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LECTURES ON DISPERSION THEORY

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Notes by G. Sudarshan and P. Signell

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Dispersion Theory and Causality

1. Classical Dispersion Theory

In classical physics, "dispersion" is the dependence of the index of refraction of light on its frequency. It is well known that for glass, blue light is refracted more than red light (in the visible region) so that white light is decomposed upon passage through a glass prism. In this century the phenomenon of dispersion has been of considerable importance in the development of physical ideas. There is nothing in Maxwell's theory itself to tell us anything about the refractive index of a homogeneous medium except the result:

$$n = \frac{c}{u}$$

where u is the phase velocity of electromagnetic waves in the medium:

$$u = \sqrt{\epsilon\mu}$$

We know that even this relation, although well-satisfied for homopolar gases and vapors, does not hold for solid and liquid media. For example, the relavent figures for water are:

$$n = 1.333$$

$$\frac{c}{\sqrt{\epsilon\mu}} \cong 12$$

Thus Maxwell's equations do not give us a clue to the dependence of n on ω (frequency). Apparently, then, dispersion provides us with information about the electrical characteristics of matter.

If we consider the scattering of light as performed by elastically bound electrons:

$$\ddot{\underline{r}} + \omega_0^2 \underline{r} = -\frac{e}{m}\underline{E}$$

is the eqn. of motion of an electron, here $\omega_0 \equiv \sqrt{k/m}$. the polarization

$$\underline{P} = N(-e\underline{r}) = \eta \underline{E}_0 e^{i\omega t} \; (= \eta \underline{E})$$

where N is the density of electrons, η is the susceptibility. substituting $\underline{P}$ into the eqn. of motion,

$$\ddot{\underline{P}} + \omega_0^2 \underline{P} = N\frac{e^2}{m}\underline{E}$$

with the solution:

$$\eta = N\frac{e^2}{m}\frac{1}{\omega_0^2 - \omega^2}$$

$$n^2(\omega) = K = 1 + 4\pi\eta = 1 + 4\pi N \frac{e^2}{m}\frac{1}{\omega_0^2 - \omega^2}$$

or if several resonant frequencies are present

$$n^2(\omega) = 1 + 4\pi \frac{e^2}{m} \sum_i \frac{N_i}{\omega_{0_i}^2 - \omega^2}$$

one can also include a radiation reaction ("damping") term in the eqn. of motion:

$$\ddot{\underline{r}} + \omega_0^2 \underline{r} = -\frac{e}{m}\underline{E} + \frac{2}{3}\frac{e^2}{mc^3}\dddot{\underline{r}}$$

then the solution becomes:

$$n^2(\omega) = 1 + 4\pi N \frac{e^2}{m} \cdot \frac{1}{\omega_0^2 - \omega^2 + i\delta} \; ; \; \delta \equiv \frac{2e^2\omega^3}{3c^3 m}$$

so n is now complex, representing absorption out of the incoming beam.

2. <u>Quantum Theory gives:</u>

$$n^2(\omega) = 1 + \frac{Ne^2}{m} \sum_i \frac{f_i}{\omega_i^2 - \omega^2}$$

here f_i = transition probability for ($i \rightarrow 0$)

f_i = "oscillator strength"

obeying $\sum_i f_i = 1$

$$\omega_i \equiv \frac{W_i - W_0}{\hbar}$$ (difference between energy levels).

This is the Rayleigh - Ritz Combination Principle in action.

3. <u>Kramers, 1927</u>

define:

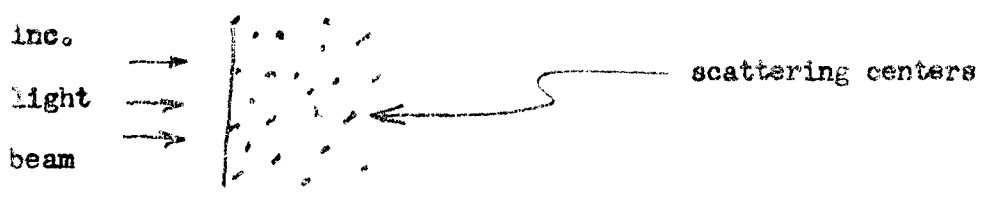
$$\frac{n^2(\omega) - 1}{4\pi} \equiv \varepsilon(\omega) + i\,\eta(\omega)$$
$$= \frac{e^2}{m} \sum_i \frac{N_i f_i}{\omega_i^2 - \omega^2 + i\delta}$$

Kramers noticed: $$\varepsilon(\omega) = \frac{2}{\pi} \mathcal{P} \int_0^\infty d\omega' \frac{\omega' \eta(\omega')}{\omega'^2 - \omega^2}$$

(1) or: $$\mathrm{Re}\, n^2(\omega) = 1 + \frac{2}{\pi} \mathcal{P} \int_0^\infty d\omega' \frac{\omega' \,\mathrm{Im}\, n^2(\omega')}{\omega'^2 - \omega^2}$$

This was the first equation relating the real and imaginary parts of the index of refraction. We call this a dispersion relation.

4. <u>Scattering Theory:</u>



various authors have derived the following relation between the index of refraction (n) and the forward scattering amplitude $f(\omega)$:

$$n^2(\omega) = 1 + \frac{4\pi f(\omega) N}{\omega^2} \quad : N = \text{density of scattering centers}$$

where ω is the frequency being considered. But does the forward scattering amplitude obey a dispersion-type equation? We will seek to relate the real to the imaginary part of the amplitude by invoking causality. We define the "Causality requirement" as:

If an incident wave packet reaches a point Z = 0 at $t = 0 \Longrightarrow$ the forward scattered outgoing waves (scatter at origin) must not be seen at a distance (r) until a time (r/c) has elapsed.

Consider an incident wave on such a scattering center, obeying the causality requirement:

$$A(z,t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} \sum_n \frac{a_n}{\omega + i\epsilon - \omega_n} \left(e^{ikz}\right)$$

$$A(0,t) = \sum_n a_n \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega + i\epsilon - \omega_n}$$

$$= 0 \quad \text{for } t < 0$$

For example, let A be a sine wave:

$$A(t) = 0 \quad \text{for } t < 0$$
$$= \sin \omega_0 t \quad : t > 0$$

then

$$A(t) = \frac{1}{4\pi} \int e^{-i\omega t} d\omega \left\{ \frac{1}{\omega + i\epsilon - \omega_0} - \frac{1}{\omega + i\epsilon + \omega_0} \right\}$$

This is of the above general form. Now look at the scattered wave: define "a (ω)" such that:

$$A(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} a(\omega) \quad : \quad a(\omega) \equiv \sum_n \frac{a_n}{\omega + i\epsilon - \omega_n}$$ in the general form.

the scattered wave: $F(t) = \int_{-\infty}^{\infty} d\omega \quad a(\omega)\, f(\omega)\, e^{-i\omega(t - r/c)}$

$= 0$ for $t < r/c$ <u>if</u> $f(\omega)$ is analytic in I^+ and I°

<u>define</u>: $I^+ \equiv$ upper (positive) imaginary plane

$I^- \equiv$ lower (negative) imaginary plane

$I^\circ \equiv$ the real axis.

Note that the scattered wave obeys the causality requirement if:

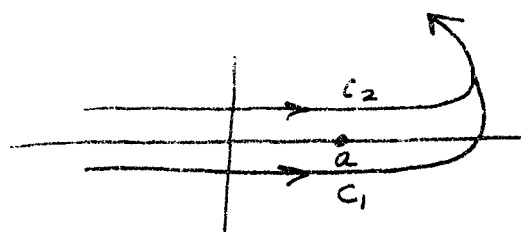
1) the incident wave is causal

and 2) the scattering amplitude is anal. in I^+ and I°

Consider a little math:

given a function $f(z)$ anal. in I^+ and I°

then: $f(a) = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{z-a}\, dz$



$0 = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z-a}\, dz$

adding:

(2) $$f(a) = \frac{1}{\pi i} \mathcal{P} \int_{-\infty}^{\infty} \frac{f(x)\, dx}{x-a}$$

this is, of course, the usual definition of a principal value integral.

providing, of course, contributions from the infinite semi-circle vanish.

Now separate f into real and imaginary parts:

$$f = g + i h$$

equate real and imaginary terms separately in the principal value equation:

(3) $$g(a) = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{h(x)\, dx}{x-a}$$

(4) $$h(a) = -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{g(x)\, dx}{x-a}$$

Such functions are called "conjugate functions" or "Hilbert Transforms" of each other. Note that if $f = e^{i\lambda x}$ then ($\sin \lambda x$) and ($\cos \lambda x$) are conjugate fn's.

Kramers' dispersion equation could have been obtained without oscillating electrons or other kinds of models. This is the prime result of dispersion theory. There is still one point (in the above) which needs to be talked about: we assumed

$$\operatorname{Im} n^2(-\omega) \equiv -\operatorname{Im} n^2(\omega)$$

but this really follows from the summetry properties of the theory as we shall show in later lectures. Given a scattering matrix $f(\omega)$ or $S(\omega)$ for $\omega > 0$ on I° find an anal. fn. (using causality and appropriate boundedness) in I^+ and I° which will go to $f(\omega)$ as $I^+ \longrightarrow I^\circ$. This will determine $S(-\omega)$. There are 2 possibilities:

(5) I: $f(-\omega) = f^*(\omega)$: this is what the Kramers eqn. uses.

(6) II: $f(-\omega) = -f^*(\omega)$

(7) I leads to:
$$g(\omega) = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{h(\omega')}{\omega' - \omega} d\omega' = \frac{2}{\pi} \mathcal{P} \int_0^{\infty} \frac{h(\omega')\, \omega' d\omega'}{(\omega'^2 - \omega^2)}$$

(8) II leads to:
$$g(\omega) = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{h(\omega')}{\omega' - \omega} d\omega' = -\frac{2\omega}{\pi} \mathcal{P} \int_0^{\infty} \frac{h(\omega')\, d\omega'}{(\omega'^2 - \omega^2)}$$

Note that II is more convergent than I, in that I contains an extra factor (ω') in the numerator. If greater convergence is needed, one can derive difference formulas:

<u>type I</u>:
$$g(\omega) - g(\omega_0) = \frac{2}{\pi} \mathcal{P} \int_0^{\infty} h(\omega')\, \omega' d\omega' \left\{ \frac{1}{\omega'^2 - \omega^2} - \frac{1}{\omega'^2 - \omega_0^2} \right\}$$

(9) <u>or</u>:
$$\frac{g(\omega) - g(\omega_0)}{\omega^2 - \omega_0^2} = \frac{2}{\pi} \mathcal{P} \int_0^{\infty} \frac{\omega'\, h(\omega')\, d\omega'}{(\omega'^2 - \omega^2)(\omega'^2 - \omega_0^2)}$$

<u>type II</u>:
$$\omega_0 g(\omega) - \omega g(\omega_0) = -\frac{2}{\pi} \mathcal{P} \int_0^{\infty} h(\omega') d\omega' \left\{ \frac{\omega_0 \omega}{\omega'^2 - \omega^2} - \frac{\omega \omega_0}{\omega'^2 - \omega_0^2} \right\}$$

(10)
$$\frac{\omega_0 g(\omega) - \omega g(\omega_0)}{\omega \omega_0 (\omega^2 - \omega_0^2)} = -\frac{2}{\pi} \mathcal{P} \int_0^{\infty} \frac{h(\omega')\, d\omega'}{(\omega'^2 - \omega^2)(\omega'^2 - \omega_0^2)}$$

Thus the pattern is set:

1) Given $S(\omega)$, defined for $\omega > 0$ on I.

2) Continue $S(\omega)$ anal. 'ly into I^+

Find $N(\omega + i\nu) \Rightarrow N(\omega + i\nu)$ is An for $\nu > 0$ and $\lim_{\nu \to 0} N(\omega + i\nu) = S(\omega)$

3) and if $N(\omega + i\nu)$ is suitably bounded:

$$\Rightarrow N(\omega + i\nu) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{N(\omega')\,d\omega'}{\omega' - (\omega + i\nu)}$$

holds for $\nu \to 0$ and also on I^*. Causality helps to find $N(\omega + i\nu)$. symmetry req.'s tell whether N falls into I or II, or neither $\mathcal{L}$ is then a mixture — $\mathcal{L}_r$ and what sort of disp. rel'n is satisfied.

Program: 1) Write down S-matrix assignment as possible.

2) Formulate causality property and study what analitic prop's are ascribed to S-matrix elen.

3) Study symm. properties in particular to decide whether it is I or II which is satisfied. Study meson-nucleon scatt. 1st we learn how to write it properly.

DISPERSION RELATIONS FOR PION-NUCLEON SCATTERING

Analytic Properties of the T-matrix

We shall start from the following general theorem on the S-matrix in terms of Heisenberg operators

$$(11) \qquad \langle \Omega | T(x_1, x_2, x_3, \ldots, x_n) \Phi^{\alpha}_{in} \rangle = -i \int K_y \langle \Omega | T(x_1, x_2, \ldots, x_n, y) \Omega \rangle f_\alpha(y)\, d^4y$$

where $T(x_1, \ldots, x_n)$ is the time ordered operator

$$T(x_1, x_2, \ldots, x_n) = T\{A(x_1)\, A(x_2) \ldots A(x_n)\}$$

and K_y is the Klein-Gordon operator

$$K_y = \Box_y^2 - m^2$$

This result is derived in the appendix.

We may now make a more detailed study of meson-nucleon scattering. To specify the initial and final states, we use the notation $|p\lambda k\rangle$ to denote a state with a nucleon of momentum (4-vector) p and a meson of momentum k and λ stands for all the other indices specifying the state (like spin, etc.) The initial and final states would then be represented in the form

$$\Phi_{in} = |p\lambda k\rangle \qquad\qquad \Phi_{out} = |p'\lambda' k'\rangle$$

The transition element for the relevant case is the:

$$(12) \qquad \langle p'\lambda' k' | p\lambda k\rangle = (-i)^2 \int K_x K_y \langle p'_{out} | T(\phi^*(x)\, \phi(y) | p_{in}\rangle f(x) f(y)\, d^4x\, d^4y$$

where $\phi(x)$ is the meson field operator. For charged mesons one has to consider complex fields ϕ_i and write, in the well known manner,

$$\phi_\pm = \frac{1}{\sqrt{2}}(\phi_1 \pm i\phi_2) \qquad\qquad \phi_0 = \phi_3$$

The corresponding transitionmatrix element would be

$$\langle p'_\pm \lambda' k' | p_\pm \lambda k\rangle = (-i)^2 \int K_x K_y \langle p'\lambda'_{out} | T(\phi^*_\pm(x)\, \phi_\pm(y)) | p\lambda_{in}\rangle\, e^{i(k'x - ky)}\, d^4x\, d^4y$$

If we write the relation between the S and T matrics (scattering and transition matrices) in the form

$$S = 1 + iT$$

we have, $$T = i\int k_x k_y \langle p'|\phi^*(x)\phi(y)|p\rangle e^{i(k'x-ky)} d^4x\, d^4y$$

If we now assume the initial and final states to be steady states we can remove the 'in', 'out' labels, as we have done in the last equation for T:- Let

$$k_x\, \phi(x) = j(x)$$

where $j(x)$ is a local interaction, ϕ being the meson field operator. The meson nucleon coupling is here not necessarily of Yukawa type, but is necessarily local.

By definition of the chronological ordering operator,

$$k_x\, T(\phi^*(x)\, \phi(y)) = k_x\left[\theta(x-y)\phi^*(x)\phi(y) + \theta(y-x)\phi(y)\phi^*(x)\right]$$
$$= T\left[j^*(x)\,\phi(y)\right] + \delta^4(x-y)\left[j^*(x), \frac{\partial\phi(y)}{\partial y_0}\right]$$

with

$$\theta(x) = \begin{matrix} 1 & x \geq 0 \\ 0 & x < 0 \end{matrix}$$

Then

$$T = i\int d^4x\, d^4y \left[\langle p'|T(j^*(x)\, j(y))|p\rangle - \delta(x_0-y_0)\langle p'|j^*(x)\frac{\partial\phi(y)}{\partial y_0}|p\rangle\right]$$

(13)

provided $j(x)$ does not depend on $\frac{\partial}{\partial x_0}\phi(x)$. In case $j(x)$ is independent of $\phi(x)$ also, the last term also vanishes. Then

$$T = i\int d^4x\, d^4y\, \langle p'|\theta(x-y)j^*(x)j(y) + \theta(y-x)j(y)j^*(x)|p\rangle e^{i(k'x-ky)}$$

If $|n\rangle$ denote a complete set of orthonormal states the projection operator

$$\sum_n |n\rangle\langle n| = 1$$

may be introduced between the two j operators on the right side. Recalling the properties of the displacement operators P_μ

$$j(x) = e^{-iP_x} j(0)\, e^{+iP_x}$$

we obtain

$$T = \sum_n i\int \theta(x-y)\langle p'|j^*(x)|n\rangle\langle n|j(y)|p\rangle e^{i(k'x-ky)} d^4x\, d^4y$$
$$+ \sum_n i\int \theta(y-x)\langle p'|j(y)|n\rangle\langle n|j^*(x)|p\rangle e^{i(k'x-ky)} d^4x\, d^4y$$

(14)

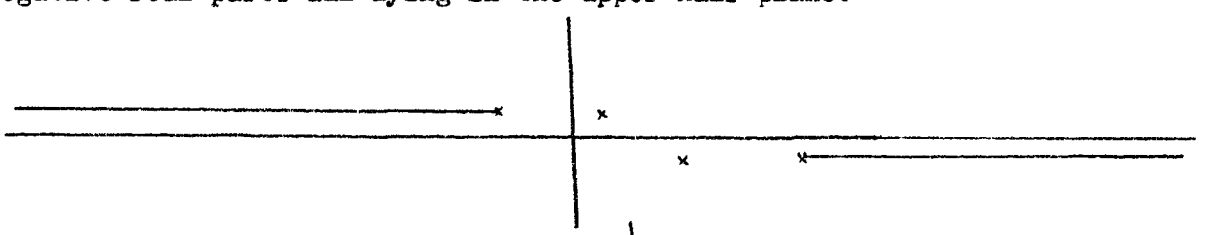
$$= \sum_n i\int \theta(x-y)\langle p'|j^*(0)|n\rangle\langle n|j(0)|p\rangle e^{-iky}\cdot e^{ik'x}\cdot e^{-i(p-n)y}\cdot e^{i(p'-n)x}\, d^4x\, d^4y$$
$$+ \sum_n i\int \theta(y-x)\langle p'|j(0)|n\rangle\langle n|j^*(0)|p\rangle e^{-iky}\cdot e^{ik'x}\cdot e^{-i(p-n)y}\cdot e^{i(p'-n)x}\, d^4x\, d^4y$$

The requisite integrations over x and y may be performed leading to the following expression for the T matrix element:

$$(15) \qquad T = (2\pi)^3 \left\{ \sum_{\substack{n_0 \\ \underline{n} = \underline{p} + \underline{k}}} \frac{\langle p' | j^*(0) | n \rangle \langle n | j(0) | p \rangle}{E_n - p_0 - k_0 - i\varepsilon} + \sum_{\substack{n_0 \\ \underline{n} = \underline{p}' - \underline{k}}} \frac{\langle p' | j(0) | n \rangle \langle n | j^*(0) | p \rangle}{E_n - p_0' + k_0 - i\varepsilon} \right\}$$

where the summation is now restricted to those states $|n\rangle$ which conserve the momentum and $- i\varepsilon$ is an infinitesimal negative quantity which ensures the correct contour for the evaluation of T.

If T is considered as an analytic function of the complex variable k_0 we may represent the singularities of T on the k_0 plane. The singularities fall into 2 groups, those with positive real parts all lying in the lower half plane; and those with negative real parts all lying in the upper half plane.



Singularities of T in the k_0 - plane

Comparing the properties so far discussed of the T matrix element with those of a classical scattering amplitude discussed earlier, we see that T is not a proper scattering amplitude for deriving dispersion relations. This is connected with the requirement, outlined earlier, that there should be no singularities in the upper half plane, for a dispersion relation to exist.

<u>The M-matrix</u>

We may however construct a new matrix element that is more or less similar to T but with the desired properties. To do this, observe that the location of the singularities of T in the k_0 plane are connected with the $- i\varepsilon$ occuring in the denominator; hence we would have a proper scattering amplitude if we alter the sign of ε in the second term in the expression for T. We will call the matrix element so obtained as M

$$(16) \qquad M = 2\pi^3 \left\{ \sum_{\substack{n_0 \\ \underline{n} = \underline{p} + \underline{k}}} \frac{\langle p' | j^*(0) | n \rangle \langle n | j(0) | p \rangle}{E_n - p_0 - k_0 - i\varepsilon} + \sum_{\substack{n_0 \\ \underline{n} = \underline{p}' - \underline{k}}} \frac{\langle p' | j(0) | n \rangle \langle n | j^*(0) | p \rangle}{E_n - p_0' + k_0 + i\varepsilon} \right\}$$

We shall denote the meson and nucleon masses consistently by μ and κ so that, always,

$$p_0 = \sqrt{\underline{p}^2 + \kappa^2} \qquad\qquad k_0 = \sqrt{\underline{k}^2 + \mu^2}$$

From the explicit expression for M it is clear that, for $k_0 > \mu$, $T \equiv M$, since in this range the singularities are unchanged between T and M and no ambiguity results by omitting the infinitesimal imaginary part in the denominator of the second term. For the entire physical range $k_0 > \mu$ and hence, for the entire physical range, T and M coincide.

One can immediately write down an integral representation for M which is

$$(17) \qquad M = i \int d^4x\, d^4y \; \langle p' | [j^*(x), j(y)] | p \rangle e^{i(k'x - ky)}$$

and is equivalent to replacing $\theta(y-x)$ by $\theta(x-y)$ in the expansion of the chronologically ordered product in the integral representation of T. We may hence form the difference T-M:

$$\begin{aligned} T - M &= -i \int d^4x\, d^4y \left\{ \theta(y-x) + \theta(x-y) \right\} \langle p' | j(y) j^*(x) | p \rangle e^{i(k'x - ky)} \\ &= -i \int d^4x\, d^4y \; \langle p' | j(y) j^*(x) | p \rangle e^{i(k'x - ky)} \\ &= -i (2\pi)^4 \sum_{\substack{n_0 = p_0' - k_0 \\ \underline{n} = \underline{p}' - \underline{k}}} \langle p' | j(0) | n \rangle \langle n | j(0) | p \rangle \end{aligned}$$

where we have used the fact that

$$\theta(x-y) + \theta(y-x) = 1 \qquad \text{for} \qquad x_0 \gtrless y_0$$

It is to be noticed that there are no energy denominators in the final summation, since the difference of two energy denominators with opposite values of ε may be replaced by a δ-function of energy differences and this we have taken care of by the restriction on the summation. For this summation the only contributing terms are those with

$$n = p' - k$$

Then,

$$n^2 = (p' - k)^2 = p'^2 + k^2 - 2p \cdot k$$

and hence

$$(n^2)_{max} = (k - \mu)^2$$

However, $|n\rangle$ are one particle states and hence

$$n^2 > k^2$$

Hence there are no states with positive energy for which both these conditions can be satisfied. Hence, as we asserted earlier (from heuristic arguements)

(18) T-M=0 for positive energies.

<u>The Negative Frequency Behaviour and the Class of the Dispersion Relations</u>

The structure and symmetry properties of M may now be investigated. The most important ones are those connected with the Lorentz group and with the charge conjugation operator. These are, respectively, related to the behaviour under rotations and (space) reflections and to the behaviour of negative energies.

Consider now the case of scattering of charged mesons (of finite mass) by a nucleon (also of finite mass). As usual, charged mesons are represented by complex fields and one would have two distinct matrix elements M^+, M^- for the scattering of positive and negative mesons:

$$M^+ = i(2\pi)^4 \delta(p'+k'-p-k) \int \theta(-y) \langle p' | [j^*(0), j(y)] | p \rangle e^{-iky} d^4y$$

$$M^- = i(2\pi)^4 \delta(p'+k'-p-k) \int \theta(-y) \langle p' | [j(0), j^*(y)] | p \rangle e^{-iky} d^4y$$

here we have used the displacement relation

$$e^{iPx} j^*(0) e^{-iPx} = j^*(x)$$

to perform one of the integrations (with respect to x). On the energy shell, the matrix element M may be written

(19) $$M^+(k; p, p') = i \int \theta(-y) \langle p' | [j^*(0), j(y)] | p \rangle e^{-iky} d^4y$$

since, on the energy shell, we have the connecting relation

$$(p+k-p')^2 = k'^2 \qquad \text{or} \qquad k \cdot (p-p') = p \cdot p' - \kappa^2$$

since $\qquad p^2 = p'^2 = \kappa^2 \qquad\qquad k^2 = k'^2 = \mu^2$

[the Lorentz metric is used in writing $x^2 = x_0^2 - \underline{x}^2$]

enabling us to express k' as a function of k, p, p .

The Matrix element being a Lorentz invariant should be expressible in the form

(20) $$M^{\pm}(k;p,p') = \bar{u}(p')\left(F^{\pm} + i\gamma\cdot k\, G^{\pm}\right)u(p)$$

where $F^{\pm}$ and $G^{\pm}$ are functions of the products $k\cdot p$, $k\cdot p'$ and $p\cdot p'$ of which only two are independent. It is best chosen as a function of $k\cdot(p+p')$ and $p\cdot p'$ so that

(21) $$F^{\pm} = F^{\pm}(k\cdot(p+p'),\ p\cdot p')$$
$$G^{\pm} = G^{\pm}(k\cdot(p+p'),\ p\cdot p')$$

We had

$$M^{+} = -i\int \theta(-y)\langle p'|[j^{*}(y), j(0)]|p\rangle e^{-iky}d^{4}y$$

So that

$$M^{+*} = i\int \theta(-y)\langle p'|[j^{*}(0), j(y)]|p\rangle e^{iky}d^{4}y$$

Hence

$$M^{+*}(k;p,p') = M^{-}(-k;p,p')$$

which, in turn, gives

(22) $$F^{+*}(k\cdot(p+p'),\ p\cdot p') = F^{-}(-k\cdot(p+p'),\ p\cdot p')$$
$$G^{+*}(k\cdot(p+p'),\ p\cdot p') = -G^{-}(-k\cdot(p+p'),\ p\cdot p')$$

If we now introduce the linear combination

(23a) $$F_1 = F^{+} + F^{-} \qquad G_1 = G^{+} + G^{-}$$
$$F_2 = F^{+} - F^{-} \qquad G_2 = G^{+} - G^{-}$$

these satisfy

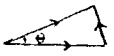
(23b) $$F_1^{*}(k\cdot(p+p'),\ p\cdot p') = +F_1(-k\cdot(p+p'),\ p\cdot p')$$
$$F_2^{*}(k\cdot(p+p'),\ p\cdot p') = -F_2(-k\cdot(p+p'),\ p\cdot p')$$
$$G_1^{*}(k\cdot(p+p'),\ p\cdot p') = -G_1(-k\cdot(p+p'),\ p\cdot p')$$
$$G_2^{*}(k\cdot(p+p'),\ p\cdot p') = +G_2(-k\cdot(p+p'),\ p\cdot p')$$

These amplitudes have hence the transformation properties of the required type for providing us with dispersion relations. These are functions of two variables $x_1 = k\cdot(p+p')$ and $x_2 = p\cdot p' - k^2$. If we keep one of these fixed and the other varied, one can obtain dispersion relations of the type

$$\mathrm{Re}\ F(x_1, x_2) = \frac{\mathcal{P}}{\pi}\int_{-\infty}^{\infty}\frac{dx_1'}{x_1'-x_1}\,\mathrm{Im}\ F(x_1', x_2)$$

In the center of mass frame one has

(24) $$x_2 = p\cdot p' - \kappa^2 = \left(2\, k_c \sin\frac{\theta_c}{2}\right)^2$$

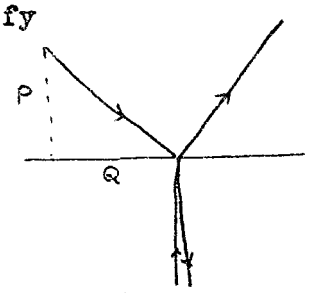
and F, G become functions of the center-of-mass variables k_c, θ_c. However, when varying x_1, keeping x_2 constant, we have to keep $k_c \sin\frac{\theta_c}{2}$ constant and not θ_c. (It may be recalled that $2k_c \sin\frac{\theta_c}{2}$ is the momentum transfer in the scattering and is represented by the third side of the momentum triangle ). Hence there exist no dispersion relations[§*] connecting scattering amplitudes for a fixed value of θ_c.

If we consider the laboratory system

$$\underline{p} = 0 \qquad p\cdot p' = \kappa p_0'$$

so that keeping $x_2 = p\cdot p'$ fixed means keeping the nucleon recoil fixed at the value x_2/κ while $x_1 = k\cdot(p+p') = \kappa k_0 + k\cdot p'$ is varied.

The frame in which the formlae appear simplest is defined by $\underline{p} + \underline{p}' = 0$. In this frame, we may specify



$$\underline{p} = 0, 0, P$$
$$\underline{p}' = 0, 0, -P$$
$$k = Q, 0, -P$$
$$k' = -Q, 0, P$$

The "Brickwall" System

so that

(25) $$p\cdot p' = p_0 p_0' - \underline{p}\cdot\underline{p}' = \kappa^2 + 2P^2$$
$$k_0 = k_0' = \sqrt{Q^2 + P^2 + \mu^2}$$

and $$k\cdot(p+p') = 2\, k_0 \sqrt{P^2 + \kappa^2}$$

The dispersion relation would now be of the form

(26) $$\mathcal{R}e\; F(k_0, P) = \frac{\mathcal{P}}{\pi}\int_{-\infty}^{\infty} \frac{\mathcal{I}m\; F(k_0', P)}{k_0' - k_0}\, dk_0'$$

---

§* Except, of course, for the single case $\theta_c = 0$, where keeping θ_c fixed is the same as keeping $k_c \sin\frac{\theta_c}{2}$ fixed.

We must now distinguish between M^{11} the no spin flip and M^{21} the spin flip amplitudes. In the "brick-wall" system, $\underline{p} + \underline{p}' = 0$, we have

(27a)
$$\left.\begin{aligned} \bar{u}^1 u^1 &= \sqrt{P^2 + k^2}/k = \sqrt{1 + P^2/k^2} \\ \bar{u}^1 i\gamma_5 k u^1 &= k_0 \end{aligned}\right\} \text{ for no spin flip}$$

and

(27b)
$$\left.\begin{aligned} \bar{u}^2 u^1 &= 0 \\ \bar{u}^2 i\gamma_5 k u^1 &= PQ/k \end{aligned}\right\} \text{ for spin flip}$$

Consider

(28) $$M^{11} = \frac{p_0}{k} F(k_0, P) + k_0 G(k_0, P)$$

M^{11} is a function of just 2 variables k_0 and P with the integral representation

$$M^{11} = i\int \theta(-y) \langle p' | [j(0), j(y)] | p \rangle e^{-iky} d^4y$$

We now notice the important fact that the entire k_0 dependence of M^{11} lies in the factor e^{-iky}. The factor $f(y,p) = \theta(-y)\langle p' | [j(0), j(y)] | p \rangle$ in the integral is a function of x and P alone and is such that it vanishes everywhere except in the backward light cone.

It can be shown that the matrix element $M(k_0,P)$ possesses a number of symmetries and as a result can be written as

$$M^{11}(k_0, P) = \int f(x,P) e^{-ik_0x_0} \cos(Qx_1) d^4x$$

where $f(x,P)$ is real and vanishes outside the backward light cone. (A. Salam: Nuovo Cimento: 1 Feb., 1956 ; A. Salam and W. Gilbert: Nuovo Cimento: 1st March, 1956).

Thus the verification that

$$\mathcal{R}e\, M^{11}(k_0, P) = \frac{\mathcal{P}}{\pi} \int_{-\infty}^{\infty} \frac{\mathcal{I}m\, M^{11}(k_0', P)}{k_0' - k_0} dk_0'$$

is really the verification that

$$\int f(x,P) \cos k_0x_0 \cos(Qx_1) d^4x \qquad x = x_0, x_1, x_2, x_3$$

$$= \frac{\mathcal{P}}{\pi} \int\int^{\infty} \frac{\sin k_0'x_0 \cos Q'x_1}{k_0' - k_0} f(x,P) d^4x\, dk_0'$$

If on the right side, the order of x and k_0' integrations can be interchanged, then this is the same as the rather "simple" verification

$$\cos k_0 x_0 \cos Q x_1 = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{\sin k_0' x_0 \cos Q' x_1}{k_0' - k_0} dk_0'$$

It can now be shown (see Goldberger: Phys. Rev. 99, 979, 1955) that this change of order of integration is valid, (if at all) only if f(x,P) is a function vanishing everywhere except in the backward light cone. We shall not repeat his arguments since the proof is very weak at this point and one has to assume that f(x,P) does not possess singularities worse than the δ-function or its derivative on the light cone. We recall that the quantum mechanical causality requirement for a relativistic field theory demands that local operators separated by a space-like interval should commute; and corresponds to the requirement that the measurement process connected with either operator cannot influence the other since no signals can be propagated with a velocity greater than the speed of light. In particular this implies in the present case

$$\left[j^*(x) , j(y) \right] = 0 \qquad \text{for x-y space-like.}$$

This guarantees the vanishing of f(x,P) for space-like x . It is particularly instructive to note that the causality requirement enters only at this stage of the theory; and further, that, the causality is demanded "in the small" and not merely for distances larger than, say, 10^{-13} cms.*

The dispersion relations are

$$(29) \qquad \mathcal{R}e \; F(k_0, P) = \frac{2}{\pi} P \int_0^{\infty} \frac{k_0' \, \mathcal{I}m \, F(k_0', P)}{k_0'^2 - k_0^2} dk_0'$$

$$(30) \qquad \mathcal{R}e \; \frac{G(k_0, P)}{k_0} = \frac{2}{\pi} P \int_0^{\infty} \frac{\mathcal{I}m \, G(k_0', P)}{k_0'^2 - k_0^2} dk_0'$$

---

* It may be shown that if we assume that causality is not valid in very small but finite regions, the dispersion relations for such a case would be mixed:

$$\mathcal{R}e \; F(k_0,P) \cos l_0 k_0 + \mathcal{I}m \; F(k_0,P) \sin l_0 k_0$$
$$= \frac{2}{\pi} P \int_0^{\infty} \frac{\mathcal{I}m \, F(k_0',P) \cos l_0 k_0' + \mathcal{R}e \; F(k_0',P) \sin l_0 k_0'}{k_0'^2 - k_0^2} k_0' dk_0'$$

(Sudarshan; to be published)
Cf. R. Oehme, Phy. Rev. 100, 1503 (1955)

Since F and G are invariant and only the spinors u, $\bar{u}$ change referred to a new system, we may now write, say, for the center of mass system,

(31) $$M^c = \bar{u}^c (F + i\gamma.k^c G) u^c$$

In this system,

$$M^c(11) = \frac{4\pi}{\hbar} \frac{p_0 + k_0}{k_0} \sum \left[l a_{l-} + (l+1) a_{l+} \right] P_l(\cos\theta_c) \qquad l_- \rightarrow J = l - \tfrac{1}{2}$$

$$M^c(21) = \frac{4\pi}{\hbar} \frac{p_0 + k_0}{k_0} \sum \left[a_{l-} - a_{l+} \right] \frac{d}{d(\cos\theta)} P_l(\cos\theta_c) \qquad l_+ \rightarrow J = l + \tfrac{1}{2}$$

The theory developed so far can be carried through to give the dispersion relations for any suitable process, say the scattering of charged pseudoscalar mesons by neutrons and protons. However for simplicity we shall consider a case which illustrates all the relevant points of the theory, namely the scattering of neutral <u>scalar</u> mesons by <u>scalar</u> nucleons. Then we have only one function F and one dispersion relation

(32) $$\mathcal{R}e \quad F(k_0, P) = \frac{2}{\pi} \mathcal{P} \int_0^\infty \frac{k_0' \, \mathcal{I}m \, F(k_0', P)}{k_0'^2 - k_0^2} \, dk_0$$

To write the relations in the center of mass frame, we may substitute the expression

(33) $$k_0 = \frac{\sqrt{k_c^2 + \mu^2}\,\sqrt{k_c^2 + k^2} + k_c^2 - P^2}{\sqrt{P^2 + k^2}} = g(k_c, P)$$

for k_0 and obtain,

(34) $$\mathcal{R}e \quad F(k_c, P) = \frac{2}{\pi} \mathcal{P} \int_0^\infty \frac{\mathcal{I}m \; F(g', P) \, g' dg'}{g'^2 - g^2}$$

where $g = g(k_c, P)$, $g' = g(k_c', P)$ and

$F(k_c, P)$ is to be taken to mean $F\left(g(k_c, P), P\right)$. Since $k_0 = \sqrt{P^2 + Q^2 + \mu^2}$,

$P < k_0 < \infty$ and $\sqrt{P^2 + \mu^2} < k_c < \infty$

Hence of the range of integration 0 to 00, the region 0 to $\sqrt{P^2 + \mu^2}$ is unphysical, ie. the threshold of the process is at the energy $\sqrt{P^2 + \mu^2}$

For the case under consideration (neutral mesons) the imaginary part comes from the infinitesmials $i\varepsilon$ in the denominator.* Usin the relation

$$\frac{1}{x \pm i\varepsilon} = \mathcal{P} \frac{1}{x} \pm i\pi \, \delta(x)$$

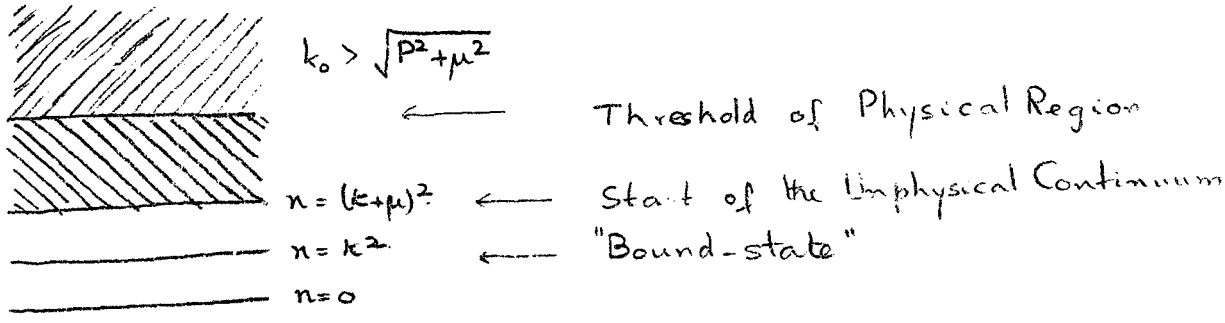
---

* This assumes that the numerator is real. Cf. Salam and Salam & Gilbert quoted earlier.

we obtain,

$$(35) \qquad \mathrm{Im}\, F = \sum_{n=p+k} \langle p' | j(0) | n \rangle \langle n | j(0) | n \rangle - \sum_{n=p'-k} \langle p' | j(0) | n \rangle \langle n | j(0) | p \rangle$$

The states $|n\rangle$ contributing to the sum may be represented in the following manner:



We have to supply the $\mathrm{Im}\, F$ for the unphysical region ourselves.

The various contributions may be separated in the fashion

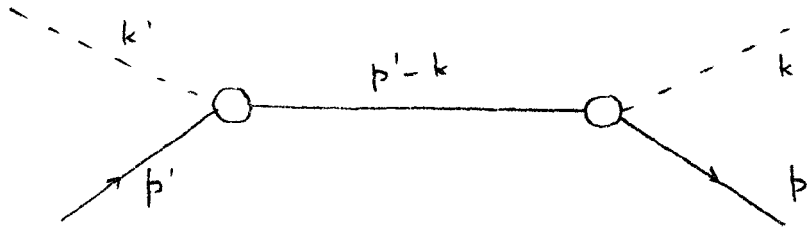
$$\mathrm{Im}\; F(k_0, P) = \int_P^\infty (\text{contr. from physical states}) + \int_0^P (\text{unphysical cont.}) + H(k_0, P)$$

where the last term is the contribution from the single state $|n\rangle$ with $n^2 = k^2$ which is usually called (somewhat inappropriately) the "bound state"

The appropriate term for $n^2=k^2$ is

$$-\sum_{n^2=(p'-k)^2=k^2} \langle p' | j(0) | n \rangle \langle n | j(0) | p \rangle$$

Graphically,



where the intermediate nucleon line represents a real nucleon. Taking radiative corrections to all orders and lumping these into a vertex modification, we may write an expression for the bound state contribution. For scalar nucleons this gives a contribution proportional to

$$G^2 \Gamma_1 \; \delta\left((p'-k)^2 - k^2\right) \Gamma_2$$

where Γ is the vertex modification:

$$\Gamma_1 = \Gamma\left(p^2, k'^2, (p'-k)^2\right) = f(\kappa^2, \mu^2, \kappa^2)$$

$$\Gamma_2 = \Gamma\left((p'-k)^2, k^2, p'^2\right) = f(\kappa^2, \mu^2, \kappa^2)$$

Thus the total contribution is

$$G^2 f^2(\kappa^2, \mu^2, \kappa^2)$$

where G is the unrenormalized coupling constant. We can now define

$$G_R = G f(\kappa^2, \mu^2, \kappa^2) \tag{36}$$

and then the total contribution is simply the Born approximation result, provided the renormalized coupling constant (*) is used.

In the full pseudoscalar theory, this gives the contributions:

$$\mathrm{Im}\, M^{11} = -\frac{G_R^2}{4\pi} 4\pi^2\, \bar{u}^1(p')\, i\gamma_5 k\, u^1(p)\, \delta\left((p'-k)^2-\kappa^2\right) \tag{37a}$$

$$\mathrm{Im}\, M^{21} = -\frac{G_R^2}{4\pi} 4\pi^2\, \bar{u}^2(p')\, i\gamma_5 k\, u^1(p)\, \delta\left((p'-k)^2-\kappa^2\right) \tag{37b}$$

So much for the "bound state". Now, the contribution from the second unphysical region is much harder. One suggestion is to differentiate the relations obtained previously with respect to P^2 and put $P = 0$. In this manner we now obtain an infinite number of integral equations. These are not merely identities. The structure of the theory (e.g. that it is local, relativistic and that the meson has odd parity) comes into what we called the bound state term. Also meson-meson attraction terms of the type $\lambda\phi^4$ were neglected. If they had been kept in, an additional term $\lambda f(P^2)$ would have appeared in the two relations (eq. 29,30) on the right hand side.

To illustrate the results obtainable by differentiation with respect to P^2, let us start with the original relation

$$\begin{aligned} &\frac{p_0^c + k_0^c}{k^c} \sum (2\ell+1) \sin\delta_\ell \cos\delta_\ell\, P_\ell\left(1-\frac{P^2}{2k_c^2}\right) \\ &= \frac{2}{\pi} \mathcal{P}\int_0^\infty \frac{\sum (2\ell+1)\sin^2\delta'_\ell}{g^2(k'_c, P) - g^2(k_c, P)} \frac{p_0'^c + k_0'^c}{k'_c} P_\ell\left(1-\frac{P^2}{2k_c'^2}\right) dk'_c \\ &- G^2 \cdot \frac{1}{2p_0} \frac{1}{g^2\left(\frac{\mu^2+2P^2}{2p_0}, P\right) - g^2(k_c, P)} \end{aligned} \tag{38}$$

(*) G_R so defined, differs by terms of order $(\mu/\kappa)^2$ from the usual definition where $G_{RN} = G f(\kappa^2, 0, \kappa^2)$

If we now differentiate this equation with respect to P^2 and then put $P^2 = 0$, we obtain

$$-\left(\frac{p_0^c + k_0^c}{2k_c^3}\right) \sum (2\ell+1) \sin\delta_\ell \cos\delta_\ell$$

$$(39a) \quad = \frac{2}{\pi} P \sum (2\ell+1) \int_0^\infty \sin^2\delta'_\ell \, \frac{p_0'^c + k_0'^c}{k'^c} \, dk_c' \cdot \frac{\left(1 + \frac{\ell(\ell+1)k^2}{4k_c'^2}\right)}{A_+(k_c')\, A_-(k_c')}$$

$$+ G^2 \left\{ \frac{k_0^*(p_0^* + k_0^*) - k_0^c(p_0^c + k_0^c)}{4k^4} + \frac{\mu^2}{4k^5}(k_0^* + p_0^*)^2 \left(1 + \frac{k^{*2}}{p_0^* k_0^*}\right) \right\}$$

where we have introduced the symbols

$$(39b) \quad \delta_\ell = \delta_\ell(k_c) \qquad \delta'_\ell = \delta_\ell(k_c')$$

$$k^* = \mu^2/2k \qquad k_0^* = \sqrt{\mu^2 + k^{*2}} \qquad p_0^* = \sqrt{k^2 + k^{*2}}$$

$$A_\pm(k_c') = (k_0^{c\,2} \pm k_0'^{c\,2}) + (p_0^c k_0^c \pm p_0'^c k_0'^c)$$

One may proceed in a similar manner to obtain further relations by repeated differentiations. The calculations are straightforward but tedious.

We have thus obtained an infinite series of integral equations which are fully relativistic, with due regard being paid to all the symmetries of the theory. No approximations have so far been made. This infinite system of equations can be made to serve as the basis of a new approximation method. For example, neglecting all except a finite number of phase shifts would correspond to a cutoff, which however is applied in a relativistically invariant fashion. In particular, by neglecting all terms except the p-wave phase shifts and takes $k \longrightarrow \infty$, one obtains the Low integral equations (The Chew-Low equations are not obtained this way, since their integral equation is for the inverted function and <u>not</u> the scattering amplitude.

One remarkable fact to be noticed is that the only place where details of the coupling come in explicitly is in evaluating the bound state contribution. In particular both the PS and the PV theories give the same bound state contributions and hence dispersion relations do not distinguish between them. Worth noticing is the fact that the PV divergences have disappeared in the dispersion relation.

To obtain scattering lengths associated with various ℓ-values is quite straightforward now. Notice that eqn. (38) starts with $\ell = 0$ on the lefthand side while eqn. (39) starts with $\ell = 1$ [illegible]

know all phase shifts on the righthand side. Consider meson-nucleon scattering where we know δ_{33} is the largest phase shift. In Cambridge Gilbert has tried to compute the zero- energy limits of $a_1, a_3, a_{11}, a_{13}, a_{31}$ using the generalized dispersion relations. The meson-meson attraction term $\lambda\phi^4$ was retained, so that the dispersion relation for no spin-flip, no isotopic spin-flip amplitudes, contain an additional term $\lambda f(p\cdot p')$ on the right hand side; (p, p' are the initial and the final nucleon 4-momenta). The crudest approximation of retaining only δ_{33} phases under the integral sign (on the right-hand side) was made, δ_{33} itself being taken to represent the experimental curve up to $\sim$ 400 Mev. The following are the results: -

(i) g^2, the pseudoscalar coupling constant is large: ~ 12.

(ii) $\frac{1}{3}(a_1 + 2a_3) \sim \left(\frac{K}{K+\mu}\right)\left[\frac{\lambda}{4\pi} f(K^2)\right] + 0.42$ Here K and μ are nucleon and meson masses.

(iii) $\frac{1}{3}(a_1 - a_3) \sim 0.3$

(iv) In the p phase-shifts an additional constant $\left[\frac{1}{f}\frac{df}{d(p\cdot p')}\right]_{p\cdot p' = K^2}$ appears due to the $\lambda\phi^4$ meson-meson attraction term. This constant could be computed, using Born Approximation. For a small value of this constant $\approx \frac{1}{K^2}$

$$a_{11} = -0.05 : a_{13} = -0.06 : a_{31} = -0.03$$

For a slightly larger value $\sim \frac{1}{K\mu}$

$$a_{11} = -0.03 : a_{13} = 0.04 : a_{31} = -0.02$$

The meson-meson attraction term, if present and large, makes life hard as far as obtaining useful information from these relations is concerned.

In closing we may recapitulate the significant points: firstly, all the symmetries of the theory come into the development naturally. The two equally

important points to notice are that the theory is essentially local and that "causality in the small" is demanded (microscopic causality). In connection with the latter, there is no fundamental length playing any significant part in the theory. The evaluation of the bound state contribution fixes a scale factor for the theory.

An essential point of generality of the theory lies in that the assumed commutation rules for the 'in' 'out' field operators do not necessarily demand canonical commutation rules for the (non-asymptotic) field operators. This is particularly of interest in view of the conjectured inconsistency (of "ghost" states etc.) of field theories starting from canonical commutation rules.

We have not yet learned how best to combine the equations, approximate to them and set up consistency relations to obtain detailed information. This requires deeper study. Causality and dispersion relations are probably destined to dominate on theories for a few years yet to come.

Appendix I Oehme: P. R. 100, 1505-6

$$N(\omega+i\nu) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{N(\omega')\,d\omega'}{\omega' - (\omega+i\nu)} \qquad \text{for } \nu > 0$$

and $N(\omega+i\nu) \longrightarrow N(\omega)$ for $\nu \to 0$

$\Longleftrightarrow$ (a) N ($\omega + i\nu$) is analytic for $\nu > 0$

(b) The lesbesque integral

$$\int_{-\infty}^{\infty} d\omega \; |N(\omega+i\nu)|^2 \qquad \text{exists and is bounded for } \nu > 0$$

(c) $\lim_{\nu \to 0} N(\omega+i\nu) \longrightarrow N(\omega)$ for $\omega \geq \mu$

Appendix II

S- Matrix in terms of Heisenberg Operators

In a program for deriving dispersion relations for a relativistic theory it is first of all necessary to write down the proper scattering amplitudes with suitable symmetry properties for this purpose we express the S-matrix in terms of Heisenberg operators. We adopt the formulation of field theory due to Lehman, Symanzik and Zimmerman and start with the following definitions and auxilary lemmas:

Take any function $f(x)$ which satisfies the Klein-Gordon equation

$$K_x f(x) = 0 \qquad K_x \equiv \Box_x^2 - m^2$$

which is normalized so that

$$-i \int \left(f(x) \frac{\partial f^*(x)}{\partial x_0} - f^*(x) \frac{\partial f(x)}{\partial x} \right) d^3x = 1$$

For <u>any</u> operator $A(x)$ we may construct the operator

$$A^f(t) = i \int_{x_0 = t} \left[A(x) \frac{\partial f(x)}{\partial x_0} - f(x) \frac{\partial A(x)}{\partial x_0} \right] d^3x$$

the space integration being carried out over the space-like surface $x_0 = t$. If in particular the operator $A(x)$ satisfies a free field equation

$$K_x A(x) = 0$$

then the operator $A^f(t)$ satisfies the relation

$$\frac{\partial A^f(t)}{\partial t} = 0$$

and hence is independent of the space-like surface $x_0 = t$. This result may be immediately verified: by direct substitution we have,

$$\frac{\partial A^f(t)}{\partial t} = i \int \left(A(x) \nabla_x^2 f(x) - f(x) \nabla_x^2 A(x) \right) d^3x$$

$$= i \int \left(A(x) \nabla_x f(x) - f(x) \nabla_x A(x) \right) dS = 0$$

where in the second step we have used Green's theorem to transformation from volume to surface integrals and the last step follows from dimensional considerations. [If L denotes the linear dimensions of the volume considered; the integral goes as $L^2 / (L^{3/2} L^{3/2})$ and hence vanishes in the limit]. Hence for a free-field operator $A(x)$, A^f is a constant of the motion.

We now introduce 2 sets of operators, labelled respectively 'in' and 'out' field operators, which satisfy the free-field equations

$$K_x \; A_{in}(x) = 0$$

$$K_x \; A_{out}(x) = 0$$

and the commutation rules

$$[A_{in}(x), A_{in}(x')] = i\,\Delta(x-x')$$

$$[A_{out}(x), A_{out}(x')] = i\,\Delta(x-x')$$

These abstract operators are endowed with physical content by demanding that for any two physical states represented by Φ, Ψ and any operator $A(t)$

$$\lim_{\tau \to -\infty} (\Phi, A^f(\tau)\Psi) = (\Phi, A^f_{in}(t)\Psi)$$

$$\lim_{\tau \to \infty} (\Phi, A^f(\tau)\Psi) = (\Phi, A^f_{out}(t)\Psi)$$

It is to be noted that the righthand side is independent of t since by virtue of the free-field equations obeyed by the 'in' and 'out' fields, the righthand sides are independent of t.

These define the formalism of the field theory under examination and constitute strong requirements on the structure of the theory. As an example it is to be noticed that this requirement does not make much sense as applied to cases when one of the two states, say Ψ is a scattering state and the other a bound state: for

$$(\Phi_B, A^f_{out}(t)\Psi_S) = \lim_{t \to \infty} (\Phi_B, A^f(x)\Psi_S) = 0$$

since $A(x)\Psi_S$ is also a scattering state and is thus orthogonal to Φ_B.

By a direct extension to derivatives we may write down corresponding results for the limits $t = \pm\infty$

We may now define the 'in' and 'out' vacuum states by the defining equations

$$A^+_{in}\,\Omega_{in} = 0 \qquad\qquad A^+_{out}\,\Omega_{out} = 0$$

where, as usual, we have split the free-field operators A_{in}, A_{out} into positive and negative frequency components A^+_{in}, A^+_{out}. One can now from the following two sets of orthonormal states,

$$\Omega_{in}$$

$$\Phi^{\alpha}_{in} = A^{\alpha}_{in}\,\Omega_{in}$$

$$\Phi^{\alpha_1 \ldots \alpha_N}_{in} = \frac{1}{\sqrt{n!\,n_1! \ldots n_N!}} A^{\alpha_1}_{in} \cdots A^{\alpha_N}_{in}\,\Omega_{in}$$

and

$$\Omega_{out}$$

$$\Phi^{\alpha}_{out} = A^{\alpha}_{out}\,\Omega_{out}$$

$$\Phi^{\alpha_1 \ldots \alpha_N}_{out} = \frac{1}{\sqrt{n_1! \ldots n_N!}}\, A^{\alpha_1}_{out} \cdots A^{\alpha_N}_{out}\,\Omega_{out}$$

where $n_1, \ldots, n_N$ are the occupation numbers and the expression involving the factorials is the usual normalization factor.

The S-matrix may be written

$$S_{(\alpha)(\beta)} = \left(\Phi^{(\alpha)}_{out}, \Phi^{(\beta)}_{in} \right)$$

where $(\alpha), (\beta)$ stand for the set of indices completely defining the states. The S-matrix so defined is unitary and has the property that

$$A_{out} = S^{+} A_{in} S$$

The invariance of the formulated theory under space-time translation together with the asymptotic properties of the field operator lead to the existence of a 4-vector operator P_{μ}, which is the generator of infinitesimal translations, with the properties:

$$-i\left[P_{\mu}, A(x)\right] = \frac{\partial A(x)}{\partial x_{\mu}}$$

$$-i\left[P_{\mu}, A_{in}(x)\right] = \frac{\partial A_{in}(x)}{\partial x_{\mu}}$$

$$-i\left[P_{\mu}, A_{out}(x)\right] = \frac{\partial A_{out}(x)}{\partial x_{\mu}}$$

The vacuum states Ω_{in} and Ω_{out} are steady states and by choosing an arbitrary phase factor we may write

$$\Omega_{in} = \Omega_{out} = \Omega$$

The steadiness of the 1 particle states lead to the relations

$$\left(\Omega, A(x)\,\Phi^{\alpha}_{in}\right) = \left(\Omega, A_{in}(x)\,\Phi^{\alpha}_{in}\right) = f_{\alpha}(x)$$

$$\left(\Omega, A(x)\,\Phi^{\alpha}_{out}\right) = \left(\Omega, A_{out}(x)\,\Phi^{\alpha}_{out}\right) = f_{\alpha}(x)$$

These results are equivalent to the Low-Gell-Mann results on adiabatic switching on of the interaction like

$$|\text{True Vacuum}\rangle = S(0,-\infty)\,|\text{Bare Vacuum}\rangle$$

$$|\text{True 1-part state}\rangle = S(0,-\infty)\,|\text{Bare 1 part. state}\rangle \qquad \text{etc.}$$

The general theorem on the expression for the S-matrix in terms of Heisenberg

operators can now be stated by the equation:

$$\left(\Omega\ ,\ T(x_1 \ldots x_n)\,\Phi^{\alpha}_{in}\right) = -i\int K_y\left(\Omega,\ T(x_1,\ldots,x_n)\,\Omega\right) f_\alpha(y)\,d^4y$$

where $T(x_1 \ldots x_n)$ stands for the chronologically ordered Wick product

$$T(x_1 \ldots x_n) = T\left\{A(x_1) \ldots\ldots A(x_n)\right\}$$

and K_y is the Klein-Gordon operator defined earlier.

To prove this result, define

$$f\,\overleftarrow{\frac{\partial}{\partial x_0}}\,g = f\frac{\partial g}{\partial x_0} - g\frac{\partial f}{\partial x_0}$$

Then form the basic postualte of the theory, one has

$$\begin{aligned}
\left(\Omega\ ,\ T(x_1 \ldots x_n)\,A^{\alpha}_{in}\,\Omega\right) &= \lim_{y\to-\infty}\left(\Omega,\ T(x_1 \ldots x_n)\,A^{\alpha}(y_0)\,\Omega\right)\\
&= i\lim_{y_0\to-\infty}\int\left(\Omega,\ T(x_1 \ldots x_n, y)\frac{\partial}{\partial y_0} f_\alpha(y)\,d^4y\right.\\
&= -i\int\frac{\partial}{\partial y_0}\left(\Omega,\ T(x_1 \ldots x_n, y_0)\,\overleftarrow{\frac{\partial}{\partial y_0}}\,f_\alpha(y_0)\,\Omega\right)d^4y\\
&= -i\int K_y\left(\Omega,\ T(x_1 \ldots x_n, y)\,\Omega\right) f_\alpha(y)\,d^4y
\end{aligned}$$

In this derivation we have used the postulated asymptotic connection between $A^{\alpha}(x)$ and $A^{\alpha}_{in}(x)$. In the last but one step we have dropped a term at $y_0 = \infty$ (coming from the y_0 integration)

$$\begin{aligned}
&i\lim_{y_0\to\infty}\int\left(\Omega,\ T(x_1 \ldots x_n, y)\,\Omega\right)\overleftarrow{\frac{\partial}{\partial y_0}}\,f_\alpha(y)\,d^3y\\
&\qquad = \left(\Omega,\ A^{\alpha}_{out}\,T(x_1 \ldots x_n, y)\,\Omega\right)
\end{aligned}$$

which vanishes by virtue of the defining property of

$$\left(A^{\alpha}_{out}\right)^{*}\Omega = 0$$

The theorem stated and proved just now can easily be extended to states containing two or more particles. The generalization for the 2-particle case, for example, would read

$$\left(\Omega,\ T(x_1 \ldots x_n)\,\Phi^{\alpha_1,\alpha_2}_{in}\right) = (-i)^2\int K_{y_1}K_{y_2}\left(\Omega,\ T(x_1 \ldots x_n, y_1, y_2)\,\Omega\right) f_{\alpha_1}(y_1)\,f_{\alpha_2}(y_2)\,d^3y_1\,d^3y_2$$

We could as well have replaced the Ω on the left side by, say, a Φ_{out} without impairing the validity of the result.