation in which isotropic scattering is assumed.

It has also been customary to ignore any inelastic scattering which may occur in collisions with moderator nuclei. This is probably a good assumption since the level spacing in light nuclei is so large ($\sim$ 0.1 Mev), that excitation of a nucleus is unlikely by inelastic scattering. However, there is no real experimental evidence to prove that the inelastic scattering in graphite, e.g., is negligible. The main justification is that range calculations based on the assumption of elastic scattering are in fair agreement with experimentally measured ranges.

The theory of slowing down which we develop here is applicable only to neutrons whose energy is well above thermal energy. A neutron which slows down to thermal energy remains almost in thermal equilibrium with the atoms of the moderator until it is captured or until it escapes from the boundary of the system. Thus by the time the neutron reaches thermal energies on the average it neither loses nor gains energy. As the neutron approaches thermal

* See, e.g., H. Bethe, Rev. Mod. Phys. 9, 103 (1937), eq. 376a.

energy the effectiveness of the moderator in slowing it down tends to become smaller and smaller; finally at thermal energies its effectiveness is null. The manner in which the moderator starts to lose its effectiveness depends on its molecular state. Thus, if the moderator is crystalline (as, e.g., graphite) at low energies the neutron loses energy only by exciting lattice vibrations. Again, if the moderator is a proton bound in a molecule, then, if the energy of the neutron is small compared to the binding energy of the molecule, the molecule acts as a whole in an elastic collision and therefore the loss in energy per collision is less than if the proton were free. At somewhat higher energies, the neutron can excite molecular vibrations -- such collisions are essentially inelastic.

A calculation of how the moderating power of graphite decreases at energies close to thermal was made by Metropolis and Teller (CP-387). Their results were used by P. Morrison (CP-506) to calculate the slowing down distance in graphite.

The Mechanics of Neutron Collisions

We consider a neutron of velocity $\underline{v}$ which strikes a moderator nucleus of mass M, velocity zero, and is scattered elastically through an angle θ as measured in the center of gravity system (Figure 1). The assumption that the moderator nucleus is at rest is justified since, except for neutrons close

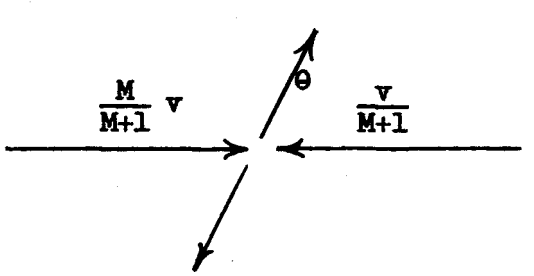

Figure 1

to thermal energy, the neutron velocity is much larger than the velocity of thermal agitation of the moderator nucleus. To calculate the slowing down of neutrons near thermal energy is very complicated and will not be attempted here. The center of gravity (c. of g.) of the system neutron+ scatterer travels in the direction of $\underline{v}$ with a velocity $\underline{v}_{c.g.} = \frac{\underline{v}}{M+1}$. Hence the velocities of neutron and of the scatterer in the c. of g. system are respectively $\frac{M}{M+1}\underline{v}$ and $-\frac{\underline{v}}{M+1}$.

After the impact the lengths of the velocity vectors of the neutron and the scatterer in the c. of g. system must be the same as before impact, and they must still lie along a straight line -- otherwise energy and momentum would not be conserved. However, the common line of motion is now rotated through the angle θ. We shall denote the cosine of θ by η.

In the laboratory system the velocity after impact is obtained by adding vectorially the velocity of the c. of g., $\frac{\underline{v}}{M+1}$, to the velocity of the scattered neutron in the c. of g. system. In this manner it is easy to see (Figure 2) that E, the kinetic energy after impact is related to η and E', the kinetic energy before impact by the formula

$$\frac{E}{E'} = \frac{1}{(1+M)^2}(M^2 + 1 + 2M\eta). \tag{3.1}$$

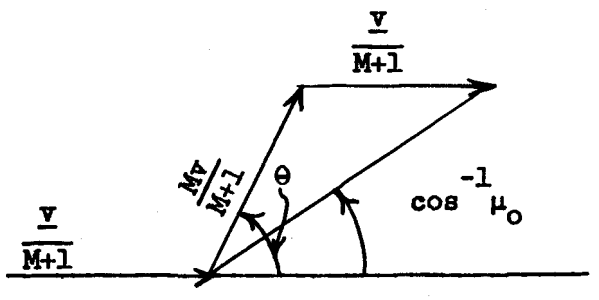

Figure 2

It is important to note the ratio of final to initial energy depends only on θ, the scattering angle in the c. of g. system. If the scattering is isotropic in the center of gravity system, then any final energy between $E = E'$ (for a glancing collision, $\eta = 1$) and $E = \alpha^2 E'$, where $\alpha^2 = \left(\frac{M-1}{M+1}\right)^2$, (for a head-on collision, $\eta = -1$) is equally probable. Furthermore, the average logarithmic energy loss per collision, ξ, defined by

$$\xi = \frac{\int_{-1}^{1} \ln \frac{E'}{E}\, d\eta}{\int_{-1}^{1} d\eta} = -\frac{1}{2}\int_{-1}^{1} \ln \frac{M^2 + 1 + 2M\eta}{(1+M)^2}\, d\eta = 1 + \frac{\alpha^2}{1-\alpha^2} \ln \alpha^2 \tag{3.2}$$

$$\approx \frac{2}{M+\frac{2}{3}} \quad \text{for } M \gg 1.$$

If the scattering is non-isotropic in the c. of g. system -- that is, if the probability of a scattering event resulting in an angular deflection between η and $\eta + d\eta$ is some function $f(E',\eta)d\eta$*, of the initial energy and the angle of scattering -- then the average logarithmic energy loss per collision is

$$\xi(E') = -\int_{-1}^{1} \left[\ln \frac{M^2 + 1 + 2M\eta}{(1+M)^2}\right] f(E',\eta)\, d\eta \ . \tag{3.3}$$

Evidently the average logarithmic energy loss will not be independent of energy if the scattering probability $f(E',\eta)$ depends on the initial energy, E'.

The average logarithmic energy loss per collision, ξ, as given by (3.2) is tabulated in Table I for several elements assuming isotropic scattering.

---

* Throughout this chapter "f" means the probability of a neutron scattering event and is therefore not always the same function of its arguments.

Since the number of collisions suffered per second by neutrons of a given energy is proportional to $N\sigma_{s_o}$, where σ_{s_o} is the total scattering cross-section, the product $N\sigma_{s_o}\xi$ is a measure of the rate at which neutrons of a given energy will be moderated. It is this product, rather than ξ itself, which determines the efficiency of a moderator in slowing down neutrons. The quantity $\sigma_{s_o}\xi$ is called the microscopic slowing down power of the moderator. Its value in the energy region of 1 to 10,000 volts for several important substances is also given in the table. In computing $\sigma_{s_o}\xi$ for this region it has been assumed that the scattering cross-section is constant throughout this range.

Table I

Average Logarithmic Energy Loss per Collision, ξ, and Microscopic Slowing Down Power $\sigma_{s_o}\xi$ in the Energy Region 1 to 10,000 ev

Element	ξ	$\sigma_{s_o}\xi$
H	1	19
D	.725	2.39
He	.425	0.637
Be	.206	1.26
C	.158	0.76
O	.120	0.50
F	.101	0.41
U	.00838	0.068

The Boltzmann Equation for Neutrons Undergoing Elastic Collisions

In order to write down the Boltzmann equation, including energy loss by elastic collision, it is necessary first to determine the number of neutrons in a unit volume which will be thrown into unit energy and solid angle range around $(E,\underline{\Omega})$ as a result of elastic scattering in that volume. Since the scattering angle in the center of gravity system is uniquely correlated with the energy loss according to (3.1), it is appropriate first to compute the number of neutrons thrown simultaneously into a particular element of solid angle in the c. of g. system, $d\underline{\omega}$, and solid angle in the laboratory system, $d\underline{\Omega}$, and then, by the usual formulas for transformation of integrals, to calculate the number thrown into an element of energy range and solid angle in the laboratory system.

Let

$$N\sigma_s(E',\underline{\omega})d\underline{\omega} = \frac{1}{2\pi}\sum_{\ell}\frac{2\ell+1}{2}P_\ell(\eta)N\sigma_{s_\ell}(E',\eta)d\eta\, d\psi \tag{3.4}$$

be the macroscopic differential scattering cross-section for deflection of a neutron of initial energy E' into the element of solid angle* $d\underline{\omega} = d\eta\, d\psi$ in the c. of g. system. The total number of collisions occurring at $\underline{r}$ involving neutrons whose original energy and direction are $E',\underline{\Omega}'$, is

$$\int_{\psi}\int_{\eta} N\sigma_s(E',\underline{\omega})\, F(\underline{r},E',\underline{\Omega}')d\eta\, d\psi = N\sigma_{s_0}(E')\, F(\underline{r},E',\underline{\Omega}'), \tag{3.5}$$

so that the probability, $f(E',\underline{\omega})d\underline{\omega}$, that a given collision will deflect a neutron into the element of solid angle (measured in the c. of g. system)

*Since solid angle elements in both laboratory and c. of g. systems will be used, it is necessary to introduce two different notations to keep them separate. In the c. of g. system, the solid angle element is $d\underline{\omega} = d\eta\, d\psi$; in the laboratory system, the solid angle is $d\underline{\Omega} = d\mu_0 d\varphi$.

$d\eta\, d\psi$ is

$$f(E',\underline{\omega})d\underline{\omega} = \frac{N\sigma_s(E',\underline{\omega})d\eta\, d\psi}{N\sigma_{s_o}(E')} \quad . \tag{3.6}$$

If the scattering is isotropic in the c. of g. system, then $N\sigma_s(E',\underline{\omega}) = \frac{N\sigma_{s_o}(E')}{4\pi}$ and $f(E',\underline{\omega}) = \frac{1}{4\pi}$, a constant. Since the scattering is always azimuthally symmetric, it is permissible to integrate over ψ; then the probability that a neutron will be deflected into an element $d\eta$ is $f(E',\eta)d\eta = 2\pi f(E',\underline{\omega})d\eta$.

The cosine of the scattering angle in the laboratory system, which we denote by $\mu_o = \underline{\Omega}' \cdot \underline{\Omega}$ ($\underline{\Omega}'$ and $\underline{\Omega}$ are the directions of the neutron velocity before and after impact) is just

$$\mu_o(\eta) = \frac{1 + M\eta}{(1 + M^2 + 2M\eta)^{1/2}} \quad . \tag{3.7}$$

This follows from Figure 2. Moreover, the probability of a particular direction after collision can depend only on $\mu_o = \underline{\Omega} \cdot \underline{\Omega}'$, not on $\underline{\Omega}$ and $\underline{\Omega}'$ separately. Any collision in which the cosine of the c. of g. scattering angle is η will have the cosine of its scattering angle measured in the laboratory system given by (3.7). Consequently the probability that a scattered neutron of initial energy E' and direction $\underline{\Omega}'$ is thrown into an element $d\eta$ (measured in the c. of g. system) and simultaneously into an element $d\mu_o$ in the laboratory system is

$$f(E',\eta,\mu_o)d\eta\, d\mu_o = f(E',\eta)\delta\left[\mu_o - \mu_o(\eta)\right] d\eta\, d\mu_o \tag{3.8}$$

where $\mu_o(\eta)$ is the function (3.7) and δ is the Dirac delta function. If (3.8) is integrated over all directions in the c. of g. system -- that is, with respect to η from -1 to +1, the result is the probability that a neutron

is scattered into the element $d\mu_o$ in the laboratory system:

$$f(E',\mu_o)d\mu_o = d\mu_o \int_{-1}^{1} f(E',\eta)\ \delta\left[\mu_o - \mu_o(\eta)\right]\ d\eta\ . \tag{3.9}$$

To evaluate the integral it is necessary to transform to $\mu_o(\eta)$ as a new variable. Then

$$\int^{1} f(E',\eta)\delta\left[\mu_o-\mu_o(\eta)\right]d\eta = \int_{\mu_o(-1)}^{\mu_o(1)} f\left[E',\eta(\mu_o)\right]\delta\left[\mu_o-\mu_o(\eta)\right]\frac{\partial\eta}{\partial\mu_o}\,d\mu_o = f\left[E',\eta(\mu_o)\right]\frac{\partial\eta}{\partial\mu_o}\ ; \tag{3.10}$$

that is, the probability $f(E',\mu_o)d\mu_o$ of scattering into the element $d\mu_o$ in the laboratory system is, in terms of the probability $f\left[E',\eta(\mu_o)\right]d\eta$ of scattering into the element $d\eta$ in the c. of g. system,

$$f(E',\mu_o)d\mu_o = f\left[E',\ \eta(\mu_o)\right]\frac{\partial\eta(\mu_o)}{\partial\mu_o}\,d\mu_o\ . \tag{3.11}$$

As an example of the significance of this result consider the case of isotropic proton scattering (M=1). Then by (3.7)

$$\mu_o(\eta) = \frac{1}{\sqrt{2}}(1+\eta)^{1/2},\quad \eta(\mu_o) = 2\mu_o^2 - 1,\quad \frac{\partial\eta}{\partial\mu_o} = 4\mu_o\ , \tag{3.12}$$

and therefore, since $f(E',\eta) = 1/2$,

$$f(E',\mu_o)d\mu_o = 2\mu_o d\mu_o; \tag{3.13}$$

i.e., the scattering in the laboratory system obeys a cosine law if the scattering in the c. of g. system is isotropic.

The foregoing discussion was given because it illustrated the use of the δ function in transforming scattering probabilities from one variable (η) to another (μ_o). Suppose now we wish to calculate the probability $f(E',E,\underline{\Omega}',\underline{\Omega})dEd\underline{\Omega}$ that a scattered neutron originally having energy E' and direction $\underline{\Omega}'$ will be thrown into $dEd\underline{\Omega}$. This is related to $f(E',\eta,\mu_o)d\eta\, d\mu_o$, the probability that a neutron with energy E' is scattered into $d\eta$ (in the c. of g. system) and $d\mu_o$ in the laboratory system, by the transformation formula

$$f(E',E,\underline{\Omega}',\underline{\Omega})dEd\underline{\Omega} = \frac{1}{2\pi} f(E',E,\mu_o)dEd\mu_o d\phi =$$

$$\frac{1}{2\pi} f\left[E',\eta(E',E),\mu_o\right] \frac{d\eta}{dE}\, dEd\mu_o d\phi, \tag{3.14}$$

the factor $\frac{1}{2\pi}$ entering because the scattering probability is independent of azimuth.

Now from (3.1)

$$\eta(E',E) = \frac{(1+M)^2\, E/E' - (M^2+1)}{2M} \tag{3.15}$$

so that

$$\frac{\partial\eta}{\partial E} = \frac{(1+M)^2}{2ME'}$$

and therefore

$$f(E',E,\underline{\Omega}',\underline{\Omega})dEd\underline{\Omega} = \frac{(1+M)^2}{4\pi ME'} f\left[E',\eta(E',E),\mu_o\right] dEd\underline{\Omega}. \tag{3.16}$$

Substituting for $f(E',\eta,\mu_o)$ from (3.8) into (3.16) we obtain

$$f(E',E,\underline{\Omega}',\underline{\Omega})dEd\underline{\Omega} = \frac{1}{2\pi} f(E',E,\mu_o)dEd\underline{\Omega} = \frac{(1+M)^2}{4\pi ME'} f\left[E',\eta(E',E)\right]\ \delta\left[\mu_o-\mu_o(E',E)\right] dEd\underline{\Omega} \tag{3.17}$$

where $\mu_o(E',E)$, which is found by substituting for η in (3.1) its value found in (3.7), is

$$\mu_o(E',E) = \frac{1}{2}\left\{(M+1)(E/E')^{\frac{1}{2}} - (M-1)(E'/E)^{\frac{1}{2}}\right\} . \tag{3.18}$$

For isotropic scattering $f(E',\eta) = \frac{1}{2}$, and therefore

$$f(E',E,\underline{\Omega}',\underline{\Omega})dEd\underline{\Omega} = \frac{(1+M)^2}{8\pi ME'}\,\delta\left[\mu_o-\mu_o(E',E)\right]\,dEd\underline{\Omega} . \tag{3.19}$$

We are finally in a position to calculate the total number of neutrons in a volume element which are thrown into $dEd\underline{\Omega}$ from all initial energies and directions E', and $\underline{\Omega}'$, as a result of scattering collisions in that volume element. This number is found by integrating the product of the total collision density $N\sigma_{s_o}(E')\ F(\underline{r},E',\underline{\Omega}')$ and the probability $f(E',E,\underline{\Omega}',\underline{\Omega})$ over all energies and directions which can lead to the energy E. Evidently a collision can result in a neutron of energy between E and E+dE only if the neutron energy before collision fell between E and E/α^2. Hence we have:

total number of neutrons thrown into $dEd\underline{\Omega}$ per second per c.c. by elastic collisions

$$= dEd\underline{\Omega}\,\frac{(1+M)^2}{4\pi M}\int_E^{E/\alpha^2}\int_{\underline{\Omega}'} N\sigma_{s_o}(E')\ F(\underline{r},E',\underline{\Omega}')\ f\left[E',\eta(E',E)\right]\,\delta\left[\mu_o-\mu_o(E',E)\right]\,\frac{dE'}{E'}\,d\underline{\Omega}' \tag{3.20}$$

which is, for isotropic scattering, $(f = 1/2)$,

$$= dEd\underline{\Omega}\,\frac{(1+M)^2}{8\pi M}\int_E^{E/\alpha^2}\int_{\underline{\Omega}'} N\sigma_{s_o}(E')\ F(\underline{r},E',\underline{\Omega}')\ \delta\left[\mu_o-\mu_o(E',E)\right]\,\frac{dE'}{E'}\,d\underline{\Omega}' . \tag{3.21}$$

We are now able to write down the Boltzmann equation for the elastic scattering of neutrons. We suppose neutrons are produced with energy E_o at the rate

$S'(\underline{r},E,\underline{\Omega})$ per second per c.c. per unit solid angle. The source strength can therefore be represented as $S'(\underline{r},E,\underline{\Omega}) = S(\underline{r},\underline{\Omega})\ \delta(E-E_o)$. Our usual material balance yields immediately

$$\nabla\cdot\ \underline{F}(\underline{r},E,\underline{\Omega}) + N\sigma F(\underline{r},E,\underline{\Omega}) = \frac{(1+M)^2}{4\pi M}\int_E^{\frac{E}{\alpha^2}}\int_{\underline{\Omega}'} N\sigma_{s_o}(E')\ F(\underline{r},E',\underline{\Omega}')\ \mathrm{x}$$

$$f\left[E',\eta(E',E)\right]\ \delta\left[\mu_o-\mu_o(E',E)\right]\ \frac{dE'}{E'}\ d\underline{\Omega}' + S(\underline{r},\underline{\Omega})\ \delta(E-E_o). \qquad (3.22)$$

If the scattering is isotropic in the c. of g. system, the Boltzmann equation reads

$$\nabla\cdot\ \underline{F}(\underline{r},E,\underline{\Omega}) + N\sigma F(\underline{r},E,\underline{\Omega}) = \frac{(1+M)^2}{8\pi M}\int_E^{\frac{E}{\alpha^2}}\int_{\underline{\Omega}'} N\sigma_{s_o}(E')\ F(\underline{r},E',\underline{\Omega}')\ \mathrm{x}$$

$$\delta\left[\mu_o-\mu_o(E',E)\right]\ \frac{dE'}{E'}\ d\underline{\Omega}' + S(\underline{r},\underline{\Omega})\ \delta(E-E_o). \qquad (3.23)$$

In the more general case of non-isotropic scattering the scattering probability $f\left[E',\eta(E',E)\right]$ is expressible as a sum of Legendre functions:

$$f\left[E',\eta(E',E)\right] = \sum_{\ell}\frac{2\ell+1}{2}\ f_{\ell}(E')\ P_{\ell}\left[\eta(E',E)\right] \qquad (3.24)$$

where $f_o = 1$. For example, if two harmonics are sufficient to represent f, then

$$f\left[E',\eta(E',E)\right] = \frac{1}{2} + \frac{3}{2}\ f_1(E')\ \frac{(1+M)^2(E/E') - (M^2+1)}{2M}\ . \qquad (3.25)$$

The Boltzmann equation with anisotropic scattering is therefore

$$\nabla\cdot \underline{F}(\underline{r},E,\underline{\Omega}) + N\sigma F(\underline{r},E,\underline{\Omega}) = \sum_{\ell} \frac{(1+M)^2}{4\pi M} \int_{E}^{\frac{E}{\alpha^2}} \int_{\underline{\Omega}'} \frac{2\ell+1}{2} N\sigma_{s_o}(E') \times$$

$$F(\underline{r},E',\underline{\Omega}')\, f_{\ell}(E')\, P_{\ell}\left[\eta(E',E)\right] \delta\left[\mu_o-\mu_o(E',E)\right] \frac{dE'}{E'}\, d\underline{\Omega}' + S\,(\underline{r},\underline{\Omega})\, \delta(E-E_o). \tag{3.26}$$

At this point it is convenient to change from the energy to the logarithm of the energy as variable. We put

$$u = \ln E_o/E \tag{3.27}$$

where E_o is the source energy. The value of u corresponding to E_o evidently is zero. Then if $\mathcal{F}(\underline{r},u,\underline{\Omega})$ is the flux density per unit u and $\underline{\Omega}$, we have

$$\mathcal{F}(\underline{r},u,\underline{\Omega})\, du d\underline{\Omega} = F(\underline{r},E,\underline{\Omega}) dE d\underline{\Omega} \tag{3.28}$$

or, by the transformation formula for integrals

$$\mathcal{F}(\underline{r},u,\underline{\Omega}) = F\left[\underline{r},E(u),\underline{\Omega}\right] \frac{\partial E}{\partial u} = -E_o e^{-u}\, F\left[\underline{r},E(u),\underline{\Omega}\right] \quad . \tag{3.29}$$

Upon substituting the new variable u, into the isotropic Boltzmann equation (3.23), and simplifying, we obtain

$$\nabla\cdot \underline{\mathcal{F}}\,(\underline{r},u,\underline{\Omega}) + N\sigma\, \mathcal{F}(\underline{r},u,\underline{\Omega}) = \frac{-(1+M)^2}{8\pi M} \int_{u}^{u+\ln\alpha^2} \int_{\underline{\Omega}'} N\sigma_{s_o}(u')\, \mathcal{F}(\underline{r},u',\underline{\Omega}') \times$$

$$e^{u'-u}\, \delta\left[\mu_o-g(u'-u)\right] d\underline{\Omega}' du' + S(\underline{r},\underline{\Omega})\, \delta(u) \tag{3.30}$$

where

$$g(u'-u) = \mu_o\left[E'(u),E(u)\right] = \frac{1}{2}\left[(M+1)e^{\frac{u'-u}{2}} - (M-1)e^{-\frac{u'-u}{2}}\right]. \qquad (3.31)$$

In the anisotropic case we define

$$h(u'-u) \equiv \eta\left[E'(u),E(u')\right] = \frac{(1+M)^2\, e^{u'-u} - (1+M^2)}{2M} \qquad (3.32)$$

and therefore we can write (3.26) thus:

$$\nabla\cdot \mathcal{F}(\underline{r},u,\underline{\Omega}) + N\sigma\, \mathcal{F}(\underline{r},u,\underline{\Omega}) = -\sum_{\ell} \frac{(1+M)^2}{4\pi M} \int_u^{u+\ell n\alpha^2} \int_{\underline{\Omega}'} \frac{2\ell+1}{2} N\sigma_{s_o}(u') \times$$

$$\mathcal{F}(\underline{r},u',\underline{\Omega}')\, f_\ell(u')\, P_\ell\left[h(u'-u)\right] \delta\left[\mu_o-g(u'-u)\right]\, e^{u'-u}\, d\underline{\Omega}'du' + S(\underline{r},\underline{\Omega})\, \delta(u) \qquad (3.33)$$

where $f_\ell(u') = f_\ell\left[E'(u')\right]$ is the ℓth coefficient in the expansion (3.24) of the scattering function considered as a function of u'.

Equation (3.30) involves μ_o which is the scalar product of the initial and final directions, $\underline{\Omega}'$ and $\underline{\Omega}$. It is therefore unsuitable for further analysis until the δ-function is reduced to an explicit function of $\underline{\Omega}$ and $\underline{\Omega}'$. This is done by applying the addition theorem for Legendre polynomials to the expanded form of the δ-function. The procedure is essentially the one used in deriving (1.28) from (1.18), and so we shall not dwell on it at great length. We have

$$\delta\left[\mu_o-g(u'-u)\right] = \sum_{\ell} \frac{2\ell+1}{2} P_\ell\left[g(u'-u)\right] P_\ell(\mu_o). \qquad (3.34)$$

By the addition theorem for Legendre polynomials we can express $P_\ell(\mu_o)$ as a series involving $P_\ell(\mu)$ and $P_\ell(\mu')$ [cf. Eq. (1.26)]. Hence

$$\delta\left[\mu_o - g(u'-u)\right] = \sum_{\ell} \frac{2\ell+1}{2} P_{\ell}\left[g(u'-u)\right] P_{\ell}(\mu)\, P_{\ell}(\mu') + \text{terms in } \varphi \text{ and } \varphi' \; ; \tag{3.35}$$

the terms in φ and φ' are unimportant if the angular distribution has a symmetry axis, since they disappear on integration over φ. If we substitute (3.35) into (3.30), and integrate over φ, we have, on the assumption that the x axis is a symmetry axis for the angular distribution, and that the neutron intensity does not depend on y or z,

$$\mu \frac{d\mathcal{F}(x,u,\mu)}{dx} + N\sigma\mathcal{F} = -\frac{(1+M)^2}{4M} \sum_{\ell} \frac{2\ell+1}{2} \int_{u}^{u+\ln\alpha^2} \int_{-1}^{+1} N\sigma_{s_o}(u')\, \mathcal{F}(x,u',\mu')\, \times$$

$$e^{u'-u} P_{\ell}\left[g(u'-u)\right] P_{\ell}(\mu)\, P_{\ell}(\mu')du'd\mu' + S(x,\mu)\, \delta(u) \tag{3.36}$$

where, as usual

$$\mathcal{F}(x,u,\mu) = \int_0^{2\pi} \mathcal{F}(x,u,\underline{\Omega})d\varphi = 2\pi\, \mathcal{F}(x,u,\underline{\Omega})$$

$$S(x,\mu) = \int_0^{2\pi} S(x,\underline{\Omega})d\varphi \; . \tag{3.37}$$

The assumption that the flux angular distribution is axially symmetric is correct only if the source $S(x,\underline{\Omega})$ emits neutrons with axial symmetry; this we suppose to be the case.

In case the scattering is anisotropic in the center of gravity system, the Boltzmann equation reads

$$\mu \frac{d\mathcal{F}(x,u,\mu)}{dx} + N\sigma\, \mathcal{F}(x,u,\mu) = \frac{-2(1+M)^2}{4M} \sum_{\ell} \sum_{\ell} \left(\frac{2\ell+1}{2}\right)\left(\frac{2\ell'+1}{2}\right) \int_{u}^{u+\ln\alpha^2} \int_{-1}^{+1} N\sigma_{s_0}(u') \times$$

$$\mathcal{F}(x,u',\mu')e^{u'-u} f_{\ell'}(u')\, P_{\ell'}\left[h(u'-u)\right]\, P_{\ell}\left[g(u'-u)\right]\, P_{\ell}(\mu)\, P_{\ell}(\mu')du'd\mu' +$$

$$S(x,\mu)\, \delta(u). \tag{3.38}$$

Equations (3.36) and (3.38) are strictly applicable only to neutrons whose energy is less than $1/\alpha^2$ times the energy, E_o, of the most energetic neutron emitted by the source. Evidently there are no neutrons with energy greater than E_o. Consequently it is necessary to add, in addition to (3.36) and (3.38), the requirement

$$\mathcal{F}(x,u,\mu) \equiv 0 \quad \text{if } u = \ln E_o/E \leqslant 0. \tag{3.39}$$

An alternative way of stating this condition is that the upper limit, $u + \ln\alpha^2$, in the integral equation must be replaced by 0 for all neutrons whose energy lies between E_o and E_o/α^2; i.e., for all neutrons whose energy can be reached from the maximum source energy in a single collision. In the case of hydrogen, assuming that the proton and neutron masses are identical, all energies below the source energy can be reached in a single collision. The upper limit of integration in the case of hydrogen is therefore 0 for all energies.

The Boltzmann Equation for Mixtures

In case the scattering medium contains several different nuclei having different atomic weights, the Boltzmann equation must be modified to take

account of the fact that the cross-sections and the maximum energy losses per collision of the constituents will in general be different. The derivation of the Boltzmann equation proceeds very much as in the case of a single scatterer, and we simply quote the result. If we denote quantities referring to the i-th constituent with a superscript $^{(i)}$ (e.g., $N^i\sigma^i_{s_o}$ is the total macroscopic scattering cross-section of the i-th constituent), then the one-dimensional isotropic Boltzmann equation is

$$\mu \frac{d\mathcal{F}(x,u,\mu)}{dx} + N\sigma\, \mathcal{F}(x,u,\mu) = \sum_i \sum_\ell \frac{(1+M^i)^2}{4M^i} \frac{2\ell+1}{2} \int_u^{u+\ln(\alpha^i)^2} \int_{-1}^{1} c^i N\sigma_{s_o}(u')\, x \tag{3.40}$$

$$\mathcal{F}(x,u',\mu')e^{u'-u}\, P_\ell\left[g^i(u'-u)\right] P_\ell(\mu)\, P_\ell(\mu')du'd\mu' + S(x,\mu)\,\delta(u)$$

where c^i, the weighted "concentration" of the i-th constituent, is

$$c^i = \frac{N^i\sigma^i_{s_o}}{N\sigma_{s_o}} \tag{3.41}$$

and

$$N\sigma_{s_o} = \sum_i N^i\sigma^i_{s_o}\,, \quad N\sigma = \sum_i N^i\sigma^i\,. \tag{3.42}$$

In the first slowing down interval for each constituent the upper limit $u + \ln(\alpha^i)^2$ must be replaced, as before, by zero.

Many moderators for nuclear chain reactions -- H_2O, D_2O, BeO, to mention a few examples -- have more than one constituent atom. The Boltzmann equation for mixtures is therefore of considerable practical importance.

The anisotropic Boltzmann equation for mixtures is easily seen to be

$$\mu \frac{d\mathcal{F}(x,u,\mu)}{dx} + N\sigma\mathcal{F}(x,u,\mu) = -\sum_i \sum_{\ell} \sum_{\ell'} \frac{(1+M^i)^2}{2M^i} \frac{2\ell+1}{2} \frac{2\ell'+1}{2} \int_u^{u+\ell n(\alpha^i)^2} \int_{-1}^{1} \times$$

$$c^i N\sigma^i_{s_o} \mathcal{F}(x,u',\mu')e^{u'-u} f^i_{\ell}(u') P_{\ell'}\left[h^i(u'-u)\right] P_{\ell}\left[s^i(u'-u)\right] P_{\ell}(\mu) P_{\ell}(\mu')du'd\mu' + S(x,\mu)\,\delta(u). \tag{3.43}$$

Reduction to Age Theory

There are two directions in which simplifying approximations in the Boltzmann equation can be made. On the one hand, the angular distribution can be expanded in spherical harmonics, and the resulting infinite series can be cut off after a finite number of terms. On the other hand, the collision density, $N\sigma_{s_o}\mathcal{F}(x,u,\mu)$ can be expanded in a Taylor's series in $(u'-u)$ under the integral, and only the first few terms kept. This is valid if $N\sigma_{s_o}\mathcal{F}$ changes very little in a single slowing down interval, $(1 - \alpha^2)E$.

An expansion in spherical harmonics evidently is valid only for neutrons which have suffered many collisions -- that is, neutrons which have lost a large amount of energy. This can be seen by considering the energy and angular distribution of neutrons which are far from a monoenergetic, isotropic point source. If the energy of the neutron is close to the source energy, either the neutron has suffered few collisions, or it has suffered many small angle collisions. In either case the angular distribution of such high energy neutrons will exhibit a strong forward bias which is poorly represented by a finite number of spherical harmonics.

At very great distances from the source there will be few neutrons which have neither reached thermal equilibrium nor have been captured. Those which still have energies appreciably greater than thermal energy must have travelled predominantly in a straight line from the source -- otherwise, had they suffered large angle collisions, their energy would have been low, and they would not have progressed very far from the source. Consequently non-thermal neutrons found far from a point source will have a predominantly forward bias, which, as we have noted, can be represented only by many spherical harmonics. For this reason we cannot expect a spherical harmonic expansion to be very good either for very high energy neutrons, or for any non-thermal neutrons at very large distances from the source.

A very similar situation was met in Chapter II in connection with the time dependent diffusion problem without energy loss. It was pointed out there that the neutron distribution at very short times after an instantaneous burst of neutrons could not be represented by a few spherical harmonics, for the neutrons which reach a given point first have suffered no collisions and are therefore travelling forward; their angular distribution tends to be singular in the forward direction. The use of only two spherical harmonics led in that case to a retardation time which was too long by a factor $\sqrt{3}$.

With these limitations in mind we proceed to simplify the Boltzmann equation. We suppose that $\mathcal{F}(x,u,\mu)$ can be represented by a sum of just the first two Legendre polynomials:

$$\mathcal{F}(x,u,\mu) \approx \frac{1}{2}\ \mathcal{F}_0(x,u) + \frac{3}{2}\ \mathcal{F}_1(x,u)\ P_1(\mu), \tag{3.44}$$

and that the source is isotropic: $S(x,\mu) = \frac{1}{2}\, S_0(x)$.

Upon substituting (3.44) into the isotropic Boltzmann equation (3.36) and equating coefficients of Legendre polynomials of the same order, we obtain

$$\frac{1}{3}\frac{d\,\mathcal{F}_o(x,u)}{dx} + N\sigma\,\mathcal{F}_1(x,u) = -\frac{(1+M)^2}{4M}\int_u^{u+\ln\alpha^2} N\sigma_{s_o}\mathcal{F}_1(x,u')e^{u'-u}\,g(u'-u)du'$$

$$\frac{d\,\mathcal{F}_1(x,u)}{dx} + N\sigma\,\mathcal{F}_o(x,u) = -\frac{(1+M)^2}{4M}\int_u^{u+\ln\alpha} N\sigma_{s_o}\mathcal{F}_o(x,u')e^{u'-u}du' + S_o(x)\,\delta(u)\ . \tag{3.45}$$

At this point it is customary to make the assumption that $N\sigma_{s_o}\mathcal{F}_o$ and $N\sigma_{s_o}\mathcal{F}_1$ change very slowly over one slowing down interval. It is then permissible to expand in Taylor's series, and keep only two terms:

$$N\sigma_{s_o}\mathcal{F}_o(x,u') = N\sigma_{s_o}\mathcal{F}_o(x,u) + (u'-u)\frac{d}{du}\left[N\sigma_{s_o}\mathcal{F}_o(x,u)\right]$$

$$N\sigma_{s_o}\mathcal{F}_1(x,u') = N\sigma_{s_o}\mathcal{F}_1(x,u) + (u'-u)\frac{d}{du}\left[N\sigma_{s_o}\mathcal{F}_1(x,u)\right] \tag{3.46}$$

The integral equations (3.45) can be converted into differential equations by substituting (3.46) into (3.45) and integrating. The resulting equation for $N\sigma_{s_o}\mathcal{F}_o$ is the basis of the so-called "age" theory first introduced by Fermi.

Before giving the remaining details of this reduction to age theory we examine the conditions under which an expansion in Taylor's series is satisfactory. The functions $N\sigma_{s_o}\mathcal{F}_o$ and $N\sigma_{s_o}\mathcal{F}_1$ are supposed to change slowly in one slowing down interval; this apparently will be the case the smaller is

the slowing down interval -- that is, the heavier is the atomic weight of the scatterer. If there is no absorption, this is true. However, if there is absorption then the Taylor's series expansion is not valid. For the heavier the nucleus, the more collisions a neutron will make in any particular energy interval. If the moderator absorbs neutrons in this energy interval, then the longer the neutron spends in the interval, the greater is the likelihood of absorption. But heavy absorption in an interval means that the neutron density changes rapidly, and therefore an expansion in Taylor's series is not justified. This fact -- that the Taylor's is not appropriate if absorption is present -- was first pointed out by E. P. Wigner.

For the reason given in the preceding paragraph, the remaining discussion will be confined to the case of a non-capturing moderator ($N\sigma = N\sigma_{s_o}$) of high atomic weight. In this case alone is the Taylor's series expansion justified. Substitution of (3.46) into (3.45) yields the two equations:

$$\frac{1}{3}\frac{\partial \mathcal{F}_o(x,u)}{\partial x} + N\sigma_{s_o}\mathcal{F}_1(x,u) = -\frac{(1+M)^2}{4M}\left[N\sigma_{s_o}\mathcal{F}_1(x,u)\int_u^{u+\ln\alpha^2} g(u'-u)e^{u'-u}du' + \frac{\partial}{\partial u}\, N\sigma_{s_o}\mathcal{F}_1(x,u)\int_u^{u+\ln\alpha^2}(u'-u)g(u'-u)e^{u'-u}du'\right]$$

$$\frac{\partial \mathcal{F}_1(x,u)}{\partial x} + N\sigma_{s_o}\mathcal{F}_o(x,u) = -\frac{(1+M)^2}{4M}\left[N\sigma_{s_o}\mathcal{F}_o(x,u)\int_u^{u+\ln\alpha^2} e^{u'-u}du' + \frac{\partial}{\partial u}\, N\sigma_{s_o}\mathcal{F}_o(x,u)\int_u^{u+\ln\alpha^2}(u'-u)e^{u'-u}du'\right] + S_o(x)\,\delta(u). \qquad (3.47)$$

Of the four integrals on the right hand side, the simplest is

$$-\frac{(1+M)^2}{4M}\int_u^{u+\ln\alpha^2} e^{u'-u}du' = 1.$$

It is of order $(1/M)^0$.

The next simplest integrals are respectively the average logarithmic energy loss per collision, and the average cosine of the scattering angle:

$$\frac{(1+M)^2}{4M}\int_u^{u+\ln\alpha^2}(u'-u)e^{u'-u}du' = \xi = 1 + \frac{\alpha^2}{1-\alpha^2}\ln\alpha^2 \approx \frac{2}{M+\frac{2}{3}} \tag{3.48}$$

$$-\frac{(1+M)^2}{4M}\int_u^{u+\ln\alpha^2} g(u'-u)e^{u'-u}du' \equiv \bar{\mu}_o = \frac{2}{3M} \quad . \tag{3.49}$$

Both quantities are of order $1/M$.

Finally, by integration by parts, it is easy to see that

$$\frac{(1+M)^2}{4M}\int_u^{u+\ln\alpha^2}(u'-u)\ g(u'-u)\ e^{u'-u}du'$$

is of order $1/M^2$.

To show that the average logarithmic energy loss, ξ, and the average cosine of the scattering angle, $\bar{\mu}_o$, are given by (3.48) and (3.49), we proceed thus:

$$\xi = \int_{\alpha^2E'}^{E'}\int_{\underline{\Omega}}\ln(E'/E)\ f(E',E,\underline{\Omega}'\underline{\Omega})dEd\underline{\Omega} = \frac{(1+M)^2}{4\pi ME'}\int_{\alpha^2E'}^{E'}\int_{\underline{\Omega}}\ln(E'/E)\ f\left[E',\eta(E',E)\right]$$

$$\delta\left[\mu_o-\mu_o(E',E)\right]\ dEd\underline{\Omega} = \frac{(1+M)^2}{2ME'}\int_{\alpha^2E'}^{E'}\ln(E'/E)\ f\left[E',\eta(E',E)\right]dE. \tag{3.50}$$

Upon changing to $u = \ln E_o/E$ as variable, this becomes

$$= \frac{(1+M)^2}{2M} \int_{u'}^{u'-\ln\alpha^2} (u-u')\, f\left[E'(u),\ h(u'-u)\right]\, e^{u'-u} du'$$

for general scattering law (3.51a)

$$= \frac{(1+M)^2}{4M} \int_{u'}^{u'-\ln\alpha^2} (u-u') e^{u'-u} du = \frac{(1+M)^2}{4M} \int_{u}^{u+\ln\alpha^2} (u'-u) e^{u'-u} du'$$

for isotropic scattering. (3.51b)

In a similar manner, the average cosine of the scattering angle is

$$\bar{\mu}_o = \int_{\alpha^2 E'}^{E'} \int_{\underline{\Omega}} \mu_o(E',E)\, f(E',E,\underline{\Omega}',\underline{\Omega})\, dE\, d\underline{\Omega} = \frac{(1+M)^2}{2ME'} \int_{\alpha^2 E'}^{E'} \mu_o(E',E)\, f\left[E', \eta(E',E)\right] dE =$$

$$= \frac{(1+M)^2}{2M} \int_{u'}^{u'-\ln\alpha^2} g(u'-u)\, f\left[E'(u),\ h(u'-u)\right] e^{u'-u} du$$

for general scattering law (3.52a)

$$= \frac{-(1+M)^2}{4M} \int_{u}^{u+\ln\alpha^2} g(u'-u) e^{u'-u} du'$$

for isotropic scattering. (3.52b)

Since

$$g(u'-u) = \frac{1}{2}\left[(M+1)\, e^{\frac{u'-u}{2}} - (M-1)\, e^{\frac{-u'-u}{2}}\right],$$

$$\bar{\mu}_o = -\frac{(1+M)^2}{4M} \int_{u}^{u+\ln\alpha^2} \frac{1}{2}\left[(M+1) e^{\frac{u'-u}{2}} - (M-1)\, e^{-\frac{u'-u}{2}}\right] e^{u'-u} du' = \frac{2}{3M} \tag{3.53}$$

provided the scattering is isotropic in the c. of g. system.

Upon substituting for the integrals their values in terms of ξ and $\bar{\mu}_o$, and neglecting the term of order $1/M^2$, we obtain

$$\frac{1}{3}\frac{\partial \mathcal{F}_o}{\partial x} + N\sigma_{s_o}\mathcal{F}_1 = \bar{\mu}_o N\sigma_{s_o}\mathcal{F}_1$$

$$\frac{\partial \mathcal{F}_1}{\partial x} + N\sigma_{s_o}\mathcal{F}_o = N\sigma_{s_o}\mathcal{F}_o - \xi\frac{\partial}{\partial u}(N\sigma_{s_o}\mathcal{F}_o) + S_o(x)\ \delta(u). \tag{3.54}$$

If the first of these equations is differentiated with respect to x and is then substituted into the second, there results a partial differential equation which approximately describes the slowing down and diffusion of the neutrons:

$$\frac{1}{3}\frac{\partial}{\partial x}\frac{1}{N\sigma_{s_o}(1-\bar{\mu}_o)}\frac{\partial \mathcal{F}_o(x,u)}{\partial x} + S_o(x)\ \delta(u) = \xi\frac{\partial}{\partial u}N\sigma_{s_o}\mathcal{F}_o(x,u)\ . \tag{3.55}$$

This equation is the fundamental equation of the so-called "age" theory.

If $N\sigma_{s_o}$ and $\bar{\mu}_o$ are independent of position, the equation can be written

$$\frac{1}{3N\sigma_{s_o}\xi N\sigma_{s_o}(1-\bar{\mu}_o)}\frac{\partial^2\left[N\sigma_{s_o}\mathcal{F}_o(x,u)\right]}{\partial x^2} + \frac{S_o(x)\ \delta(u)}{\xi} = \frac{\partial}{\partial u}\left[N\sigma_{s_o}\mathcal{F}_o(x,u)\right]\ ; \tag{3.56}$$

i.e., the collision density (number of collisions per c.c. per logarithmic energy interval) satisfies an equation like the time dependent heat conduction equation with u replacing t, and $\dfrac{1}{3N\sigma_{s_o}\xi N\sigma_{s_o}(1-\bar{\mu}_o)}$ replacing the thermal diffusivity (thermal conductivity ÷ specific heat x density). The name "age theory" is suggested by this analogy between logarithmic energy loss u, and time, t.

If the scattering is isotropic in the c. of g. system, ξ is constant, and it can be put under the differential operator in Eq. (3.56). Thus the one-dimensional slowing down equation for isotropic scattering can also be written

$$\frac{1}{3N\sigma_{s_o}\xi N\sigma_{s_o}(1-\bar{\mu}_o)}\ \frac{\partial^2\left[N\sigma_{s_o}\xi\mathcal{F}_o(x,u)\right]}{\partial^2 x} + S_o(x)\ \delta(u) = \frac{\partial}{\partial u} N\sigma_{s_o}\xi\mathcal{F}_o(x,u) \qquad (3.56a)$$

The quantity

$$q(x,u) = N\sigma_{s_o}\xi\mathcal{F}_o(x,u)$$

is called the "slowing down density". As will be shown later, q(x,u) is the number of neutrons which, in the course of slowing down and diffusing, cross energy E per c.c. per second.

In the case of a mixture, the procedure which led to (3.56) must be modified slightly. We now expand each of the functions $c^i N\sigma_{s_o}\mathcal{F}_{o,1}(x,u)$ in Taylor's series and insert in (3.40). The result, correct to terms in the first derivative with respect to u, is

$$\frac{1}{3(N\sigma_{s_o})^2\sum c^i\xi^i\sum c^i(1-\bar{\mu}_o^i)}\ \frac{\partial^2\sum c^i\xi^i N\sigma_{s_o}\mathcal{F}_o(x,u)}{\partial x^2} + S_o(x)\ \delta(u) = \frac{\partial}{\partial u}\sum c^i\xi^i N\sigma_{s_o}\mathcal{F}_o\ ; \qquad (3.57)$$

i.e., the quantity

$$N\sigma_{s_o}\sum_i c^i\xi^i\ \mathcal{F}_o(x,u).$$

in which ξ^i is the average logarithmic decrement of the ith substance, satisfies a heat conduction type equation in which the average logarithmic energy loss and the average cosine of the scattering angle are

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$$\bar{\xi} = \sum c^i \xi^i \quad \text{and} \quad \bar{\bar{\mu}}_o = \sum c^i \bar{\mu}_o^i \,, \tag{3.58}$$

provided, of course, the scattering is isotropic. The slowing down density in the mixture case can therefore be written

$$q(x,u) = N\sigma_{s_o} \sum c^i \xi^i \mathcal{F}_o(x,u) = N\sigma_{s_o} \bar{\xi} \mathcal{F}_o(x,u).$$

There is nothing in the derivation of (3.55) which involves the assumption of isotropic scattering in the c. of g. system. It is equally valid for a general scattering law; however, ξ and $\bar{\mu}_o$ in that case, have the values given in (3.51a) and (3.52a), and are, in general, energy dependent. It may be noticed that ξ and $\bar{\mu}_o$ are energy dependent in the mixture case also; in this respect slowing down in a mixture is analogous to slowing down in a non-isotropically scattering single element moderator.

Finally, the treatment given here has been confined to diffusion in one dimension. In the general three-dimensional case (3.55) must be written

$$\frac{1}{3}\nabla \cdot \frac{1}{N\sigma_{s_o}(1-\bar{\mu}_o)} \nabla \mathcal{F}_o(\underline{r},u) + S_o(\underline{r})\,\delta(u) = \xi \frac{\partial}{\partial u}\left[N\sigma_{s_o} \mathcal{F}_o(\underline{r},u)\right], \tag{3.59}$$

and, for a uniform medium,

$$\frac{1}{3N\sigma_{s_o}\xi N\sigma_{s_o}(1-\bar{\mu}_o)} \Delta N\sigma_{s_o} \mathcal{F}_o(\underline{r},u) + \frac{S_o(\underline{r})\,\delta(u)}{\xi} = \frac{\partial}{\partial u} N\sigma_{s_o} \mathcal{F}_o(\underline{r},u). \tag{3.60}$$

It is well at this point to recapitulate the assumptions which underly a differential equation description [for example, Eq. (3.56)] of the slowing down process. These assumptions are:

1) Heavy moderator; i.e., average logarithmic energy loss per collision is small. It may be noticed that the terms retained in (3.47) are of order 1/M; in fact, the Taylor expansion leads to a power series in 1/M; as is suggested already by a comparison of (3.48), (3.49) and (3.50). The important case of moderation in a hydrogenous medium cannot be handled by this formalism.

2) No absorption.

3) Small distance from source; i.e., neutron angular distribution is fairly isotropic.

4) Mean free path varies slowly in an energy range comparable to ξ. If this were not so, the Taylor's expansion would not be valid. This restriction, like (1), renders this simple theory inapplicable for hydrogen.

Simplified Derivation of Age Theory Equations; the Slowing Down Density

Placzek has given a very simple but instructive derivation of the age theory equation which shows why the term "age" is appropriate. We consider a monoenergetic point source of neutrons which emits S neutrons per second at energy E_o. Let $n(\underline{r},t)dt$ represent the number of neutrons at $\underline{r}$ which have been emitted by the source at a time between t and t+dt before the interval dt of observation. We assume that the loss of energy by collision, instead of being a statistical process, occurs in a continuous, non-statistical manner so that there is a definite correlation between the time that a neutron has diffused and the energy which it has lost. In fact, if a neutron had diffused for a time t, it has suffered

$$\int_0^t N\sigma_{s_o}\, v(t)\,dt$$

collisions. At each collision we assume the neutron loses ξ units of logarithmic energy; hence the value of $u = \ln E_o/E$ which the neutron has at time t after leaving the source is

$$u(t) = \int_0^t N\sigma_{s_o} \xi v dt.$$

The neutrons are assumed to diffuse according to a time dependent diffusion equation; the diffusion coefficient $D = \frac{v}{3N\sigma_{tr}}$ is a function of t because the energy of a neutron changes as it diffuses. Hence, provided the diffusion coefficient changes but little during the time between collisions,

$$\frac{v(t)}{3N\sigma_{tr}} \Delta n(\underline{r},t) = \frac{\partial n}{\partial t}(\underline{r},t). \tag{3.62}$$

We now change to u as variable. In the usual manner, we have

$$n(\underline{r},t) = n\left[\underline{r},u(t)\right] \frac{\partial u}{\partial t} = N\sigma_{s_o} \xi v n(\underline{r},u), \tag{3.63}$$

where $n(\underline{r},u)$ is the neutron density per unit u. Also,

$$\frac{\partial}{\partial t} = \frac{\partial u}{\partial t}\frac{\partial}{\partial u} = N\sigma_{s_o} \xi v \frac{\partial}{\partial u}; \tag{3.64}$$

hence, substituting (3.63) and (3.64) into (3.62),

$$\frac{1}{3N\sigma_{tr}} \Delta N\sigma_{s_o} \xi \mathcal{F}_o(\underline{r},u) = N\sigma_{s_o} \xi \frac{\partial}{\partial u} N\sigma_{s_o} \xi \mathcal{F}_o(\underline{r},u). \tag{3.65}$$

In the case of isotropic scattering (ξ = constant), (3.65) and (3.56) are equivalent for all $u > 0$.

The neutron density $n(\underline{r},t)dt$ comprises all those neutrons in $\underline{r}$ which have diffused between t and t+dt seconds. During this time they have experienced a continuous degradation of energy and at time t their logarithmic energy is about to fall below the value of u which corresponds to the diffusion time t. Thus, $n(\underline{r},t)dt$ is the number of neutrons at $\underline{r}$ which cross the logarithmic energy u (or the energy E) in the time interval dt; i.e., $n(\underline{r},t)$ is the number of neutrons crossing energy E per sec. per c.c. at $\underline{r}$. The quantity $n(\underline{r},t)$ is called the "slowing down density", and is denoted by q.

The slowing down density is expressible in terms of the neutron density per logarithmic energy interval, according to (3.63) as

$$q = N\sigma_{s_o} \xi v\, n(\underline{r},u) = N\sigma_{s_o} \xi \mathcal{F}_o(\underline{r},u).$$

The meaning of this formula is the following: $N\sigma_{s_o} nv$ is the number of collisions occurring per c.c. per second at $\underline{r}$. On each collision a "distance" ξ in logarithmic energy space is travelled by the neutron. Hence q = number of neutrons crossing E per second per c.c. = collision density x distance in logarithmic energy travelled per second = $N\sigma_{s_o} \xi v\, n(\underline{r},u)$.

Equation (3.65) can be further simplified by a change of variable. We set

$$\tau(u) = \int_0^{t(u)} \frac{v\,dt}{3N\sigma_{tr}} = \int^{u} \frac{du}{3N\sigma_{tr}\, N\sigma_{s_o} \xi}. \tag{3.66}$$

The quantity $\tau(u)$ is called the "age" of the neutron; it is the chronological age, t, of the neutron since its birth, weighted with $\bar{D}$, the average of the diffusion coefficient during the time of diffusion: $\bar{D} = \frac{1}{t} \int_0^t \frac{v}{3N\sigma_{tr}} dt$:

$$\tau(u) = \bar{D}\, t(u).$$

The age has units cm^2, <u>not</u> seconds.

In terms of τ

$$n(\underline{r},u) = n\left[\underline{r}, \tau(u)\right] \frac{\partial\tau}{\partial u} = \frac{n(\underline{r},\tau)}{3N\sigma_{tr}\, N\sigma_{s_o}\xi}$$

$$\frac{\partial}{\partial u} = \frac{\partial}{\partial\tau} \cdot \frac{\partial\tau}{\partial u} = \frac{1}{3N\sigma_{tr}\, N\sigma_{s_o}\xi} \frac{\partial}{\partial\tau}$$

and therefore (3.65) becomes

$$\Delta\left[\frac{\mathcal{F}_o(\underline{r},\tau)}{3N\sigma_{tr}}\right] = \frac{\partial}{\partial\tau}\left[\frac{\mathcal{F}_o(\underline{r},\tau)}{3N\sigma_{tr}}\right], \tag{3.67}$$

where $\mathcal{F}_o(\underline{r},\tau)$ is the flux per unit τ.

The slowing down density in terms of the neutron density per unit τ is

$$q(\underline{r},\tau) = \frac{v}{3N\sigma_{tr}}\, n(\underline{r},\tau) = \frac{\mathcal{F}_o(\underline{r},\tau)}{3N\sigma_{tr}} \tag{3.68}$$

and therefore the age equation can be written

$$\Delta q(\underline{r},\tau) = \frac{\partial q(\underline{r},\tau)}{\partial\tau}. \tag{3.69}$$

The analogy between (3.69) and the heat conduction equation is apparent.

<u>The Space Independent Boltzmann Equation; Energy Distribution of Neutrons</u>

If the production of neutrons throughout all space is uniform, then the neutron density cannot depend on x. The general Boltzmann equation (3.38) reduces to

$$N\sigma \mathcal{F}(u,\mu) = -\frac{(1+M)^2}{2M} \sum_{\ell} \sum_{\ell'} \frac{2\ell+1}{2} \frac{2\ell'+1}{2} \int_{u}^{u+\ln\alpha^2} \int_{-1}^{1} N\sigma_{s_o}(u')\, \mathcal{F}(u',\mu') e^{u'-u} \tag{3.70}$$

$$f_{\ell'}(u')\, P_{\ell}\left[h(u'-u)\right] P_{\ell}\left[g(u'-u)\right] P_{\ell}(\mu)\, P_{\ell}(\mu') du' d\mu' + S(\mu)\, \delta(u-u_o) .$$

The total neutron flux per logarithmic energy interval, $\mathcal{F}_o(u)$, and the total number of neutrons produced per c.c. per second, S_o, are

$$\mathcal{F}_o(u) = \int_{-1}^{1} \mathcal{F}(u,\mu) d\mu$$

$$S_o = \int_{-1}^{1} S(\mu) d\mu .$$

Thus, upon integrating (3.70) over μ and μ', there results

$$N\sigma \mathcal{F}_o(u) = -\frac{(1+M)^2}{2M} \int_{u}^{u+\ln\alpha^2} N\sigma_{s_o} \mathcal{F}_o(u') e^{u'-u} f\left[u', h(u'-u)\right] du' + S_o\, \delta(u-u_o). \tag{3.71}$$

In the isotropic scattering case, $f = 1/2$, this reduces to

$$N\sigma \mathcal{F}_o(u) = -\frac{(1+M)^2}{4M} \int_{u}^{u+\ln\alpha^2} N\sigma_{s_o} \mathcal{F}_o(u') e^{u'-u} du' + S_o\, \delta(u-u_o).$$

It is to be understood, as before, that $\mathcal{F}_o(u) \equiv 0$ if $u < u_o$.

The distribution $\mathcal{F}_o(u)$ for large values of u (i.e., at energies far from the source energy) is easy to calculate provided there is no absorption ($N\sigma = N\sigma_{s_o}$). In this case the distribution equation (3.71) is

$$N\sigma_{s_o} \mathcal{F}_o(u) = -\frac{(1+M)^2}{2M} \int_u^{u+\ln\alpha^2} N\sigma_{s_o} \mathcal{F}_o(u') e^{u'-u} f\left[u', h(u'-u)\right] du' . \tag{3.72}$$

Now, since $f(E', E, \underline{\Omega}', \underline{\Omega})$ is the probability that a collision by a neutron of energy E' and direction $\underline{\Omega}'$ results in an energy E, and a direction $\underline{\Omega}$,

$$\int_{\underline{\Omega}} \int_{E'}^{\alpha^2 E'} f(E', E, \underline{\Omega}', \underline{\Omega}) dE d\underline{\Omega} = 1; \tag{3.73}$$

hence, on transforming to the variable u, and integrating over angle, we find

$$-\frac{(1+M)^2}{2M} \int_u^{u+\ln\alpha^2} e^{u'-u} f\left[u', h(u'-u)\right] du' = 1. \tag{3.74}$$

Consequently

$$N\sigma_{s_o} \mathcal{F}_o(u) = C, \text{ a constant,}$$

satisfies (3.72). However, since this solution does not satisfy the initial condition [which is derived from (3.71)], namely,

$$\lim_{u \to u_o} N\sigma \mathcal{F}_o(u) = S_o \delta(u-u_o),$$

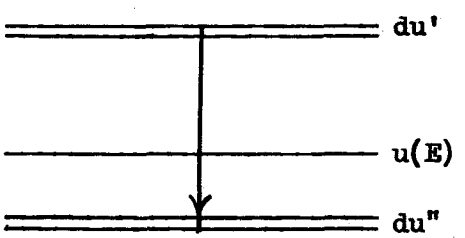
it cannot be correct close to the source energy u_o. Thus $N\sigma_{s_o} \mathcal{F}_o(u) = C$ is only the asymptotic solution of (3.71), correct at energies u which differ from u_o by several logarithmic slowing down intervals $\ln\alpha^2$.

We compute the value of the constant C in the case of isotropic scattering, i.e., f = 1/2. This is of most practical interest since the asymptotic solution

applies only to neutrons which have lost considerable energy, and therefore the scattering by the moderator will usually have become isotropic by the time the asymptotic solution becomes valid. To compute C we equate the number of neutrons which cross a given energy E per second per c.c. in the course of slowing down, to the number of neutrons produced per second per c.c.

To calculate the number of neutrons which cross E per second per c.c. we observe that all neutrons from logarithmic interval du' which enter a logarithmic energy interval du" lying below u(E) will have crossed E. The number of collisions per logarithmic energy interval du' per second is

$$N\sigma_{s_o} \mathcal{F}_o(u')du';$$



the probability that these collisions will result in neutrons being thrown into energy interval dE" is

$$\frac{(1+M)^2}{2M} f\left[E', \eta(E',E")\right] dE"$$

or, in the logarithmic energy variable,

$$\frac{(1+M)^2}{2M} e^{u'-u"} f\left[u',h(u'-u")\right] du".$$

Hence the total number of neutrons thrown across E per second per c.c. is

$$\frac{(1+M)^2}{2M} \int_{u}^{u+\ln\alpha^2} \int_{u'-\ln\alpha^2}^{u} N\sigma_{s_o} \mathcal{F}_o(u') e^{u'-u"} f\left[u',h(u'-u")\right] du" du' \quad . \qquad (3.75)$$

Since the scattering is assumed isotropic, f = 1/2. Hence the number of neutrons crossing energy E per second per c.c., i.e., the slowing down

density q(E), which is to be equated to S_o, is

$$q(E) = S_o = \frac{(1+M)^2}{4M} C \int_{u}^{u+\ln\alpha^2} \int_{u'-\ln\alpha^2}^{u} e^{u'-u''} du'' du' = C\xi = N\sigma_{s_o} \mathcal{F}_o(u)\,\xi. \qquad (3.76)$$

Thus

$$C = \frac{S_o}{\xi}$$

and the energy distribution is

$$\mathcal{F}_o(u) = \frac{S_o}{N\sigma_{s_o}\xi} \quad . \qquad (3.77)$$

In terms of E, rather than u, the distribution is

$$F_o(E) = \frac{S_o}{N\sigma_{s_o}\xi E} \qquad (3.78)$$

or in terms of v,

$$n(v) = \frac{2S_o}{N\sigma_{s_o}\xi v^2} \quad . \qquad (3.79)$$

The energy distribution (3.77) can, of course, be derived from the approximate equations (3.56a) and (3.57). In the case of isotropic scattering in a mixture, the slowing down density in an infinite medium in which neutrons are produced uniformly satisfies the equation

$$\frac{\partial}{\partial u}\left[N\sigma_{s_o}\bar{\xi}\mathcal{F}_o(u)\right] = S_o\,\delta(u) \qquad (3.80)$$

where

$$\bar{\xi} = \sum_i c^i \xi^i.$$

Now the solution of (3.80) is

$$N\sigma_{s_o}\bar{\xi}\mathcal{F}_o(u) = S_o \qquad u > 0$$
$$= 0 \qquad u < 0 \tag{3.81}$$

and the neutron energy distribution implied by (3.81) is the same as that of (3.78).

The assumption of a single substance was not necessary to obtain (3.81); on the other hand the differential equation which led to (3.81) is an approximation which was valid, because it involved a Taylor series expansion, only if c^i (in the case of mixtures) varied slowly over one slowing down interval. Actually the energy distribution (3.81) is a rigorous asymptotic solution of the space-independent Boltzmann equation only for isotropic scattering in a single substance. For mixtures or anisotropic scattering, (3.81) is only approximately correct.

Spatial Distribution of Slowed Neutrons; the Slowing Down Kernels

The slowing down density satisfies

$$\Delta q(\underline{r}, \tau) = \frac{\partial q(\underline{r}, \tau)}{\partial \tau} \tag{3.69}$$

with the initial condition

$$\lim_{\tau \to 0} \Delta q(\underline{r}, \tau) - \frac{\partial q(\underline{r}, \tau)}{\partial \tau} = S_o(\underline{r})\, \delta(\tau).$$

Suppose a point source emits one fast neutron per second at $\underline{r} = 0$ in an infinite medium. The slowing down density at some lower energy corresponding to age τ will be the solution of

$$\Delta q(\underline{r}, \tau) + \delta(\tau)\, \delta(\underline{r}) = \frac{\partial q(\underline{r}, \tau)}{\partial \tau}\,. \tag{3.82}$$

This equation is identical in form with the time dependent diffusion equation discussed in Chapter I. The solution, as was found there, is

$$q(\underline{r}, \tau) = \frac{e^{-r^2/4\tau}}{(4\pi\tau)^{3/2}} ; \tag{3.83}$$

i.e., the slowing down density of neutrons from a point monoenergetic source is distributed around the point according to a Gaussian function. The range r_o of the Gaussian (i.e., the distance at which the density falls to 1/e of its value at the source) is

$$r_o = 2\sqrt{\tau}. \tag{3.84}$$

For many purposes it is important to know the second spatial moment of the slowing down density. If the slowing down density is Gaussian, then the second moment of neutrons slowed to age τ, which we denote by $\overline{r^2}(\tau)$, is

$$\overline{r^2}(\tau) = \frac{\int_0^\infty q(r,\tau) r^2 \cdot 4\pi r^2 dr}{\int_0^\infty q(r,\tau) \cdot 4\pi r^2 dr} = \frac{\int_0^\infty r^4 \, e^{-r^2/4\tau} dr}{\int_0^\infty r^2 \, e^{-r^2/4\tau} dr} = 6\tau .$$

For a Gaussian distribution, the following relation holds between the age, the second moment, and the range:

$$\tau = \overline{r^2}/6 = r_o^2/4 .$$

The relation between τ and the second moment of the slowing down distribution is the same as the relation between the square of the diffusion length, L, and the second moment of the distribution of thermal neutrons

around a point source. For this reason $\sqrt{\tau}$ is often called the slowing down length.

The age τ is related to the logarithmic energy according to (3.66) by

$$\tau(u) = \int_0^u \frac{du}{3N\sigma_{tr}N\sigma_{s_o}\xi}, \tag{3.85}$$

and is therefore a monotone increasing function of u. Thus the spatial distribution of slowed neutrons keeps a Gaussian shape as the neutrons lose energy, but the neutron distribution gradually spreads out since the range r_o increases with u. The distribution of neutrons slowing down from an energetic source is in this approximation exactly the same as the distribution of heat from an instantaneous heat source.

The energy distribution of the neutrons slowed from a point source is, according to (3.83)

$$\mathcal{F}_o(r,u) = \frac{1}{N\sigma_{s_o}\xi} \frac{e^{-r^2/4\tau(u)}}{\left[4\pi\tau(u)\right]^{3/2}},$$

$\tau(u)$ being the function expressed in (3.85). If all cross-sections are constant, then

$$\tau(u) = \frac{u}{3N\sigma_{tr}N\sigma_{s_o}\xi}$$

and

$$\mathcal{F}_o(r,u) = \frac{1}{N\sigma_{s_o}\xi} \frac{e^{-3N\sigma_{tr}N\sigma_{s_o}\xi r^2/4u}}{\left[\frac{4\pi u}{3N\sigma_{tr}N\sigma_{s_o}\xi}\right]^{3/2}}. \tag{3.86}$$

At a given point the flux as a function of u waxes and then wanes; the maximum occurs at the logarithmic energy u_{max} given by

$$u_{max} = \frac{N\sigma_{tr} N\sigma_{s_o} \xi r^2}{2} \quad . \tag{3.87}$$

The slowing down density (3.83) from a point monoenergetic neutron source may be designated the "point slowing down kernel". Following the procedure used in the discussion of the time dependent diffusion equation, we can write down the corresponding kernels in other geometries.

Slowing Down Kernels

Geometry	Notation	Source Normalization	P
Plane	$P_{p\ell}(x,\tau;x',\tau')$	1 neut/cm^2/sec at (x',τ')	$\dfrac{e^{-\lvert x-x'\rvert^2/4(\tau-\tau')}}{[4\pi(\tau-\tau')]^{1/2}}$
Point	$P_p(\underline{r},\tau;\underline{r}',\tau')$	1 neut/sec at $(\underline{r}',\tau')$	$\dfrac{e^{-\lvert \underline{r}-\underline{r}'\rvert^2/4(\tau-\tau')}}{[4\pi(\tau-\tau')]^{3/2}}$
Line	$P_\ell(r,\varphi,\tau;r',\varphi',\tau')$	1 neut/cm/sec at (r',φ',τ')	$\dfrac{e^{-\lvert\rho-\rho'\rvert^2/4(\tau-\tau')}}{4\pi(\tau-\tau')}$, $\rho^2 = r^2+r'^2 - 2rr'\cos(\varphi-\varphi')$
Spherical Shell	$P_S(r,\tau;r',\tau')$	1 neut/sec per shell of radius r' at age τ'.	$\dfrac{1}{4\pi rr'}\left[\dfrac{e^{-\lvert r-r'\rvert^2/4(\tau-\tau')}}{4\pi(\tau-\tau')^{1/2}} - \dfrac{e^{-\lvert r+r'\rvert^2/4(\tau-\tau')}}{4\pi(\tau-\tau')^{1/2}}\right]$
Cylindrical Shell	$P_c(r,\tau;r',\tau')$	1 neut/sec/cm of shell of radius r' at age τ'	$\dfrac{e^{-\lvert r-r'\rvert/4(\tau-\tau')}}{4\pi(\tau-\tau')} I_o\left(\dfrac{rr'}{2(\tau-\tau')}\right)$

From the slowing down density arising from a monochromatic energy source it is a trivial matter to compute the slowing down density from a fission neutron source which is polyenergetic. If the number of neutrons emitted per second between energy E' and E'+dE' from a point source at the origin is

$$f(E')dE' ,$$

then the slowing down density from such a source is evidently

$$q\left[\underline{r},\tau(E)\right] = \int_{E}^{\infty} \frac{e^{-r^2/4\left[\tau(E)-\tau(E')\right]}}{\left\{4\pi\left[\tau(E)-\tau(E')\right]\right\}^{3/2}} f(E')dE' , \qquad (3.88)$$

where $q\left[\underline{r},\tau(E)\right]$ is the number of neutrons crossing energy E per second per c.c. at $\underline{r}$. Since in a chain reaction the neutrons originate from a fission spectrum, the slowing down distribution as given by (3.88) is the one appropriate to a chain reactor in which the moderator is non-hydrogenous.

Elementary Improvements on Age Theory

The age approximation, and the Gaussian slowing down distribution which it yields, resulted from a spherical harmonic expansion of the angular distribution, and a Taylor's series expansion of the energy distribution. As has already been pointed out, the Taylor's series expansion is valid only if the mean free path varies slowly over one slowing down interval, while the spherical harmonic expansion could be expected to be good only fairly near the source. Thus the age approximation is poor in hydrogenous media (where the mean free path changes rapidly), or at large distances from the source in any medium.

That the Gaussian cannot be correct at large distances is evident from the following physical argument: Consider neutrons which have made no collisions at all. These will be distributed like $S_o \frac{N\sigma e^{-N\sigma r}}{4\pi r^2}$, where $N\sigma$ is the macroscopic scattering cross-section and S_o is the source strength. Now at small distances the Gaussian slowing down distribution will exceed this exponential; at large distances, however, the ratio

$$\frac{\text{source neutrons}}{\text{Gaussian moderated neutrons}} = \left(\frac{N\sigma e^{-N\sigma r}}{4\pi r^2}\right)\left(\frac{(4\pi\tau)^{3/2}}{e^{-r^2/4\tau}}\right)$$

approaches ∞, since the Gaussian falls off faster than the exponential. Thus at large distances, the distribution is more exponential than Gaussian.

An improvement on the age theory distribution which at least is free from this "first collision paradox" can be expected if the "aging" process, which leads to the Gaussian, is assumed to begin only after the neutrons have made their first collision. The points at which first collisions occur act as "sources" for the slowing down process.

The first collisions are distributed as

$$\frac{N\sigma(0)\, e^{-N\sigma(0) r}}{4\pi r^2},$$

the 0 referring to $u = 0$, the source energy.

According to this picture the slowing down distribution should therefore be

$$q(\underline{r},\tau) = \int_0^\infty \frac{N\sigma(0)\, e^{-N\sigma(0)|\underline{r}'|}}{4\pi r'^2} \; \frac{e^{-|\underline{r}'-\underline{r}|^2/4\tau}}{(4\pi\tau)^{3/2}}\, d\underline{r}' .$$

A further improvement can be made by taking into account the fact that after a neutron has suffered a collision which throws it across energy E, it experiences a "free ride", without changing its energy, until it suffers its next collision. To take this free ride after the last collision into account, it is plausible to include another exponential with mean free path appropriate to the lower energy. Thus the slowing down distribution, including both first and last collisions is

$$q(\underline{r},\tau) = \int_0^\infty \int_0^\infty \frac{N\sigma(0)e^{-N\sigma(0)|\underline{r}'|}}{4\pi r'^2} \; \frac{e^{-|\underline{r}''-\underline{r}'|^2/4\tau}}{(4\pi\tau)^{3/2}} \; \frac{e^{-N\sigma(u)|\underline{r}''-\underline{r}|}}{4\pi|\underline{r}''-\underline{r}'|^2} \, d\underline{r}'d\underline{r}'' \qquad (3.89)$$

where $\sigma(0)$ and $\sigma(u)$ are cross-sections at the initial and final log energies respectively. Formulas like (3.89) are of course not rigorous; they are rather more plausible than the simple Gaussian and have been used to represent the slowing down distribution from a point monoenergetic source.

In order to compute the second moment of the distribution (3.89) we first state the well known result that the second moment of the distribution from a plane source is just 1/3 the second moment from a point. This follows from the relation between a point kernel and the corresponding plane kernel,

$$P_p(r) = -\frac{1}{2\pi r} P'_{p\ell}(r).$$

Hence

$$\overline{r^2} = \frac{\int_0^\infty r^2 P_p(r) r^2 dr}{\int_0^\infty P_p(r) r^2 dr} = \frac{\int_0^\infty z^3 P'_{p\ell}(z) dz}{\int_0^\infty z\, P'_{p\ell}(z) dz} .$$

Upon integrating by parts, and using the fact that $r^3 P_{p\ell}(r) \to 0$ as $r \to \infty$ for any kernels of interest, we obtain

$$\overline{r^2} = 3\overline{z^2},$$

$\overline{z^2}$ being the second moment of the plane distribution.

With this preliminary we compute $\overline{r^2}$ for the distribution (3.89) by computing the corresponding plane second moment, and multiplying by 3. The plane distribution corresponding to (3.89) we write as

$$q(z,\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_1(z')P_{p\ell}(|z'-z''|)K_2(|z''-z|)dz'dz'' \tag{3.90}$$

where $K_1(z)$ and $K_2(z)$ are plane transport kernels and $P_{p\ell}$ is the plane Gaussian kernel. The quantity $q(z,\tau)$ is the convolution of the three kernels $K_1, P_{p\ell}$, and K_2.

Now, if $\overline{K}_1(B^2)$ is the Fourier transform of $K_1(z)$, i.e., if

$$\overline{K}_1(B^2) = \int_{-\infty}^{\infty} K_1(z)\, e^{iBz} dz,$$

then $\overline{z^2_{K_1}}$, the second moment of the distribution defined by $K_1(z)$ is

$$\overline{z^2_{K_1}} = - \left. \frac{\dfrac{d^2\overline{K}_1(B^2)}{dB^2}}{\overline{K}_1(B^2)} \right|_{B=0} . \tag{3.91}$$

Furthermore, the Fourier transform of $q(z,\tau)$ is

$$\overline{q}(B^2,\tau) = \overline{K}_1(B^2)\, \overline{P}_{p\ell}(B^2)\, \overline{K}_2(B^2), \tag{3.92}$$

as follows from (3.90) and the definition of the Fourier transform. Hence

$$\left.\frac{\dfrac{d^2\bar{q}(B^2,\tau)}{dB^2}}{\bar{q}(B^2,\tau)}\right|_{B=0} = \left.\frac{\dfrac{d^2\bar{K}_1(B^2)}{dB^2}}{\bar{K}_1(B^2)}\right|_{B=0} + \left.\frac{\dfrac{d^2\bar{P}_{p\ell}(B^2)}{dB^2}}{\bar{P}_{p\ell}(B^2)}\right|_{B=0} + \left.\frac{\dfrac{d^2\bar{K}_2(B^2)}{dB^2}}{\bar{K}_2(B^2)}\right|_{B=0}$$

that is,

$$\overline{z_q^2} = \overline{z_{K_1}^2} + \overline{z_{P_{p\ell}}^2} + \overline{z_{K_2}^2} \quad , \tag{3.93}$$

and

$$\overline{r_q^2} = \overline{r_{K_1}^2} + \overline{r_{P_p}^2} + \overline{r_{K_2}^2} \quad .$$

In other words, <u>the second moment of a distribution which is the convolution of several kernels is the sum of the second moments of each kernel</u>.

We now apply this result to the distribution (3.90). The second moment of the transport kernel $\dfrac{N\sigma(0)e^{-N\sigma(0)r}}{4\pi r^2}$ is

$$\frac{2}{[N\sigma(0)]^2} ,$$

and the second moment of the Gaussian is 6τ. Hence the second moment of the slowing down distribution corrected for first and last collisions is

$$\overline{r^2}(\tau) = \frac{2}{[N\sigma(0)]^2} + 6\tau + \frac{2}{[N\sigma(u)]^2} \tag{3.94}$$

or, upon using the formula (3.66) for τ,

$$\overline{r^2}(\tau) = \frac{2}{[N\sigma(0)]^2} + 2\int_0^u \frac{du}{N\sigma_{tr}N\sigma_{s_0}\xi} + \frac{2}{[N\sigma(u)]^2} \, . \tag{3.95}$$

The distribution (3.90) is unwieldy analytically, and it has therefore been customary to replace it by a single Gaussian (for a point distribution)

$$\frac{e^{-r^2/4\tau'}}{(4\pi\tau')^{3/2}}$$

where τ', the corrected age, is chosen so as to give the same second moment as (3.95). Thus the corrected age is

$$\tau'(u) = \frac{\overline{r^2}}{6} = \frac{1}{3\left[N\sigma(0)\right]^2} + \frac{1}{3}\int_0^u \frac{du}{N\sigma_{tr}N\sigma_{s_o}\xi} + \frac{1}{3\left[N\sigma(u)\right]^2}, \tag{3.96}$$

and it is this age, together with the simple Gaussian, which is usually used to represent the slowing down density in a heavy moderator.

The Group Picture

The Gaussian slowing down distribution with the corrected age (3.96) is a fairly satisfactory representation of the slowing down process in heavy moderators. However, for certain problems, e.g., those involving slowing down in composite media, even the elegant Gaussian age theory becomes very unwieldy. The analytical difficulties arise because the age theory equation is a partial differential equation. To avoid these complications a simplified formulation of the slowing down problem which describes the process by a sequence of ordinary differential equations has been used very widely in pile theory.

The general idea of this method, called the method of groups, is to divide the total logarithmic energy interval through which the neutrons pass into a finite number of energy subintervals. Neutrons in a given energy group are supposed to diffuse without energy loss until they have experienced a number of collisions equal to the average number of collisions actually required to pass through the energy interval; at this time they pass into the next lower

energy interval. Thus removal from an energy interval is treated as an "absorption" process, the "absorption" cross-section σ_a^* being determined from the relation

$$\frac{\sigma_s}{\sigma_a^*} = \text{number of collisions before removal from energy range.}$$

The cross-section for removal of neutrons from one group is also the cross-section for creation of neutrons in the next lower group. The slowing down density, i.e., the number of neutrons passing from the νth energy group to the $\nu+1$st group, is therefore

$$q_\nu = N\sigma_a^* \Phi_\nu \tag{3.97}$$

where Φ_ν is the flux of neutrons in the νth energy group. If the magnitude of a logarithmic energy interval is denoted by u_ν, and ξ is the logarithmic energy decrement per collision, then the number of collisions required to cross u_ν is u_ν/ξ, and the "absorption" cross-section for the νth group is

$$\sigma_a^* = \frac{\sigma_{s\nu}\xi}{u_\nu} \tag{3.98}$$

where $\sigma_{s\nu}$ is the average scattering cross-section in the u_ν energy interval.

If D_ν is the average diffusion coefficient in the νth energy group, then the neutron flux $\Phi_\nu(\underline{r})$ in the νth group satisfies the diffusion equation

$$D_\nu \triangle\Phi_\nu(\underline{r}) - N\sigma_{a\nu}^*\Phi(\underline{r}) + N\sigma_{a\,\nu-1}^*\Phi_{\nu-1}(\underline{r}) + S_\nu(\underline{r}) = 0, \tag{3.99}$$

$S_\nu(\underline{r})$ being the number of neutrons produced by an external source per unit volume at $\underline{r}$ in the νth energy interval.

In a one group picture, $\nu = 1$, in which the external source is a δ-function at the origin, the group equation is

$$D_1 \Delta\Phi_1 - N\sigma^*_{a_1}\Phi_1 + \delta(r) = 0; \tag{3.100}$$

this has the solution

$$N\sigma^*_{a_1}\Phi_1(r) = \frac{e^{-r/L_1^*}}{4\pi r L_1^{*2}} \tag{3.101}$$

where

$$L_1^{*2} = D_1/N\ \sigma^*_{a_1}\ . \tag{3.102}$$

The second moment of this distribution is $6L_1^{*2}$, and the slowing down length is given by

$$\frac{\overline{r^2}}{6} = L_1^{*2} = \frac{D_1}{N\sigma^*_{a_1}}\ . \tag{3.103}$$

If we substitute for $N\sigma^*_{a_1}$ and D_1 their expressions in terms of cross-sections, we obtain

$$L_1^{*2} = \frac{u_1}{3N\sigma_{tr_1}N\sigma_{s_1}\xi}\ . \tag{3.104}$$

This is identical with the age theory expression for one sixth of the second moment provided the product $N\sigma_{tr_1}N\sigma_{s_1}$ is chosen as

$$\frac{1}{N\sigma_{tr_1}N\sigma_{s_1}} = \frac{1}{u_1}\int_0^{u_1}\frac{du}{N\sigma_{tr_1}(u)N\sigma_{s_1}(u)}\ . \tag{3.105}$$

With this choice of average cross-sections, the one-group picture is seen to give a spatial distribution which is exponential instead of Gaussian, but which has the same second moment as the age theory.

In the n-group picture the distribution from a localized source of high energy neutrons is readily found by solving (3.99). For simplicity we deal with the problem with plane symmetry; the solution from a point source is then found by differentiating according to (1.141). The differential equations to be solved are

$$\left.\begin{aligned} D_1 \frac{d^2\Phi_1}{dx^2} - N\sigma^*_{a_1}\Phi_1 + \delta(x) &= 0 \\ D_\nu \frac{d^2\Phi_\nu}{dx^2} - N\sigma^*_{a\nu}\Phi_\nu + N\sigma^*_{a\nu-1}\Phi_{\nu-1} &= 0 \qquad \nu = 2 \ldots n \end{aligned}\right\} \quad (3.106)$$

To solve these equations we make a Fourier transformation,

$$\Phi_\nu(x) = \int_{-\infty}^{\infty} \bar{\Phi}_\nu(\omega) e^{i\omega x} d\omega, \qquad \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} d\omega.$$

The transforms satisfy

$$(D_1\omega^2 + N\sigma^*_{a_1})\,\bar{\Phi}_1 = \frac{1}{2\pi}$$

$$-(D_\nu\omega^2 + N\sigma^*_{a\nu})\,\bar{\Phi}_\nu + N\sigma^*_{a\nu-1}\,\bar{\Phi}_{\nu-1} = 0 \qquad \nu > 1 .$$

Hence

$$\bar{\Phi}_1(\omega) = \frac{1}{2\pi(D_1\omega^2 + N\sigma^*_{a_1})}; \quad \bar{\Phi}_k(\omega) = \bar{\Phi}_{k-1}\frac{N\sigma^*_{ak-1}}{D_k\omega^2 + N\sigma^*_{ak}} =$$

$$\frac{1}{2\pi(D_1\omega^2 + N\sigma^*_{a_1})}\prod_{\nu=2}^{k}\frac{N\sigma^*_{a\nu-1}}{(D_\nu\omega^2 + N\sigma^*_{a\nu})}$$

and therefore

$$\Phi_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{D_1\omega^2 + N\sigma_{a_1}^*}\, d\omega$$

$$\Phi_k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{D_1\omega^2 + N\sigma_{a_1}^*} \prod_{\nu=2}^{k} \frac{N\sigma_{a_{\nu-1}}^*}{(D_\nu\omega^2 + N\sigma_{a_\nu}^*)}\, d\omega. \tag{3.107}$$

Since $\overline{\Phi}_k(\omega)$ is the product of the Fourier transforms of the preceding Φ_ν, the distribution $\Phi_k(x)$ in a given group must be the convolution of the distribution of the previous groups. Physically this means that each group acts as a source for the succeeding group.

To actually compute $\Phi_\nu(x)$, it is necessary to evaluate the integrals in (3.107). The integrands have poles in the upper half plane at

$$\omega = i\sqrt{\frac{N\sigma_{a_\nu}^*}{D_\nu}} = i/L_\nu^*, \tag{3.108}$$

and we assume for simplicity that all roots are simple. Hence, according to the residue theorem.

$$\Phi_k(x) = \frac{1}{N\sigma_{a_k}^*}\sum_{j=1}^{k} \frac{1}{2L_j^*} \frac{e^{-x/L_j^*}}{(1-L_1^{*2}/L_j^{*2})(1-L_2^{*2}/L_j^{*2})\ldots(1-L_k^{*2}/L_j^{*2})} \tag{3.109}$$

where the term $(1-L_j^{*2}/L_j^{*2})$ is omitted from the sum. If $k = 1$ (one-group picture), (3.109) reduces to

$$N\sigma_{a_1}^*\Phi_1(x) = \frac{e^{-x/L_1^*}}{2L_1^*} \tag{3.110}$$

which is the plane equivalent of (3.101).

Since $\Phi_k(x)$ is the convolution of all the previous $\Phi_\nu(x)$, the total slowing down length for the neutrons slowed out of the kth group must be the sum of the squares of the slowing down lengths in each group individually:

$$\frac{\overline{r_k^2}}{6} = L_1^{*2} + L_2^{*2} + \ldots\ldots + L_k^{*2}. \tag{3.111}$$

If the number of groups becomes infinite, but each L_ν is reduced so that

$$\frac{\overline{r_\infty^2}}{6} = \sum_{\nu=1}^{\infty} L_\nu^{*2} \equiv \tau \tag{3.112}$$

where τ remains finite, then the group picture should go over into the continuous age theory. The slowing down density in the ∞-group case can be computed most readily by first passing to the limit in the integrand of (3.107) and then evaluating the integral. From (3.107)

$$N\sigma_{a_k}^* \Phi_k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{(1 + L_1^{*2}\omega^2)(1 + L_2^{*2}\omega^2)\ldots(1 + L_k^{*2}\omega^2)}\, d\omega .$$

Now

$$\lim_{\substack{L_\nu^* \to 0 \\ \sum_\nu L_\nu^{*2} \to \tau}} \prod_{\nu=1}^{\infty} \frac{1}{(1 + L_\nu^{*2}\omega^2)} = e^{-\omega^2 \sum_\nu L_\nu^{*2}} = e^{-\omega^2\tau}, \tag{3.113}$$

as is verified by taking logarithms of both sides. Hence

$$\lim_{k\to\infty} N\sigma_{a_k}^* \Phi_k(x) = \frac{1}{2\pi}\int_{\infty}^{\infty} e^{i\omega x}\, e^{-\omega^2\tau} d\omega = \frac{e^{-x^2/4\tau}}{(4\pi\tau)^{1/2}} ; \tag{3.114}$$

that is, the group picture and the age picture merge when the number of groups becomes infinite.

The great merit of the group method is that it involves ordinary instead of partial differential equations. By taking enough groups it is possible to approximate the age theory slowing down function to any degree of accuracy, and still deal only with ordinary equations. The approximate slowing down functions which are constructed out of group picture exponentials are called "synthetic" kernels. In pile problems involving H_2O as moderator, it is customary to use one or two fast neutron groups in addition to the thermal neutron group; in piles moderated by heavier materials, as many as five or six groups have been used.

In assessing the relative accuracy of the group method and the age theory, it must be remembered that the slowing down function from a point fission source, even in, say, graphite, is not a Gaussian because of the energy spread of the source neutrons. Thus in graphite the three group model is only slightly less accurate than the single Gaussian while in H_2O, because of the very long mean free path at high energies, the slowing down is more nearly represented by a single group picture than by a Gaussian.

Average Transport Cross-Section in Group Method

In order to obtain a one-group distribution which has the same second moment as the Gaussian, it is necessary to average the product of the transport and scattering cross-sections according to (3.105). In problems involving only one medium it is only this product which determines the neutron distribution. However, in problems involving composite media, since one of the boundary conditions across an interface is continuity of the net current, and the current

is proportional to the transport mean free path, it is necessary to find an appropriate average for the transport mean free path separately.

To calculate an average transport mean free path which will insure continuity of the net neutron current in a group, it is necessary to make some assumption with regard to the actual energy distribution of the neutrons in a given group. Evidently the energy distribution will depend on the particular arrangement and properties of the slowing down media on each side of the boundary. However, as a simple approximation, it is useful to assume that the energy distribution of the neutrons is the asymptotic distribution

$$\Phi(x,E)\,dE = f(x)\,\frac{dE}{N\sigma_{s_o}\xi E}\,, \tag{3.115}$$

where σ_{s_o} is the scattering cross-section.

The total flux of neutrons in a group from energy E_1 to E_2 is

$$\Phi(x) = \int_{E_1}^{E_2} \Phi(x,E)\,dE = f(x)\int_{E_1}^{E_2} \frac{dE}{N\sigma_{s_o}\xi E} \tag{3.116}$$

and the net current is

$$\frac{1}{3}\,\bar{\lambda}_{tr}\,\frac{d\Phi(x)}{dx} = \frac{1}{3}\int_{E_1}^{E_2} \lambda_{tr}(E)\,\frac{d\Phi}{dx}\,(x,E)\,dE = \frac{1}{3}\,\frac{df}{dx}\int_{E_1}^{E_2} \frac{\lambda_{tr}(E)\,dE}{N\sigma_{s_o}\xi E}\,, \tag{3.117}$$

where $\bar{\lambda}_{tr}$ is the correct average transport mean free path. Thus combining (3.116) and (3.117), we obtain

$$\bar{\lambda}_{tr} = \frac{\displaystyle\int_{E_1}^{E_2} \lambda_{tr}(E)\,\frac{dE}{N\sigma_{s_o}E}}{\displaystyle\int_{E_1}^{E_2} \frac{dE}{N\sigma_{s_o}E}} \tag{3.118}$$

if ξ is constant; i.e., the average transport mean free path which will give continuity of flow and density in a group in which the asymptotic energy distribution holds is an average over $1/Nc_{s_0}E$.

The Energy Transfer Distribution of Slowed Neutrons

It is a matter of some practical importance to calculate the manner in which the energy transferred to a moderator by elastic collisions of fast neutrons is distributed in space as the neutrons slow down from a plane source. If the flux of neutrons of log energy $u = \ln E_0/E$ is $\mathcal{F}(x,u)$ (plane symmetry), then the number of elastic collisions per c.c. per second at energy $E = E_0e^{-u}$ is

$$N\sigma_{s_0}(u)\,\mathcal{F}(x,u)$$

Since the logarithm of the ratio of the average energies E' and E after two successive collisions is

$$\ln E'/E = \xi,$$

the average energy loss per collision, ΔE, is

$$\Delta E = E' - E = E(e^{\xi} - 1); \tag{3.119}$$

i.e., if the moderator is heavy,

$$\Delta E \approx \xi E. \tag{3.120}$$

This energy increment appears as kinetic energy of the moderator atom. Hence $\mathcal{E}(x)$, the energy released per c.c. per second to the moderator by elastic collisions, is, for heavy moderators,

$$\mathcal{E}(x) = \int_0^\infty N\sigma_{s_0}(u)\,\xi E\,\mathcal{F}(x,u)du = E_0\int_0^\infty N\sigma_{s_0}(u)\,\xi e^{-u}\,\mathcal{F}(x,u)du. \tag{3.121}$$

To evaluate this integral an assumption must be made with respect to the neutron distribution $\mathcal{F}(x,u)$. This we take to be Gaussian:

$$\mathcal{F}(x,u) = \frac{q(x,u)}{N\sigma_{s_o}\xi} = \frac{S_o}{N\sigma_{s_o}\xi}\,\frac{e^{-x^2/4\tau(u)}}{\left[4\pi\,\tau(u)\right]^{1/2}}\,, \tag{3.122}$$

where S_o is the number of neutrons emitted per sq. cm. per second by the source. Hence we obtain for the integral (3.121),

$$\mathcal{E}(x) = SE_o\int_0^\infty \frac{e^{-\left[u+x^2/4\,\tau(u)\right]}}{\left[4\pi\,\tau(u)\right]^{1/2}}\,du. \tag{3.123}$$

This integral can in general be evaluated only by numerical methods. However, if all cross-sections are constant, then, in simplest approximation,

$$\tau(u) = \frac{\lambda_{tr}\lambda}{3\xi}\,u \tag{3.124}$$

and

$$\mathcal{E}(x) = S_oE_o\int_0^\infty \frac{e^{-\left[u\,+\,3x^2\xi/4\lambda_{tr}\lambda u\right]}}{\left[\frac{4}{3}\,\pi\,\frac{\lambda_{tr}\,\lambda u}{\xi}\right]^{1/2}}\,du. \tag{3.125}$$

Evaluating the integral according to Watson's Bessel Functions, p. 183, we obtain

$$\mathcal{E}(x) = \frac{\alpha S_oE_o}{2}\,e^{-\alpha x} \tag{3.126}$$

where

$$\alpha = \left[3\xi/\lambda\lambda_{tr}\right]^{1/2}. \tag{3.127}$$

The total energy emitted from one side of the source plane per sq. cm per second is $S_oE_o/2$; thus, according to (3.126), the fractional energy release in each cubic centimeter falls off exponentially with length constant $1/\alpha$.

Slowing Down Distribution in a Finite Block

In order to measure the slowing down distribution from a source it is customary to place the source on the axis of a long parallelopiped and measure the activity of Cd covered In foils placed along the long axis of the parallelopiped. Since In has a deep resonance at 1.44 ev, the activity of such a foil will in good part be proportional to the flux of 1.44 ev neutrons. Actually, because of higher resonances, the reading of the In foil is not quite proportional to the 1.44 ev flux; according to Hill and Roberts, at points close to a source of 30 kv neutrons in graphite, almost 40% of the activation of In is due to absorption above 1.44 ev. Farther from the source the perturbation due to higher resonances becomes less so that the mean sqaure distance to 1.44 ev as measured by In foils is in error by much less than 40%. The theory of this experiment is a good illustration of the usefulness of the age approximation, and we give the details in the following paragraphs.

Suppose a monoenergetic unit source is placed at the point $x = 0$, $y = 0$, $z = 0$ in an infinitely long moderating prism of sides $2a'$. The slowing down density satisfies

$$\Delta q(x,y,z,\tau) = \frac{\partial q(x,y,z,\tau)}{\partial \tau} \tag{3.128}$$

$$\lim_{\tau \to 0} q(x,y,z,\tau) = \delta(x,y,z) \tag{3.129}$$

where we have assumed the long direction is along z. The boundary conditions may be taken with sufficient accuracy (provided the width of the block is much larger than the mean free path),

$$q \equiv 0 \quad \text{on the extrapolated boundary,} \tag{3.130}$$

the extrapolated boundary being the geometric boundary a' augmented by the

extrapolation distance $0.71\,\lambda_{tr}$. We denote $a' + 0.71\,\lambda_{tr}$ by a. It is convenient to assume λ_{tr} independent of energy; again this is an unimportant assumption provided the block dimension is large compared to a mean free path.

The solution of (3.88) which satisfies the boundary conditions is

$$q(x,y,z,\tau) = \frac{1}{a^2}\sum_{m,n\,o}^{\infty} \cos B_m x \cos B_n y \; e^{-\left[B_m^2 + B_n^2\right]\tau} \frac{e^{-z^2/4\tau}}{(4\pi\tau)^{1/2}} \tag{3.131}$$

where

$$B_m = (2m+1)\frac{\pi}{2a}, \quad B_n = (2n+1)\frac{\pi}{2a}. \tag{3.132}$$

The sine solution is not used because of the symmetry of the source distribution.

The shape of the distribution along the z-direction is the same as from an infinite plane. As the neutrons age (τ increases) the intensity of the distribution falls because of the exponential factor, $e^{-(B_m^2 + B_n^2)\tau}$. This factor accounts for leakage out of the block. Its dependence on τ arises from the circumstance that neutrons with large τ must have diffused for a relatively long time and therefore must have had a good chance to leak out of the sides. The magnitude of the leakage is determined by the ratio τ/a^2.

The distribution (3.131) is represented as a sum of characteristic functions. The slowing down density can, of course, also be computed by observing that the neutron distribution from a point source in a finite block can be viewed as the superposition of distributions from point sources and sinks appropriately distributed in an infinite medium. The mathematical relation between the source-wise and characteristic function representations of the

distributions is established by means of the Poisson summation formula (Courant-Hilbert, Methoden der Mathematischen Physik, p. 65):

$$\sum_{m=-\infty}^{\infty}\sum_{n}^{\infty} \varphi(m,n) = \sum_{\lambda=-\infty}^{\infty}\sum_{\mu}^{\infty}\iint_{-\infty}^{\infty}\varphi(u,v)\, e^{-2\pi i(\lambda u + \mu v)}\,du\,dv. \qquad (3.133)$$

Upon applying this transformation to the series (3.131) with

$$\varphi\,(m,n) = \frac{1}{4a^2}\cos B_m x \cos B_n y e^{-(B_m^2 + B_n^2)\tau}$$

we obtain

$$q(x,y,z,\tau) = \frac{1}{(4\pi\tau)^{3/2}}\sum_{-\infty}^{\infty}\sum_{\lambda,\mu}(-)^{\lambda+\mu}\exp -\left[(x-2\lambda a)^2 + (y-2\mu a)^2 + z^2\right]\Big/4\tau. \qquad (3.134)$$

Each term in (3.134) represents a source or sink of unit strength situated at the point $(2\lambda a, 2\mu a, 0)$. The source-wise representation of the slowing down distribution converges better than the characteristic function representation at points close to the source; at points far from the source the characteristic function form is the better converging.

Measurement of Slowing Down Length

The second moment of the distribution (3.131) is

$$\overline{z^2} = \frac{\int_0^{\infty} q(x,y,z,\tau)z^2dz}{\int_0^{\infty} q(x,y,z,\tau)dz} = 2\tau;$$

that is, the second moment in a finite block is the same as in an infinite block. Hence foil measurements in a block of finite width yield the same

second moment as measurements in an infinite medium. This result is independent of the relative importance of the various harmonics contained in (3.131) and holds provided only that the distribution is strictly Gaussian.

Most neutron sources are not monoenergetic, nor is the slowing down intrinsically Gaussian. For both these reasons the mean square distance measured in a finite block is not strictly the same as the mean square distance in an infinite system. For example, if the energy distribution of the source is $f(\tau')d\tau'$, then

$$q(x,y,z,\tau) = \frac{1}{a^2}\sum_{m,n} \cos B_m x \cos B_n y \int_0^\infty \frac{e^{-\left[B_m^2 + B_n^2\right](\tau-\tau')} f(\tau')\, e^{-z^2/4(\tau-\tau')}}{\left[4\pi(\tau-\tau')\right]^{1/2}}\, d\tau' .$$

The second moment of this distribution along the z axis ($x = y = 0$) is

$$\overline{z^2} = \frac{\sum_{m,n}\int_\tau^\infty\int_0^\infty \frac{e^{-\left[B_m^2 + B_n^2\right](\tau-\tau')} f(\tau') z^2\, e^{-z^2/4(\tau-\tau')}}{\left[4\pi(\tau-\tau')\right]^{1/2}}\, dz\, d\tau'}{\sum_{m,n}\int_\tau^\infty\int_0^\infty \frac{e^{-\left[B_m^2 + B_n^2\right](\tau-\tau')} f(\tau')\, e^{-z^2/4(\tau-\tau')}}{\left[4\pi(\tau-\tau')\right]^{1/2}}\, dz\, d\tau'} . \qquad (3.135)$$

In general this second moment will differ from the second moment $\overline{z_\infty^2}$, measured in an infinite medium:

$$\overline{z_\infty^2} = \frac{\int_\tau^\infty\int_0^\infty f(\tau') z^2 \frac{e^{-z^2/4(\tau-\tau')}}{\left[4\pi(\tau-\tau')\right]^{1/2}}\, dz\, d\tau'}{\int_\tau^\infty\int_0^\infty f(\tau') \frac{e^{-z^2/4(\tau-\tau')}}{\left[4\pi(\tau-\tau')\right]^{1/2}}\, dz\, d\tau'} . \qquad (3.136)$$

Corrections must therefore be made to the observed finite system $\overline{z^2}$ in order to obtain the true infinite system $\overline{z^2_\infty}$. It is possible to compute these corrections for a completely general kernel and this will be done in the remainder of this section.*

The corrections will be made by observing that the neutron distribution in a finite block can be considered as the sum total of effects from a suitable distribution of positive and negative sources in an infinite medium, provided, as we shall assume, the extrapolation distance can be neglected compared to the block size, or is independent of neutron energy.

Now a point source at the center of the z = 0 plane in a long block of sides 2a is equivalent to a sequence of positive and negative sources spaced at intervals of 2a in the z = 0 plane. Such a sequence can be represented as

$$\delta(x)\ \delta(y)\ \delta(z) = S(x,y)\ \delta(z) = \delta(z) \sum_{m,n} \cos B_m x \cos B_n y \tag{3.137}$$

where x and y are allowed to have any value from $-\infty$ to $+\infty$.

We consider the function

$$q(x,y,z) = \iint_{-\infty}^{\infty} S(x',y')\ P\left[\sqrt{(x-x')^2 + (y-y')^2 + z^2}\right] dx'dy' \quad \sum_{m,n} \iint_{0}^{\infty} \cos B_m x' \cos B_n y' \tag{3.138}$$

$$\times\ P\left[\sqrt{(x-x')^2 + (y-y')^2 + z^2}\right] dx'dy' - \sum_{m,n} \cos B_m x \cos B_n y\ \overline{P}(B_m^2 + B_n^2, z)$$

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* M. E. Rose and A. M. Weinberg, MonP-297.

where $\overline{P}(B_m^2 + B_n^2, z)$ is the two-dimensional Fourier transform of the point slowing down kernel, $P(\underline{r})$:

$$\overline{P}(B_m^2 + B_n^2, z) = \iint_{-\infty}^{\infty} e^{-i\left[B_m\xi + B_n\eta\right]} P\left[\sqrt{\xi^2 + \eta^2 + z^2}\right] d\xi\, d\eta \,. \qquad (3.139)$$

The function $q(x,y,z)$ can be viewed as the slowing down density in an infinite medium in which the infinite array of positive and negative sources defined by (3.137) is situated. Since, according to (3.138), $q(x,y,z)$ vanishes on the boundary of the block, it can also be viewed as the slowing down density in the finite system due to a single point source at $x = y = z = 0$, provided the extrapolation distance is energy independent. A range measurement results in the observed $2k^{th}$ moment $\overline{z^{2k}}$:

$$\overline{z^{2k}} = \frac{\int_{-\infty}^{\infty} z^{2k} q(o,o,z)dz}{\int_{-\infty}^{\infty} q(o,o,z)dz} = \frac{\sum_{m,n} \iiint_{-\infty}^{\infty} z^{2k} \cos B_m x' \cos B_n y' \, P\left(\sqrt{x'^2+y'^2+z^2}\right) dx'dy'dz}{\sum_{m,n} \iiint_{-\infty}^{\infty} \cos B_m x' \cos B_n y' \, P\left(\sqrt{x'^2+y'^2+z^2}\right) dx'dy'dz} \,, \qquad (3.140)$$

We now show how the moments in an infinite system can be expressed in terms of the observed moments in the finite system. Since $P(x,y,z)$ is an even function of x,y,z, we can replace $\cos B_m x' \cos B_n y'$ by $\cos(B_m x + B_n y)$ in (3.140). Now retaining only even functions

$$\cos(B_m x + B_n y) = \sum_{\nu} \sum_{s=0}^{\nu} \frac{(-1)^{\nu}}{(2\nu)!} \; {}_{2\nu}C_{2s} \; B_m^{2(\nu-s)} B_n^{2s} x^{2(\nu-s)} y^{2s} \qquad (3.141)$$

where ${}_{2\nu}C_{2s}$ is the binomial coefficient

$$ {}_{2\nu}C_{2s} = \frac{(2\nu)!}{[2(\nu - s)]!\,(2s)!} . $$

Upon substituting (3.141) into (3.140) we find

$$ \overline{z^{2k}} = \frac{\sum_m \sum_n \sum_\nu \sum_{s=0}^{\nu} \frac{(-1)^\nu}{[2(\nu - s)]!\,(2s)!} B_m^{2(\nu - s)} B_n^{2s} \iiint_{-\infty}^{\infty} x^{2(\nu - s)} y^{2s} z^{2k} P(\underline{r})dxdydz}{\sum_m \sum_n \sum_\nu \sum_{s=0}^{\nu} \frac{(-1)^\nu}{[2(\nu - s)]!\,(2s)!} B_m^{2(\nu - s)} B_n^{2s} \iiint_{-\infty}^{\infty} x^{2(\nu - s)} y^{2s} P(\underline{r})dxdydz} . \tag{3.142} $$

The integrals which appear in (3.142) are of the form

$$ \overline{x_\infty^{2i} y_\infty^{2j} z_\infty^{2L}} = \iiint_{-\infty}^{\infty} x^{2i} y^{2j} z^{2L} P(\underline{r})dxdydz, $$

and can be evaluated by shifting to polar coordinates. Thus

$$ \overline{x_\infty^{2i} y_\infty^{2j} z_\infty^{2L}} = \frac{(2i)!\,(2j)!\,(2L)!\,(i + j + L)!}{i!\,j!\,L!\,(2i + 2j + 2L)!} \overline{z_\infty^{2i+2j+2L}} . \tag{3.143} $$

Upon substituting (3.143) into (3.142) we obtain

$$ \overline{z^{2k}} = \frac{\sum_m \sum_n \sum_\nu \sum_{s=0}^{\nu} (-1)^\nu B_m^{2(\nu - s)} B_n^{2s} \frac{(2k)!\,(\nu + k)!\,\overline{z_\infty^{2(\nu + k)}}}{s!\,(\nu - s)!\,k!\,[2(\nu + k)]!}}{\sum_m \sum_n \sum_\nu \sum_{s=0}^{\nu} (-1)^\nu B_m^{2(\nu - s)} B_n^{2s} \frac{\nu!\,\overline{z_\infty^2}}{s!\,(\nu - s)!\,(2\nu)!}} \tag{3.144} $$

which is an infinite system of linear equations relating the observed moments $\overline{z^{2k}}$ to the infinite system moments $\overline{z_\infty^{2k}}$. The system can be solved for each $\overline{z_\infty^{2k}}$ in terms of the $\overline{z^{2k}}$ by successive approximations, in which, at each stage of the approximation only a finite number of equations and unknowns are used. Such a process will converge well if the block dimension is large compared to the slowing down range.

If the source, instead of being concentrated in a point, is distributed over the $z = 0$ plane like $\cos B_o x \cos B_o y$, only the term $m = n = 0$ appears in (3.144). The infinite second moment can then be expressed explicitly in terms of the measured finite system moments:

$$\overline{z_\infty^2} = \frac{\overline{z^2} + \frac{1}{3} B_o^2 \overline{z^4} + \frac{1}{30} B_o^4 \overline{z^6} \ldots}{1 + B_o^2 \overline{z^2} + \frac{1}{6} B_o^4 \overline{z^4} + \frac{1}{90} B_o^6 \overline{z^6} \ldots} \quad ; \qquad (3.145)$$

and this expression gives the correction for converting $\overline{z^2}$ into $\overline{z_\infty^2}$. Equation (3.145) is of practical importance since measurement of fission neutron ranges are sometimes performed by using the thermal neutrons from a thermal column which are distributed like $\cos B_o x \cos B_o y$ to produce fissions in a flat plate of fissionable material. The fission neutrons in such an arrangement will be distributed also as $\cos B_o x \cos B_o y$, and therefore the results of this section are applicable.