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MAGNETOELASTIC ENERGY CALCULATIONS FOR FINITE ELEMENT ANALYSIS OF SUPERCONDUCTORS

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Summary

It has been shown that the high current density and magnetic flux density associated with superconductors can make the magnetoelastic energy a significant portion of the total energy in a structural system. The present work presents a procedure for evaluating this magnetoelastic energy for use in the finite element analysis of the structural dynamics and stability of the superconductor. A simple, special case of the element matrices is illustrated.

Introduction

The utilization of superconducting materials is becoming increasingly common. In the areas of plasma physics and fusion energy research the utilization of superconducting magnetics is often necessary for the generation of the confining magnetic fields. The high current density and magnetic flux density associated with such applications contribute a magnetoelastic energy term that must be considered in an accurate evaluation of the structural stability and/or dynamic response of the superconducting structure. Experimental results and simplified analytical approximations have shown that the magnetoelastic contributions can have a significant effect on the structural behavior. In some cases the estimated critical current density, necessary to induce buckling, is less than published design densities.

The current state of the art is most accurately described in the several experimental and analytical studies by Moon, et al[4-9]. They have shown the magnetoelastic contributions to be quite significant. However, these studies have practical limitations that make it difficult to apply in general. Their approximate analyses are based on a single degree of freedom Ritz approximation. For circular superconducting coils their accuracy is fairly good. This is due to the utilization of the known exact analytical form of the mode shapes for given symmetrical support conditions. This information clearly will not be available for a general structure.

The single unknown constant in the Ritz analysis represents the magnitude of the maximum deflection. The approach used by Moon should yield good engineering approximations of the magnetoelastic energy and the strain energy. The latter quantity was evaluated using the analytic expression for the strain energy in a curved beam. The magnetic energy was based on an empirically determined relation. As expected, both forms reduce to a quadratic expression involving the unknown maximum deflection.

These approximations were substituted into the definition of the Lagrangian, L:

$$L = T + W - V \qquad (1)$$

where T = kinetic energy, W = magnetic energy and V = strain energy. The new term, the magnetic energy, is usually defined for a rigid circuit as $W_0 = i^2 L$ where i is the current density and L is the inductance. The inductance depends on the geometry of the circuit and thus is affected by the structural deflections. Therefore, the magnetic energy could be written as $W = W_0(i) + W(i,\delta)$ where δ is the displacement from the original shape. Only the latter term will vary when one searches for the displacement field that would render the Lagrangian stationary.

To determine the buckling current, i_b, causing structural instability, Moon utilized the following analytic procedure (since it closely followed his experimental procedure). Assume a current density, i. Minimize the Lagrangian to find the natural frequency, ω_n, for a simple harmonic excitation. Repeat the procedure for several i and generate $\omega^2 - i^2$ plot. Use this plot to estimate the buckling current magnitude, i_b, where $\omega_n = 0$.

Present Procedure

The contribution of Moon and his colleagues is important. However, there are a number of obvious disadvantages with the classical Ritz type of approximation. First, one usually requires an analytic expression, or approximation, for the mode shapes. Thus, it is difficult to treat general shapes and boundary conditions. The calculation of the strain energy in terms of given analytical mode shapes is restrictive. The extrapolation procedure using the $\omega^2 - i^2$ plot is expensive and inaccurate. However, Southwell plots can be used to improve the accuracy. Finally, the analytic procedure makes it difficult to introduce the geometric stiffness so as to consider the axial loads and moments that exist in most applications.

By utilizing the finite element method the shortcomings of the current procedures can be overcome. However, a major problem still needs investigation. That is the finite element formulation of the deflection dependent portion of the magnetic energy. First, the necessary integral form must be developed and then it must be implemented into the standard finite element methodology. While a number of formulations of the magnetic energy are possible the initial emphasis has been on the procedure described below.

Let Greek subscripts denote a circuit number. For simplicity, consider conductors that can be approximated as line circuits. The displacement dependent portion of the magnetic energy density, per unit length of circuit α, due to the displacement of that circuit is the negative of the work done by the magnetic forces. That is

$$dW_{\alpha\beta} = \vec{B}_\alpha \cdot d\vec{F}_{\alpha\beta} \qquad (2)$$

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where $\vec{\delta}_\alpha$ is the displacement of a point on circuit α and $d\vec{F}_{\alpha\beta}$ is the force on α due to a circuit β. Thus the total displacement dependent magnetic energy of circuit α due to circuit β is

$$W_{\alpha\beta} = \oint_\alpha dW_{\alpha\beta} \ . \tag{3}$$

At a point, P, on α the force due to circuit β is given by

$$d\vec{F}_{\alpha\beta} = i_\alpha \, \vec{ds}_\alpha \times \vec{B}_{\alpha\beta} \tag{4}$$

where $\vec{B}_{\alpha\beta}$ is the magnetic flux density vector at point P on α due to circuit β, i_α is the current in α and $\vec{ds}_\alpha$ is the unit tangent vector at P. The magnetic flux density in the deformed configuration is

$$\vec{B}_{\alpha\beta} = \frac{\mu_0 i_\beta}{4\pi} \oint_\beta \frac{\vec{ds}_\beta \times \vec{r}_{\alpha\beta}}{r^3} \ . \tag{5}$$

where from Figure 1,

$$\vec{r}_{\alpha\beta} = \vec{R}_{\alpha\beta} + \vec{\delta}_\alpha - \vec{\delta}_\beta \tag{6}$$

and $r = |\vec{r}|$. This step is important since it separates data about the original geometry, $\vec{R}$, and the circuit deformations, $\vec{\delta}$. Usually $\delta \ll R$ so $r^3 \simeq R^3$. Substituting the above identities gives

$$W_{\alpha\beta} = \frac{\mu_0 i_\alpha i_\beta}{4\pi} \left(\oint \vec{\delta}_\alpha \cdot \left[\vec{ds}_\alpha \times \left[\oint_\beta \frac{\vec{R}_{\alpha\beta} \times \vec{ds}_\beta}{R^3} \right] \right] \right.$$

$$+ \oint_\alpha \vec{\delta}_\alpha \cdot \left[\vec{ds}_\alpha \times \left[\vec{\delta}_\alpha \times \oint_\beta \frac{\vec{ds}_\beta}{R^3} \right] \right]$$

$$\left. + \oint \vec{\delta}_\alpha \cdot \left[\vec{ds}_\alpha \times \left[\oint_\beta \frac{\vec{ds}_\beta \times \vec{\delta}_\beta}{R^3} \right] \right] \right) \ . \tag{7}$$

There are two common special cases of interest in structural applications. The first is where the forcing circuit β is rigidly supported so that $\vec{\delta}_\beta = \vec{0}$. For self-induced magnetic forces $\alpha = \beta$. In such cases the ratio of δ/R is not small over all regions of integration. It may be necessary to use the identity that

$$\frac{1}{r} = \frac{1}{R} \left[1 + (\delta/R) \cos\Theta + \frac{1}{2}(\delta/R)^2 (3 \cos^2\Theta - 1) + \ldots \right]$$

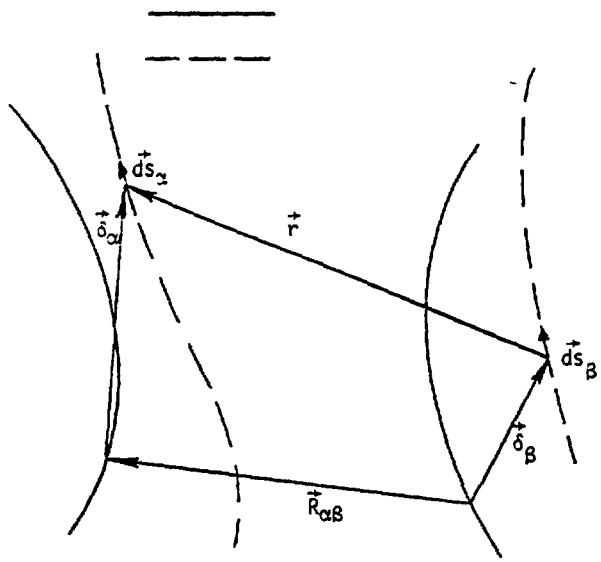

Figure 1. Definitions of Displacement Vectors

or

$$\frac{1}{r} = \frac{1}{R} \sum_{n=0}^{\infty} (\delta/R)^n \, P_n(\cos\Theta) \tag{8}$$

where P_n is the Legendre Polynomial of order n, and $\vec{\delta} \cdot \vec{R} = \delta R \cos\Theta$.

The finite element theory enters in defining the displacement vector components in terms of unknown nodal displacements and a set of element interpolation functions. The application of such an approach leads to the definition of an "element geometric inductance" matrix, $[L_G^e]$. Thus for a typical application the stationary Lagrangian leads to the eigenproblem

$$\left| \omega^2 [M] \pm i^2 [L_G] - [K_E] - [K_G] \right| = 0 \tag{9}$$

where

$[M]$ = system mass matrix

$[L_G]$ = system geometric inductance matrix

$[K_E]$ = system elastic stiffness matrix

$[K_G]$ = system geometric stiffness matrix

and where i is the current density and ω is the frequency of the simple harmonic motion. The sign of the current term in the above expression depends on the directions of the currents but will usually be positive. Thus the magnetoelastic effects are destabilizing.

For simplicity assume that $\omega = 0$ and that one is interested in a stability analysis. The presence of

the geometric stiffness introduces the axial loads, P. If the current density, i, is known then the above problem can be solved for the critical buckling load P_b. However, it is more likely that in a practical application the axial load would be a tension (stabilizing) force that is directly proportional to the square of the current. That is $P = ki^2$ where k is a known constant. In such a case, it is possible to solve for the critical buckling current, i_b.

Example: Parallel Conductors

Consider a typical conducting element, of length ℓ^e, centered between two parallel, infinite, rigid conductors. For the system shown in Figure 2, the quantities defined above are

$$\vec{\delta}_\alpha = u(x)j, \quad \vec{\delta}_\beta = \vec{\delta}_\gamma = \vec{0} ,$$

$$\vec{R}_\beta = (\Delta x i - D_1 j), \quad \vec{R}_\gamma = (\Delta x i + D_2 j)$$

$$\vec{ds}_\alpha = \vec{ds}_\beta = \vec{ds}_\gamma = dxi, \; i_\alpha = i_\beta = i \quad . \tag{10}$$

Thus the magnetic energy of the element is

$$W = W_o + W_{\alpha\beta} + W_{\alpha\gamma} \tag{11}$$

where

$$W_{\alpha\beta} = \frac{\mu_o i^2}{4\pi} \int_{\ell^e} \left[-u \int_{-\infty}^{\infty} \frac{Ddx}{R^3} + u^2 \int_{-\infty}^{\infty} \frac{dx}{R^3} \right] dx$$

$$W_{\alpha\gamma} = \frac{\mu_o i^2}{4\pi} \int_{\ell^e} \left[+u \int_{-\infty}^{\infty} \frac{Ddx}{R^3} + u^2 \int_{-\infty}^{\infty} \frac{dx}{R^3} \right] dx \quad . \tag{12}$$

The integrals from $-\infty$ to ∞ could be evaluated numerically by existing inductance calculation codes. However, in this simple case an analytic solution is available. Noting that $D = R \sin \Theta = Rd\Theta$, and $-\infty \leq x \leq \infty$ corresponds to $\pi \leq \Theta \leq 0$ this reduces to

$$W = W_o + \frac{\mu_o i^2}{4\pi} \int_{\ell^e} \left[\frac{u - u}{D} + \frac{u^2 + u^2}{D^2} \right] dx$$

or

$$W = W_o + \frac{\mu_o i^2}{2\pi D^2} \int_{\ell^e} u^2(x) dx \quad . \tag{13}$$

Introducing a standard finite element model such that the displacement, u(x), is given by

$$u(x) = [H(x)]\{\delta^e\} \tag{14}$$

where [H] and $\{\delta^e\}$ denote the interpolation functions and nodal displacements, respectively, gives

$$W = W_o + \frac{1}{2}\{\delta^e\}^T [L_G^e] \{\delta^e\} \tag{15}$$

where the symmetric geometric inductance matrix for the element, $[L_G^e]$, is defined as

$$[L_G^e] \equiv \frac{\mu_o i^2}{\pi D^2} \int_{\ell^e} [H(x)]^T [H(x)] dx \quad . \tag{16}$$

This form is very similar to the consistent mass matrix for the element.

Assume circuit α is divided into NE finite elements. The total Lagrangian of that circuit is the sum of the element contributions, i.e.,

$$L = \sum_{e=1}^{NE} L^e \quad , \tag{17}$$

where

$$L^e = T^e + W^e - V^e \tag{18}$$

and where the kinetic, magnetoelastic and strain energies are

$$T^e = \frac{1}{2} \{\dot{\delta}^e\}^T [M^e] \{\dot{\delta}^e\}$$

$$W^e = \frac{1}{2} \{\delta^e\}^T [L_G^e] \{\delta^e\}$$

$$V^e = \frac{1}{2} \{\delta^e\}^T \left[[K_E^e] + [K_G^e] \right] \{\delta^e\}. \tag{19}$$

If the element is a cubic beam element [11] subject to an axial tension load P then the typical terms are:

$$[M^e] = m^e [\lambda] \; , \; [L_G^e] = \frac{\mu_o i^2}{\pi D^2} [\lambda] \tag{20}$$

$$[\lambda] = \frac{1}{420} \begin{bmatrix} 156 & & & \text{sym} \\ 22\ell & 4\ell^2 & & \\ 54 & 13\ell & 156 & \\ -13\ell & -3\ell & -22\ell & 4\ell^2 \end{bmatrix} \tag{21}$$

$$[K_E^e] = \frac{EI}{\ell^3} \begin{bmatrix} 12 & & & \text{sym} \\ 6\ell & 4\ell^2 & & \\ -12 & -6\ell & 12 & \\ 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix} \tag{22}$$

$$[K_G^e] = \frac{p}{30\ell}\begin{bmatrix} 36 & & & \text{sym} \\ 3 & 4\ell^2 & & \\ -36 & -3\ell & 36 & \\ 3 & -\ell^2 & -3\ell & 4\ell^2 \end{bmatrix}. \quad (23)$$

Following the procedure of Przemieniecki [11], one would expect a two element model analytical solution to give critical values accurate to within 20%. Of course, one can easily increase the accuracy by increasing the number of elements and carrying out the eigenanalysis numerically.

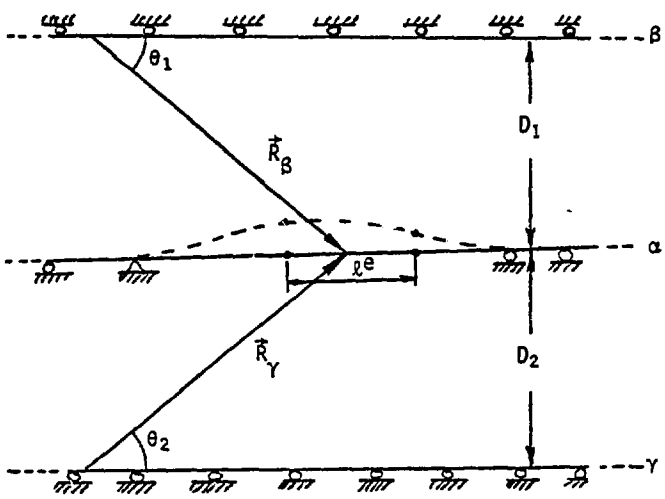

Figure 2. Example of a Three Wire System

Numerical Results

The above special case has been implemented in a finite element analysis. It has been applied to the buckling of a simply supported center conductor with no dynamic or axial loads (i.e., $\omega = 0$, $p = 0$). The problem parameters were selected to provide a rough approximation of the ORMAK system. The free span was $\ell = 6$ m and the distance between the conductors was $D = 1$ m. The elastic modulus and moment of inertia were $E = 1 \times 10^{11}$ N/m^2 and $I = 9 \times 10^{-4}$ m^4, respectively. The system was analyzed using 1,2,3 and 4 cubic beam element models. The single element model was included for completeness only since experience suggests a minimum of two elements is required for an accuracy of about 30% in the buckling values. The calculated mode shapes and buckling currents seem reasonable but have not yet been verified by alternate solution techniques.

The calculated buckling currents for the first three modes are given in the table below. They illustrate the importance of increasing the number of degrees of freedom (D.O.F.) in a stability analysis.

Table 1. Critical Buckling Currents

Number of Elements	Active D.O.F.	Calculated i_b, 10^6A		
		Mode 1	Mode 2	Mode 3
1	2	7.906	36.23	-
2	4	5.056	22.36	56.21
3	6	4.116	16.64	41.08
4	8	3.562	14.30	34.74

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