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ANALYTIC PROPERTIES OF EXPECTATION VALUES OF PRODUCTS OF FIELD OPERATORS

by

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ANALYTIC PROPERTIES OF EXPECTATION VALUES OF PRODUCTS OF FIELD OPERATORS

Lectures given by Dr. A. Salam

at

The Department of Physics and Astronomy
University of Rochester
Rochester, New York

with

Appendix by Dr. S. Okubo

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PREFACE

These notes cover the material of lectures given by Dr. A. Salam from the Imperial College of London at the end of July 1958 at the Physics Department, University of Rochester. The purpose of these lectures was to acquaint the graduate students with the latest work on the axiomatic approach of field theory. The reader is assumed to be familiar with the fundamental ideas of Field Theory and Dispersion relations.

The appendix of Chapter IV is due to Dr. S. Okubo. We would like to thank Dr. Okubo for proof reading the manuscript.

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INTRODUCTION

The aim of these lectures is to give an account of recent work in "Dispersion Theory".

Dispersion relations are integral formulas connecting Real and Imaginary parts of scattering amplitudes. In establishing these, one wishes to use the minimum number of physical assumptions. One assumes that

1. Local field operators A(x), B(x) exist in Hilbert space
2. There exist no negative energy states.
3. The field operators have suitable transformation properties for Lorentz transformations.
4. One assumes that these satisfy local causality requirements

$$[A(x), B(y)] = 0 \qquad (x-y)^2 < 0$$

5. Asymptotic condition*

With these minimum of assumptions it appears that one can go a long way in specifying what the character of a scattering matrix should be as a function of its relevant variables. Whereas axiom (3) specifies the group theoretic properties associated with space time (parity, angular momentum, etc.), axioms (2) and (4) tell us that one can consider scattering amplitudes as boundary values of analytic functions in configuration or momentum space. One is thus enabled to use the very powerful machinery of functions of complex variables.

The dispersion relations are merely a reflection of these analyticity properties and are rather trivial deductions from them.

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* To establish the dispersion relations in a physically useful form, one also needs knowledge of spectrum of physical states. (This will be loosely called "Spectral relations".)

One rather simple remark: It is equally feasible to consider scattering amplitudes or vacuum expectation values of field operators. This is because the 5th axiom allows us now to obtain from vacuum expectation values, the appropriate scattering amplitudes. For scattering process with two incident and two outgoing particles, it is sufficient to consider vacuum expectation values involving four operators.

The central problem, in the context of dispersion relations, then, is to discover the analyticity properties of vacuum expectation values considered as functions of many complex variables.

To study these properties, two methods have been adopted. One is to use directly the theory of complex variables. This is the approach of

1. Boguliubov and
2. Bremmerman, Oehme and Taylor for proving dispersion relations and
3. The computation of the analyticity domain for the vacuum expectation of three operators by Kallen and Wightman (preprint)

These papers rely on theorems in complex variable theory--theorems which are not already part of the Physicists' heritage and consequently the acceptance of proofs remains still somewhat an act of faith for most of us.

The second approach is to use axiom (2), (3) and particularly (4) to obtain integral representation for vacuum expectation values of the scattering amplitudes for real values of the arguments. The analytic continuation to complex values becomes trivial and one can study the analyticity properties by actually staring at these

expressions and the proofs do not have the status of acts of faith.

This approach was first used by Nambu (N. C. 6 1064 (1957)). He made use of perturbation theory to obtain these representations. Kallen, Lehmann, Low and Gell-Mann. (Physical Review) were the first to obtain general representations for vacuum expectation value of two field operators. Dyson (P.R. 110, 1460 (1958) solved the problem of constructing a general integral representation for a scattering matrix element following work of Jost I. Lehmann (N.C. 5, 1598 (1957)) and Lehmann has used the Dyson representation to obtain a proof of dispersion relations, following the original idea of Bogolyubov.

Thus, the aim of these lectures is to lead to a simple proof of dispersion relations using Dyson-Lehmann method. We thus obtain all results of Bogoliubov, Taylor, Oehme and Bremmerman. Unfortunately we do not seem to do any better than the last named authors. The lectures really attempt to present the ideas. Proofs rigorous enough to be acceptable to a mathematician are not attempted.

The plan of the lectures is as follows:

We first discuss general properties of vacuum expectation values and consider the magnitude of the problem from the point of view of theory of many complex variables.

Next, we write down Nambu's integral representation.

Next, we consider Dyson's integral representation for the scattering amplitude and its analytical properties following Lehmann.

Then we complete the proof of dispersion relations and summarize the results so far obtained for elastic scattering problems.

DISCUSSION

What are the problems and the outlook.

First, dispersion theory: -

We have succeeded proving dispersion relations for ($\pi\ \pi$), (π N), ($\pi\ \Xi$) cases only , with restrictions on momentum transfer.

We succeed in proving (N N) relations for $\Delta = 0$ only if $M_\pi >$ ($\sqrt{2}$ - 1) M_N which is unrealistic, so that really no proof for N - N Case exists.

We cannot prove the relations for (N - K) (π, Σ).,.(KΞ) etc. . . . . Cases at all, not even for forward scattering. This is rather a sad picture.

In using the Dyson integral representation, we remarked that the representation is powerful in studying analyticity properties of the scattering amplitude as a function of 1 variable while 5 variables are held fixed. This is a limitation. Again we saw that by using the completeness relation, Lehmann was able to obtain larger analyticity domains for momentum transfer for the case of Im T. Here one is essentially studying Im T more closely and in more detail. One may possibly repeat this, use unitarity condition which has not been exploited fully and thus be able to prove some of the unproven results. The field is completely open in this respect.

The other interesting thing which one may remark upon is the consideration of where the lower limit should be in a dispersion integral. This is the type of problem we first discussed using perturbation theory for W_3 (see report on work of Karplus, preprint). This again points to the inadequacy of the information fed into dispersion relations. This may be made more explicit. Dispersion re-

lations (and all our considerations in these lectures) refer to space-time (or momentum space). Conservation laws (like those of charge, nucleon number etc.), and all things corresponding to internal degrees of freedom come in rather surreptitiously. Take N- N scattering. What one shows first of all is that there are 10 scalar functions which describe the scattering. The fact that there are 10 such functions reflects the spin, charge, nucleon number state (strangeness if any) etc. of the system. But as far as dispersion relations are concerned, the heart of the matter lies in these 10 scalar functions. They may and indeed do have interrelations through unitarity but dispersion relations and all our work in these lectures disclaim having anything to do with this aspect of the physical situation.

How can these 10 functions still "know" of the conservation laws? Only through the mass values fed into the spectral relations. It appears that one of our inadequacies may be in not feeding this information properly and this may be the core of the difficulty in specifying lower limits in dispersion integrals. One suspicion is that all this may be connected with what fields are elementary and what are not. Nambu (preprint) has some considerations on this problem. Also Mandelstam has conjectured dispersion relations where one may fix energy and vary momentum transfer. But, I would not like to start on these topics now.

I lectured here 2 years ago when dispersion relations were new and I said " . . . . these relations will dominate our thinking for many years to come". I would like to repeat that statement.

CHAPTER I

In this lecture we consider vac. exp. values of field operators. The most famous product is the T-product of Feynman. For our purposes we shall find it advantageous to concentrate on two types of products; (1) a straight product of field operators - the Wightman product - in configuration space and (2) Fourier transform (in momentum space) of retarded products.

We consider these two types of products and try to deduce their general properties from the axioms. No integral representations are found in this lecture. For those interested only in the integral representations this first lecture can be omitted. Some of the remarks are rather stiff.

§1. Wightman Product

Definition

$$W_n(x_1, x_2, \ldots, x_n) = \langle 0 | \phi_1(x_1) \cdots\cdots \phi_n(x_n) | 0 \rangle \qquad 1.1$$

We restate the field theoretic Axiomatic assumptions and study their consequences systematically for W_n.

(1) $$U(a,\Lambda)\,\phi(x)\,U(a,\Lambda)^{-1} = \phi(\Lambda x + a) \qquad \text{Relativistic invariance} \qquad 1.2$$

(a,Λ) is an element of the inhomogeneous proper Lorentz Group $\mathcal{L}^{\uparrow}$, meaning the operation of transforming by the homogeneous Lorentz transformation Λ $(\det\Lambda = 1)$ followed by translating by a.

$U(a,\Lambda)$ is the corresponding unitary (or anti-unitary) transformation operator.

(2) no negative energy states exist.

Wightman shows (Phys. Rev. 101, 860 (1956)) that given a set of W_n one can construct a theory of a neutral scalar field.

For the present space time reflections will not be included in

our considerations. In §4, the role of reflections is discussed.

Consequences of assumptions

(1) L.I. gives

$$W_n(x_1,\dots,x_n) = W_n(\Lambda x_1+a, \Lambda x_2+a, \dots, \Lambda x_n+a) \tag{1.3}$$

since $\quad U(a,\Lambda)|0\rangle = |0\rangle$

(2) From non existence of negative energy states one can show that W_n are boundary values of analytic functions:

Write

$$\xi_i = x_i - x_{i+1} \tag{1.4}$$

$$W_n = W_n(\xi_1,\dots,\xi_{n-1})$$

Assume W_n posess Fourier transforms:

$$W_n(\xi_1,\dots,\xi_n) = \int \exp\left(-i\sum_1^{n-1} p_i\cdot\xi_i\right)\widetilde{W}_n(p_1,\dots,p_{n-1})\,d^4p_1\cdots d^4p_{n-1} \tag{1.5}$$

We first show that:

$$\widetilde{W}_n = 0 \quad \text{unless} \quad p_i^2 \geq 0 \quad , \quad p_{i0} \geq 0 \tag{1.6}$$

Write

$$W_n(x_1,\dots,x_n) = \sum_k \langle 0|\phi_1(x_1)\cdots\phi_k(x_k)|k\rangle\langle k|\phi_{k+1}(x_{k+1})\cdots\phi_n(x_n)|0\rangle$$

Here $\quad P_\mu|k\rangle = k_\mu|k\rangle \qquad \sum_k |k\rangle\langle k| = 1$

$$W_n(x_1,\dots,x_n) = \sum_k \langle 0|\phi_1(x_1)\cdots e^{+iP_\mu x_k}\phi_k(0)|k\rangle\langle k|\phi_{k+1}(0)e^{-iP_\mu x_{k+1}}\cdots\phi_n(x_n)|0\rangle\cdot$$
$$\cdot e^{-ik(x_k-x_{k+1})} =$$
$$= \sum_k \langle 0|\phi_1(x_1)\cdots e^{+iP_\mu x_k}\phi_k(0)|k\rangle\langle k|\phi_{k+1}(0)e^{-iP_\mu x_{k+1}}\cdots\phi_n(x_n)|0\rangle e^{-ik\xi_k} \tag{1.7}$$

Comparison of 1.5 and 1.7 establishes the result of 1.6.

Consequence: $W_n(x)$ are boundary values of analytic functions.

Write

$$\zeta = \xi - i\eta$$

This theorem is displayed in and at the same time proved by the formula

$$W_n(\zeta_1, \ldots, \zeta_{n-1}) = \int \exp\left(-i \sum_1^{n-1} p_j \cdot \zeta_j\right) \widetilde{W}_n(p_1, \ldots, p_{n-1})\, d^4p_1 \cdots d^4p_{n-1} \qquad (1.8)$$

where η_j lies in the future light cone. The 8(n-1) dimensional open region thus defined in 8(n-1) dimensional space of the components of the ζ_j is called the future tube.

§2: Study of the Analytic Function $W_n(\zeta)$

In order to study further the properties of W_n we need, on the mathematical side, a basic theorem of Wightman, Hall and Bargmann, (Kon Danske Vid, Selsk Mat. Fys. Medd 31, No. 5, 1957) and on the physical side either invariance for various reflections of local commutativity of field operators. These concepts will be defined as we go along.

Wightman, Hall, Bargmann Theorem

A function $f(\zeta_1, \ldots, \zeta_n)$ of n 4-vectors $\zeta_1 \ldots \zeta_n$ analytic in the tube $-\infty < Re\, \zeta_i < \infty$, $Im\, \zeta_i$ in the forward cone and invariant under the proper homogeneous Lorentz group $L^{\uparrow}_{+}$ is also analytic in the extended domain obtained by making the complex Lorentz transformation $\zeta \to \Lambda \zeta$ where $\det \Lambda = 1$ (Λ can have complex matrix elements rather than real ones). If further $f(\zeta_1, \ldots \zeta_n)$ is invariant under space reflections also (group $L^{\uparrow}$) then $f(\zeta_1, \ldots, \zeta_n)$ is a function of the scalar products $\zeta_i \cdot \zeta_k$ and is analytic in the corresponding domain of variables $\zeta_i \cdot \zeta_k$.

The theorem tells us that

$$W_{n+1}(\zeta_1, \ldots, \zeta_n) = W_{n+1}(\zeta_i \cdot \zeta_j) \tag{1.9}$$

There are in all (n-1) ζ's and $\frac{n(n-1)}{2}$ scalar products.

For greater compactness it is more convenient to introduce a set of new $\frac{n(n-1)}{2}$ variables defined thus: $Z_{ij} = \sum_{i}^{j-1} \zeta_k$, $i < j$.

e.g. for n = 3, there are 2 ζ's; thus

$$\zeta_1 = x_1 - x_2 \qquad \zeta_1 = \xi_1 - i\eta_1$$

$$\zeta_2 = x_2 - x_3 \qquad \zeta_2 = \xi_2 - i\eta_2$$

there are three scalar products $\zeta_1^2, \zeta_2^2, \zeta_1 \cdot \zeta_2$. Define three z's thus

$$z_{12} = \zeta_1 \qquad z_{23} = \zeta_2 \qquad z_{13} = \zeta_1 + \zeta_2$$

The three scalar products can be replaced by three squares.

$$z_{12}^2 \quad , \quad z_{23}^2 \quad , \quad z_{13}^2$$

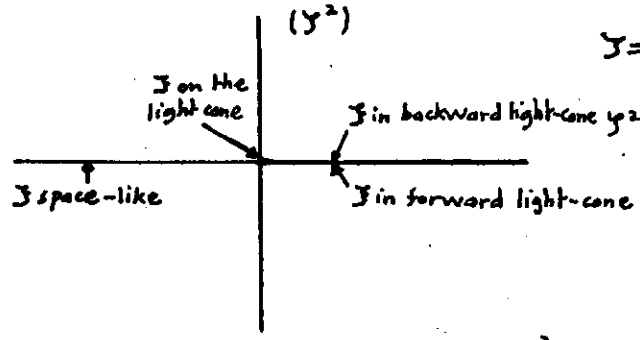
The theorem can be restated.

$$W_n = W_n(z_{ij}^2)$$

As the variables ζ vary over the tube, each variable z_{ij} varies in an open set of the complex plane. Since the z_{ij}^2 are not independent (all z_{ij} depend on $(n-1)$ ζ's) it is not a trivial problem to find the manifold which is defined by z_{ij}^2 .

<u>Examples</u>:

1) Consider n = 2. As we have only one vector ζ in this case $L^{\updownarrow}$ and $L^{\uparrow}$ are equivalent. Here



$$\zeta = \xi - i\eta \ , \ \xi = x_1 - x_2$$

$$\zeta^2 = \xi^2 - \eta^2 - 2i\,\xi\cdot\eta$$

It is easy to check that the set ζ^2 fills the entire plane except for the real axis and the origin. (This will be called the cut-plane).

2) For the case n = 3, Kallen and Wightman (preprint) have

discovered what the domain looks like. In this case also, $\mathcal{L}_+^\uparrow$ is sufficient for the validity of (1.9).

Write

$$Z_1 = \zeta_1^2 \qquad Z_2 = \zeta_2^2 \qquad Z_3 = (\zeta_1 + \zeta_2)^2 \tag{1.10}$$

Here $\zeta = \xi - i\eta$, η lies in the forward light cone while $-\infty < \xi < \infty$

The problem is to find an explicit determination of point Z_1, Z_2, Z_3 which can be written as follows.

<u>Answer</u>: The boundary of the domain Z_1, Z_2, Z_3 space is composed of four analytic hypersurfaces. The cuts $Z_1 = \rho \geq 0$, $Z_2 = \rho \geq 0$

$$Z_3 = Z_1 + Z_2 + r + \frac{Z_1 Z_2}{r} \qquad 0 < r < \infty \tag{1.11}$$

(For the case when $(\mathrm{Im}\, Z_1)(\mathrm{Im}\, Z_2) > 0$)

$$Z_3 = Z_1(1-k) + Z_2\left(1 - \frac{1}{k}\right) \qquad 0 < k < \infty$$

For the case when $(\mathrm{Im}\, Z_1)(\mathrm{Im}\, Z_2) < 0$. For W_4 and higher products no determination of this domain has been made.

We wish now to make an extension of this domain by using local commutativity of field operators.

§3. Local Commutativity

We do not wish to take $\phi_1, \phi_2, \ldots, \phi_n$ as identical fields. To simplify life consider only W_2 and W_3.

We wish to show that as a consequence of local commutativity of fields one can extend the analyticity domains.

Take W_2

$$W_{AB}(x_1, x_2) = \langle 0 | \phi_A(x_1)\, \phi_B(x_2) | 0 \rangle \tag{1.12}$$

We have seen that $W_{AB}(\zeta^2)$ is an analytic function in cut ζ^2 plane. Consider

$$W_{BA}(x_2, x_1) = \langle 0 | \phi_B(x_2)\, \phi_A(x_1) | 0 \rangle \tag{1.13}$$

Using same proof $W_{BA}(\zeta^2)$ is another function analytic in cut plane. From local commutativitv

$$[\phi_A(x_1), \phi_B(x_2)] = 0 \tag{1.14}$$

(We are using L.C. for the first time), we infer that the two functions $W_{AB}(\zeta^2)$ and $W_{BA}(\zeta^2)$ are equal for real negative values of ζ^2. Thus

$$W_{AB}(\zeta^2) = W_{BA}(\zeta^2) \tag{1.15}$$

in the cut plane. This is a powerful result. Note the cut-plane excludes the positive real axis, so that there is no statement about the points when x_1 and x_2 are time-like. However, one still says that for the Fourier transforms

$$\tilde{W}_{AB}(p) = \tilde{W}_{BA}(p) \tag{1.16}$$

Let us now consider n = 3.

$$W_{ABC}(x_1, x_2, x_3) = \langle 0 | \phi_A(x_1)\, \phi_B(x_2)\, \phi_C(x_3) | 0 \rangle \tag{1.17}$$

W_{ABC} is B. V. (Boundary Value) of $W_{ABC}(z_1,z_2,z_3)$ where Z_1, Z_2, Z_3 have been defined in (1.10). Likewise

$W_{BAC}(x_2,x_1,x_3)$ is B.V. of

$$W_{BAC}(Z_1, Z_3, Z_2) \tag{1.18}$$

If $x_1 \sim x_2$ ($\sim$ space-like to)

$$\text{B. V. of } W_{ABC}(Z_1,Z_2,Z_3) = B.V \text{ of } W_{BAC}(Z_1,Z_3,Z_2) \quad \text{if } (x_1-x_2)^2 < a \tag{1.19}$$

It is easy to show (Wightman and Kallen) (similar to the case of W_{AB} and W_{BA}) that the above is sufficient to give us the result

$$W_{ABC}(Z_1,Z_2,Z_3) = W_{BAC}(Z_1,Z_3,Z_2)$$

for those regions where both functions are analytic. This also means that wherever one function is not defined, the other can be considered as its analytic continuation. Local commutativity permits us to extend the domain of analyticity from the original domain in (1.11) of $\mathcal{S}_2$, to the domain formed from it by permutation of Z_2 and Z_3. In a similar way, we find, by considering the function $W_{ACB}(Z_3,Z_2,Z_1)$ that it is equal to $W_{ABC}(Z_1,Z_2,Z_3)$ wherever both are defined so that one is analytic continuation of the other into the domain where only one of them is defined. In particular it follows that $W_{ABC}(Z_1,Z_2,Z_3)$ is analytic also in the domain obtained by permuting Z_1, Z_3 in (1.11) of $\mathcal{S}_2$. Further we get a similar result for $W_{CBA}(Z_2,Z_1,Z_3)$ etc... It happens that (1.11) defines a domain which is symmetric for permutation of Z_1, Z_2 so that there is no gain in extending the domain.

The final result of application of local commutativity is the following. All the six different vacuum expectation values from the three operators ϕ_A, ϕ_B, ϕ_C are boundary values of the <u>same</u> "master" analytic function $W_{ABC}(z_1, z_2, z_3)$.

This function is analytic in the union of domain in equation (1.11) and the two domains obtained from it by permuting Z_1 with Z_2 and Z_3.

§4: Weak Local Commutativity.

In this section we wish to consider the deep correspondence which exists between

(1) local commutativity

(2) TCP invariance

(3) Existence of analyticity domains of W-products.

Consider complex Lorentz transformations: In particular one may choose Λ so that

$$\Lambda \xi = -\xi \tag{1.21}$$

(See Jost, Helv.Phys. Acta 30, 5, 409 (1957)). Thus

$$W(\xi) = W(-\xi) \tag{1.22}$$

At this stage we now introduce reflections; time reflection T, space reflection P and also Assumption 4 charge-conjugation. If we assume that the W-products posess invariance for the combined operation TCP then

$$W_n(x_1, \cdots, x_n) = \langle 0 | \phi_1(x_1) \cdots \phi_n(x_n) | 0 \rangle = \langle 0 | \phi_n(-x_n) \cdots \phi_1(-x_1) | 0 \rangle \tag{1.23}$$

Specializing again to n = 2, TCP invariance gives-:

$$W_{AB}(\xi) = \langle 0 | \phi_A(x_1) \phi_B(x_2) | 0 \rangle = \langle 0 | \phi_B(-x_2) \phi_A(-x_1) | 0 \rangle = W_{BA}(\xi) \tag{1.24}$$

Now putting equations (1.22) and (1.24) together

(1) $$W_{AB}(\xi) = W_{AB}(-\xi) \qquad \text{(1.22 B-H-W)}$$

(2) $$W_{AB}(\xi) = W_{BA}(\xi) \qquad \text{(1.24 TCP)}$$

We would like now to find those points ξ for which $\mathrm{Im}\,\xi = 0$ so that for such points, putting (1) and (2) together

$$W_{AB}(\xi) = W_{BA}(-\xi) \tag{1.25}$$

This would be nearly local commutativity for such ξ. To prove this all one needs to show is that there exists a set of <u>real points</u> for which (1) holds (i.e. the set of real points where complex Lorentz transformation matrix Λ ($\det\Lambda = 1$) can be chosen so that $\Lambda\xi = -\xi$). One can show that such ξ exist and for these $\xi^2 < 0$. The proof is given by Jost and we shall not repeat it here. Repeating the steps in this chain of arguments is the following:

Use B-H-W to show W_{AB} is an analytic function of ξ^2.

Since ξ^2 is invariant for $\xi \to \Lambda\xi$ where Λ is a complex L.T. ($\det\Lambda = 1$) and since W is invariant for these transformations.

$$W_{AB}(\xi) = W_{BA}(\xi) \tag{1.25}$$

In particular consider Λ such that $\Lambda\xi = -\xi$. The only <u>real</u> points for which it is possible to find Λ are such that $\xi^2 < 0$.

Thus

$$W_{AB}(\xi) = W_{AB}(-\xi) \tag{1.26}$$

i.e.

$$\langle \phi_A(x_1)\,\phi_B(x_2)\rangle_0 = \langle \phi_A(-x_1)\,\phi_B(-x_2)\rangle_0 \quad , (x_1 - x_2)^2 < 0 \tag{1.27}$$

From TCP invariance

$$\langle \phi_A(x_1)\,\phi_B(x_2)\rangle_0 = \langle \phi_B(-x_2)\,\phi_A(-x_1)\rangle_0 \tag{1.28}$$

Thus

$$\langle [\phi_A(x_1)\,\phi_B(x_2)]\rangle_0 = 0 \qquad (x_1 - x_2)^2 < 0 \tag{1.29}$$

This is still not local commutativity in the operator sense; it is a statement referring only to the vacuum expectation values. Jost calls this weak local commutativity (WLC). Below we give the precise

definitions but the important point made by Jost is that WLC is necessary and sufficient to guarantee TCP invariance of the Wightman product.

Definition of WLC:

Let

$$\langle 0|\phi_1(x_1)\cdots\phi_n(x_n)\rangle_0 = \langle \phi_n(x_n)\cdots\phi_1(x_1)\rangle_0 \tag{1.30}$$

for all $x_1,\ldots,x_n$ such that

$$\left(\sum_1^{n-1}\lambda_i\xi_i\right)^2 < 0 \qquad \sum\lambda_i = 1 \qquad \lambda_i \geqslant 0 \tag{1.30}$$

(where $\xi_i = x_i - x_{i+1}$). Then $\phi_1,\ldots,\phi_n$ are said to satisfy WLC.

Jost Theorem

TCP invariance which implies

$$\langle \phi_1(x_1)\cdots\phi_n(x_n) = \langle \phi_n(-x_n)\cdots\phi_1(-x_1)\rangle_0 \tag{1.31}$$

is completely equivalent to WLC. The heart of the proof lies in observing that $W(\xi) = W(\Lambda\xi)$

and the set of real points for which the complex Lorentz-matrix can be chosen such that $\Lambda\xi = -\xi$ is precisely the set of points where $\left(\sum\lambda_i\xi_i\right)^2 < 0 \quad \sum\lambda_i = 1 \quad \lambda_i \geqslant 0$

so that for this set of points

$$W(\xi) = W(-\xi)$$

i.e.

$$\langle \phi_1(x_1)\cdots\phi_n(x_n)\rangle_0 = \langle \phi_1(-x_1)\cdots\phi_n(-x_n)\rangle_0 \tag{1.32}$$

If WLC holds then L.H.S. of (1.32) by definition equals

$$\langle 0|\,\phi_n(x_n)\cdots\phi_1(x_1)\,|0\rangle \tag{1.33}$$

Comparing (1.32) and (1.33) we have the statement of TCP invariance and vice versa.

The connection of analyticity with TCP and WLC has been made more explicit by Dyson. Dyson (Preprint)

We have seen from B-H-W that W (ζ) is analytic inside the domain where ζ lies in the future tube and also the extension of this domain given by the complex L.T. Λ (i.e. the set of points $\Lambda\zeta$.) Dyson asks the question:

What is the maximal set of <u>real</u> points inside this extended domain, so that $W(\zeta)$ is regular and one-valued.

For the case $W_2 = \langle \phi_A(x_1)\phi_B(x_2)\rangle_0$ this set is the set of points for which $x_1 \sim x_2$. In general Dyson finds that this set of points is the same set at which WLC holds. (From Jost's theorem we know that WLC holds at these points if theory TCP invariant.)

If we ask, under what conditions will W_n be analytic at all real points $x_1, \ldots, x_n$ for which the vectors $x_i - x_j$ are space-like. The answer is, this will be so if WLC holds at the same points.

§ 5: Retarded Products

So far we have considered just one type of product--the Wightman product. Recapitulating, Wightman product turned out to be the B.V. of an analytic function in x- space on account of the property of field theory that there are no negative energy states. Local causality came in to show that, for example, W_{AB} and W_{BA} are different boundary values of this same function. In fact, as we saw the suggestion is that local causality is too much--all we need is WLC or TCP for this to hold.

Now we study retarded products. Here we shall find that not the product itself but its Fourier transform in p- space is a B.V. of an analytic function--this time as a consequence of local causality. Thus, there is a reciprocity relation.

1. $W(x)$ = B.V. of $W(\zeta)$ because $p_\mu^2 \geq 0$, $p_0 \geq 0$.	1. $\tilde{R}(p)$ = B.V. of $\tilde{R}(r)$, $r = p - i\eta$ because $[\phi(x)\,\phi(y)] = 0$ for $(x-y)^2 < 0$.
2. There is one master function in x- space if the theory is TCP invariant or WLC holds.	2. There is one master function in p- space due to $p_\mu^2 \geq 0$, $p_0 \geq 0$.

In fact, it turns out the domains of analyticity of $W(\zeta)$ and $\tilde{R}(r)$ are identical. Consider

$$R_2 = \theta(x_1 - x_2) \langle 0 | [\phi_A(x_1)\,\phi_B(x_2)] | 0 \rangle = \int e^{-ip(x_1 - x_2)}\, \tilde{R}_2(p)\, d^4p \quad (1.34)$$

we are interested in $\tilde{R}_2(p)$. Clearly $\tilde{R}_2(p)$ is B.V. of an analytic function $r = p - i\eta$ (η in forward light-cone) From B - H - W; $\tilde{R}_2 = \tilde{R}_2(r^2)$ i.e. $\tilde{R}_2$ is regular in the cut-plane r^2. We can equally consider

$$R_2^1(x_1, x_2) = \theta(x_1 - x_2) \langle 0 | [\phi_B(x_1)\,\phi_A(x_2)] | 0 \rangle \quad (1.35)$$

note

$$R_2(x_1,x_2)-R_2'(x_2,x_1)=\langle 0|[\phi_A(x_1),\phi_B(x_2)]|0\rangle \tag{1.36}$$

Fourier transforming both sides (note R_2 is a function of x_1-x_2 and R_2' of x_2-x_1 and remarking that the Fourier transform of the casual commutator = 0 for $p^2<0$ * , we have

$$\tilde{R}_2(p)=R_2'(p) \qquad p^2<0 \tag{1.37}$$

Since R_2 and R both are analytic in the cut plane r^2, and they coincide on the negative real axis, we have the same situation as in x-space. We have one master function in r^2 plane.

An analogous situation exists for R_3. Consider

$$R_A(x_1,x_2,x_3)=\theta(x_1-x_2)\theta(x_2-x_3)\langle 0|[\phi_C(x_3)[\phi_B(x_2)\phi_A(x_1)]]|0\rangle+\theta(x_1-x_3)\theta(x_3-x_2)\cdot$$

$$\cdot\langle 0|[\phi_B(x_2)[\phi_C(x_3)\phi_A(x_1)]]|0\rangle=\int dp\,dp'\,e^{-ip(x_2-x_1)+ip'(x_3-x_1)}\tilde{R}_A(p,p') \tag{1.38}$$

Clearly

$$R_A(x_1,x_2,x_3)=0 \tag{1.39}$$

unless the two vectors x_1-x_2 and x_1-x_3 both lie in the forward cone. Thus $\tilde{R}_A(p,p')$ is the boundary value of an analytic function $\tilde{R}_A(z_1,z_2,z_3)$ regular in the domain defined by

$$z_1=(p-i\eta)^2$$

$$z_2=(p'-i\eta')^2$$

$$z_3=(p+p'-i(\eta+\eta'))^2 \tag{1.40}$$

This is the <u>same</u> domain in p-space as was found for $W(x_1,x_2,x_3)$ in x-space. We have written the function as $\tilde{R}_A$ to emphasize that $R_A(x_1,x_2,x_3)$ issymmetrical for ϕ_B and ϕ_C but ϕ_A plays a special role. We could also write R_B and R_C and define $\tilde{R}_B$, $\tilde{R}_C$. It is easy to check that if these functions are expressed in terms of the same variables z_i that were used in $\tilde{R}_A$ these new functions are analytic in the two domains that are

* For proof see next section Lehmann representation

obtained by permuting the above domain of Z_3 with Z_1 or Z_2. Thus, the situation is analogous to the situation in x- space except for the fact that there is no local commutativity to assure us that the three functions are equal in the common domain. Källen and Wightman go into the structure of $R_A(x_1, x_2, x_3)$ in some detail and discover that in fact there does exist a relation which can serve as a substitute for local commutativity. In fact, they show that $\widetilde{R}_A = \widetilde{R}_B = \widetilde{R}_C$ for all Z space-like. Thus, there is the remarkable fact that both in x- space and in p- space we have only one master function (i.e. one in each space) and the domains of analyticity of these functions are the same.

§ 6: Analytic Continuation

The $W(\gamma)$ products or $R(p)$ products are analytic functions of many complex variables. The very first problem one can tackle is to determine the domain of analyticity of these functions. In this respect one must remember that the theory of analytic functions of n complex variables $(n > 1)$ differs profoundly from the corresponding theory for functions of one complex variable.

For functions of one complex variable there exists for every domain D, a function $f(z)$ such that the domain is the "natural boundary" of the function and $f(z)$ cannot be continued outside D. For functions of more than one complex variable any arbitrary domain D cannot be a domain of analyticity for an analytic function. The important point is there exists the possibility that <u>all</u> functions (of many complex variables) which are analytic in D can be continued simultaneously into larger domains. The intersection of all such domains is called the envelope of holomorphy $E(D)$ of the domain D. A domain which is equal to its holomorphic envelope is called a "natural domain" of analyticity.

In an earlier section we considered the domain of holomorphy of $W_3\ (Z_1, Z_2, Z_3)$. The next problem would be to compute the envelope of holomorphy of the domain. Once one knew the envelope one could set up some kind of integral representation for the most general function analytic in the domain and interalies prove dispersion relations (For example, Bergmann-Weil Formula).

One can always express the analytic function for arbitrary values in the points Z_i inside the domain of analyticity in terms of the boundary values of the function on a certain subset of the so called <u>distinguished</u> boundary.

For W_3 Wightman and Kallen* compute the holomorphy envelope as well as the distinguished boundary; and thus in principle it is possible to generate the whole analytic function with the aid of an integral.

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* Their result is that the envelope is bounded by seven analytic hypersurfaces

$$Z_k = \rho \geq 0 \qquad k = 1, 2, 3$$

$$Z_m = Z_k + Z_\ell - \rho - \frac{Z_k Z_\ell}{\rho} \qquad (\mathrm{Im}\, Z_m)(\mathrm{Im}\, Z_k) \leq 0$$

$$0 < \rho < \infty \qquad (\mathrm{Im}\, Z_m)(\mathrm{Im}\, Z_\ell) \leq 0$$

$$Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3 - \rho(Z_1 + Z_2 + Z_3) + \rho^2 = 0 \qquad (\mathrm{Im}\, Z_1)(\mathrm{Im}\, Z_2) \geq 0$$

$$0 < \rho < \infty \qquad (\mathrm{Im}\, Z_1)(\mathrm{Im}\, Z_3) \geq 0$$

This is the largest domain in which <u>every</u> function W_3 (Z_1, Z_2, Z_3) satisfying the requirements of local causality is analytic. If the class of W_3 is further restricted by imposing new physical requirements (e.g. spectral requirements) Then this will enlarge the domain. They also compute the "distinguished boundary" and show that this consists of points of the form

$$Z_1 = (x_1 - x_2)^2$$
$$Z_2 = (x_2 - x_3)^2$$
$$Z_3 = (x_3 - x_1)^2$$

with x_1, x_2, x_3 real vectors such that at least two of them are time-like or light-like.

One could do likewise for W_4 which is the product of field operators relevant for scattering processes involving two incoming particles and two outgoing particles. W_4 would be a function of six complex variables. The general problem of obtaining the holomorphy envelope has not been tackled. Bremmermann, Oehme and Taylor* however have considered the problem of computing the holomorphy envelope for a suitable sub-domain (called the "semi-tube") of these six variables. They are thus able to demonstrate enough analyticity properties of W_4 to be able to demonstrate dispersion relations for π - N scattering cases and several other cases with restrictions for momentum transfer and in some cases restrictions also on the ratio of masses of the scattering particle and the target.

We shall not follow these general considerations any more for to make any headway one needs the theory of functions of many complex variables. Instead we use the simpler and converse approach of trying to obtain integral representation for the scattering matrix elements we are interested in and then study their analyticity properties.

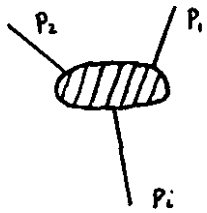
* Phys. Rev. 109, 2178 (1958)

CHAPTER II

From this lecture we start considering integral representations. We first consider Perturbation theory - that is to say - we are assuming a Lagrangian exists and has specified form. As before we avoid complications due to spin etc. and work with neutral scalar fields. The perturbation theory results are to obtain an orientation of what to expect.

§ 1: Integral Representation for T- products in Perturbation Theory:[2]

It is simplest when using perturbation theory to work with the familiar T- products. Consider ${}_0\langle T\,\phi(1)\cdots\phi(E)\rangle_0$ represented by the diagram



with E external lines

Consider a particular Feynmann graph with

E external lines momenta p_i

r internal lines momenta k_i

n vertices

ℓ independent momenta

We have the relation $\ell = r - n - 1$

At each vertex there is an energy-momentum conserving δ- function. For each internal line we have the propagator $\Delta_F(k)$ The matrix element is proportional to

$$M = \int d^4k_1 \ldots d^4k_\ell \; \Delta_F(k_1)\; \Delta_F(k_2) \ldots \Delta_F(k_\ell + p_i) \tag{2.1}$$

there are r factors in the integral corresponding to the r internal lines.

Using Feynmann's method [3] to evaluate the integral we have

$$M \sim I = (r-1)! \int_0^1 \delta\left(\sum_1^r u_r - 1\right)(du)^r \int (dk)^\ell \frac{1}{[(k_1^2-m_1^2)u_1 + \cdots + [(k_\ell+p_i)^2-m^2]u_r]^r} \tag{2.2}$$

The integrand is of the form $\frac{1}{[Q(k_1 \cdots k_\ell)]^r}$ where $Q = \sum_{s,t} A_{st} k_s \cdot k_t + \sum_s \bar{A}_s \cdot k_s + C$ is a quadratic form in the k_i

A_{st} are linear sums of u_i

$\bar{A}_s$ are linear combinations of u_i and p_j (2.3)

C is a sum of terms $-m_j^2 u_i$ and $p_j^2 u_i$

The integration in $(dk)^\ell$ is performed in the appendix. The result is

$$I = (i\pi^2)^\ell (r-2\ell-1)! \int_0^1 \delta\left(\sum_1^r u_i - 1\right)\left(\frac{\Lambda}{\chi}\right)^{r-2\ell} \cdot \frac{1}{\Lambda^2}(du)^r \tag{2.4}$$

where

$$\Lambda = |A_{st}| = F(u) \qquad \chi = \begin{vmatrix} A_{st} & \bar{A}_s \\ \bar{A}_s & C \end{vmatrix} \tag{2.5}$$

are two invariants of the quadratic form.

$$\frac{\chi}{\Lambda} = \frac{C\Lambda + \bar{A}_s \cdot \bar{A}_t \frac{\partial \Lambda}{\partial A_{st}}}{\Lambda} = C(m^2 u, p^2 u) + p_i \cdot p_j \; C_{ij}(u) = -\sum m^2 u + d_{ij}(u)\, p_i \cdot p_j \tag{2.6}$$

so that

$$I = (i\pi^2)^\ell (r-2\ell-1)! \int_0^1 (du)^r \delta\left(\sum_1^r u_i - 1\right) \Lambda^2(u) \frac{1}{[p_i \cdot p_j d_{ij}(u) - \sum m^2 u]^{r-2\ell}} \tag{2.7}$$

which is the representation wanted [*1].

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*1. Chisholm, Proc. Cambridge Soc., 48, 300 (1952)

2. Nambu, N. C., VI no. 5 1064 (1957)

3. Jauch and Rohrlich. "Theory of Photons and Electrons", Appendix A.5.

Another method to evaluate (2.1) is given in the appendix. It makes use of an integral representation of $\Delta_F(k)$.

EXAMPLE:

Let us make use of the Chisholm-Nambu representation (2.7) to obtain the analytic properties of T(3) calculated in the lowest order Feynmann diagram. Assume there are three fields ϕ_1, ϕ_2, ϕ_3 coupled to three intermediate fields ϕ_a, ϕ_b, ϕ_c as follows

$$\mathcal{L}_{int} = g_1 \phi_1 \phi_b \phi_c + g_2 \phi_2 \phi_a \phi_c + g_3 \phi_3 \phi_a \phi_b \tag{2.8}$$

The masses of corresponding particles are m_1, m_2, m_3 and m_a, m_b, m_c where the stability criteria are

$$m_2 < m_a + m_c \qquad m_3 < m_a + m_b \tag{2.9}$$

Then

$$I(p_1^2, p_2^2, p_3^2) = \text{constant} \times \int_0^1 \frac{\delta(1-\alpha-\beta-\gamma)\, d\alpha\, d\beta\, d\gamma}{\left[p_1^2\beta\gamma + p_2^2\gamma\alpha + p_3^2\alpha\beta - m_a^2\alpha - m_b^2\beta - m_c^2\gamma\right]} \tag{2.10}$$

Specialize to mass shell i.e. $p_2^2 = m_2^2$, $p_3^2 = m_3^2$; and also set for simplicity $m_2^2 = m_3^2 = M^2$ and $m_b^2 = m_c^2 = m^2$.

$$I(p_1^2) = \text{const.} \times \int_0^1 \frac{\delta(1-\alpha-\beta-\gamma)\, d\alpha\, d\beta\, d\gamma}{\left[p_1^2\beta\gamma - m_a^2\alpha\right] + (\beta+\gamma)\left[M^2\alpha - m^2\right]} \tag{2.11}$$

At this stage there are three masses in the problem, the intermediate masses m_a and m and the external mass M. Dropping the suffix in p_1^2, $I(p^2)$ is regular for complex p^2 because the denominator

$$D = (p^2\beta\gamma - m_a^2\alpha) + (\beta+\gamma)\left[M^2\alpha - m^2\right] \tag{2.12}$$

does not vanish. On the real axis D = 0 for $p^2 = p_0^2, \alpha = \alpha_0, \beta = \beta_0, \gamma = \gamma_0$

when

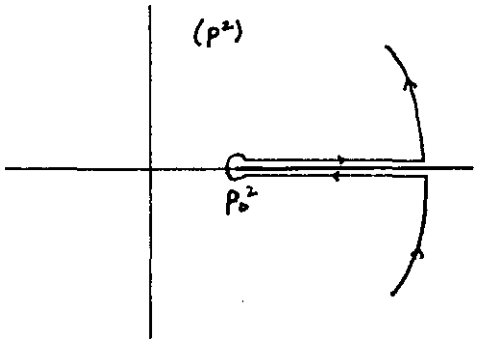
$$\alpha_0+\beta_0+\gamma_0=1$$

$$p_0^2\beta_0\gamma_0-m_a^2\alpha_0+(1-\alpha_0)(M^2\alpha_0-m^2)=0 \qquad \alpha_0,\beta_0,\gamma_0\geq 0 \tag{2.13}$$

The function $I(p^2)$ is therefore analytic in the cut p^2 plane. The cut extends along the real axis from $p_0^2=\min p_i^2$ to ∞. (Note p_0^2 cannot be negative because in the relation,

$$\beta_0\gamma_0 p_0^2=m^2+\alpha_0(m_a^2-m^2-M^2)+\alpha_0^2M^2=f(\alpha_0)$$

the r.h.s. is always positive*). Thus we expect a dispersion relation,



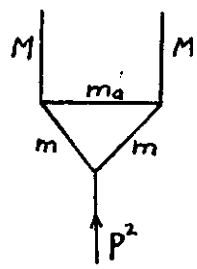
$$I(p^2)=\frac{i}{2\pi}\int_{p_0^2}^{\infty}dk^2\,\frac{I(k^2+i\epsilon)-I(k^2-i\epsilon)}{p^2-k^2}=$$

$$=\frac{1}{\pi}\int_{p_0^2}^{\infty}dk^2\,\frac{\mathrm{Im}\,I(k^2)}{p^2-k^2} \tag{2.14}$$

The important question now is, what is p_0^2 ? One would expect that the branch point occurs at $p_0^2\simeq(2m)^2$ because this is the threshold for the production by the quantum ϕ_1, of a real intermediate state of particles ϕ_b and ϕ_c to which it is coupled. Let us see if this expectation is justified.

$$p_0^2=\mathrm{Min}\,\frac{1}{\beta\gamma}\left\{m_a^2\alpha+(m^2-M^2\alpha)(\beta+\gamma)\right\}$$

$$=\mathrm{Min}\,\frac{1}{\beta\gamma}\left\{m_a^2+(\beta+\gamma)(m^2-m_a^2-M^2)+(\beta+\gamma)^2M^2\right\} \tag{2.15}$$



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*To show this, $f(0)>0, f(1)>1$ and the discriminant of $f(\alpha_0)$ is $-(m_a+m+M)(m_a+m-M)(m_a+M-m)(m+M-m_a)$. Stability criteria are $m_a+m>M, m_a+M>m, m+M>m_a$. Hence the discriminant is negative. So $f(\alpha)>0$.

for $\beta, \gamma \geq 0$ $\qquad \beta + \gamma = 1 - \alpha \leq 1$

A straightforward minimization of the right hand side gives for

$$\beta_0 = \gamma_0 = \frac{m_a^2}{M^2 + m_a^2 - m^2} \tag{2.16}$$

Since $\beta_0 = \gamma_0 \leq \frac{1}{2}$ in order that β_0, γ_0 should lie within the range of integration, we have

(1) $M^2 \geq m_a^2 + m^2$: In this case

$$p_0^2 = \frac{1}{m_a^2}\left[m^2 - (M - m_a)^2\right]\left[(M + m_a)^2 - m^2\right] < 4m^2 \tag{2.17}$$

(2) when $M^2 \leq m_a^2 + m^2$ then $\beta_0 = \gamma_0 = \frac{1}{2}$, $\alpha_0 = 0$

$$p_0^2 = 4m^2 \tag{2.18}$$

Thus the expectation on general grounds for the value of p_0^2 is verified provided $M^2 \leq m_a^2 + m^2 < (m + m_a)^2$. It is not if $m_a^2 + m^2 < M^2 \leq (m + m_a)^2$. This is an important result. Before we discuss its significance, let us note that for the <u>photon nucleon vertex</u>

$$m = m_\pi$$
$$m_a = m_N$$
$$M = m_N$$

so that $M^2 < m_a^2 + m^2$: Case (2) applies and $\underline{p_0^2 = 4m_\pi^2}$.

The conclusion from above is that the general arguments based on thresholds leave out an important case which is taken no cognizance of at all. This does not mean that for case (2) a general proof of the dispersion relations can be supplied. In fact it appears that proceeding from general arguments one can obtain <u>only</u> case (1) i.e. $p_0^2 = 4m^2$. This means that intermediate states <u>must</u> be such that $M^2 \leq m_a^2 + m^2$. The last resort for this is $M^2 = m_a^2 + m^2$ (2.18)

Chapter III Integral Representations From General Considerations:

There is a second class of integral representations which are derived from general principles, and these differ from the Chisholm-Nambu type of representations. The only known ones are representations for the vacuum expectation values of products of two field operators and for the general matrix elements of the commutator of two field operators.

§1. Two Fold Vacuum Expectation Values (1).

We want to find integral representations for the quantities

$$\begin{aligned} &\langle \varphi(x)\,\varphi(x')\rangle = i\,\Delta^{(+)}(x-x') \\ &\langle [\varphi(x)\,\varphi(x')]\rangle_0 = i\,\Delta'(x-x') = -2i\,\epsilon(x_0-x_0')\,\bar{\Delta}'(x-x') \\ &\langle T\,\varphi(x)\,\varphi(x')\rangle_0 = \tfrac{1}{2}\,\Delta_F'(x-x') \end{aligned} \tag{3.1}$$

where $\varphi(x)$ is a scalar real field. We suppose the existence of the vacuum defined by $P_\mu|\rangle_0 = 0$ and a complete set of eigen-states of P_μ

$P_\mu|k\rangle = k_\mu|k\rangle$ with $k_0>0$. (k may be degenerate).

We assume invariance under the inhomogeneous Lorentz-transformation. We have

$$\begin{aligned} \langle \varphi(x)\,\varphi(x')\rangle_0 &= \sum_{n,k} \langle 0|\varphi(x)|nk\rangle\langle nk|\varphi(x')|0\rangle = \\ &= \sum_{n,k} \langle 0|\varphi(0)|nk\rangle\langle nk|\varphi(0)|0\rangle\, e^{-ik(x-x')} \end{aligned} \tag{3.2}$$

using translation invariance; n represents all the other quantum numbers necessary to specify the states. Defining

$$\rho(k^2) = (2\pi)^3 \sum_n \langle 0|\varphi(0)|kn\rangle\langle kn|\varphi(0)\rangle = (2\pi)^3 \sum_n |\langle 0|\varphi(0)|kn\rangle|^2 \tag{3.3}$$

(1) H. Lehmann. Nuovo Cimento, 11, 342 (1954)

$\rho(k^2)$ depends only on k^2 as a consequence of Lorentz invariance. We have so

$$\Delta^{(+)}(x-x') = -\frac{i}{(2\pi)^3}\int \theta(k_0)\rho(k^2)e^{-ik(x-x')}d^4k \tag{3.4}$$

where $\theta(k_0)$ enters because we sum only on positive energy states $k_0 > 0$. Formally

$$\rho(k^2) = \int_0^\infty \rho(\varkappa^2)\delta(k^2-\varkappa^2)d\varkappa^2 \tag{3.5}$$

Using the integral representation of the free field $\Delta^+(x-x')$

$$\Delta^+(x-x';m) = -\frac{i}{(2\pi)^3}\int \theta(k_0)\delta(k^2-m^2)e^{-ik(x-x')}d^4k \tag{3.6}$$

$$\Delta^{(+)}(x-x') = \int_0^\infty \rho(\varkappa^2)\,\Delta^+(x-x',\varkappa^2)\,d\varkappa^2 \tag{3.7}$$

Similarly

$$\Delta^{(-)}(x-x') = \int_0^\infty \rho(\varkappa^2)\,\Delta^-(x-x',\varkappa^2)d\varkappa^2 \tag{3.8}$$

This is the integral representation wanted. In fact it could have been obtained as a special case of the Dyson representation treated below.

From (3.3) we have $\rho(\varkappa^2) \geq 0$ as a consequence of the definite metric.

In the case of two different fields $\varphi_1(x), \varphi_2(x)$ we would find the same integral representations but we need WLC or TCP (Ch. I) to be able to use the same density functions for all the functions in (3.8).

§2. Dyson's General Integral Representation for the Commutator of Two Field Operators.(2)

Dyson has completely solved the problem of finding integral representation for $f(x) = \langle Q|[A(\frac{x}{2}), B(-\frac{x}{2})]|P\rangle$ by an extremely elegant trick; not only has he solved the problem of finding an integral representation for any function $\tilde{f}(x)$ which is 0 for $x^2 < 0$, but also is able to incorporate into this representation the knowledge given by the spectral relations. This knowledge can be stated in the form of a condition on $f(q)$ the Fourier transform of $\tilde{f}(x)$. In fact our problem is the following:

(1) Given $\tilde{f}(x) = 0$, $x^2 < 0$

(2) Given that $f(q)$, the Fourier transform of $\tilde{f}(x)$ vanishes in a specified region in q-space: R

Find an integral representation for $\tilde{f}(x)$.

Dyson's Solution:

Let

$$z = x_0, x_1, x_2, x_3, y_1, y_2 = x, y \qquad z^2 = x^2 - y^2 \tag{3.9}$$

$$r = q_0, q_1, q_2, q_3, k_1, k_2 = q, k \qquad r^2 = q^2 - k^2 \tag{3.10}$$

Consider

$$\tilde{F}(z) = 4\pi \tilde{f}(x)\, \delta(x^2 - y^2) \tag{3.11}$$

Then

$$\tilde{f}(x) = \frac{1}{4\pi}\int_0^\infty \tilde{F}(z)\, dy^2 = \frac{1}{4\pi^2}\int \tilde{F}(z)\, d^2y \tag{3.12}$$

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(2) Phys. Rev. 110, 1460 (1958)

Further if $F(r)$ is the F.T. of $\tilde{F}(z)$ and $f(q)$ is the F.T. of $\tilde{f}(x)$, it is easy to check that $f(q)=F(q,0)$ *. (3.13)

The crucial remark is that

$$\Box_6^2 F(r)=0$$

Here
$$\Box_6^2=\frac{\partial^2}{\partial r_0^2}-\frac{\partial^2}{\partial r_1^2}-\frac{\partial^2}{\partial r_2^2}-\frac{\partial^2}{\partial r_3^2}-\frac{\partial^2}{\partial r_4^2}-\frac{\partial^2}{\partial r_5^2} \tag{3.14}$$

so that if $\Box_6^2 D(r)=0$ where $D(r)$ is the conventional odd invariant function

$$D(r)=\frac{1}{2\pi^2}\epsilon(r_0)\,\delta'(r^2) \tag{3.15}$$

$F(r)$ can be represented in terms of a surface integral

$$F(r)=\int d\sigma_\alpha'\left[F(r'),\frac{\partial}{\partial r_\alpha'}D(r-r')\right] \tag{3.16}$$

Here

$$\left[F,\frac{\partial}{\partial r}D\right]=F\frac{\partial}{\partial r}D-D\frac{\partial}{\partial r}F \tag{3.17}$$

In particular

$$f(q)=\int d\sigma_\alpha'\left[F(q',k'),\frac{\partial}{\partial r_\alpha'}D(q'-q,k')\right] \tag{3.18}$$

* This follows using (3.12). Thus

$$f(q)=\frac{1}{(2\pi)^4}\int\tilde{f}(x)e^{-iqx}d^4x=\frac{1}{(2\pi)^4}\int\tilde{F}(z)e^{-iqx}d^4x\,d^2y=F(q,0)$$

Dyson's definition of F.T. of $\tilde{f}(x)$ is: $f(q)=\frac{1}{(2\pi)^4}\int\tilde{f}(x)e^{-iqx}d^4x$

Formula (3.18) is the basis of the integral representation being sought. Let us choose the surface σ' to be the surface $q_0' = 0$. Then

$$d\sigma_0' = d^3q' dk_1' dk_2'$$

$$f(q) = \frac{1}{2\pi}\int dk'^2 \int d^3q' \Big\{ F(q',k') \frac{\partial}{\partial q_0'}\left[\epsilon(q_0'-q_0)\,\delta'((q'-q)^2-k'^2)\right] -$$

$$- \epsilon(q_0'-q_0)\,\delta((q'-q)^2-k'^2)\frac{\partial F(q',k')}{\partial q_0'}\Big\}_{q_0'=0} \qquad (3.19)$$

Formula (3.19) can be recast by partial integrations to read

$$f(q) = \frac{1}{2\pi}\int dk^2 \int d^3u\, \epsilon(q_0)\,\delta(q_0^2-(\underline{q}-\underline{u})^2-k^2)\left[\Phi_1(u,k^2)+q_0\Phi_2(u,k^2)\right] \qquad (3.20)$$

The crucial point here is that Φ_1 , Φ_2 themselves are fully determinate in terms of the original function $f(q)$ *. More symetrically

$$f(q) = \int dk^2 \int d^4u\, \epsilon(q_0-u_0)\,\delta((q-u)^2-k^2)\,\Psi(u,k^2) =$$

$$= -i(2\pi)^3 \int dk^2 \int d^4u\, \Delta(q-u,k^2)\,\Psi(u,k^2) \qquad (3.21)$$

In configuration space

$$\tilde{f}(x) = -i(2\pi)^3 \int dk^2 \Delta(x;k^2)\,\tilde{\Psi}(x,k^2) \qquad (3.22)$$

---

* In fact

$$\Phi_1(u,k^2) = \frac{1}{\pi}\frac{d}{dk^2}\left[\theta(k^2)F_1(u,k^2)\right], \quad \Phi_2(u,k^2) = \frac{1}{\pi}\frac{d^2}{(dk^2)^2}F_2(u,k^2)$$

where

$$F_1(u,k^2) = \frac{1}{\pi^3}\int f(q)\frac{2q_0\,d^4q}{[q_0^2-(\underline{u}-\underline{q})^2-k^2]^3}, \quad F_2(u,k^2) = \frac{1}{\pi^3}\int \frac{f(q)}{[q_0^2-(\underline{u}-\underline{q})^2-k^2]}\,d^4q$$

The explicit form for $\Psi(u,k^2)$ is :

$$\Psi(u,k^2) = \frac{1}{6\pi^4}\int d^4q\, f(q)\,q_0\left[\frac{4\theta(k^2)+3[q_0^2-(\underline{u}-\underline{q})^2-k^2]}{[q_0^2-(\underline{u}-\underline{q})^2-k^2]^4} + \frac{6\,\delta(k^2)}{[q_0^2-(\underline{u}-\underline{q})^2-k^2]^3}\right]$$

So much for the solution of the first part of the problem. We now wish to impose restrictions on $\Psi(u,k^2)$ to correspond to restrictions on $f(q)$. Dyson proves the following result.

Let

$$\tilde{f}(x) = \langle Q | [A(\tfrac{x}{2}), B(-\tfrac{x}{2})] | P \rangle \tag{3.23}$$

Write $R = \frac{P+Q}{2}$ and let the Fourier transform of $\tilde{f}(x)$ be $f(q)$. $f(q) = 0$ unless $R+q$, $R-q$ both lie in the forward cone, and

$$(R+q)^2 \geq m_1^2 \quad \text{or} \quad (R-q)^2 \geq m_2^2 \tag{3.24}$$

Then $\Psi(u,k^2) = 0$ except in the region S given below. Assume without loss of generality that $0 \leq m_1 \leq m_2$

$$\left.\begin{array}{ll} \left.\begin{array}{l}(m_1-k)^2 \leq (R-u)^2 \\ (m_2-k)^2 \leq (R+u)^2\end{array}\right\} & 0 \leq k^2 \leq m_1^2 \quad (3.25) \\ \left.\begin{array}{l}0 \leq (R-u)^2 \\ (m_2-k)^2 \leq (R+u)^2\end{array}\right\} & m_1^2 \leq k^2 \leq m_2^2 \quad (3.26) \\ 0 \leq (R-u)^2 \quad 0 \leq (R+u)^2 & k^2 \geq m_2^2 \quad (3.27) \end{array}\right\} (S)$$

§3. Analyticity Properties of Dyson's Integral Representation.

In the last section we have seen that

$$\tilde{f}(x) = (2\pi i)^3 \int \Delta(x,k^2)\, \tilde{\Psi}(x,k^2,P,Q)\, dk^2 \tag{3.22}$$

where

$$\tilde{\Psi}(x) = \int e^{iqx} \Psi(q,k^2)\, d^4q$$

clearly

$$f_r = \theta(x)\tilde{f}(x) = (2\pi i)^3 \int \Delta_r(x,k^2)\Psi(x,k^2)\, dk^2 \tag{3.28}$$

so that

$$f_r = \int \frac{\Psi(u,k^2,P,Q)}{(q-u)^2 - k^2}\, dk^2\, d^4u \tag{3.29}$$

Set

$$R = \frac{P+Q}{2} \qquad R' = \frac{P-Q}{2} \tag{3.30}$$

Note that with three vectors there are six scalar products q^2, $q\cdot R$, $q\cdot R'$ and R^2, R'^2, $R\cdot R'$. f_r is a function of these six scalar products. Ψ depends directly on $R^2, R'^2, R\cdot R'$ while the denominator $(q-u)^2-k^2$ in (3.29) contains q^2 directly. Note that R occurs also in the specification of the regions R and S outside which f_r and Ψ vanish. Ideally one would like to have the domain of analyticity of f_r for the simultaneous variation of all the six scalar quantities. The Dyson representation is quite powerful but not powerful enough to give this. However, holding five of these variables fixed it is possible to get the analyticity region in the sixth variable.

For most purposes f_r is considered as function of $q\cdot R, q\cdot R'$. We shall find in succeeding pages the analyticity region of f_r in the variable $q\cdot R'$ while all other variables ($q\cdot R$ ... etc) are held fixed.

Specializing to the frame $\underline{R}=0$ (time-like vector) it is easy to show that $R'\cdot q$ is directly proportional to cos θ where θ is the angle between vectors $\underline{P}$ and $\underline{q}$ **.

Thus
$$f_r(q\cdot R, q\cdot R') = f_r(q_0, \underline{P}\cdot\underline{q}) \qquad (3.31)^*$$

---

* The invariant expressions for these quantities are

$$q_0 = \frac{q\cdot R}{\sqrt{R^2}} \qquad \underline{P}\cdot\underline{q} = P_0 q_0 - P\cdot q = \frac{(P\cdot R)(q\cdot R)}{R^2} - P\cdot q = -\underline{Q}\cdot\underline{q} =$$

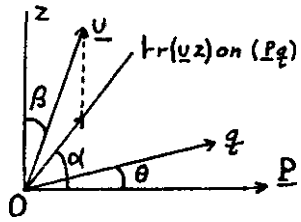
$$= Q\cdot q - Q_0 q_0 = Q\cdot q - \frac{(Q\cdot R)(q\cdot R)}{R^2} = q\cdot\frac{Q-P}{2} - \frac{(q\cdot R)}{R^2}\{R\cdot R'\} =$$

$$= q\cdot R' - \frac{(q\cdot R)(R\cdot R')}{R^2}$$

**This holds only if the states are the same one-particle state or $P_0=Q_0$.

The analytic behavior of f_r as a function of $\underline{P}\cdot\underline{q}$ can be derived easily. Use spherical polar coordinates

$$\hat{\underline{P}}\cdot\hat{\underline{q}} = \cos\theta$$
$$\hat{\underline{P}}\cdot\hat{\underline{u}} = \cos\alpha\sin\beta$$
$$\hat{\underline{q}}\cdot\hat{\underline{u}} = \cos(\alpha-\theta)\sin\beta$$



The denominator is equal to

$$q^2+u^2-2q_0u_0+2|\underline{q}||\underline{u}|\cos(\theta-\alpha)-k^2 \tag{3.32}$$

Write

$$x = -\frac{q^2+u^2-2q_0u_0-k^2}{2|\underline{q}||\underline{u}|\sin\beta} \tag{3.33}$$

Thus

$$f_r = \int_{x_0}^{\infty} dx \int_0^{2\pi} \frac{\Phi(x,\alpha)}{x-\cos(\theta-\alpha)}\,d\alpha \tag{3.34}$$

where

$$\Phi(x,\alpha) = -\frac{1}{4\pi}\int \delta\left(x+\frac{q^2+u^2-2q_0u_0-k^2}{2|\underline{q}||\underline{u}|\sin\beta}\right)\frac{|\underline{u}|}{|\underline{P}|}\,du_0\,d|\underline{u}|\int_0^{\pi} d\beta \int dk^2\,\psi(u,k^2) \tag{3.35}$$

$x_0 = \text{Min } x$ when u and k vary over the region S .(See appendix)

$$x_0 = \text{Min} -\frac{q^2+u^2-2q_0u_0-k^2}{2|\underline{q}||\underline{u}|} = \left[1 + \frac{(m_2^2-\overline{R-q}^2)(m_1^2-\overline{R+q}^2)}{|\underline{q}|^2[4R^2-(m_1-m_2)^2]}\right]^{1/2} \tag{3.36}$$

and $\underline{q}^2$ invariantly equals

$$\frac{(Rq)^2-R^2q^2}{R^2} \tag{3.37}$$

If we expand the denominator of (3.24) we have $x - \cos\alpha\cdot\cos\theta - \sin\alpha\cdot\sin\theta$. $\sin\theta = \sqrt{1-\cos^2\theta}$. It seems as if we must cut the $\cos\theta$-plane because of the double valuedness of $\sin\theta$. Since the denominator contains only $\cos\alpha$ and not $\sin\alpha$

$$J = \int_0^{2\pi} \frac{\Phi(\cos\alpha)}{x-\cos\theta\cos\alpha-\sin\theta\sin\alpha}\,d\alpha = \int_0^{\pi} \frac{\Phi(\cos\alpha)}{x-\cos(\theta-\alpha)}\,d\alpha + \int_{\pi}^{2\pi} \frac{\Phi(\cos\alpha)}{x-\cos(\theta-\alpha)}\,d\alpha$$

In the second integral put $\alpha = 2\pi - \alpha'$. We have:

$$\int_{\pi}^{2\pi} \frac{\Phi(\cos\alpha)}{x - \cos(\theta - \alpha)} d\alpha = \int_{0}^{\pi} \frac{\Phi(\cos\alpha')}{x - \cos(\theta + \alpha')} d\alpha'$$

so that

$$J = 2\int_{0}^{\pi} \frac{\Phi(\cos\alpha)(x - \cos\theta\cos\alpha)}{(x - \cos\theta\cos\alpha)^2 - \sin^2\theta\sin^2\alpha} d\alpha \tag{3.38}$$

As $\sin^2\theta = 1 - \cos^2\theta$ the integrand is function of $\cos\theta$ only and there is no branch introduced by $\sin\theta$.

Singularities of f_r as function of $\cos\theta$ occur only if the denominator vanishes. That is, for $\cos\theta = x\cos\alpha \pm i\sqrt{x^2-1}\sin\alpha = u + iv$

Splitting into real and imaginary part, we find

$$\frac{u^2}{x^2} + \frac{v^2}{x^2-1} = 1$$

that is the singularities lie on or outside the ellipse. We can assert that the real and imaginary parts of f_r are analytic functions of $\cos\theta$ in an ellipse with center at the origin and semi-axis x_0 and $\sqrt{x_0^2-1}$. Note that x_0 is a function of (R, q, m_1, m_2) i.e. of five variables $R^2, q^2, R\cdot q$ and m_1, m_2; $\cos\theta$ involves besides $R^2, R\cdot q$ also $R'\cdot q$. Thus we have essentially obtained the analyticity domain for the variable $R'\cdot q$ in terms of q^2, R^2, $q\cdot R$ and m_1, m_2.

§4. Integral Representation of a Double Commutator:

The problem of finding an integral representation for the vacuum expectation value, of products of three fields is still unsolved.

Consider the double commutator

$$D_{CBA}(y,z) = \langle 0 | [C(x_3)[B(x_2), A(x_1)]] | 0 \rangle \tag{3.38}$$

Introduce $y = x_1 - x_2$, $z = x_2 - x_3$. Dyson has (preprint) solved the problem of finding an integral representation for D which satisfies the causality and mass spectrum conditions, that is:

$$D(y,z) = 0 \quad \text{for} \quad y^2 < 0 \quad \text{or both} \quad z^2 < 0 \;, (y+z)^2 < 0 \tag{3.39}$$

and $F(p,q) = 0$ if $q^2 < 0$ or both $p^2 < 0, (p-q)^2 < 0$ where $F(p,q)$ is the F.T. of $D(y,z)$. The result is

$$D(y,z) = \int_0^\infty ds \int_0^\infty dt \int_0^1 d\lambda \; \Psi(s,t,\lambda)\, \Delta(y,s)\, \Delta(z+\lambda y, t) \tag{3.40}$$

We have the property $F(p,q) = D(p,-q)$. This integral representation fails to satisfy a basic property of the double commutator: The Jacoby identity:

$$D_{CBA}(y,z) + D_{BAC}(-y-z,y) + D_{ACB}(z,y-z) = 0 \tag{3.41}$$

Chapter IV Application of Dyson's integral representation to obtain a proof of "Dispersion Relations" for scattering amplitudes.

§1. Kinematics:

Consider the elastic scattering of two particles

	Initial state	final state	field	current
scatterer	k	k'	$A(x)$	$(\Box^2 - \mu^2)A = j$
target	p	p'	$\psi(x)$	$(\Box^2 - m^2)\psi = f$

On the energy shell : $p+k = p'+k'$ (4.1)

On the Mass shell : $p^2 = p'^2 = m^2$ (4.2)

$k^2 = k'^2 = \mu^2$ (4.3)

The main complications always arise from the kinematic specifications of scattering so that it is worthwhile to devote careful attention to this.

From (4.1) only three vectors are independent p, k, p' $(k' = k+p-p')$

There are six scalar products $p.k$, $p.p'$, $k.p'$, p^2, p'^2, k^2. It is convenient to use $k'^2 = (k+p-p')^2$ to re-express $k.p'$ in terms of k'^2, and the other five products so that the independent scalars are $k.p$, $p.p'$, k^2, k'^2, p^2, p'^2. Since $k^2 = k'^2$, and $p^2 = p'^2$, the number is reduced to four. We shall always specialize to the mass-shell $p^2=p'^2=m^2$, but keep k^2 arbitrary. Thus there are three independent covariants

$$k.p,\ p.p',\ k^2 \tag{4.4}$$

Alternatively one uses the set

$$W^2,\ \Delta^2,\ k^2 \tag{4.5}$$

Here $W^2 = (p + k)^2$, $\Delta^2 = -\left(\frac{p-p'}{2}\right)^2$ (4.6)

W is the total c.m. energy, Δ the momentum transfer. Subsidiary relations for c.m. frame are

$$k_c^2 = \frac{(W^2+m^2-k^2)^2}{4W^2} - m^2 \tag{4.7}$$

$$\cos\theta_c = 1 - \frac{2\Delta^2}{k_c^2} \tag{4.8}$$

There are two further variables which are useful. These can be expressed in terms of W^2, Δ^2 and k^2. But it is simpler sometimes to have a separate symbol for them. These are

$$\omega = \frac{(p+p')(k+k')}{2\sqrt{(p+p')^2}} \tag{4.9}$$

The connection with W^2 is as follows:

$$W^2 = 2\omega\sqrt{\Delta^2+m^2} + 2\Delta^2 + m^2 + k^2 \tag{410}$$

one further variable will be introduced:

$$\overline{W}^2 = (p'-k)^2 = (p-k')^2 = -2\omega\sqrt{\Delta^2+m^2} + 2\Delta^2 + k^2 + m^2 \tag{4.11}$$

So much for kinematics.

§2. Analyticity of scattering amplitude: (Lehmann, preprint)

Before discussing dispersion relations, let us use results of the last section to obtain analyticity domain in a number of variables for the scattering amplitudes.

To simplify discussion we assume that the scatterer is a meson and the target a nucleon, while all states congruent to the scatterer (i.e. states having same quantum numbers) have mass 2μ. All states congruent to the target particle have mass $m + \mu$ $(\mu < m)$.

Using LZS,* one can write $\langle out; p', k' | p, k\, in\rangle = \langle in\, p', k' | p, k\, in\rangle + 2\pi i\,\delta(p+k-p'-k')T$

$$T = -\int d^4x\, e^{i\frac{k'-p'}{2}x} \langle 0|\theta(x)[j(\tfrac{x}{2}), f(-\tfrac{x}{2})]|p, k\, in\rangle \tag{4.12}**$$

* Lehmann, Zimmermann, Simanzik, Nuovo Cimento.

** Omitting an unimportant kinematical factor.

Write

$$\frac{p+k}{2} = R \qquad \frac{k'-p}{2} = q \qquad \frac{p-k}{2} = R' \tag{4.13}$$

T is a function of $R^2 = (\frac{W^2}{4})$, Δ^2 and k^2. The frame defined by $\underline{R} = 0$ happens to be directly the c.m. frame. Using results of the last chapter both (Re T) and (Im T) are analytic functions of $\cos\theta_c =$ $= 1 - \frac{2\Delta^2}{|k_c|^2}$ regular inside an ellipse with center at the origin and semi- axes $x_0, \sqrt{x_0^2-1}$ Here noting $(R+q)^2 = k^2$, $(R-q)^2 = m^2$, we find

$$x_0^2(W, k^2) = 1 + \frac{(m_1^2-k^2)(m_2^2-m^2)}{[W^2-(m_1-m_2)^2]|k_c|^2} \quad , |k_c|^2 = \frac{(W^2+m^2-k^2)^2}{4W^2} - m^2 \tag{4.14}$$

Thus one has directly the result that T is regular in a certain region in Δ^2 plane, the boundary being expressed in terms of variables W^2 and k^2.

This method tells us nothing about the analyticity regions for the variables W^2 and k^2.

This is because in the last section we concentrated on the singularities introduced by the vanishing of the denominator in (3.35) while the numerator $\psi(u, \varkappa^2, W^2, k^2)$ may itself introduce singularities for W^2 and k^2.

For Im T it is possible to construct a Dyson integral-representation, which contains more information. This representation is such that essentially a simultaneous variation of variables k^2 and Δ^2 (for fixed W^2) can be studied. For the variation of Δ^2 alone, the region of analyticity will turn out to be even wider than the one previously given.

To this purpose we use the alternative form of Im T given by LZS.

$$\mathrm{Im}\, T = \frac{1}{2}\left[M(W^2, k^2, \Delta^2) - M(\overline{W}^2, k^2, \Delta^2)\right] \tag{4.15}$$

This follows from the fact that

$$\mathrm{Im}\, T = \frac{1}{2}\int d^4x\, e^{i\frac{k+k'}{2}x} \cdot \langle p'|[j(\tfrac{x}{2}), j(-\tfrac{x}{2})]|p\rangle \tag{4.16}$$

M (W^2, k^2, Δ^2) is defined by

$$M(W^2, k^2, \Delta^2) = \int d^4x\, e^{i\frac{k+k'}{2}x} \cdot \langle p'|j(\tfrac{x}{2}), j(-\tfrac{x}{2})|p\rangle =$$

$$= (2\pi)^4 \sum_{\gamma} \langle p'|j(0)|p+k,\gamma\rangle\langle p+k,\gamma|j(0)|p\rangle \tag{4.17}$$

One can cast M in a form where a Dyson-representation can be written for it. One can show* that indeed

$$M(W^2, k^2, \Delta^2) = 2\pi \int d^4x_1 \int d^4x_2\, e^{i\frac{k'-p'}{2}x_1 - i\frac{k-p}{2}x_2} \cdot \tag{4.18}$$

$$\cdot \sum_{\gamma} \langle 0|\theta(x_1)[j(\tfrac{x_1}{2})\psi(-\tfrac{x_1}{2})]|p+k,\gamma\rangle\langle p+k,\gamma|\theta(x_2)[j(\tfrac{x_2}{2})\psi^+(-\tfrac{x_2}{2})]|0\rangle$$

$M(\overline{W}^2, k^2, \Delta^2)$ can be obtained from above by changing

$$k \to -k' \;,\; k' \to -k \;,\; p \to p' \;,\; p' \to p \tag{4.19}$$

For this form of M, a Dyson-representation is immediate

$$M(W^2, k^2, \Delta^2) = \frac{1}{2\pi} \int \frac{d^4u_1\, d^4u_2\, dx_1^2\, dx_2^2\, \Phi(u_1, u_2, x_1^2, x_2^2)}{\left[\left(\frac{k'-p'}{2} - u_1\right)^2 - x_1^2\right]\left[\left(\frac{k-p}{2} - u_2\right)^2 - x_1^2\right]} \tag{4.20}$$

where

$$\Phi = \sum_{\gamma} \psi_{\gamma}(u_1, x_1^2, p+k)\, \psi_{\gamma}^{*}(u_2, x_2^2, p+k)$$

The crucial point here is that Φ is a real invariant function of $(p+k)^2 = W^2$ alone and does not independently depend on k^2. This, as we remarked before, is not true of ψ_{γ}.

To prove this let the annihilation operator for incoming nucleon be

$\psi_{in}(p)$, then

$$\langle p' | j(0) | p+k, \gamma \rangle = \langle 0 | \psi_{in}(p') j(0) | p+k, \gamma \rangle = \langle 0 | [\psi_{in}(p'), j(0)] | p+k, \gamma \rangle$$

if $(p+k-p')^2 = k'^2 < m_1^2 > \mu^2$.

since $\langle 0 | j(0)\, \psi_{in}(p') | p+k, \gamma \rangle = 0$

in this case. For mesons $m_1 \geq 3\mu$.

Using the relation $[\psi_{in}(p'), j(0)] = \frac{1}{(2\pi)^{3/2}} \int d^4x\, e^{ip'x}\, \theta(x) [j(0)\, \psi(x)]$

and an analogous treatment of the second factor in (4.17), one obtains after some rearrangement the stated result.

The analyticity properties of M in k^2 as well as Δ^2 are contained in the denominator. Specializing in the frame $\underline{p} + \underline{k} = 0$ (c.m. frame) one can again study the points where the denominator vanishes. One finds that one obtains once again an allipse in complex cos θ plane, with semi-axis given by $2x_0^2 - 1$ and $2x_0\sqrt{x_0^2-1}$ wherin the denominator does not vanish.

x_0 is given by the same expression as previously. The proof is elementary and given in the appendix.

§3. Proof of Dispersion Relations.

Consider the expression for the scattering amplitude:

$$T = -\int d^4x\, e^{i\frac{k+k'}{2}x} \langle p' | \theta(x)[j(\tfrac{x}{2})\, j(-\tfrac{x}{2})] | p \rangle \tag{4.21}$$

In this section it will be useful to set down some of the formulae in terms of the variable ω, instead of W^2 and $\overline{W}^2$. In the frame $\underline{p} + \underline{p}' = 0$ set

$$\frac{k+k'}{2} = \left\{ \omega, \sqrt{\omega^2 - (\Delta^2 + k^2)}\, \hat{\underline{e}} \right\} \tag{4.22}$$

where $\hat{\underline{e}}$ is a unit vector $\hat{\underline{e}} \cdot \underline{p} = 0$. Then

$$T(\omega, \Delta^2, k^2) = -\int d^4x\, e^{i\omega x_0 - i\hat{\underline{e}}\cdot\underline{x}\sqrt{\omega^2 - \Delta^2 - k^2}}\, f(p, p', x) \tag{4.23}$$

Notice that all the ω dependence of T lies in the exponential. Put

$$k^2 = k_1^2 + i k_2^2 \quad ; \quad \omega = \omega_1 + i\omega_2$$

then (4.23) defines an analytic function $T(\omega, \Delta^2, k^2)$ of the complex variables ω and k^2 regular for

$$\mathrm{Im}\, \omega > \mathrm{Im}\sqrt{\omega^2 - \Delta^2 - k^2} \qquad \omega_2 > 0 \qquad \omega_1^2 > k_1^2 + \Delta^2 \tag{4.24}$$

or

$$2\omega_2\left(\omega_1 - \sqrt{\omega_1^2 - k_1^2 - \Delta^2}\right) < k_2^2 < 2\omega_2\left(\omega_1 + \sqrt{\omega_1^2 - k_1^2 - \Delta^2}\right) \tag{4.25}$$

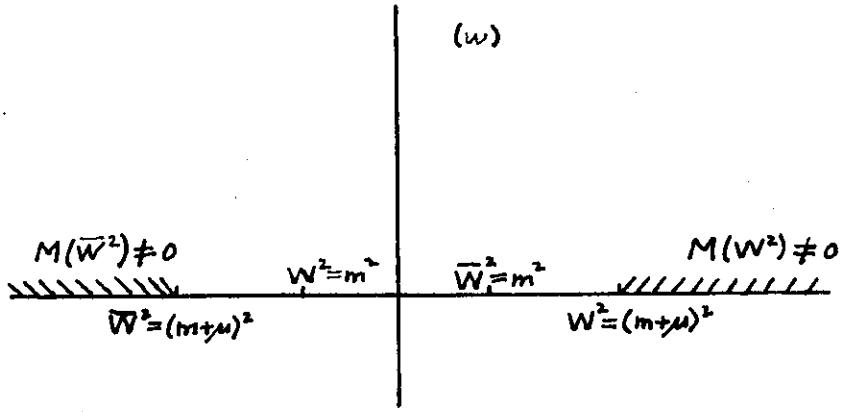
Take $k^2 = k_1^2 < -\Delta^2$ Then T is regular in $\mathrm{Im}\, \omega > 0$ and if T is bounded we can write the relation:

$$T(\omega, \Delta^2, k^2) = \frac{1}{\pi}\int_{-\infty}^{+\infty} d\omega' \frac{\mathrm{Im}\, T(\omega', \Delta^2, k^2)}{\omega' - \omega} \tag{4.26}$$

Using (4.16) and the relation $M(\overline{W}^2(-\omega)) = M(W^2(\omega))$ we can write

$$T(\omega,\Delta^2,k^2)=\frac{1}{\pi}\int_0^\infty d\omega'\,\mathrm{Im}\,T(\omega',\Delta^2,k^2)\left[\frac{1}{\omega'-\omega}+\frac{1}{\omega'+\omega}\right] \tag{4.27}$$

Returning to the variables W^2 and $\overline{W}^2$, we can now write the spectral conditions*. From (4.17) we see that $M(W^2,k^2,\Delta^2)$ has a pole for $W^2=(p+k)^2=m^2$ and is zero unless $(p+k)^2\geq(\mu+m)^2$ that is $W^2\geq(\mu+m)^2$ and $W_0>0$. From (4.19) we deduce that $M(\overline{W}^2,k^2,\Delta^2)$ has a pole for $\overline{W}^2=m^2=(p-k)^2$ and is zero unless $(p-k)^2\geq(\mu+m)^2$ that is $\overline{W}^2\geq(\mu+m)^2$ and $\overline{W}_0>0$. In the ω plane



The bound state contribution can be shown to be given by

$$M(W^2,k^2,\Delta^2)=g^2(k^2)\,\delta(W^2-m^2)$$

$$M(\overline{W}^2,k^2,\Delta^2)=g^2(k^2)\,\delta(\overline{W}^2-m^2)$$

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*The point about shifting back to W^2 is that the results of last lecture were all proven, holding W^2 fixed and not ω fixed. It is in fact, not really necessary to introduce the variable ω. One does it for relation (4.26) looks rather nice.

From (4.16) and the spectral conditions we have

$$T(\omega,\Delta^2,k^2)=\frac{1}{2\pi}\int_{(m+\mu)^2}^{\infty} dW'^2 M(W'^2,\Delta^2,k^2)\left[\frac{1}{W'^2-2\Delta^2-m^2-k^2-2\omega\sqrt{\Delta^2+m^2}}+\frac{1}{W'^2-2\Delta^2-m^2-k^2+2\omega\sqrt{\Delta^2+k^2}}\right]+$$

$$+\frac{g^2(k^2)}{2\pi}\left[\frac{1}{W^2-m^2}+\frac{1}{W^2-m^2-2k^2-4\Delta^2}\right] \qquad (4.28)$$

This relation has been written for $k^2<-\Delta^2$. We want to continue this relation to the mass shell $k^2=\mu^2$ *. If it is possible (4.26) would become

$$T(\omega,\Delta^2,\mu^2)=\frac{1}{\pi}\int_{-\infty}^{+\infty} d\omega' \frac{\mathrm{Im}\, T(\omega',\Delta^2,\mu^2)}{\omega'-\omega}$$

which leads to the usual dispersion relation

$$\mathrm{Re}\, T(\omega,\Delta^2,\mu^2)=\frac{P.V}{\pi}\int_{-\infty}^{+\infty} d\omega' \frac{\mathrm{Im}\, T(\omega',\Delta^2,\mu^2)}{\omega'-\omega} \qquad (4.29)$$

If $M(W'^2,\Delta^2,k^2)$ is an analytic function of k^2 in the domain $|\mathrm{Im}\, k^2|<\delta$, δ arbitrarily small, $\mathrm{Re}\, k^2\leq\mu^2$ with $\Delta^2\geq 0$ and $W'^2\geq(m+\mu)^2$ then the integral in (4.28) defines an analytic function of ω and k^2 in the domain:

$$k_1^2\leq\mu^2 \qquad |k_2^2|<2|\omega_2|\sqrt{\Delta^2+M^2} \qquad (4.30)$$

The domain is a wedge and $k_2=\omega_2=0$ is on the edge. At this point we need the edge of the wedge theorem which tells that the function will be analytic in a neighborhood of the edge, that is for $|k_2^2|<\delta_1$ and $|\omega_2|<\delta_2$ (4.31)

The next step is to prove that $g^2(k^2)$ is analytic for $k^2\leq\mu^2$.

*This method of taking $k^2+\Delta^2<0$ and then continuing to the value $k^2=\mu^2$ has been invented by Bogolyubow.

In (4.20) the bound-state term is given by

$$M_b(k^2) = g^2(k^2)\left[\delta(W^2-m^2)+\delta(\overline{W}^2-m^2)\right]$$

with

$$g(k^2) = \int \frac{\Phi(u,x^2,p+k)}{(\frac{k-p}{2}-v)^2-x^2}\, d^4u\, dx^2 \qquad , (p+k)^2=m^2.$$

Let us choose the CM system with $\underline{p}+\underline{k}=0$, $p_0+k_0=m$

$$g(k^2) = \int \frac{\Phi(u,x^2,m,u_0)}{\frac{k^2}{2}+\frac{m^2}{4}+u^2+mu_0-2k\cdot u-x^2}\, d^4u\, dx^2$$

The analyticity can be studied in the same way as for the complete T-matrix. $g(k^2)$ will be analytic as long as the denominator does not vanish in the region S with $W=m$.

This condition has been investigated by Oehme and Taylor (preprint). Their result is that $g(k^2)$ is analytic for real k^2 as long as

$$k^2 < k^2_{max} = \frac{2mm_1(m_2-m)}{m+m_2-m_1}$$

For considered cases we have $k^2_{max} > \mu^2$ which is sufficient foe the proof of dispersion relations.

To sum the results, the right hand side of (4.28) defines for complex k^2 an analytic function in the strip $k_1^2 \leq \mu^2$, $k_2^2 < \delta_1$. $T(k^2)$ is an analytic function equal to this right hand side for $k^2 \leq -\Delta^2$ which is inside the strip. So the two functions are identical and

(4.28) is proved.*

The proof above merely makes clear the main ideas involved. To see the same ideas developed with precise rigour, see the papers of Bogolyubow and Bremmerman, Oehme and Taylor (Physical Review, 109, 2180-2181 (1958)).

We must remember that we have supposed that $M(k^2)$ was analytic for $k^2 \leq \mu^2$. Using the result of §2 and of the appendix we know that this is possible only if $\Delta \leq \Delta_{max}$.

The value of Δ_{max} is computed in the appendix and applied to three interesting cases.

---

*The dispersion relation can be written

$$T(W^2,\Delta^2,k^2) = \frac{1}{2\pi}\int_{(m+\mu)^2}^{\infty} dW'^2 \frac{M(W'^2,\Delta^2,k^2)}{W^2-W'^2} + \frac{1}{2\pi}\int_{(m+\mu)^2}^{\infty} d\overline{W}'^2 \frac{M(\overline{W}'^2,\Delta^2,k^2)}{\overline{W}^2-\overline{W}'^2} + \frac{g^2(k^2)}{2\pi}\left(\frac{1}{W^2-m^2}+\frac{1}{\overline{W}^2-m^2}\right)$$

with $W^2+\overline{W}^2 = 4\Delta^2+2m^2+2k^2$. It is not different, but looks nicer than (4.28).

§4 Application to Dispersion Relations

Here, we consider three cases.

1) pion-nucleon scattering.

m = nucleon mass

μ = pion mass

$m_1 = 3\mu$, $m_2 = m+\mu$ and $2R \geq m+\mu$

We can check that the minimum of X is given by X_0 . Then, according to the procedure by Lehman, the region of analyticity of the imaginary part of the amplitude with respect to k^2 is given by

$$\Delta^2 \leq |k_c|^2 + \frac{(m_1^2-\mu^2)(m_2^2-m^2)}{4R^2-(m_1-m_2)^2} \qquad \text{for} \quad k^2=\mu^2.$$

By inserting the various values, we have the condition

$$\Delta^2 \leq \frac{8}{3}\mu^2 \frac{2m+\mu}{2m-\mu}$$

2) nucleon-nucleon scattering.

m = nucleon mass

μ = m $\qquad m_1 = m+m_\pi \qquad m_2 = m+m_\pi$

$2R \geq 2m_\pi$ if $m \geq m_\pi$

$2R \geq 2m$ if $m_\pi \geq m$

Similarly, the minimum X is given by X_0 and the region of analyticity is

$$\Delta^2 \leq \frac{16R^4-4(m^2+\mu^2)R^2+(m^2-\mu^2)^2}{16R^2} + \frac{(m_2^2-m^2)(m_1^2-\mu^2)}{4R^2-(m_1-m_2)^2}$$

Taking the minimum with respect to R, we have

$$\Delta^2 \leq m_\pi^2 + 2mm_\pi - m^2$$

In order to have $\Delta^2 = 0$ this requires

$$m_\pi \geqslant (\sqrt{2}-1)m$$

3) <u>K-N scattering</u>.

To have the dispersion relation hold, we must have

$$\Delta^2 \geqslant 0 \; ; \; \Delta^2 = \frac{[4R^2-(m+\mu)^2][4R^2-(m-\mu)^2]}{16R^2} + \frac{(m_1^2-\mu^2)(m_2^2-m^2)}{4R^2-(m_1-m_2)^2}$$

m = nucleon mass

$\mu = m_K$

$m_1 = m_K + 2m_\pi \qquad m_2 = m + m_\pi$

It is rather difficult to get a necessary and sufficient condition for $\Delta^2 \geqslant 0$ since $\Delta^2 = 0$ gives an algebraic equation of the third order with respect to R^2 and the final result is rather complicated, and does not seem useful. Instead, we calculate numerically, the necessary condition for $\Delta^2 \geqslant 0$. Using

$m = 6.719$

$m_K = 3.538\, m_\pi$

We can show by numerical calculation that only if $m_\Lambda > 8.573\, m_\pi$ then $\Delta^2 > 0$ for $2R \geqslant m_\Lambda + m_\pi$. Thus, we can prove the dispersion relation, only in the case of $m_\Lambda > 8.573\, m_\pi$. Actually, $m_\Lambda = 7.986\, m_\pi$ and this does not satisfy the above condition, and hence we cannot prove the dispersion relation for K-N scattering. If m_Λ were slightly bigger than the actual value, by about $0.6 m_\pi$ then we could prove the dispersion relation. (In this case, of course, m_Σ must be bigger than m_Λ) (actual m_Σ is $8.522\, m_\pi$.)

Appendix to Chapter II

By a translation and a rotation in the k space we can put the quadratic form in the form:

$$Q = \sum A'_{ss} k_s^2 + C' \qquad A'_{ss} > 0$$

The invariants are in the new system:

$$\Lambda = \begin{vmatrix} A_{11} & & 0 \\ & A_{22} & \\ 0 & & \ddots \; A_{ss} \end{vmatrix} = \prod_s A_{ss} \qquad x = \Lambda C' \tag{A.1}$$

We know the formula (Jauch and Rohrlich p. 457)

$$\int d^4k \frac{1}{(k^2+L)^q} = \frac{i\pi^2}{(q-1)(q-2)L^{q-2}}$$

With our choice of the time-like metric L is negative in our case. The integral has to be understood as having a small imaginary part in the denominator $i\epsilon$. We perform the integration successively on k_1, k_2, ... k_ℓ. We find:

$$I = (r-1)! \int_0^1 \delta\left(\sum_1^r u_i - 1\right)(du)^r (i\pi^2)^\ell \frac{(r-2\ell-1)!}{(r-1)!} \cdot \frac{1}{(\prod A_{ss})^2 C'^{r-2\ell}}$$

which can be written using (A.1).

$$I = (i\pi^2)^\ell (r-2\ell-1)! \int_0^1 (du)^r \, \delta\left(\sum_1^r u_i - 1\right) \left(\frac{\Lambda}{x}\right)^{r-2\ell} \frac{1}{\Lambda^2}$$

The result can be obtained in a different manner*, using an integral representation for the $\Delta_F(k)$ function.

$$\Delta_F(k) = \frac{1}{k^2 - m^2} = \frac{i}{2} \int_0^\infty e^{-i\frac{b}{2}(k^2 - m^2)} \, db$$

The matrix element can now be rewritten:

$$I = \frac{i^r}{2^r} \int_0^\infty (db)^r \, e^{-\frac{i}{2} Q(k)} (dk)^\ell$$

where Q(k) is exactly the quadratic form (1.3) with b instead of u.

We have the formula

$$\int_0^\infty e^{-i\frac{ax^2}{2}} d^4x = \frac{4\pi^2 i}{a^2}$$

We diagonalize again the quadratic form

$$Q(k) = \sum A'_{ss} k_s^2 + c'$$

We perform now the integration in the k^ℓ.

$$I = \frac{i^r}{2^r}\int (db)^r e^{-i\frac{c'}{2}} \int d^4k_1 e^{-i\frac{A'_{11}k_1^2}{2}} \cdots\cdots \int d^4k_\ell\, e^{-i\frac{A'_{\ell\ell}k_\ell^2}{2}} =$$

$$= \frac{i^r}{2^r}\int_0^\infty (db)^r e^{-i\frac{c'}{2}} \frac{(4\pi^2 i)^\ell}{(\pi A'_{ss})^2} = \frac{i^r(4\pi^2 i)^\ell}{2^r}\int_0^\infty (db)^r \frac{e^{-i\frac{\varkappa}{2\Lambda}}}{\Lambda^2}$$

$\varkappa$ and Λ have been defined in equat. (2.5). We notice that if we replace the b's by bt where t is a common factor, $\varkappa$ and Λ are multiplied respectively by $t^{\ell+1}$ and t^ℓ. This is a simple consequence of (2.3). We make the change of variables

$$\begin{aligned} b_1 &= \alpha_1 t \\ b_2 &= \alpha_2 t \\ &\vdots \\ b_{r-1} &= \alpha_{r-1} t \\ b_r &= (1-\sum^{r-1}\alpha_j)\, t \end{aligned}$$

The Jacobian is equal to t^{r-1}

We introduce an additional variable α_r defined by $\alpha_r = 1-\sum_1^{r-1}\alpha_j$. The domain of variation of the new variable is $0 \leq \alpha_j \leq 1$ $0 \leq t \leq \infty$

So that

$$I = \frac{i^r(4\pi^2 i)^\ell}{2^r}\int_0^1 (d\alpha)^{r-1}\int_0^\infty \frac{e^{-i\frac{\varkappa'(\alpha_j;t)}{2\Lambda(\alpha_j;t)}}}{\Lambda'^2(\alpha_j;t)}\, t^{r-1}\, dt =$$

$$= \frac{i^r(4\pi^2 i)^\ell}{2^r}\int_0^1 (d\alpha)^r\, \delta(\sum_1^r \alpha_j - 1)\int_0^\infty \frac{e^{-i\frac{\varkappa'(\alpha_j;t)}{2\Lambda'(\alpha_j;t)}}}{\Lambda'^2(\alpha_j;t)}\, t^{r-1}\, dt$$

j = 1, ... r-1 in the first line and j = 1 to r in the last. Using the preceding remark it can be written

$$I = \frac{i^r (4\pi^2 i)^\ell}{2^r} \int_0^1 (d\alpha)^r \delta\left(\sum_1^r \alpha_j - 1\right) \int_0^\infty \frac{e^{-i\frac{x}{2\Lambda}t}}{\Lambda^2 t^{2\ell}} t^{r-1} dt$$

Integrating r-2ℓ-1 times by part we find adding a small imaginary term to make the exp. convergent at ∞.

$$I = \frac{i^r (4\pi^2 i)^\ell}{2^r} \int_0^1 (d\alpha)^r \delta\left(\sum_1^r \alpha_j - 1\right) (r-2\ell-1)! \frac{1}{\Lambda^2} \frac{1}{\left(-i\frac{x}{2\Lambda}\right)^{r-2\ell}} =$$

$$= (i\pi^2)^\ell (r-2\ell-1)! \int_0^1 (d\alpha)^r \delta\left(\sum_1^r \alpha_j - 1\right) \left(\frac{\Lambda}{x}\right)^{r-2\ell} \cdot \frac{1}{\Lambda^2}$$

This is the Chisholm formula.

Appendix to Chapter III. Alternative Ways of Specifying the Region $\mathcal{S}$ in Dyson's Integral Representation.

$$(R-u)^2 \geqslant (m_1-k)^2 > 0 \qquad 0 \leq k^2 \leq m_1^2$$

$$(R-u)^2 \geqslant 0 \qquad k^2 \geqslant m_1^2$$

$$(R+u)^2 \geqslant (m_2-k)^2 > 0 \qquad 0 \leq k^2 \leq m_2^2$$

$$(R+u)^2 \geqslant 0 \qquad k^2 \geqslant m_2^2$$

If $\underline{R} = 0$ (this is always the most convenient choice) then this means that so far as $|\underline{u}|$ and k are concerned, $|\underline{u}| \leq R$, and $k \geqslant f(\underline{u})$. Where

$$f(\underline{u}) = \frac{1}{2}(m_1+m_2) - R\left(1 - \frac{\underline{u}^2}{B^2}\right)^{1/2}$$

for $|\underline{u}| < \frac{B^2}{R}$, $f(\underline{u}) = m_2 - 2[R(R-|\underline{u}|)]^{1/2}$ for $\frac{B^2}{R} \leq |\underline{u}| \leq R$

Here $B^2 = R^2 - \frac{1}{4}(m_1-m_2)^2$; when $B^2 < 0$, $f(u)$ is given by the second equation for all $|\underline{u}| \leq R$. For u_0 this reads

$$\sqrt{\underline{u}^2 + (m_2-k)^2} - R \leq u_0 \leq R - \sqrt{\underline{u}^2 + (m_1-k)^2} \qquad 0 \leq k \leq m_1$$

$$\sqrt{\underline{u}^2 + (m_2-k)^2} - R \leq u_0 \leq R - |\underline{u}| \qquad m_1 \leq k \leq m_2$$

$$|\underline{u}| - R \leq u_0 \leq R - |\underline{u}| \qquad k \geqslant m_2$$

These equations can be written:

$$|\underline{u}| \leq R \qquad |\underline{u}| - R \leq u_0 \leq R - |\underline{u}|$$

$$k \geqslant \max\left\{0,\ m_1 - \sqrt{(R+u_0)^2 - |\underline{u}|^2},\ m_2 - \sqrt{(R-u_0)^2 - |\underline{u}|^2}\right\}$$

Appendix Chapter IV

$$\underline{k}+\underline{p}=0 \qquad q_1=\frac{\underline{k}'-\underline{p}'}{2} \qquad q_2=\frac{\underline{k}-\underline{p}}{2}$$

$$M(W^2,k^2,\Delta^2)=\frac{1}{8\pi|\underline{k}_d|^2}\int du_{i0}\,|\underline{u}_i|\,d|\underline{u}_i|\,dx_i^2\int_0^{2\pi}d\alpha\int_0^{\pi}d\beta_1\int_0^{\pi}d\beta_2\int_0^{2\pi}dx \;\cdot$$

$$\cdot\;\frac{\Phi\left(u_{i0},|\underline{u}_i|^2,x_i^2,\cos\alpha\sin\beta_1\sin\beta_2+\cos\beta_1\cos\beta_2,W\right)}{\left[x_1(k^2)-\cos(\theta-x)\right]\left[x_2(k^2)-\cos(x-\alpha)\right]}$$

Here

$$x_i(k^2)=\frac{2q_{i0}u_{i0}-v_i^2-q_i^2-x_i^2}{2|k_c||\underline{u}_i|\sin\beta_i}$$

$$\hat{k}'=1,0,0 \qquad \hat{v}_1=\cos\varphi_1\sin\beta_1,\ \sin\varphi_1\sin\beta_1,\ \cos\beta_1$$

$$\hat{k}=\cos\theta,\sin\theta,0 \qquad \hat{v}_2=\cos\varphi_2\sin\beta_2,\ \sin\varphi_2\sin\beta_2,\ \cos\beta_2$$

$$x=\varphi_1 \qquad \alpha=\varphi_1-\varphi_2$$

note that

$$\int_0^{2\pi}\frac{dx}{\left[x_1-\cos(\theta-x)\right]\left[x_2-\cos(x-\alpha)\right]}=$$

$$=2\pi\cdot\frac{\dfrac{x_1}{\sqrt{x_1^2-1}}+\dfrac{x_2}{\sqrt{x_2^2-1}}}{x_1x_2+\sqrt{x_1^2-1}\sqrt{x_2^2-1}-\cos(\theta-\alpha)}$$

Introduce $z_i=|k_c|x_i\sin\beta_i$. Then

$$M(W^2,\Delta^2,k^2)=\int du_{i0}\,d|\underline{u}_i|\,|\underline{u}_i|\,dx_i^2\int_0^{2\pi}d\alpha\,\bar{\varphi}\left(u_{i0},|\underline{u}_i|,x_i^2,\cos\alpha,W\right)\cdot$$

$$\cdot\left\{\frac{\dfrac{z_1}{\sqrt{z_1^2-k_c^2}}+\dfrac{z_2}{\sqrt{z_2^2-k_c^2}}}{z_1z_2+\sqrt{z_1^2-k_c^2}\sqrt{z_2^2-k_c^2}-k_c^2\cos(\theta-\alpha)}\right\}$$

Note that the entire Δ^2, and k^2 dependence is contained inside the factor in brackets. To study this function we restrict ourselves to $\Delta^2 \geq 0$ and $k^2 \leq \mu^2$. M is singular if

(1) $z_i^2 - k_c^2 = 0$

(2) $z_1 z_2 + \sqrt{z_1^2 - k_c^2}\sqrt{z_2^2 - k_c^2} - k_c^2 \cos(\theta - \alpha) = 0$

(Note that k_c^2 can become negative in the so-called unphysical region). It will be shown in the next appendix that for ν, x^2 varying over the region S, under some restrictions, and for $k^2 \leq \mu^2$

$$\min(z_i^2 - k_c^2) = \frac{(m_1^2 - \mu^2)(m_2^2 - m^2)}{W^2 - (m_1 - m_2)^2}$$

So $z_i^2 - k_c^2 \neq 0$ if $m_1 > \mu, m_2 > m, W^2 > (m_1 - m_2)^2$ what has been assumed in the calculation of the min. This is always so in all applications. The second term equals zero when

$$k_c^2 \cos\theta = z \cos\alpha \pm i\sqrt{z^2 - k_c^2}\sin\alpha$$

Here $z = z_1 z_2 + \sqrt{z_1^2 - k_c^2}\sqrt{z_2^2 - k_c^2}$

Note $z_i \geq 0$

Two cases arise $z_i > 0$ so $z > |k_c^2|$

then singularity occurs when $k_c^2 \cos\theta = (k_c^2 - 2\Delta^2) = \pm z$

Since $z > |k_c^2|$ only the minus sign is possible, so that the singularity arises when

$$\Delta^2 = \frac{1}{2}(k_c^2 + z) = \frac{1}{2}\left(k_c^2 + z_1 z_2 + \sqrt{z_1^2 - k_c^2}\sqrt{z_2^2 - k_c^2}\right)$$

This is irrespective of the sign of k_c^2.

Thus

$$\Delta^2 \leq \Delta^2_{max} = \frac{1}{2}\min(k_c^2+2z^2-k_c^2) = \min z^2 = k_c^2 + \frac{(m_1^2-\mu^2)(m_2^2-m^2)}{W^2-(m_1-m_2)^2}$$

In the second case $z_i = 0$. Then the ellipse in $\cos\theta$ plane which gave the analyticity domain in $\cos\theta$ just collapses into the real axis.

<u>Evaluation of the Minimum X_0</u> :

The problem, which we want to solve here, is to find the minimum of the following quantity:

$$X = \frac{|\underline{u}|^2+|\underline{q}|^2+x^2-(q_0-u_0)^2}{2|\underline{k}_c||\underline{u}|} \tag{A.1}$$

where u_0 and $|\underline{u}|$ are the variational variables, subject to

$$0 \leq |\underline{u}| \leq R$$

$$-R+|\underline{u}| \leq u_0 \leq R-|\underline{u}| \tag{A.2}$$

and

$$x \geq \max\left\{0, m_1-\sqrt{(R+u_0)^2-|\underline{u}|^2}\ , m_2-\sqrt{(R-u_0)^2-|\underline{u}|^2}\right\} \tag{A.3}$$

From (A.1) it is clear we must take the equality in (A.3) to have a minimum. Obviously, the minimum X_0 of X is a function of $R, |k_c|$ and q_0 . However, these are not so convenient for our discussion. Noticing that at the center-of-mass system $\underline{p}+\underline{k}=\underline{R}=0$,

$$q_0 = -\frac{m^2-k^2}{4R}$$

$$|\underline{k}_c|^2 = |\underline{q}|^2 = \frac{1}{16R^2}\left[16R^4-4(m^2+k^2)R^2+(m^2-k^2)^2\right] \tag{A.4}$$

By (A.4), instead of q_0 and $|k_c|$, we can use m^2 and k^2. The physical meaning of m and $\sqrt{k^2}$ is that they are the mass of the scatterer and the projectile, respectively. Since we will want to make an analytic continuation later, with respect to k^2, from a negative value to the positive one; we do not assume that k^2 must be positive in this note. The notation of the Ch. $\underline{\text{IV}}$ is related by $W = 2R$.

Furthermore, it is convenient to use the following x and y rather than $|\underline{u}|$ and u_0.

$$x = \sqrt{(R+u_0)^2 - |\underline{u}|^2}$$

$$y = \sqrt{(R-u_0)^2 - |\underline{u}|^2} \tag{A.5}$$

or

$$u_0 = \frac{1}{4R}(x^2 - y^2)$$

$$|\underline{u}|^2 = \frac{1}{16R^2}\left[4R^2 - (x+y)^2\right]\left[4R^2 - (x-y)^2\right] \tag{A.6}$$

Then, Eq. (A.1) is readily transferred into

$$X = \frac{2R}{|k_c|} \frac{\frac{1}{2}\left[4R^2 - (m^2+k^2)\right] + x^2 - \frac{1}{2}\left[x^2 + y^2 + \frac{m^2 - k^2}{4R^2}(x^2 - y^2)\right]}{\sqrt{\left[4R^2 - (x+y)^2\right]\left[4R^2 - (x-y)^2\right]}} \tag{A.7}$$

and

$$\varkappa = \max\{0, m_1 - x, m_2 - y\} \tag{A.8}$$

As we can easily check, then the conditions Eq. (A.2) become simply

$$x + y < 2R, \quad x > 0, \quad y > 0 \tag{A.9}$$

Thus, our problem is simply reduced to finding the minimum X_0 of X from Eqns. (A.7), (A.8) and (A.9). Because of Eq. (A.8), we must classify

three cases according to

(a) $2R > m_1 + m_2$

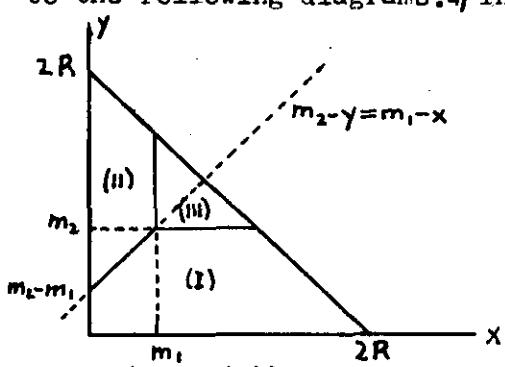
(b) $m_1 + m_2 > 2R > |m_2 - m_1|$

(c) $|m_2 - m_1| > 2R > 0$

The physically interesting cases are the first two (a) and (b). In what follows, we assume

$$m_2 > m_1 > 0 \tag{A.10}$$

Generally, this does not change the final result, and whenever the difference comes, we will specify it. Furthermore, the variational region of x and y are subdivided into several regions, according to the following diagrams.



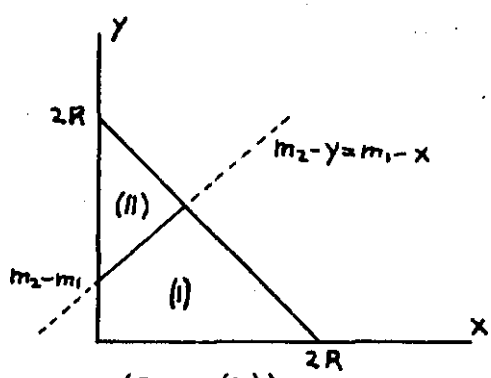
(Case (a))

a) In this case, the point $(x=m_1, y=m_2)$ is inside the region Eq. (A.9). So, according to Eq. (A.8), we divide the region by the three lines; $m_2-y=m_1-x$, $x=m_1$ and $y=m_2$ and we get the three regions, (I), (II) and (III). Then, by (A.8),

at (I) $\quad x = m_2 - y$

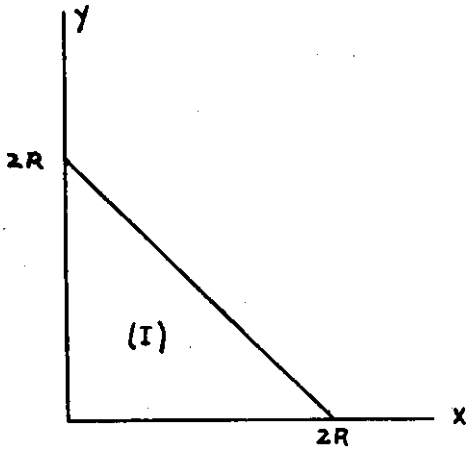
at (II) $\quad x = m_1 - x$

at (III) $\quad x = 0$ $\qquad$ (A.11)



(Case (b))

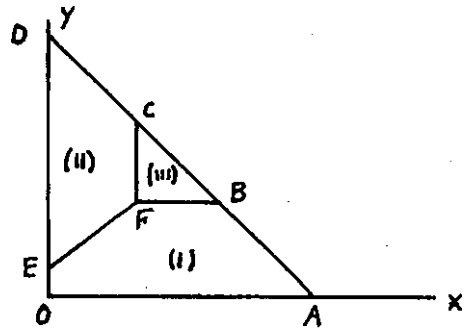
b) In this case, the point $(x=m_1, y=m_2)$ is outside the region Eq. (A.9) and the region (III) of the previous case disappears. The value of x is still given by Eq. (A.11).



Case (c)

In this case, the line $m_2 - y = m_1 - x$ does not intersect with the region Eq. (A.9) and the region (II) of the previous case disappears, also. Thus, the region (I) covers the whole. By far, the case (a) is the most complicated. First, we will investigate the case (a).

Investigation of the case (a): We will name the points of the previous diagrams as indicated here. Furthermore it is sometimes convenient to use



$$x + y = t$$
$$x - y = s \tag{A.12}$$

This means to rotate the triangle by 45° in the counter-clockwise direction. First, consider the region (I). By (A.11) and (A.12), Eq. (A.7) becomes

$$X = \frac{2R}{|k_c|} \cdot \frac{\frac{1}{2}\left[4R^2 - (m^2 + \mu^2)\right] + m_2^2 + m_2(s-t) - \frac{4R^2 + m^2 - R^2}{8R^2} st}{\sqrt{(4R^2 - s^2)(4R^2 - t^2)}} \tag{A.13}$$

It is easy to check that we have no solution of the next equation inside the region (I). If $m_2^2 \neq m^2$

$$\frac{\partial X}{\partial s} = \frac{\partial X}{\partial t} \equiv 0$$

As we will see later, when the relation

$$m_2^2 - m^2 > 0 \quad , \quad m_1^2 - R^2 > 0 \tag{A.14}$$

is not satisfied, generally, there is no minimum of X, namely $x_0 = -\infty$. Thus, in what follows, we assume (A.14). Actually, in the physically interesting case, (A.14) is automatically satisfied. Then we have no minimum of X inside the region (I). This is also true for the regions (II) and (III), if (A.14) holds. Thus, the minimum X_0 of X must be obtained from the boundary, if it exists. So we must investigate the boundaries.

A) The boundary EF

The line EF is obviously defined by

$$s = -(m_2 - m_1) \qquad m_2 - m_1 < t < m_1 + m_2 \tag{A.15}$$

Then, Eq. (A.13) gives

$$X = \frac{2R}{|k_c|\sqrt{4R^2 - (m_2 - m_1)^2}} \cdot \frac{\frac{1}{2}[4R^2 - m^2 - k^2] + m_1 m_2 - \left[\frac{1}{2}(m_1 + m_2) - \frac{m^2 - k^2}{8R^2}(m_2 - m_1)\right]t}{\sqrt{4R^2 - t^2}} \tag{A.16}$$

The minimum point of (A.16) is given by

$$t = t_0 = \frac{4(m_1 + m_2)R^2 - (m_2 - m_1)(m^2 - k^2)}{4R^2 - (m^2 + k^2) + 2m_1 m_2} \tag{A.17}$$

In order to have Eq. (A.15) satisfied by this too, we must have

$$4R^2 > \left(m_2 - \frac{k^2}{m_1}\right)(m_2 - m_1) \tag{A.18}$$

This is surely satisfied for $k^2 > 0$ because $2R > m_1 + m_2$. For the negative k^2 one sufficient condition is

$$-k^2 < \frac{3m_2 + m_1}{m_2 - m_1} m_1^2 \tag{A.19}$$

Inserting (A.17) into (A.16), and after some algebra, we find

$$X_0 = \left[1 + \frac{(m_2^2 - m^2)(m_1^2 - k^2)}{|k_c|^2 [4R^2 - (m_2 - m_1)^2]}\right]^{1/2} \tag{A.20}$$

B) The line AD:

X diverges on this line as we can see from Eq. (A.13) by putting $t=W$. However, it may be possible to have $X=-\infty$ there, instead of $X=+\infty$. So, we must investigate the sign of the numerator of Eq. (A.13), carefully in this case. We can check that as far as the condition (A.14) is satisfied, $X=\infty$. But if the condition (A.14) is violated, then it may give $-\infty$, which means there is no finite minimum. (See D and E).

C) The lines CF and BF:

In this case, it is convenient to use the original x and y variables. For example, on the line BF, $y=m_2$, $m_1<x<2R-m_2$

Then, the expression (A.7) gives

$$X=\frac{2R}{|k_c|}\cdot\frac{\frac{1}{2}(4R^2-m^2-k^2)-\frac{1}{2}\left[x^2+m_2^2+\frac{m^2-k^2}{4R^2}(x^2-m_2^2)\right]}{\sqrt{(4R^2-m_2^2-x^2)^2-4m_2^2x^2}}$$

By direct calculation, we can check that

$$\frac{dX}{dx}<0 \qquad \text{for } m_1<x<2R-m_2,$$

So, the minimum of X on the line BF occurs at the end-point F. However, this point F is also the end-point of the line FE, where the minimum is proved to exist in the middle of it (See A.). Thus, the minimum on the line BF is not smaller than that no the line FE, and we are not necessary to consider the case further. The same is also true on the line CF.

D) The line DE.

$x=0$, $2R>y>m_2-m_1$ The formula (A.7) gives.

$$X=\frac{2R(m_1^2-k^2)}{|k_c|(4R^2-y^2)}+\frac{R}{|k_c|}\left(1-\frac{m^2-k^2}{4R^2}\right)$$

So, according to the condition (A.14), the minimum occurs at the end point E. So, by the same reason as was given in the previous case, this minimum is not smaller than that of Eq. (A.20). It must be noted that if $m_1^2-k^2<0$ then $X=-\infty$ at $y=2R$. The condition (A.14) is essential.

E) The line AO:

Similarly, Eq. (A.7) gives

$$X=\frac{2R}{|k_c|}\cdot\frac{(m_2^2-m^2)}{4R^2-x^2}+\frac{R}{|k_c|}\left(1+\frac{m^2-k^2}{4R^2}\right),\quad 2R\geqslant x\geqslant 0$$

By the condition (A.14) the minimum occurs at $x=0$.

$$X_1=X_{min}=\frac{1}{4R|k_c|}\left(4R^2-m^2-k^2+2m_2^2\right) \tag{A.21}$$

Similarly, if $m_2^2-m^2<0$ then $X=-\infty$ at $x=2R$.

F) The line EO:

We can check that the minimum of X occurs at the origin and it is given by Eq. (A.21). In the discussion of the cases (D), (E) and (F), it is necessary to use the fact $m_2>m_1$. If $m_2<m_1$, then m_2^2 in Eq. (A.21) must be changed into m_1^2. So, (A.21) must be written as

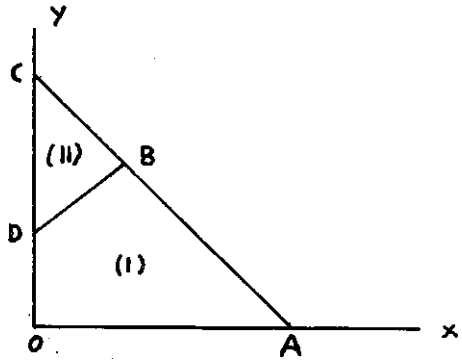
$$X_1=\frac{1}{4R|k_c|}\left(4R^2-m^2-k^2+2\max(m_1^2,m_2^2)\right) \tag{A.22}$$

Thus, we obtained two minimum (A.20) and (A.21). We must compare both. The requirement $X_1\geqslant X_0$, and so X_0 is the entire minimum gives the following conditions on R.

$$4R^2(m_2^2-m^2)(m_1^2-k^2)<\left[4R^2-(m_2-m_1)^2\right]\left\{4m_2^2R^2+m_2^4+m^2k^2-m_2^2(m^2+k^2)\right\} \tag{A.23}$$

In our case $2R > m_1 + m_2$; this is satisfied always if $k^2 > 0$. Next, let us investigate the case (b)

Investigation of the case (b)



$m_2 + m_1 > 2R > m_2 - m_1 > 0$

In this case, we can follow the procedure similar to the previous one.

A) The line BD:

In this case, we have the formula (A.20) under the following conditions.

$$4R^2 > \left(m_2 - \frac{k^2}{m_1}\right)(m_2 - m_1) \tag{A.18}$$

$$2R\left\{4R^2 - m^2 - k^2 + 2m_1 m_2\right\} > (m_1 + m_2)4R^2 - (m^2 - k^2)(m_2 - m_1) \tag{A.24}$$

In the actual application of dispersion relation, k_c^2 may be negative, also. In that case, we must interpret Eq. (A. 20) as

$$X_0 = \frac{1}{k_c}\left[k_c^2 + \frac{(m_2^2 - m^2)(m_1^2 - k^2)}{4R^2 - (m_2 - m_1)^2}\right]^{1/2} \tag{A.25}$$

Then X_0 is purely imaginary, and the condition (A.24) means that the inside of the square root of (A.25) is not negative; and we are looking for the minimum of iX in its rigorous sense.

B) The line AC:

If the conditions (A14) and (A.24) are, then we can show that $X = +\infty$ is there. However, if one of (A.14) or (A.24)is not satisfied,

then generally $X = -\infty$. Note that the new condition (A.24) is necessary in our case, though it is automatically satisfied in the case of $2R > m_1 + m_2$.

C) The line CD, DO and AO:

The arguments are exactly the same as in the previous case (a) and we do not respect our arguments here. We have the same minimum (A.22).

c) The final case (c), $(2R \leq |m_2 - m_1|)$ is the most simple, and its minimum is given by (A.22). Summarizing these results, the minimum of X is given by

$$\mathrm{Min}\, X = \mathrm{Min}(X_0, X_1)$$

where

$$X_0 = \left[1 + \frac{(m_2^2 - m^2)(m_1^2 - k^2)}{|k_c|^2 [4R^2 - (m_2 - m_1)^2]} \right]^{1/2}$$

$$X_1 = \frac{1}{4R|k_c|} \left[4R^2 - (m^2 + k^2) + 2\,\mathrm{Max}(m_1^2, m_2^2) \right]$$

and if the conditions

$$4R^2 > \left(m_2 - \frac{k^2}{m_1}\right)(m_2 - m_1)$$

$$2R[4R^2 - m^2 - k^2 + 2m_1 m_2] > (m_1 + m_2)4R^2 - (m^2 - k^2)(m_2 - m_1)$$

and

$$4R^2(m_2^2 - m^2)(m_1^2 - k^2) < [4R^2 - (m_2 - m_1)^2][4m_2^2 R^2 + m_2^4 + m^2 k^2 - m_2^2(m^2 + k^2)]$$

are satisfied, then $\mathrm{Min}\, X = X_0$. If the one of the first and the third is violated, then $\mathrm{Min}\, X = X_1$. If the second condition is violated, or (A.14), then $\mathrm{Min}\, X = -\infty$