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Mathematical Deductions From Some Rules Concerning High Energy Total Cross Sections*

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1. In 1956 Pomeranchuk[1] and Okun and Pomeranchuk[2] discussed a very interesting rule, which we shall call the Pomeranchuk-Okun rule. It states that at very high energies, in the forward direction, the exchange scattering amplitudes (such as charge exchange, strangness exchange, etc.) are infinitesimal compared with the elastic amplitude. Two years later Pomeranchuk[3] discussed another very interesting rule, which we shall call the Pomeranchuk rule. It states that the total cross section in an A+B collision is the same as that of $\bar{A}$+B at very high energies. Both rules seem to have increasing experimental support in recent years.

In this note we want to discuss the mathematical consequences of each of the two rules, when one takes into consideration the isotopic spin symmetry of the strong interactions. The results are valid only in so far as the effects of the electromagnetic and weak interactions

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are negligible. Higher symmetries are touched upon briefly in Sec.5.

2. We shall use slightly generalized versions of the two rules: Each of the two colliding particles is supposed to be a coherent mixture of the various components of an isotopic spin multiplet, (e.g. 70 percent proton plus 30 percent neutron). The same reasons for believing the original rules support this generalized version of the rules.

3. Theorem 1: The Pomeranchuk-Okun rule implies that at ∞ energy the total cross section in a collision A+B is independent of the total isotopic spin. I.e.

$$\sigma_t(A+B) = \sigma_t(A'+B')$$

if A and A', B and B' belong to the same isotopic multiplet. Proof: Let $a = 1,2,\ldots$, $b = 1,2,\ldots$ denote the m-quantum numbers of the particles A and B respectively. Let

$$< a',b'|\alpha|a,b > \qquad (1)$$

denote the forward scattering amplitude (times $4\pi\lambda$) from A+B to A'+B'. Then the Pomeranchuk-Okun rules states that

$$< a',b'|\alpha|ab > = \text{diagonal.}$$

The rule states in fact that (1) still holds if one changes the repre-

sentations of the particles A by a unitary transformation U and that of B by a unitary transformation V:

$$< a',b'|UV\alpha V^{-1}U^{-1}|ab > \; = \; \text{diagonal} \; . \tag{2}$$

It follows easily from (2) that α is a multiple of the unit matrix. Taking its imaginary part one obtains the theorem.

4. In this section we shall discuss the consequences of the Pomeranchuk rule irrespective of the validity of the Pomeranchuk-Okun rule.

<u>Theorem 2</u>: The Pomeranchuk rule is equivalent to the rule that at ∞ energy, $\sigma_J = \sum_{J'} M'_{JJ'}\bar{\sigma}_{J'}$ where $M'_{JJ'} = (-1)^{2A+2B}(2J'+1)\left\{ \begin{matrix} ABJ \\ ABJ' \end{matrix} \right\}$, A and B stand for the total isotopic spins of particles A and B (as well as for the particles themselves), the $\{\;\}$ for the 6J symbol,[4] and σ_J, $\bar{\sigma}_J$ respectively for the total cross sections with total isotopic spin J in the collisions A+B and A+$\bar{B}$. <u>Proof</u>: (a) Take the forward scattering amplitude matrix α of (1) and let

$$\beta \; = \; -\tfrac{1}{2}i(\alpha-\alpha^{+}) \; . \tag{3}$$

β is thus the imaginary part of the forward scattering amplitude. Its diagonal elements are the total cross sections. We define the <u>corresponding</u> matrices $\bar{\alpha}$ and $\bar{\beta}$ for A+$\bar{B}$ collisions. The Pomeranchuk rule states that

$$\text{diagonal elements of } (\beta-\bar{\beta}) \; = \; 0 \; . \tag{4}$$

Under a unitary transformation U on the states of A and a unitary transformation V on the states of B, the matrices β and $\bar{\beta}$ transform by:

$$\beta \to UV\beta V^{-1}U^{-1}, \quad \bar{\beta} \to U\, V^{*}\bar{\beta}V^{*-1}U^{-1} \quad .$$

Thus the generalized Pomeranchuk rule states that

$$\text{diagonal element of } (UV\beta V^{-1}U^{-1} - UV^{*}\bar{\beta}V^{*-1}U^{-1}) = 0 \ . \tag{5}$$

Since U is an arbitrary unitary matrix operating on the states of A, (5) is equivalent to

$$< ab|V\beta V^{-1} - V^{*}\bar{\beta}V^{*-1}|a'b > = 0 \ . \tag{6}$$

Since V is unitary, (6) is equivalent to

$$< ab|V\beta V^{-1}|a'b > = < a'b|V\bar{\bar{\beta}}V^{-1}|ab > \tag{7}$$

Since U is an arbitrary unitary matrix operating on the states of b (7) is equivalent to

$$< ab|\beta|a'b' > = < a'b|\bar{\bar{\beta}}|ab' > \ .$$

Or to

$$< ab|\beta|a'b' > = < ab'|\bar{\beta}|a'b > \quad . \tag{8}$$

(b) We can express β in terms of σ_J and Clebsch Gordon coefficients

$$< ab|\beta|a'b' > = \sum_{Jm} < ab|Jm > \sigma_J < Jm|a'b' > . \qquad (9)$$

We can also express $\bar{\beta}$ in terms of $\bar{\sigma}_J$, Clebsch Gordon coefficients and the symbols[4] $\binom{J}{mm'}$:

$$< ab'|\bar{\beta}|a'b > = \sum_{\substack{b''b''' \\ J'm'}} \binom{B}{b'b''}\binom{B}{bb'''} < ab''|J'M' > \bar{\sigma}_{J'} < J'm'|a'b''' > . \qquad (10)$$

The occurrence of the symbols $\binom{J}{mm'}$ is due to the definition of $\bar{\beta}$ in which the rows and columns are labelled to correspond to the rows and columns of β.

(c) Substituting (9) and (10) into (8) one obtains after some straightforward algebra the relationship between σ_J and $\overline{\sigma_J}$, given by theorem 2. The use of 6J symbols for crossing relations was made by Wick.[5]

We now discuss in detail the consequences of theorem 2 for the following three cases:

(A) A and $\bar{A}$ do not belong to the same isotopic spin multiplet, and B and $\bar{B}$ do not belong to the same isotopic spin multiplet. In this case we have four sets of σ_J's: for A+B, A+$\bar{B}$, $\bar{A}$+B and $\bar{A}$+$\bar{B}$. Be repeated application of theorem 2 one obtains

$$\sigma_J(A+B) = \sigma_J(\bar{A}+\bar{B}), \quad \sigma_J(A+\bar{B}) = \sigma_J(\bar{A}+B).$$

We have made use of the fact that the square matrix M' multiplied by itself is unity, as can be easily verified from the properties of the 6J symbols. Furthermore, $\sigma_J(A+B)$ and $\sigma_J(A+\bar{B})$ are related by theorem 2. There are thus altogether 2I+1 linearly independent total cross sections

where I is the smaller of the total isotopic spins of A and B.

(B) A and $\bar{A}$ do not belong to the same isotopic spin multiplet, B and $\bar{B}$ do. In this case, by definition,

$$\sigma_J(A+B) = \sigma_J(A+\bar{B}) \ .$$

Thus theorem 2 shows that $\sigma_J(=\overline{\sigma_J})$ satisfies a set of linear homogeneous equations. These equations have been completely and explicitly solved. 6 In particular the number of linearly independent total cross sections is

$$\begin{array}{ll} B+1 & \text{for } A \geq B, \\ A+\frac{1}{2} & \text{for } A < B, \quad 2A = \text{odd}, \\ A+1 & \text{for } A < B, \quad 2A = \text{even}. \end{array}$$

It is obvious by repeated application of theorem 2 that

$$\sigma_J(A+B) = \sigma_J(\bar{A}+B).$$

(c) A and $\bar{A}$ belong to the same isotopic spin multiplet. B and $\bar{B}$ belong to the same isotopic spin multiplet. In this case, the same conclusions as in case (B) obtain.

5. If one takes both the Pomeranchuk and the Pomeranchuk-Okun rule, one obtains, of course, trivially that all σ_J are the same for all J and for A+B, $\bar{A}$+B, etc.

6. The above considerations can clearly be made also for any compact group of symmetries of the strong interactions. To obtain a general solution, however, one must first have a proper defintion of the 6J symbols, and one must know certain symmetry and summation properties of the 6J symbols. This is an unsolved problem.[7]

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