LONG–RANGE INTERAC
OF SMALL COLOR DIP

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We study the scattering of small color dipoles (e.g.,
at low energies. We find that even though the coupli
gluon field can be described in perturbation theory, at
tion becomes totally non–perturbative. The structure o
however, is fixed by the (broken) chiral and scale sym
ing long–distance contribution arises from the correlat
use the spectral representation technique to evaluate b
perturbative contributions to the scattering amplitude.
rule which relates the overall strength of the non–pertu
color dipoles to the energy density of QCD vacuum.

1 Introduction

In a 1972 article entitled "Zero pion mass limit i
energies" [1], A.A. Anselm and V.N. Gribov posed an
is the total cross section of hadron scattering in t
On one hand, as everyone believes since the pioneer
range of strong interactions is determined by the n
i.e. is proportional to $\sim m_\pi^{-1}$. The total cross secti
to scale as $\sim m_\pi^{-2}$, and would tend to infinity as m
soft–pion theorems, which proved to be very use
energy hadronic phenomena, state that hadronic
singularities in the limit $m_\pi \to 0$, and one expects t
self-consistent in the limit of the vanishing pion ma

At first glance, the advent of QCD has not m
ier; on the contrary, the presence of massless gluon
introduces another long–range interaction. In this
this problem considering the scattering of small col

choosing this, somewhat specific, example is simple: asymptotic freedom dictates that strong coupling becomes weak at short distances, and since the small size of dipoles r introduces a natural infrared cut-off $r \ll \Lambda_{\rm QCD}^{-1}$, one expects that their interactions can be systematically treated in QCD. This approach was pursued both at low[2,3,4,5,6] and high[7] energies.

It has been found, however, that at sufficiently high energies the perturbative description of "onium–onium" scattering breaks down[8,9]. The physical reason for this is easy to understand: the higher the energy, the larger impact parameters contribute to the scattering, and at large transverse distances the perturbation theory inevitably fails, since the virtualities of partons in the ladder diffuse to small values ("Gribov diffusion"). At this point, the following questions arise: Does this mean that the problem becomes untreatable? Does the same difficulty appear at large distances in low–energy scattering? And, finally, what (if any) is the role played by pions?

2 Perturbation Theory

The Wilson operator product expansion[10] allows one to write down the scattering amplitude (in the Born approximation) of two small color dipoles in the following form[3]:

$$V(R) \;=\; -i\int dt\langle 0|\mathrm{T}\left(\sum_i c_i O_i(0)\right)\left(\sum_j c_j O_j(x)\right)|0\rangle, \qquad (1)$$

where $x=(t,R)$, $O_i(x)$ is the set of local gauge-invariant operators expressible in terms of gluon fields, and c_i are the coefficients which reflect the structure of the color dipole. At small (compared to the binding energy of the dipole) energies, the leading operator in (1) is the square of the chromo-electric field $(1/2)g^2\mathbf{E}^2$[2,3] — other twist–two operators contain covariant derivatives leading to the powers of momentum in the amplitude and are therefore suppressed at small energies.

Keeping only this leading operator, we can rewrite (1) in a simple form

$$V(R) \;=\; -i\Big(\bar{d}_2\frac{a_0^2}{\epsilon_0}\Big)^2\int dt\langle 0|\mathrm{T}\,\frac{1}{2}g^2\mathbf{E}^2(0)\,\frac{1}{2}g^2\mathbf{E}^2(t,R)|0\rangle, \qquad (2)$$

where $\bar{d}_2$ is the corresponding Wilson coeffi

$$\bar{d}_2\frac{a_0^2}{\epsilon_0} = \frac{1}{3N}\langle\phi|r^i\overline{H}$$

where we have explicitly factored out the de
radius a_0 and the Rydberg energy ϵ_0; N is t
quarkonium wave function, which is Coulo
factors a_0 and $\tau\sim 1/\epsilon_0$ represent the chara
time of the color dipole, respectively. The
tified when the wavelength of gluon fields
change slowly compared to $1/\epsilon_0$.

In physical terms, the structure of (2
elastic scattering of two dipoles which act
dipole fields; color neutrality permits only
the elastic scattering.

The amplitude (2) was evaluated before
tional methods. For our purposes, however
technique based on the spectral representa
approach, we will reproduce the result of[3]

It is convenient, as a first step, to expre
strength tensor[12]:

$$g^2\mathbf{E}^2 = -\frac{1}{4}g^2G_{\alpha\beta}G^{\alpha\beta} + g^2(-G_{0\alpha}G_0^\alpha + \frac{1}{4}g$$

where

$$\theta^\mu_\mu \equiv \frac{\beta(g)}{2g}G^{\alpha\beta a}G^a_{\alpha\beta} = -\frac{bg}{32}$$

Note that as a consequence of scale anomal
momentum tensor of QCD in the chiral limi
and the β function in (5) does not contain t
(*i.e.* $b=\frac{1}{3}(11N-2N_f)=9$).

[a]The Wilson coefficients $\bar{d}_2$, evaluated in the larg
quarkonium states.

Let us now write down the spectral representation for the correlator of the trace of energy-momentum tensor:

$$\langle 0|\mathrm{T}\theta^\mu_\mu(0)\theta^\mu_\mu(x)|0\rangle = \int d\sigma^2 \rho_\theta(\sigma^2)\Delta_F(x;\sigma^2), \tag{6}$$

where the spectral density is defined by

$$\rho_\theta(k^2)\theta(k_0) = \sum_n (2\pi)^3\delta^4(p_n - k)|\langle 0|\theta^\mu_\mu|n\rangle|^2, \tag{7}$$

and

$$i\Delta_F(x;\sigma^2) \;=\; i\int \frac{d^4k}{(2\pi)^3}\delta(k^2-\sigma^2)\theta(k_0)(e^{-ikx}\theta(x^0) + e^{ikx}\theta(-x^0)) \tag{8}$$

is the Feynman propagator of a scalar field.

Using the representation (6) in (2), we get

$$\begin{aligned} V_\theta(R) &= -i\Big(\bar{d}_2\frac{a_0^2}{\epsilon_0}\Big)^2\Big(\frac{4\pi^2}{b}\Big)^2\int dt\int d\sigma^2\rho_\theta(\sigma^2)\Delta_F(x;\sigma^2) \\ &= -\Big(\bar{d}_2\frac{a_0^2}{\epsilon_0}\Big)^2\Big(\frac{4\pi^2}{b}\Big)^2\int d\sigma^2\rho_\theta(\sigma^2)\frac{1}{4\pi R}e^{-\sigma R}. \end{aligned} \tag{9}$$

We see that the potential (9) can be represented as a superposition of Yukawa potentials corresponding to the exchange of scalar quanta of mass σ.

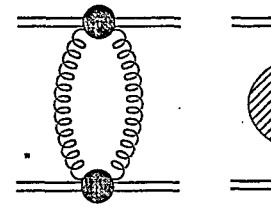
Our analysis so far has been completely general; the dynamics enters through the spectral density (7). Let us first evaluate this quantity in perturbation theory. In this case we have to evaluate the contribution of two–gluon states (see Fig. 1(a)) defined by

$$\rho^{\mathrm{pt}}_\theta(q^2) \;\equiv\; \sum(2\pi)^3\delta^4(p_1+p_2-q)|\langle 0|\theta^\alpha_\alpha|p_1\varepsilon_1 a, p_2\varepsilon_2 b\rangle|^2, \tag{10}$$

where the phase-space integral should be understood as well as the summations over the polarization and the color indices of the gluons.

The evaluation of (10) is straightforward: for $SU(N)$, one has

$$\rho^{\mathrm{pt}}_\theta(q^2) \;=\; \left(\frac{bg^2}{32\pi^2}\right)^2\frac{N^2-1}{4\pi^2}q^4. \tag{11}$$



(a)

Figure 1: Contributions to the scattering amplitude
correlated two pion ex

Then, substituting (11) into (9) and performi
mass σ^2, we get, for $N = 3$

$$V_\theta(R) \;=\; -g^4\Big(\bar{d}_2\frac{a_0^2}{\epsilon_0}$$

The $\propto R^{-7}$ dependence of the potential
from atomic physics [15]; as is apparent in our
gration in (9)), the extra R^{-1} as compared
$\propto R^{-6}$ is the consequence of the fact that the
time, and the characteristic fluctuation time τ
spatial separation of the "onia" : $\tau \ll R$.

Finally we note that in perturbation theor
the matrix element of the θ^μ_μ, as will be disc
perturbative), and accordingly the potential
the second term in (4) gives the contributic
contribution is due to the tensor 2^{++} state of
in a completely analogous way. Adding this co
the result of Ref. [3]:

$$V(R) \;=\; -g^4\Big(\bar{d}_2\frac{a_0^2}{\epsilon_0}\Big)$$

note that our $\bar{d}_2$ is related to the d_2 of Ref. [3]

3 Beyond the Perturbation Theory: The Role of Pions

At large distances, the perturbative description breaks down, because, as can be clearly seen from (9), the potential becomes determined by the spectral density at small q^2, where the transverse momenta of the gluons become small. To see this explicitly in the dispersive language, let us consider the correlator

$$\Pi(q^2) = \int d^4x e^{iqx} \langle 0|\mathrm{T}\theta^\mu_\mu(x)\theta^\mu_\mu(0)|0\rangle = \int d\sigma^2 \frac{\rho_\theta(\sigma^2)}{\sigma^2 - q^2 - i\epsilon}. \qquad (14)$$

An important low energy theorem[16] states that, as a consequence of the broken scale invariance,

$$\Pi(0) = -4\langle 0|\frac{\beta(g)}{2g} G^{\alpha\beta a} G^a_{\alpha\beta}(0)|0\rangle. \qquad (15)$$

The operator on the r.h.s. of (15) is regularized by subtracting the contribution of perturbation theory. The vacuum expectation value of this operator therefore measures the energy density of non–perturbative fluctuations in the QCD vacuum[17]. The low–energy theorem (15) therefore implies the sum rule for the spectral density[16]:

$$\int \frac{d\sigma^2}{\sigma^2}[\rho_\theta^{\rm phys}(\sigma^2) - \rho_\theta^{\rm pt}(\sigma^2)] = -4\langle 0|\frac{\beta(g)}{2g} G^{\alpha\beta a} G^a_{\alpha\beta}|0\rangle = -16\epsilon_{\rm vac} \neq 0, \qquad (16)$$

where $\rho_\theta^{\rm pt}(\sigma^2)$ is given by (11), and the vacuum energy density is $\epsilon_{\rm vac} = (1/4)\langle\theta^\mu_\mu\rangle \simeq -(0.24\ {\rm GeV})^4$ [17]. Since the physical spectral density, $\rho_\theta^{\rm phys}$, should approach the perturbative one at high σ^2, the integral in (16) can accumulate its value required by the r.h.s. only in the region of relatively small σ^2 [b].

At small invariant masses, we have to saturate the physical spectral density in the sum rule (16) by the lightest state allowed in the scalar channel – two pions:

$$\rho_\theta^{\rm phys}(q^2) \quad = \quad \sum (2\pi)^3 \delta^4(p_1 + p_2 - q)|\langle 0|\theta^\alpha_\alpha|\pi(p_1)\pi(p_2)\rangle|^2, \qquad (17)$$

where, as in (10), the phase–space integral is understood.

Since, according to (5), θ^α_α is gluonic operator, the evaluation of (17) requires the knowledge of the coupling of gluons to pions. This is a purely

[b] The analysis of sum rules shows however that the approach to the asymptotic freedom in the scalar channel is rather slow[17].

non–perturbative problem, but it can nevertheless be rigorously solved, as it was shown in Ref.[18] (see also[12]). The idea of[18] is the following: at small pion momenta, the energy–momentum tensor can be accurately computed using the low–energy chiral Lagrangian:

$$\theta^\mu_\mu = -\partial_\mu\pi^a\partial^\mu\pi^a + 2m_\pi^2\pi^a\pi^a + \cdots \qquad (18)$$

Substituting this expression into (17), in the chiral limit of vanishing pion mass one gets an elegant result[18]

$$\langle 0|\frac{\beta(g)}{2g} G^{\alpha\beta a} G^a_{\alpha\beta}|\pi^+\pi^-\rangle = q^2. \qquad (19)$$

The result (19) can actually be generalized for the coupling of gluons to *any* number of pions. Indeed, consider the Lagrangian of non–linear σ model,

$$\mathcal{L} = \frac{f_\pi^2}{4}\ {\rm tr}\ \partial_\mu U \partial^\mu U^\dagger \ - \ \Lambda\ {\rm tr}\left(MU + U^\dagger M^\dagger\right), \qquad (20)$$

where $U = \exp(2i\pi/f_\pi)$, $\pi \equiv \pi^a T^a$, tr $T^aT^b = \frac{1}{2}\delta^{ab}$, M is the quark mass matrix, and $\Lambda = m_\pi^2 f_\pi^2/\hat{m}$, with $\hat{m}$ being the average light quark mass. Using the mathematical identity for a generic matrix A,

$$\partial^\mu[\exp(A)] = \int_0^1 ds\ \exp(sA)\ \partial^\mu A\ \exp((1-s)A), \qquad (21)$$

one can explicitly evaluate the trace of the energy–momentum tensor for the Lagrangian (20), with the result

$$\theta^\mu_\mu = -2\ \frac{f_\pi^2}{4}\ {\rm tr}\ \partial_\mu U \partial^\mu U^\dagger \ + \ 4\Lambda\ {\rm tr}\left(MU + U^\dagger M^\dagger\right). \qquad (22)$$

Expanding (22) in powers of pion field, one can generalize (18) for any (even) number of pions; to lowest order, we reproduce (18).

Now that we know the coupling of gluons to the two pion state, the pion–pair contribution to the spectral density (17) can be easily computed by performing the simple phase space integration, with the result

$$\rho_\theta^{\pi\pi}(q^2) = \frac{3}{32\pi^2} q^4. \qquad (23)$$

Multi–pion contribution can be evaluated using (22); its influence will be discussed elsewhere. However the dominant contribution at small invariant masses σ, in which we are primarily interested here, comes from the $\pi\pi$ state.

Coming back to the initial expressions (2), (4) we find that to complete our derivation of the scattering amplitude we need to evaluate also the transition matrix element[12] of the second term in (4), $\langle 0|g^2\theta_{00}^{(G)}|\pi\pi\rangle$. This tensor operator was discussed in the previous Section, where we have found that it contributes a substantial fraction, 8/23, to the complete perturbative result. However, unlike the scalar operator, the tensor term is not coupled to the anomaly, and therefore $\langle 0|g^2\theta_{00}^{(G)}|\pi\pi\rangle \sim g^2$, where the coupling has to be evaluated at the heavy quarkonium scale. This contribution therefore is of higher order in the strong coupling, and will vanish in the heavy quark limit. Omitting it, we come to the following low–energy expression,

$$\langle 0|g^2\mathbf{E}^{a2}|\pi\pi\rangle = \left(\frac{8\pi^2}{b}\right) q^2 + O(\alpha_s, m_\pi^2). \tag{24}$$

Thus, in the heavy quark limit, the matrix element in question is known up to α_s and m_π^2 corrections.

The result (24) has been derived in the chiral limit; the most important correction coming from the finite mass of the pion is the phase space threshold; we correct for it by writing down the spectral density in the form ($q^2 \geq 4m_\pi^2$),

$$\rho_\theta^{\pi\pi}(q^2) \;=\; \frac{3}{32\pi^2}\left(\frac{q^2-4m_\pi^2}{q^2}\right)^{1/2} q^4, \tag{25}$$

which should be valid at small q^2. Substituting (25) in (9), we get the potential due to the $\pi\pi$ exchange; at large R

$$V^{\pi\pi}(R) \to -\left(\bar{d}_2\frac{a_0^2}{\epsilon_0}\right)^2\left(\frac{4\pi^2}{b}\right)^2\frac{3}{2}(2m_\pi)^4\frac{m_\pi^{1/2}}{(4\pi R)^{5/2}}e^{-2m_\pi R}. \tag{26}$$

The same functional dependence of $\pi\pi$ exchange at large R has been obtained previously in Ref.[3], but up to an unknown constant; in our approach, the overall strength of the interaction is fixed. Note that, unlike the perturbative result (13) which is manifestly $\sim g^4$, the amplitude (26) is $\sim g^0$ – this "anomalously" strong interaction is the consequence of scale anomaly[c].

The low–energy theorems[16,18] not only allow us to evaluate explicitly the contribution of uncorrelated $\pi\pi$ exchange; they also tell us that this contribution alone is not the full story yet. Indeed, the $\pi\pi$ spectral density (25) alone cannot saturate the sum rule (16) – at high σ^2, the physical spectral density approaches the spectral density of perturbation theory, so the integral in (16) does not get any contribution; at small σ^2, the $\pi\pi$ spectral density (25), according to the chiral and scale symmetries is suppressed by $\sim \sigma^4$. The low energy theorems *require* the presence of resonant enhancement(s)[19] in the 0^{++} $\pi\pi$, and perhaps multi-pion, $\bar{K}K$ and $\eta\eta$ channels as well. Here we will leave this interesting problem aside, and study only the influence of these resonances on the potential between the color dipoles. To do this, we introduce the pion scalar form factor $F(q^2)$ and write down the spectral density as

$$\rho_\theta^{\pi\pi}(q^2) \;=\; \frac{3}{32\pi^2}\left(\frac{q^2-4m_\pi^2}{q^2}\right)^{1/2} q^4|F(q^2)|^2. \tag{27}$$

The form factor is directly related to the experimental $\pi\pi$ phase shifts by the Omnès–Muskhelishvili equation with the solution

$$F(t) = \exp\left[\frac{t}{\pi}\int ds\frac{\delta_0^0(s)}{s(s-t-i\epsilon)}\right]\;; \tag{28}$$

with this formula we can make a full use of the experimental information on the $\pi\pi$ correlation. For our calculation we have used a simple analytic form[20] for the phase shift δ_0^0 which was shown to fit the experimental data up to $s_{\pi\pi} \simeq (1.2\ \mathrm{GeV})^2$. There are two main contributions to the spectral density, $\rho_\theta^{\pi\pi}$, which may be interpreted as the broad σ and narrow f_0 resonances.

In Fig. 2 we show the resulting potential between two J/ψ states. In computing it, we assumed that Coulomb relations for the Bohr radius and the Rydberg energy $a_0 = 4/(3\,\alpha_s m)$ and $\epsilon_0 = (4/3\,\alpha_s)^{-2}m = 1/a_0^2\,m$ hold for the J/ψ. Using as an input $\epsilon_0 = 2M_D - M(J/\psi)$=642 MeV and m=1.5 GeV, we get α_s=0.87 and a_0=0.20 fm.

It can be clearly seen from Fig. 2 that at large distances the non–perturbative interaction dominates over the perturbative one. To evaluate the amplitude,

[c]Of course, in the heavy quark limit the amplitude (26) will nevertheless vanish, since $a_0 \to 0$ and $\epsilon_0 \to \infty$.

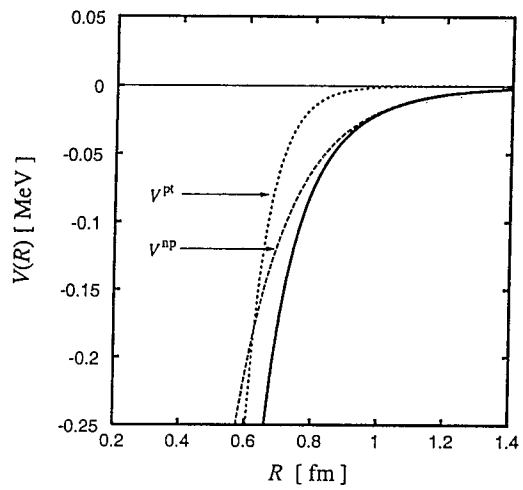


Figure 2: The potential between two J/ψ's (bold solid line); the perturbative contribution V^{pt} (dashed line) was evaluated within the invariant mass range $\sigma > 2$ GeV in the spectral density; V^{np} (long-dashed line) is the non-perturbative contribution.

we had to use an experimental input – the $\pi\pi$ phase shifts, and the detailed shape of the potential depends on this input. However, as we will now see, the dominance of the non-perturbative interaction is a model-independent consequence of the low-energy theorems. Moreover, its overall strength can be shown to be completely determined by the energy density of non-perturbative vacuum of QCD.

4 The Sum Rule

Consider the integral over the non-perturbative part of the potential; according to (9), it can be written down as

$$\int_a^\infty d^3\mathbf{R}\,(V(R) - V^{pt}(R))$$

$$= -\left(\bar{d}_2\frac{a_0^2}{\epsilon_0}\right)^2\left(\frac{4\pi^2}{b}\right)^2\int d\sigma^2\left(\rho(\sigma^2) - \rho^{pt}(\sigma^2)\right)\int_a^\infty dR R^2\frac{1}{R}e^{-\sigma R}$$

$$= -\left(\bar{d}_2\frac{a_0^2}{\epsilon_0}\right)^2\left(\frac{4\pi^2}{b}\right)^2\int\frac{d\sigma^2}{\sigma^2}\left(\rho(\sigma^2) - \rho^{pt}(\sigma^2)\right)\Gamma(2,\sigma a). \qquad (29)$$

As a consequence of asymptotic freedom, $\rho(\sigma^2) \to \rho^{pt}(\sigma^2)$ at high σ, so the integral in (29) attains its value in the region of $\sigma < \sigma_0$, where σ_0 is some characteristic scale at which the perturbative regime sets in. In the heavy quark limit the size of quarkonium $a \sim (\alpha_s m)^{-1} \to 0$, and when $\sigma_0 a \ll 1$, $\Gamma(2,\sigma a) \simeq 1$ in the entire region of integration. The integral in (29) then, up to perturbative corrections $\sim O(g^4)$, coincides with the integral in (16). Therefore we can re-write (29), in the heavy quark limit, as a sum rule

$$\int d^3\mathbf{R}\,(V(R) - V^{pt}(R)) = -\left(\bar{d}_2\frac{a_0^2}{\epsilon_0}\right)^2\left(\frac{4\pi^2}{b}\right)^2 16\,|\epsilon_{vac}|\,, \qquad (30)$$

which expresses the overall strength of the interaction between two color dipoles in terms of the energy density of the non-perturbative QCD vacuum.

5 Final Remarks

What are the implications of our results? First, the pion clouds which dominate interactions of small color dipoles at low energies, as revealed by our analysis, may as well be important in high-energy scattering; this was suggested long time ago on general grounds [1,21]. However it is not yet clear if one can extend our approach to the scattering at high energies[d].

Second, the fact that pions (and therefore light quarks) dominate the long-distance interactions of heavy quark systems is important for the lattice QCD simulations. Our findings suggest that to extract the information on the properties and interactions of heavy quarkonia from lattice QCD one should go beyond the "quenched" approximation.

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[d]For a very interesting related discussion see Ref. [22].

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