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Testing the Surrogate Zeta-Function Method

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ABSTRACT

Use of the surrogate zeta-function method was crucial in calculating the Casimir energy in non-Abelian Kaluza-Klein theories. We establish the validity of this method for the case that the background metric is (Euclidean space)×(N-sphere). Our techniques do not apply to the case where the background is (Minkowski-space)×(N-sphere).

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The motivation for this work is to understand in more detail certain subtleties of the zeta-function regularization procedure[1–2] with application to the computation of the gravitational Casimir energy in non-Abelian Kaluza-Klein theories.[3–6] This Casimir energy has been ofinterest recently in the literature; it is expected to play an important role in the dynamics of spontaneous compactification.[7]

The problem to be addressed arises in the following way. We begin with the action for quantum gravity in an $m+N$-dimensional space, which we shall take to be the direct product of m-dimensional Euclidean space with coordinates x^a and the N-sphere S^N with coordinates y^i. (Following the prescription of Hawking,[8] we define the theory in Euclidean space; physical quantities are then obtained by analytic continuation back to Minkowski space.) The action is then

$$S\,[g] = \frac{-1}{16\pi G_D}\int d^m x\, d^N y\,\sqrt{|g|}\,(R - 2\Lambda + g.f. + gh.)\,, \tag{1}$$

where "$g.f.$" stands for the gauge-fixing terms, and "$gh.$" is the ghost action. We expand

$$g_{\mu\nu} = \mathring{g}_{\mu\nu} + h_{\mu\nu}, \tag{2}$$

where $\mathring{g}_{\mu\nu}$ is the metric of the above-mentioned background, and $h_{\mu\nu}$ is the quantum fluctuation. Greek indices run over the entire $m+N$ dimensional manifold. To compute the Casimir energy, or equivalently the one-loop effective potential, we keep only the quadratic pieces in $h_{\mu\nu}$:

$$S^{(2)}\,[g] \equiv \frac{1}{2}\int d^m x\, d^N y\,\left(h^{\alpha\beta} S_{\alpha\beta;\mu\nu}\, h^{\mu\nu}\right), \tag{3}$$

and similarly we compute the matrix $S_{\rm Gh}$ as the quadratic piece in the ghost fields. The one-loop effective action is then

$$\Gamma_{\rm eff} = S_{\rm cl} + \frac{1}{2}\ln {\rm Det}\, S - \ln {\rm Det}\, S_{\rm Gh} \tag{4}$$

where $S_{\rm cl}$ is the classical action evaluated at $\mathring{g}$. Explicitly, for a convenient gauge choice the matrix S is given by

$$\begin{aligned} S^{(2)}\,[gf] = \int d^m x\, d^N y\,\sqrt{|g|}\Big\{ &\left[\tfrac{1}{2}\mathring{g}^{\mu\nu} h - h^{\mu\nu}\right]\tfrac{1}{2}\mathring{\nabla}^2 h_{\mu\nu} \\ &- \left(h^{\mu\alpha}h^{\nu\alpha} - h\,h^{\mu\nu}\right)\mathring{R}_{\mu\nu} + h^{\mu\nu}h^{\lambda\sigma}\mathring{R}_{\lambda\mu\sigma\nu} \\ &- \tfrac{1}{4}\left(h^2 - \tfrac{1}{2}h^{\alpha\beta}h_{\alpha\beta}\right)\left(\mathring{R} - 2\Lambda\right)\Big\}. \end{aligned} \tag{5}$$

Here indices on $h_{\mu\nu}$ are raised with $\mathring{g}^{\mu\nu}$, and $h \equiv \mathring{g}^{\mu\nu} h_{\mu\nu}$; $\mathring{R}_{\mu\nu}$ and $\mathring{R}_{\lambda\mu\sigma\nu}$ are respectively the Ricci tensor and Riemann tensor associated with $\mathring{g}_{\mu\nu}$.

It is useful to write

$$h_{\mu\nu} = p_{\mu\nu} + \varphi \mathring{g}_{\mu\nu} + \chi k_{\mu\nu} \tag{6}$$

where

$$k_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & k_{ij} \end{pmatrix} \tag{7}$$

and k_{ij} is the metric on S^N, and where $p_{\mu\nu}$ is traceless on both S^N and on the full manifold, i.e.,

$$\mathring{g}^{\mu\nu} p_{\mu\nu} = k^{\mu\nu} p_{\mu\nu} = 0. \tag{8}$$

The determinant of S can be obtained by solving the eigenvalue problem

$$S^{\alpha\beta}{}_{\mu\nu} h_{\alpha\beta} = \lambda h_{\mu\nu}. \tag{9}$$

The traceless part of $h_{\alpha\beta}$ (i.e. $p_{\alpha\beta}$) is relatively straightforward although not simple to handle. The problem comes when one studies the trace modes φ and χ. As a consequence of Eq. (9), they satisfy the coupled equations

$$\frac{1}{2}(N+m-2)\,\tilde{\Delta}\varphi - \frac{N(N-1)}{r^2}\varphi + \frac{1}{2}N\,\tilde{\Delta}\chi - \frac{N(N-1)}{r^2}\chi = \lambda\varphi \tag{10}$$

and

$$-\tilde{\Delta}\chi - \frac{(N-1)(N-4)}{r^2}\chi - \frac{(N-1)(N+m-4)}{r^2}\varphi = \lambda\chi. \tag{11}$$

where r is the radius of S^N and $\tilde{\Delta}$ is the operator

$$\tilde{\Delta} = R - 2\Lambda - \nabla^2 \quad ; \quad \nabla^2 = \nabla_M^2 + \frac{1}{r^2}\nabla_S^2. \tag{12}$$

After some computations involving the spectrum of the laplacian ∇_S^2 on the unit N-sphere, one arrives at the formal expression[3]

$$\text{Det } S_{coupled} = \prod_k \prod_{l=0}^{\infty} \left\{ (N+m-2)\left[\frac{l^2+(N-1)(l+N-2)}{r^2} - 2\Lambda + k^2\right]^2 + \frac{2(N-2)^2(N-2)(m-2)}{r^4} \right\}^{d_s(l)} \tag{13}$$

where the product on k comes from ∇_M^2 (the laplacian on m-dimensional Euclidean space), the product on l comes from ∇_S^2, and

$$d_s(l) = \binom{N+l}{l} - \binom{N+l-2}{l-2} \tag{14}$$

is the degeneracy of scalar eigenvalues of ∇_S^2, and k^2 is of course $k^a k_a$. By contrast, the traceless modes $p_{\mu\nu}$ give rise to formal determinants whose factors are only quadratic in k and l:

$$\text{Det } S_{traceless} = \prod_{i=1}^{T} \prod_{k} \prod_{l=l_{0_i}}^{\infty} \left[\frac{l^2 + (N-1)\,l + c_i}{r^2} + k^2\right]^{d_i(l)}, \tag{15}$$

where the product on i is finite and comes from the various kinds of eigenmodes of ∇_S^2 that are present. The determinants in Eq. (13) and (15) are ill-defined, since they correspond to products over infinitely many eigenvalues. A standard technique to handle this is the zeta-function method. Given an operator $\mathcal{O}$ with an increasing set of eigenvalues $\{\lambda_i\}$, one defines the associated zeta-function

$$\zeta(s) = \sum_i \left(\frac{1}{\lambda_i}\right)^s. \tag{16}$$

For s sufficiently large, this sum will converge. In the region where the right-hand side does not converge one can define $\zeta(s)$ by analytic continuation, and one then takes advantage of the formal equation

$$-\left.\frac{d\zeta}{ds}\right|_{s=0} = \sum_i \ln \lambda_i \tag{17}$$

to define

$$\ln \text{Det } \mathcal{O} = -\left.\frac{d\zeta}{ds}\right|_{s=0}. \tag{18}$$

Thus for the traceless case, one has to evaluate a zeta-function of the form

$$\zeta_{traceless}(s) = \int \frac{d^m k}{(2\pi)^m} \sum_{l=l_0}^{\infty} d_i(l) \left[\frac{l^2 + l\,(N-1) + c}{r^2} + k^2\right]^{-s}, \tag{19}$$

whereas for the coupled scalars the analogous expression is

$$\zeta_{coupled}(s) = \int \frac{d^m k}{(2\pi)\,m} \sum_{l=0}^{\infty} d_i(l) \left\{\left[\frac{l^2 + (N-1)\,l + c - 1}{r^2} + k^2\right]^2 + \frac{{c_2}^2}{r^4}\right\}^{-s}. \tag{20}$$

The values of c_1 and c_2 can be read off from Eq. (13).

For the traceless case, when $m+N$ is odd, there are techniques available, which we shall review below, for performing the analytic continuation and thereby obtaining a well-defined expression for the determinant. In order to take advantage of this knowledge in the coupled case, the procedure that has been adopted in the literature[3,5,6] is as follows: one first makes the observation that

$$\begin{aligned} &\left[\frac{l^2+(N-1)\,l+c_1}{r^2}+k^2\right]^2+\frac{c_2{}^2}{r^4} \\ &\quad = \left[k^2+\frac{l^2+(N-1)\,l+c_1+ic_2}{r^2}\right]\left[k^2+\frac{l^2+(N-1)\,l+c_1-ic_2}{r^2}\right], \end{aligned} \tag{21}$$

and one uses the "theorem" that

$$\ln \mathrm{Det}\ MN = \ln \mathrm{Det}\ M + \ln \mathrm{Det}\ N \tag{22}$$

to define two "surrogate" zeta-functions

$$\zeta_{\pm}(s) = \int \frac{d^m k}{(2\pi)^m} \sum_{l=0}^{\infty} d_s(l) \left\{\frac{l^2+(N-1)\,l+c_1 \pm ic_2}{r^2}+k^2\right\}^{-s} \tag{23}$$

and to assert that, as a consequence of Eq. (22),

$$\zeta'(0) = \zeta'_{+}(0) + \zeta'_{-}(0)\,. \tag{24}$$

The surrogate zeta-functions $\zeta_{\pm}(s)$ can be handled by exactly the same techniques as in the traceless case. Thus if the above theorem is true, as it surely is for finite matrices, this factorization trick will lead to an evaluation of $\zeta'_{coupled}(0)$. It is perhaps worth noting, however, that the two "eigenvalues" into which the coupled determinant has been factorized are not the eigenvalues of the pair of equations Eqs. (10) and (11). (The true eigenvalues are real.) Rather, they correspond to a different factorization which is more convenient for the present analysis. Another point is that for the Euclidean background that we are using, it can be shown that $\zeta_{-}(s) = \zeta_{+}^{*}(s)$, so that the full answer for the coupled zeta-function is

$$\zeta'_{coupled}(0) = 2\,\mathrm{Re}\,\zeta'_{+}(0)\,. \tag{25}$$

The same Kaluza-Klein Casimir energy has also been studied by others[5] using techniques that enable them to perform the computation in a Minkowski background. They use the same surrogate zeta-function trick, but the difference between Euclidean and Minkowski space manifests itself in a crucial phase difference:

$$\zeta_{-}(s) = -\zeta_{+}^{*}(s) \tag{26}$$

so that for them

$$\zeta'_{coupled}(0) = 2i\,\mathrm{Im}\,\zeta'_{+}(0)\,, \tag{27}$$

(the function $\zeta_{+}(s)$ is the same in the two methods.) Thus these two apparently equally valid ways of calculating the same physically measurable quantity give different results. We shall return to this point later.

We turn now to the central question of this paper: is Eq. (24) true for the case at hand? As we mentioned above, the answer is undoubtedly yes for finite matrices, although even in that case it is not a relation among the zeta-functions themselves,

$$\zeta_{coupled}(s) \neq \zeta_{+}(s) + \zeta_{-}(s)\,, \tag{28}$$

but only among their derivatives at the origin, which correspond to the relevant determinants. Also, as shown by an example due to Allen,[9] Eq. (24) is not true in general for infinitematrices. Allen considers an operator with an infinite discrete spectrum:

$$\begin{aligned}\lambda_n &= n^2 + \alpha n + \beta \\ &\equiv (n + \alpha/2)^2 + \gamma\end{aligned} \tag{29}$$

and a degeneracy at the n^{th} level of

$$g_n = (n + \alpha/2)^3 + c_1 (n + \alpha/2)^2 + c_2 (n + \alpha/2) + c_3. \tag{30}$$

Thus the zeta-function to be evaluated is

$$\zeta(s) = \sum_{n=0}^{\infty} g_n \lambda_n^{-s}. \tag{31}$$

Now

$$\lambda_n = (n + \alpha/2 + i\sqrt{\gamma})(n + \alpha/2 - i\sqrt{\gamma}) \tag{32}$$

so that we can define the surrogate zeta-functions

$$\zeta_{\pm}(s) = \sum_{n=0}^{\infty} g_n \,(n + \alpha/2 \pm i\sqrt{\gamma})^{-s}. \tag{33}$$

Using Allen's techniques to obtain the analytic continuation of $\zeta(s)$ gives

$$\zeta'(0) = \zeta'_+(0) + \zeta'_-(0) + \gamma c_2 - \tfrac{3}{8}\gamma^2. \tag{34}$$

indicating the failure of Eq. (24) in this case.

We now present a quick review of how we perform the analytic continuation in the traceless (i.e. quadratic) case. A fuller discussion may be found in Ref. 4. We use

$$\int d^m k \left(k^2 + \alpha^2\right)^{-s} = \frac{\Gamma(s - m/2)}{\Gamma(s)} \pi^{m/2} \left(\alpha^2\right)^{-s+m/2} \tag{35}$$

where $s > m/2$, to write Eq. (19) as

$$\zeta(s) = \frac{\pi^{m/2} r^{2s}}{(2\pi r)^m} \frac{\Gamma(s - m/2)}{\Gamma(s)} \sum_{l=0}^{\infty} d_s(l) \left[\left(l + \left(\frac{N-1}{2}\right)\right)^2 - \beta^2\right]^{m/2-s}, \tag{36}$$

where $\beta^2 = \frac{1}{4}(N-1)^2 - c$. We then use the Laplace transform

$$\frac{1}{(z^2 - \beta^2)^\nu} = \frac{\sqrt{\pi}}{\Gamma(\nu)} \int_0^\infty e^{-zt} \left(\frac{t}{2\beta}\right)^{\nu - \frac{1}{2}} I_{\nu - \frac{1}{2}}(\beta t)\, dt, \tag{37}$$

where $z = l + \frac{1}{2}(N-1)$, $\nu = s - m/2 > 0$, and I_ν is the modified Bessel function. We perform the sum on l using

$$\sum_{l=0}^{\infty} \binom{N+l}{l} e^{-lt} = \frac{1}{[1 - e^{-t}]^{N+1}} \tag{38}$$

The result is

$$\zeta(s) = \frac{1}{4} \frac{\kappa_i}{(2\pi r)^m} \frac{r^{2s}\sqrt{\pi}}{\Gamma(s)} \int_0^\infty \frac{\sinh t}{(2\sinh\frac{1}{2}t)^{N+1}} \left(\frac{t}{2\beta}\right)^{\nu - \frac{1}{2}} I_{\nu - \frac{1}{2}}(\beta t)\, dt \tag{39}$$

where κ_i is an overall degeneracy factor that depends upon the modes being considered (e.g., scalar, vector, etc...). A complete list of the values of the κ_i are given in Ref. 4; for the surrogate zeta functions in Eq. (24) $\kappa = 4\pi^2$.

The reason that Eq. (39) is not valid for small s is that near $t = 0$ the integrand behaves as $t^{2s-m-N-1}$, so as defined $\zeta(s)$ is singular for $\mathrm{Re}(s) < m + N$. To improve the situation, one notes that the integrand can be written as $t^p f(t^2)$, with $p = 2s - m - N - 1$, and that hence

$$\begin{aligned} \mathrm{I} &\equiv \int_0^\infty dt\, t^p f\left(t^2\right) \\ &= \frac{1}{1 + e^{\mp i\pi p}} \int_{-\infty \pm i\epsilon}^{\infty} dt\, t^p f\left(t^2\right) \\ &= \frac{1}{1 + e^{\pm i\pi p}} \int_{\infty \pm i\Delta}^{\infty \pm i\Delta} dt\, t^p f\left(t^2\right), \end{aligned} \tag{40}$$

where $0 < \Delta < 2\pi$. (The upper bound on Δ arises because $f(t^2)$ has a pole at $t = 2\pi i$). Now that the contour of integration no longer passes through $t = 0$, it is safe to continue s to a neighborhood of $s = 0$. In fact, for the purpose of computing $\zeta'(0)$ we note that $\zeta(s)$ is of the form $\zeta(s) = \tilde{\zeta}(s)/\Gamma(s)$ where $\tilde{\zeta}(s)$ is regular at $s = 0$. Thus $\zeta'(0) = \tilde{\zeta}(0)$, and we have, explicitly,

$$\zeta'(0) = \frac{\kappa(N,m)}{(2\pi r)^m}\frac{1}{1+e^{\mp i\pi p}}\int_{-\infty\pm i\Delta}^{\infty\pm i\Delta}\frac{\sinh t}{(\sinh\frac{1}{2}t)^{N+1}}\left(\frac{t}{2\beta}\right)^{-(m+1)/2} I_{(m+1)/2}(\beta t)\,dt \tag{41}$$

where now $p = -(m+N+1)$. It should be noted that this method fails when $m+N$ is even, because then $e^{\mp i\pi p} = -1$. We shall henceforth restrict the discussion to the case $m+N$ odd, although the the $m+N$ even case has also been treated in Ref. 10.We now return to the coupled zeta-function, Eq. (20), which we shall rewrite as

$$\zeta_{coupled}(s) = \int\frac{d^m k}{(2\pi)^m}\sum_{l=0}^{\infty} d_s(l)\left[\left(k^2+\alpha_l{}^2\right)^2+\gamma^2\right]^{-s} \tag{42}$$

where α_l and γ can be read off from Eq. (20). From Eq. (13) we see that $\alpha_l{}^2$ depends on the cosmological constant Λ, and that there will exist a range of Λ for which

$$\gamma^2 < \left(k^2+\alpha_l{}^2\right)^2 \tag{43}$$

for all values of k^2 and l. [This is only true in Euclidean space. If we work in Minkowski space, where k^2 can be negative, it will always be possible to have $(k^2+\alpha_l{}^2)^2 = 0$.]

Assuming Λ is such that the inequality (43) holds, we can employ the binomial expansion:

$$\left[\left(k^2+\alpha_l{}^2\right)^2+\gamma^2\right]^{-s} = \sum_{q=0}^{\infty}\frac{\Gamma(1-s)}{\Gamma(q+1)\,\Gamma(1-q-s)}\rho^q \tag{44}$$

where

$$\rho \equiv \frac{\gamma^2}{\left(k^2+\alpha_l{}^2\right)^2} < 1. \tag{45}$$

Using Eq. (35) we can perform the k^2 integral, and using

$$\frac{\Gamma(1-s)}{\Gamma(1-s-q)} = (-1)^q\frac{\Gamma(s+q)}{\Gamma(s)} \tag{46}$$

we arrive at

$$\zeta_{coupled}(s) = \frac{\pi^{m/2}}{(2\pi)^m}\frac{1}{\Gamma(s)}\sum_{q=0}^{\infty}\frac{(-1)^q}{q!}\gamma^{2q}\frac{\Gamma(s+q)\,\Gamma(2s+2q-m/2)}{\Gamma(2s+2q)} \times\left\{\sum_{l=0}^{\infty} d_s(l)\left(\alpha_l{}^2\right)^{m/2-(2s+2q)}\right\}. \tag{47}$$

We can now apply the same analysis as in Eqs. (37) - (41) to the sum on l, with s replaced by $(2s+2q)$. Defining

$$\tilde{c}^2 = [l + \tfrac{1}{2}(N-1)]^2 - {\alpha_l}^2 \tag{48}$$

($\tilde{c}$ is the analogue of β in the quadratic case), and $\nu_0 = -\frac{1}{2}(m+1)$ we arrive, by much the same route as in the quadratic case, at the expression

$$\zeta'_{coupled}(0) = \frac{1}{2^N}\frac{\pi^{\frac{m+1}{2}}}{(2\pi)^m}\tilde{c}^{(m+1)/2}\int\limits_{-\infty\pm i\Delta}^{\infty\pm i\Delta} dt\left(\frac{2}{t}\right)^{(m+1)/2}\frac{\sinh t}{(\sinh t/2)^{N+1}}S(t) \tag{49}$$

where

$$S(t) \equiv \sum_{q=0}^{\infty}\left(\frac{-i\gamma}{\tilde{c}^2}\right)^{2q}\frac{1}{(2q)!}\left(\frac{\tilde{c}t}{2}\right)^{2q} I_{\nu_0+2q}(\tilde{c}t)\,. \tag{50}$$

At this point the form for $\zeta'_{coupled}(0)$ looks vaguely like the quadratic case, except that we have an infinite sum, given by $S(t)$, instead of the pair of terms we were hoping for, [cf. Eq. (24)]. To remedy this situation we first express $S(t)$ as

$$\begin{aligned} S(t) = \frac{1}{2}\sum_{q=0}^{\infty}\Bigg[&\left(\frac{-i\gamma}{\tilde{c}^2}\right)^{q}\frac{1}{q!}\left(\frac{\tilde{c}t}{2}\right)^{q} I_{\nu_0+q}(\tilde{c}t) \\ &+\left(\frac{i\gamma}{\tilde{c}^2}\right)^{q}\frac{1}{q!}\left(\frac{\tilde{c}t}{2}\right)^{q} I_{\nu_0+q}(\tilde{c}t)\Bigg] \end{aligned} \tag{51}$$

and then use the multiplication formula for modified Bessel functions:

$$\lambda^{-\nu} I_\nu(\lambda z) = \sum_{k=0}^{\infty}\frac{(\lambda^2-1)^k}{k!}\left(\frac{z}{2}\right)^k I_{\nu+k}(z)\,. \tag{52}$$

The first term in Eq. (51) is of the appropriate form to apply Eq. (52) with

$$\lambda = \beta_- = (\tilde{c}^2 - i\gamma)^{\frac{1}{2}} \tag{53}$$

and likewise the second term with

$$\lambda = \beta_+ = (\tilde{c}^2 + i\gamma)^{\frac{1}{2}}\,. \tag{54}$$

One then obtains

$$\begin{aligned} \zeta'(0) &= \frac{1}{2^{N+1}}\frac{\pi^{(m+1)/2}}{(2\pi)^m}\int\limits_{-\infty\pm i\Delta}^{\infty\pm i\Delta} dt\Bigg[\left(\frac{t}{2\beta_+}\right)^{-(m+1)/2} I_{-(m+1)/2}(\beta_+ t) \\ &\qquad + \left(\frac{t}{2\beta_-}\right)^{-(m+1)/2} I_{-(m+1)/2}(\beta_- t)\Bigg]\frac{\sinh t}{(\sinh\frac{1}{2}t)^{N+1}} \\ &= \zeta'_+(0) + \zeta'_-(0)\,. \end{aligned} \tag{55}$$

In obtaining this result it was assumed that A was restricted to a range for which inequality (43) was satisfied, but by analytic continuation in A the result should hold for all A. Thus Eq. (24) holds, and the use of the zeta-function method to compute the gravitational Casimir energy on a Euclidian background has been explicitly justified.

It is clear that the methods used here cannot be applied directly to the same kind of calculation performed in a Minkowski space background. It is to be hoped that some means of analytically continuing the Minkowski space equivalent of Eq. (20) can be found, so that we may know if the surrogate method also holds in Minkowski space. If it does hold then there is a real difference between the Casimir energy computed in Euclidian space and that computed in Minkowski space. On the other hand, if the surrogate method breaks down in Minkowski space we might expect to obtain the Euclidian result from the exact (i.e. non-surrogate) calculation, although this is by no means guaranteed. In either case the difference between the results of the two methods of calculation (either the Euclidian/Minkowski difference or the surrogate/non-surrogate difference) will need to be understood further.

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