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OPTIMUM FLOW DISTRIBUTION FOR MULTICOMPONENT ISOTOPE SEPARATION IN A SINGLE CASCADE

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Oak Ridge, Tennessee

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Author: J. E. Murphy

SUMMARY

This paper concerns the separation of isotopic mixtures containing three components in a gaseous diffusion cascade operating over an infinite reservoir. The flow distribution yielding the smallest total flow per unit product is found and is compared with the flow distributions in some other theoretical cascades. Curves are presented to show the number of stages required, the total flow and the concentration gradients for both light and middle isotope separation.

INTRODUCTION--DESCRIPTION OF THE PROBLEM

The physical separation of isotopic mixtures may be accomplished by various processes, e.g., thermal diffusion and gaseous diffusion. The mathematical expressions of the separative behavior of these various processes are very similar. These developments are found in Ref. (2).

If a binary mixture is to be separated by gaseous diffusion, the minimum power (total flow) requirements and the associated interstage flow distribution may be found by using the so-called ideal cascade theory. This requires that any streams which are mixed have the same composition, i.e., no mixing loss is to occur. When more than two isotopes are involved, it is not possible to satisfy this no-mixing requirement. It is possible to require that one component of the mixed streams be present in equal concentrations, and this matched mol fraction cascade is used as a basis for design considerations. The cascades which result from the requirement of matching abundance ratios have been considered in Ref. (1). In the binary case the match mol fraction and matched abundance ratio cascades are the same.

This paper is concerned with the ternary problem and provides comparisons of:

1) The matched mol fraction (x) cascade.

2) The matched abundance ratio (R) cascade.

3) The minimum power cascade.

The total power requirements, number of stages, and concentration gradients are demonstrated for some specific problems with single cascades operating over an infinite reservoir under the above conditions.

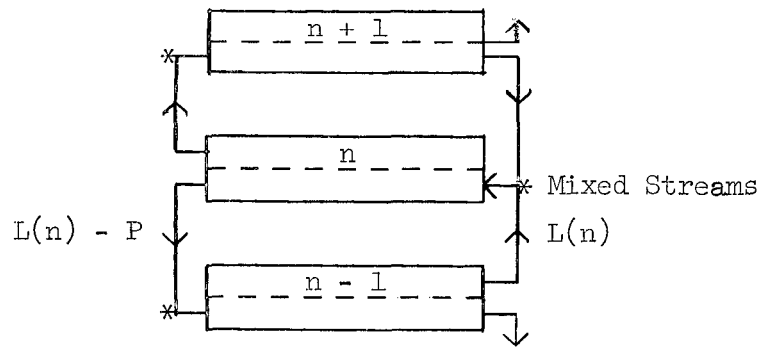


Figure 1

The stages of the gaseous diffusion cascade are assumed to be connected as shown in Figure 1.

Notation:

x, y, z	=	mol fractions of isotopes in ternary mixture
R	=	abundance ratio $x/(1-x-y) = x/z$
S	=	abundance ratio y/z
ψ_1	=	separation constant of x relative to z
ψ_2	=	separation constant of y relative to z
k	=	ψ_2/ψ_1
P	=	product withdrawal rate
n	=	stage number
x_o, y_o	=	mol fractions of product stream
y_p, x_p	=	mol fractions of product stream
$L(n)$	=	interstage flow rate at stage n

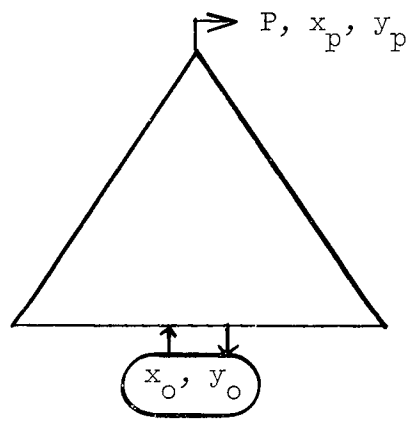


Figure 2

The cascades considered are assumed to be operating over an infinite reservoir of concentrations (x_o, y_o) as shown in Figure 2. For all problems,

$$P = 1,$$

and the interstage flow rates are measured in units of production.

The difference equations for the concentrations at each stage are found by combining material balance and equilibrium relations. These difference equations are approximated by differential equations. This assumption has been shown to yield good results when the separation factors are small as they are in gaseous diffusion.

The differential equations are:

$$\frac{dx}{dn} = \psi_1 x(1 - x) - \psi_2 xy - \frac{(x_p - x)P}{L}, \qquad \begin{array}{l} x(0) = x_o \\ x(N) = x_p \end{array} \tag{1}$$

and

$$\frac{dy}{dn} = \psi_2 y(1 - y) - \psi_1 xy - \frac{(y_p - y)P}{L}, \qquad \begin{array}{l} y(0) = y_o \\ y(N) = y_p \end{array} \tag{2}$$

where N is the total number of stages in the cascade.

A single stage of an isotope separation cascade will produce concentration differences in some or all of the components. These differences depend on equilibrium relations only and may bear little relation to the desired cascade separation. The proper flow distribution in a cascade depends on the degree and nature of the separation required.

Two separation problems are considered in this paper. The first is the problem of enriching the lightest of three isotopes. The model chosen is the U_{235}, U_{236}, U_{238} system and the binary U_{235}, U_{238} system is used as a comparison. The effect of contamination by U_{236} is studied.

In the second problem the middle isotope is to be enriched. Again the U_{235}, U_{236}, U_{238} system is used but this time the U_{236}, U_{238} binary system is used as a basis for comparison, and the U_{235} is treated as a contaminant.

DISCUSSION OF RESULTS

Light Isotope Separation Problem*

The conditions are:

$$x_o = 1\%$$

$$x_p = 60\%$$

$$k = 2/3,$$

with the cascade operating over an infinite reservoir with concentrations x_o, y_o. The y_o concentration is varied and the effect of this contamination on the total flow and number of stages is calculated.

Figure 3 shows the effect of y_o on total flow for the matched x, matched R, and minimum power cascades. There is little difference in the power requirements for the three cascades for $y_o \leq 1.5\%$. The behavior of the matched R cascade at $y_o = 2\%$ is explained in Appendix I.

---

* x is the mol fraction of U_{235}, y is the mol fraction of U_{236}.

Figure 4 shows the effect of y_o on the number of stages and shows a greater difference between the proposed modes of operation than was the case for the total flow. The matched x cascade does somewhat better than the matched R cascade in this respect. It is of interest to note that the minimum power cascades give the fewest stages, thus indicating that the number of stages required for minimum power separation of ternary mixtures increases moderately with third isotope contamination.

Figure 5 represents x concentration versus stage number for one specific y_o with the associated binary ideal gradient plotted to provide a comparison. The matched x gradient is always less steep than the binary as may be seen from the differential equation. The minimum power cascade has steeper gradients at the bottom of the cascade than does the ideal binary.

Figure 6 represents y concentration versus stage number for the same case as Figure 3. The maximum concentration of y in the matched x cascade is seen to be considerably higher than in the minimum power cascade.

Figure 7 shows the flow versus stage number for the case of Figures 5 and 6 with the binary ideal taper represented for comparison. The minimum power cascade has higher flows at the bottom than the matched x cascade. This is related to the fact that the minimum power cascade requires fewer stages.

Figures 8 and 9 show the power requirements and stages needed for matched x and minimum power separation as a function of y_o from $0 \leq y_o \leq 99\%$. The difference in the number of stages for the two modes of operation is of interest since it indicates the engineering requirement of fewer stages is not incompatible with increased efficiency. The minimum power curves were not obtained beyond $y_o = 30\%$.

Figure 10 is a plot of $\rho = dx/dn$ (operating) $\div$ dx/dn (total reflux). For a matched x cascade, $\rho(x) = 1/2$. The point of minimum ρ for the matched R cascade is the point of maximum y. The minimum power cascade has less flow in the top and more in the bottom than does the matched x cascade. The matched R cascade in this plot is operating with $y = 1.975\%$ which is close to the limit of 2%. The curve of ρ versus x for the matched R cascade with smaller y_o has the same shape as shown except that its departure from the line $\rho = 1/2$ is smaller.

Middle Isotope Problem*

The conditions are:

$$x_o = 1\%$$
$$x_p = 40\%$$
$$k = 3/2,$$

---

* x is the U_{236} mol fraction and y is the U_{235} mol fraction.

with the plant operating over an infinite reservoir with concentrations x_o, y_o. The y_o concentration is varied and the effect of this contamination on the total flow and number of stages is calculated. This problem is studied only for those concentrations y_o for which a positive dx/dn can be obtained throughout the cascade.

Figure 11 shows the effect of y_o on total flow for matched x, matched R, and minimum power cascades. There is not much difference in the flow requirements for the three modes shown. The matched R cascade is slightly better than the matched x cascade. For the middle isotope problem, the matched R cascade has higher interstage flow rates than the matched x cascade. In the light isotope situation, the reverse is true.

Figure 12 shows the effect of y_o on stage requirements. This is again the most striking difference between the proposed methods of operation. For this case, the matched R cascade is markedly better than the matched x cascade.

Figure 13 shows the x concentration versus stage number obtained for matched x, matched R and minimum power cascades with the same y_o. Note the marked similarity between the matched R and the minimum power cascade.

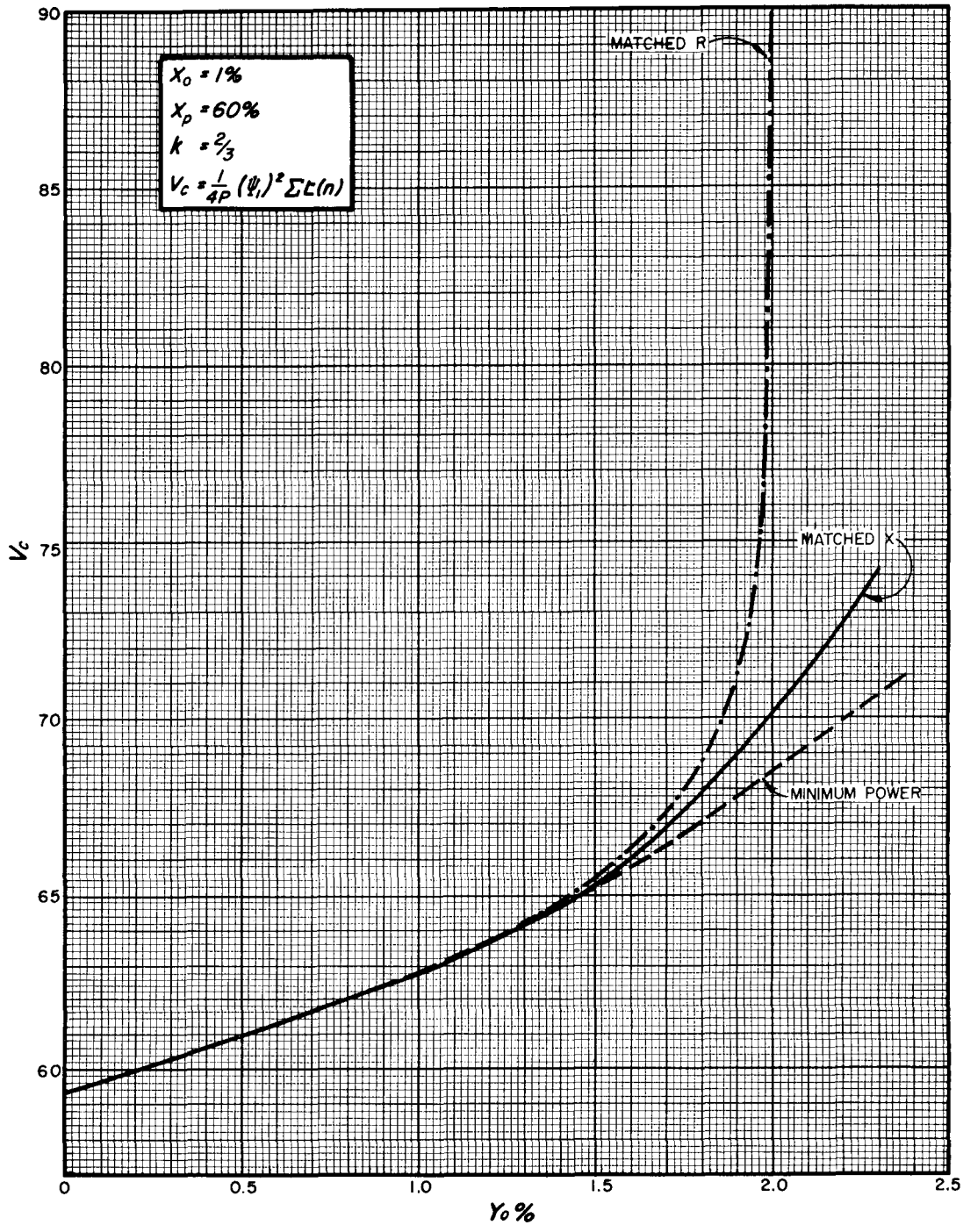
Figure 15 shows the y concentration versus stage number obtained for the same three cascades.

Figure 15 shows the flow versus stage number for the above case number and demonstrates the remarkable correspondence between the minimum power and matched R cascade.

Figure 16 is a plot of ρ versus x for the three cascades and further shows the correspondence between the matched R, matched x, and minimum power cascades for the middle isotope problem.

GENERAL COMMENTS

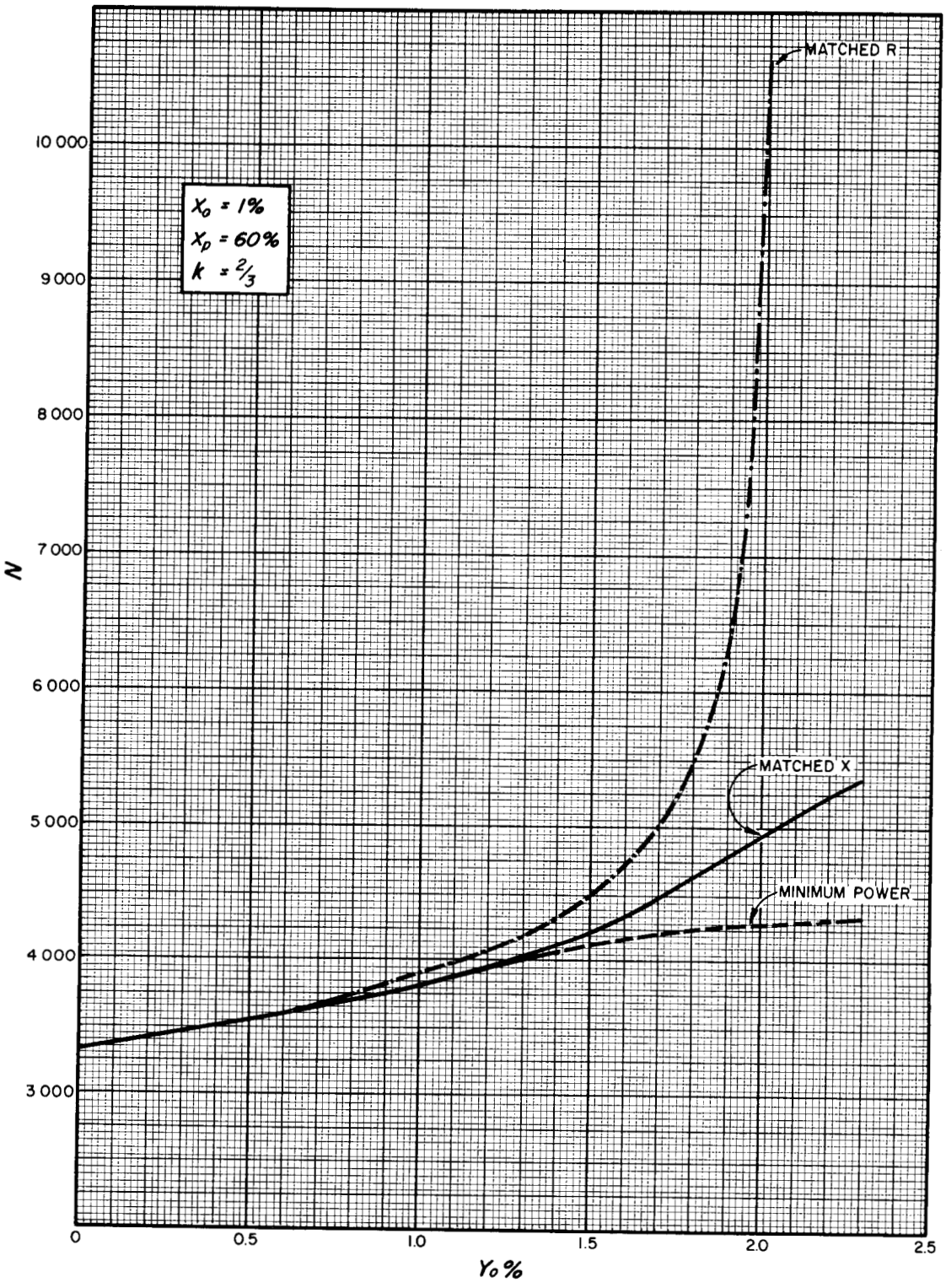
For the cases studied, the results indicate that in a light isotope problem the matched x cascade is better basis for design than the matched R and that the reverse is true for the middle isotope problem. In any case the matched R cascade is easily calculated using the methods for Ref. (1) and should be used as a comparison in any design problem. The relatively good agreement between the total flow requirements of all three cascades indicates that the losses incurred by squaring off sections for an actual plant should be small, as they are in a cascade used for binary separation. The results also indicate that the optimum flow distribution in an actual cascade should be calculated with the effect of all isotopes considered.



LIGHT ISOTOPE PROBLEM

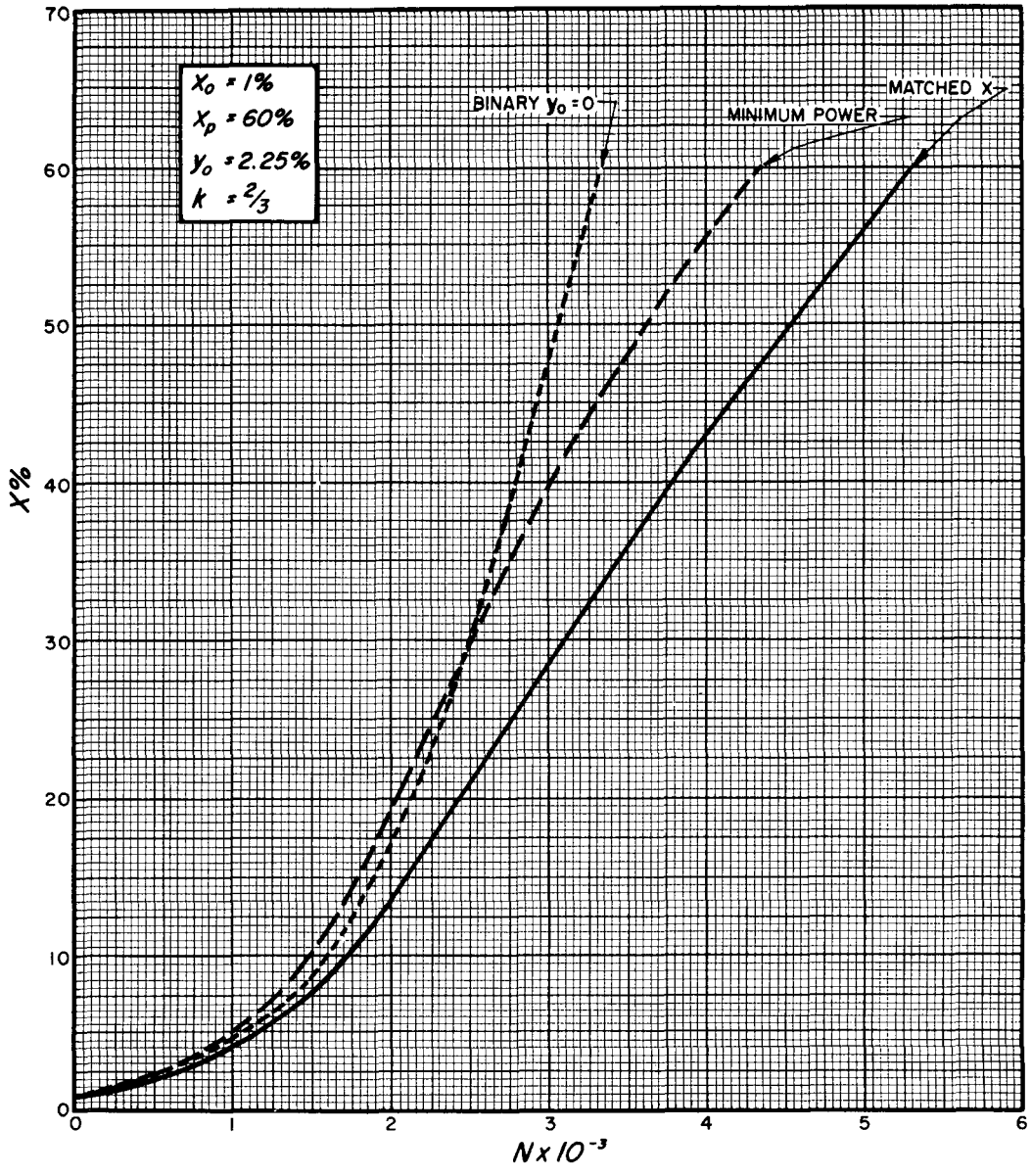
TOTAL FLOW FOR MATCHED X, MATCHED R, AND MINIMUM POWER CASCADES vs BOTTOM CONCENTRATION OF MIDDLE ISOTOPE

Figure 3



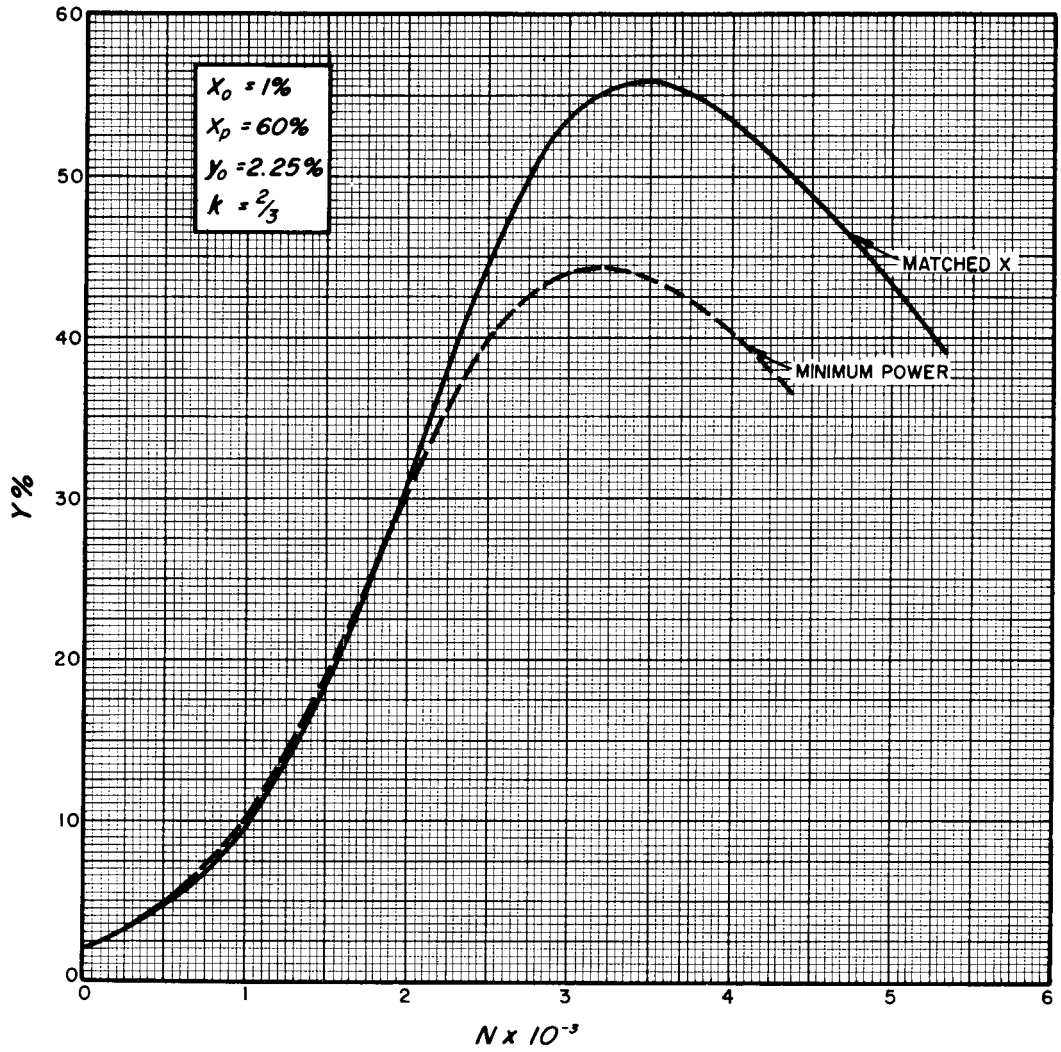
LIGHT ISOTOPE PROBLEM
NUMBER OF STAGES FOR MATCHED X, MATCHED R, AND MINIMUM POWER CASCADES vs BOTTOM CONCENTRATION OF MIDDLE ISOTOPE

Figure 4



LIGHT ISOTOPE PROBLEM
CONCENTRATION GRADIENTS FOR MATCHED X AND MINIMUM POWER CASCADES AND THE RELATED BINARY IDEAL CASCADE

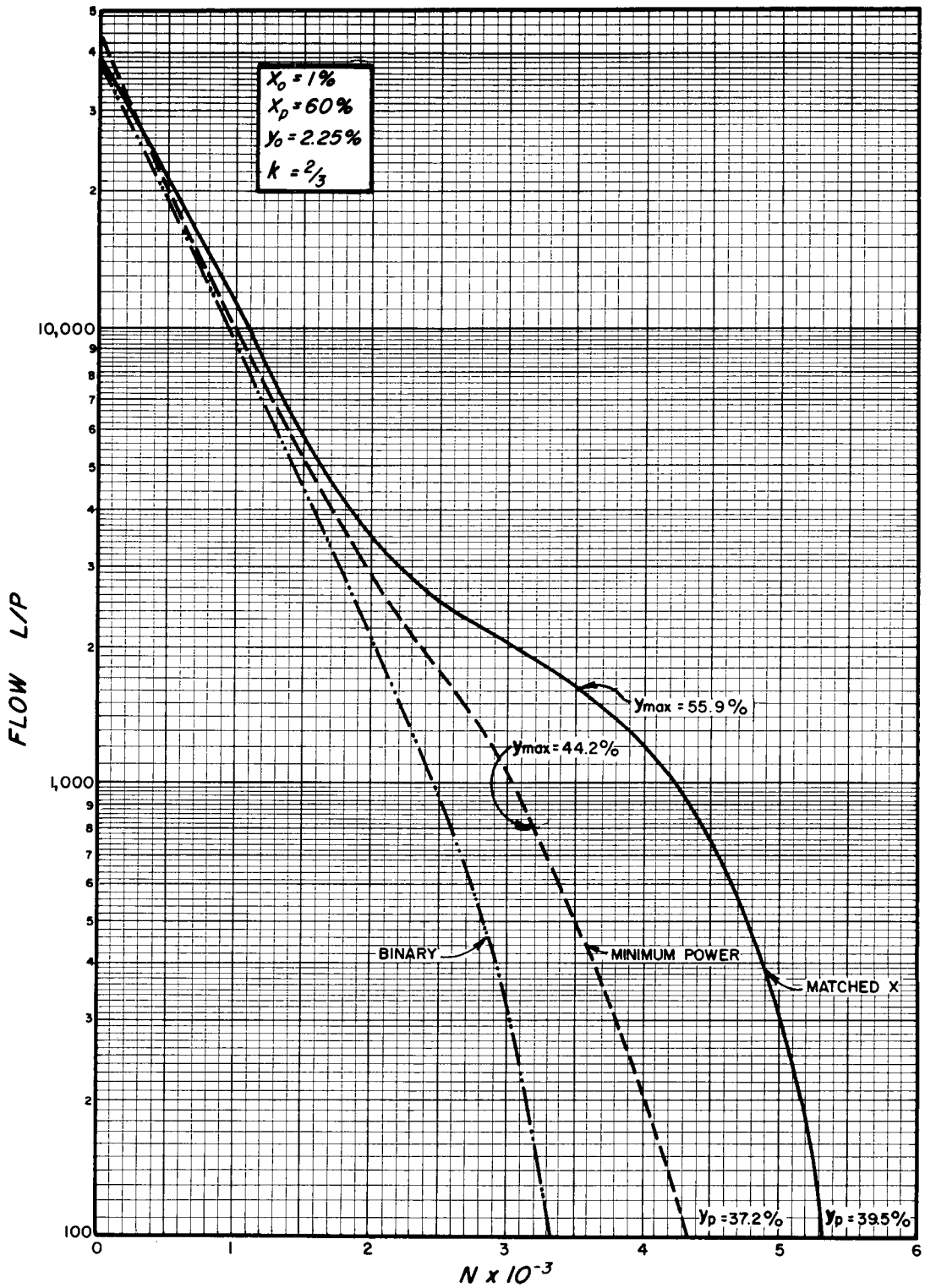
Figure 5



LIGHT ISOTOPE PROBLEM

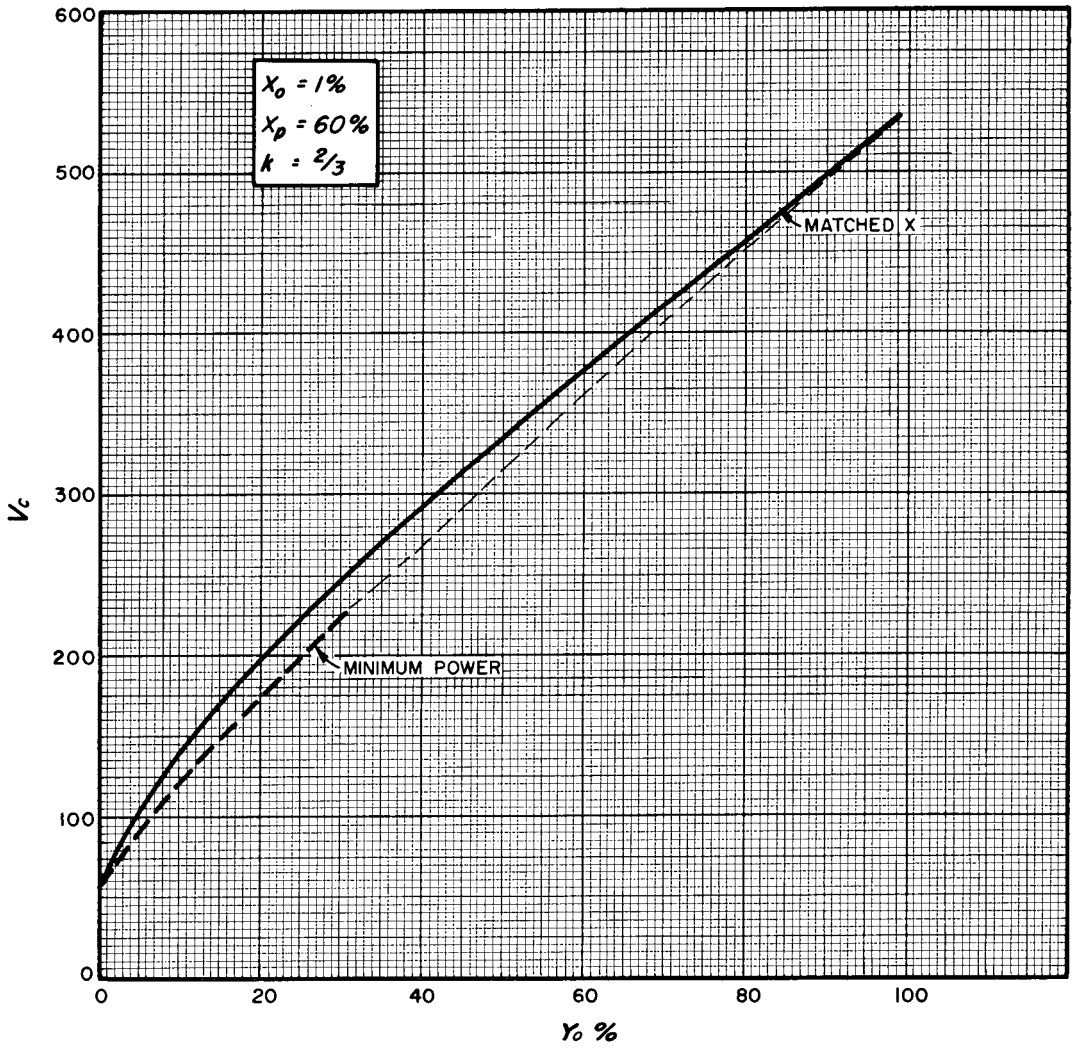
MIDDLE ISOTOPE CONCENTRATION GRADIENTS FOR MATCHED X AND MINIMUM POWER CASCADES

Figure 6



LIGHT ISOTOPE PROBLEM
FLOW vs STAGE NUMBER FOR MATCHED X, MINIMUM POWER, AND IDEAL BINARY CASCADE

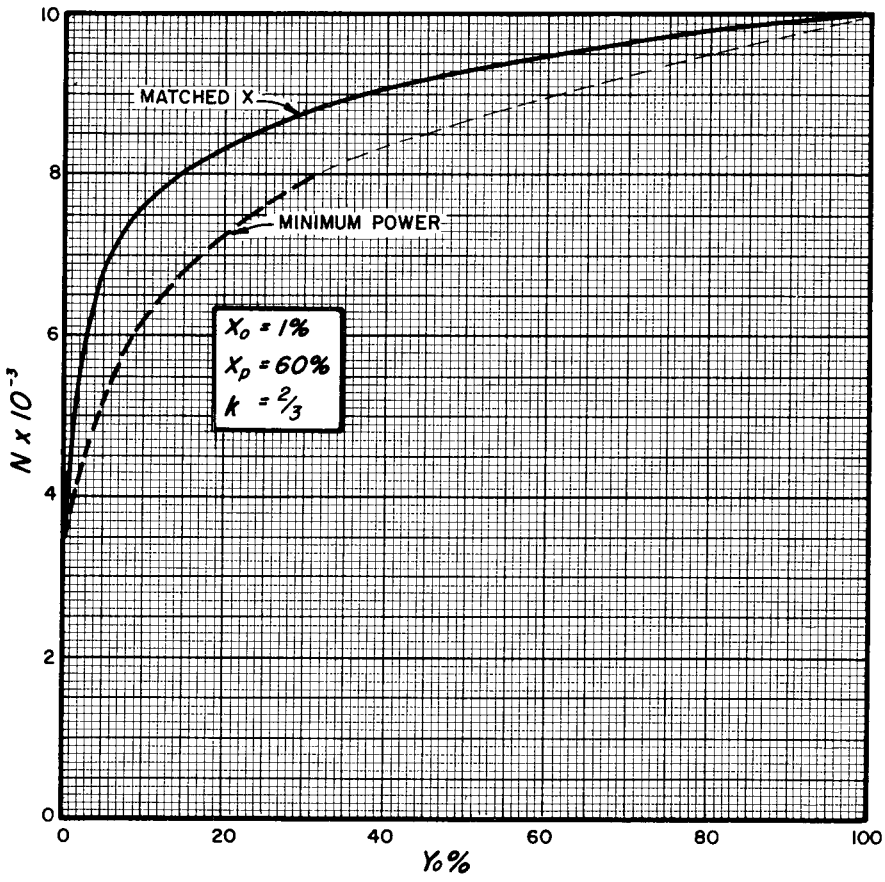
Figure 7



LIGHT ISOTOPE PROBLEM

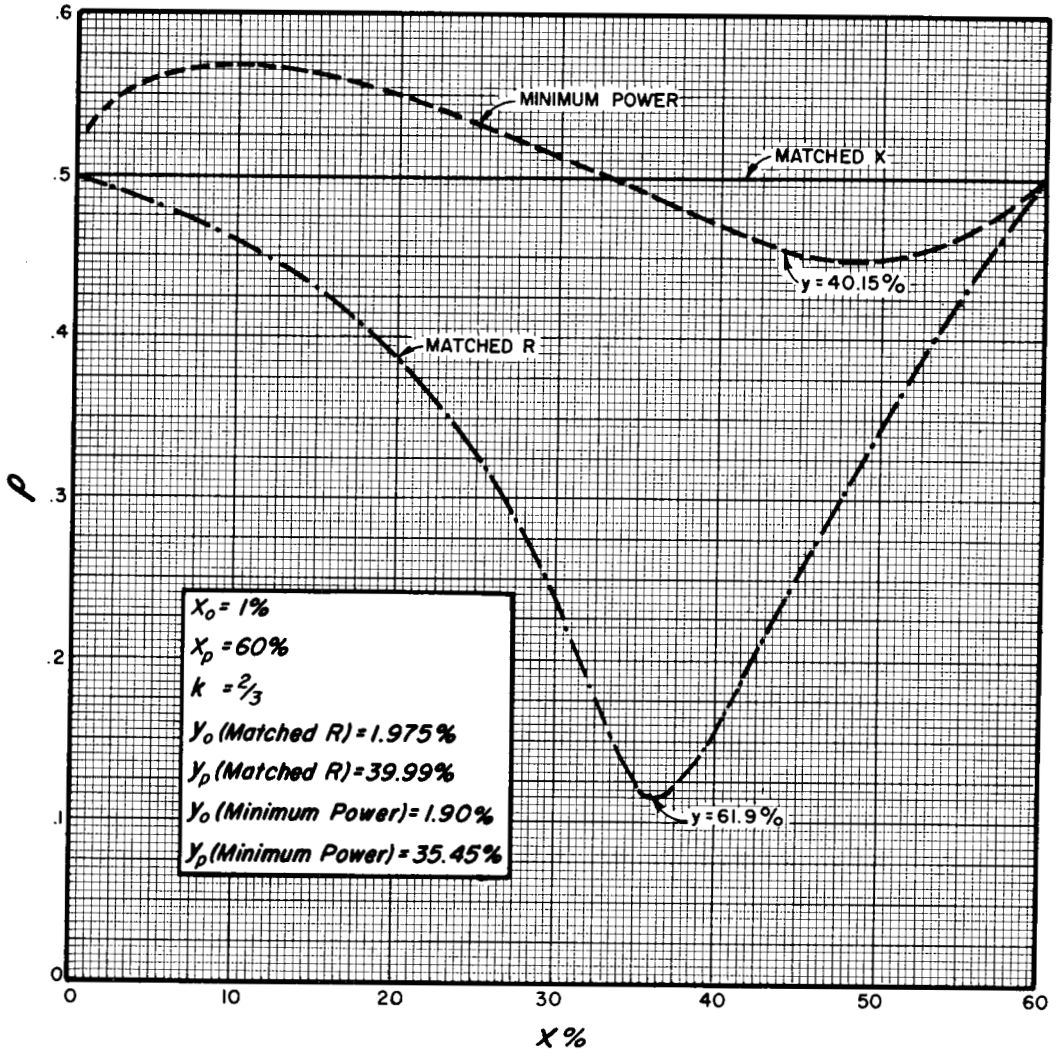
TOTAL FLOW REQUIREMENTS FOR MATCHED X AND MINIMUM POWER CASCADES vs BOTTOM CONCENTRATION OF MIDDLE ISOTOPE

Figure 8



LIGHT ISOTOPE PROBLEM
NUMBER OF STAGES REQUIRED FOR MATCHED X AND MINIMUM POWER CASCADES vs BOTTOM CONCENTRATION OF MIDDLE ISOTOPE

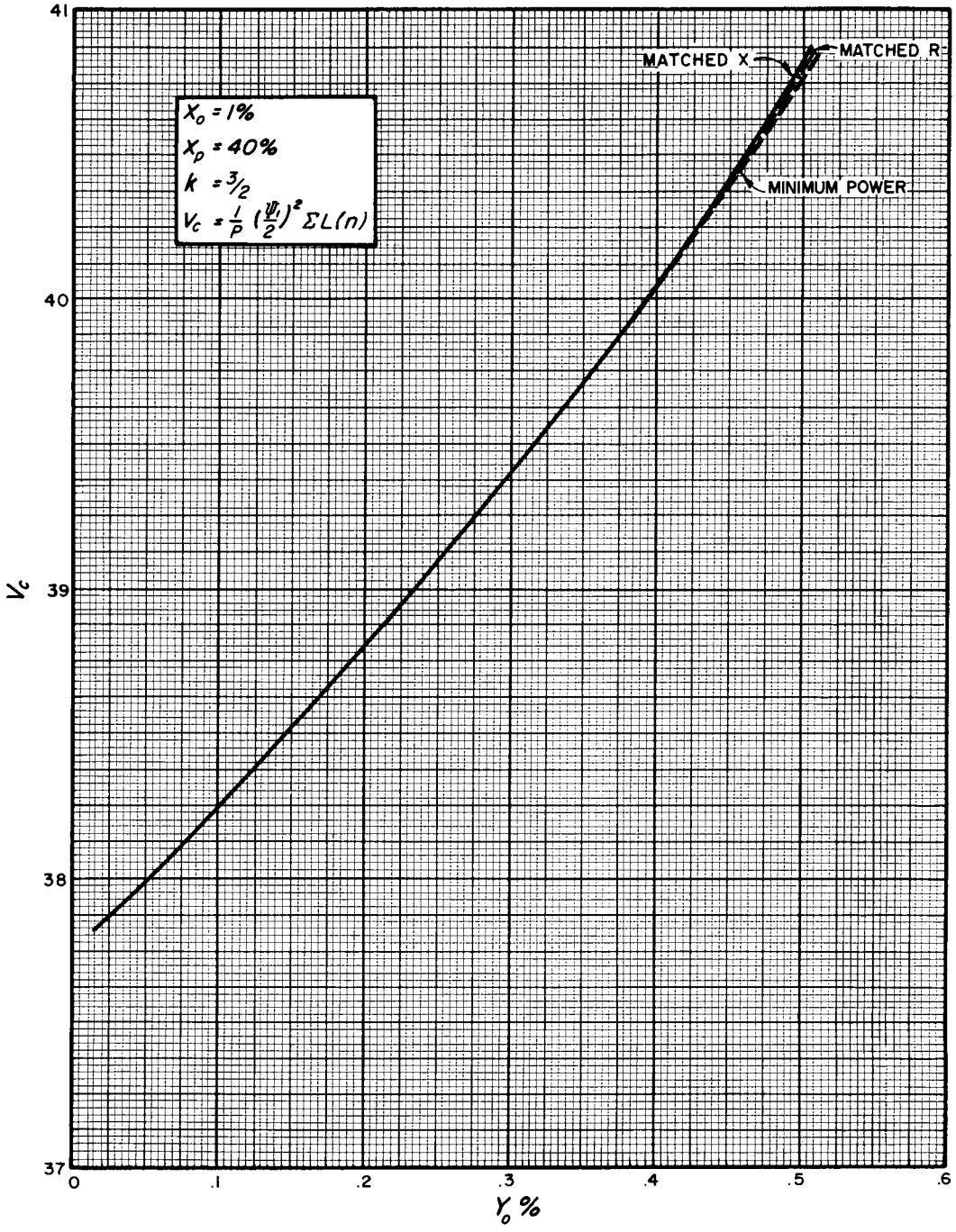
Figure 9



LIGHT ISOTOPE PROBLEM

$$\rho = \frac{dx}{dn}\,(\text{OPERATING}) \Big/ \frac{dx}{dn}\,(\text{TOTAL REFLUX})$$

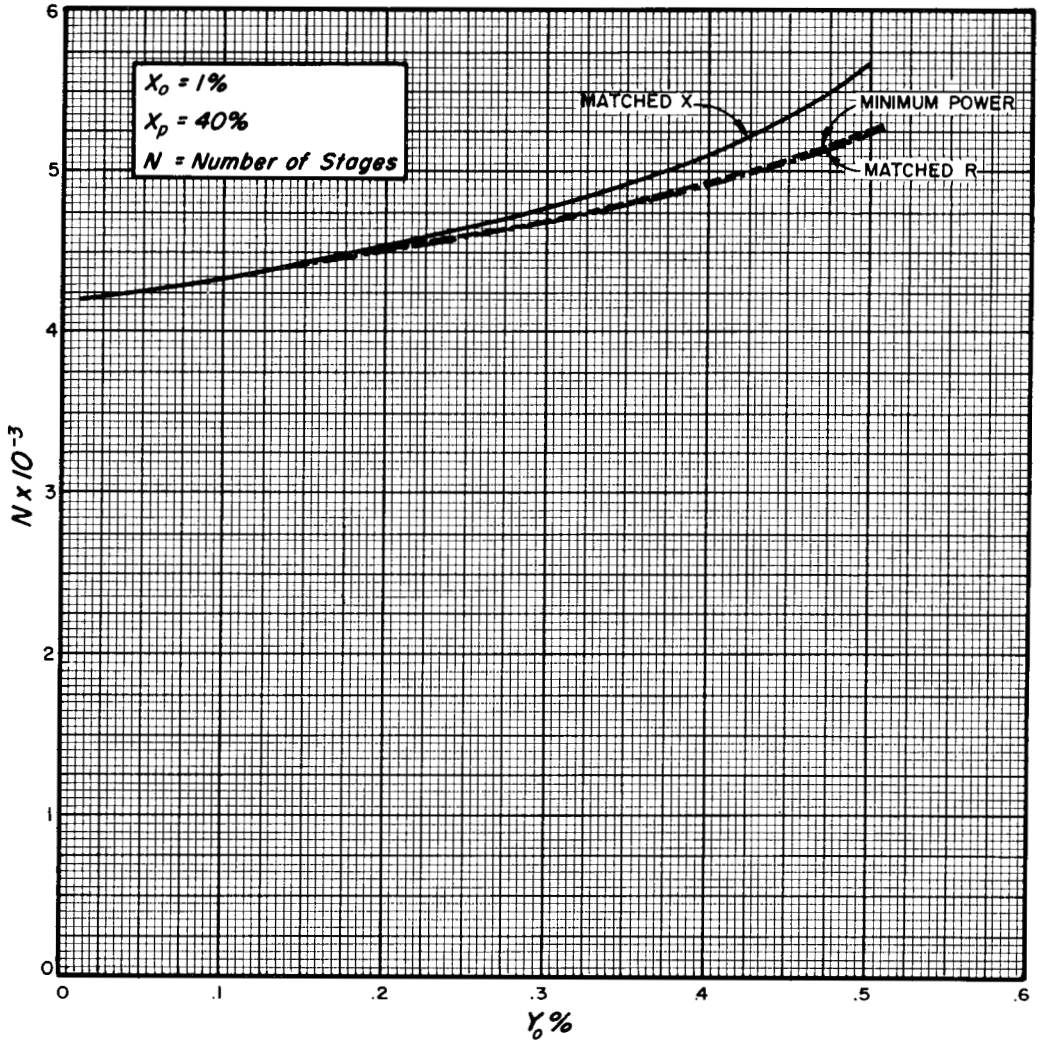
Figure 10



MIDDLE ISOTOPE PROBLEM

TOTAL FLOW vs BOTTOM CONCENTRATION OF LIGHT ISOTOPE

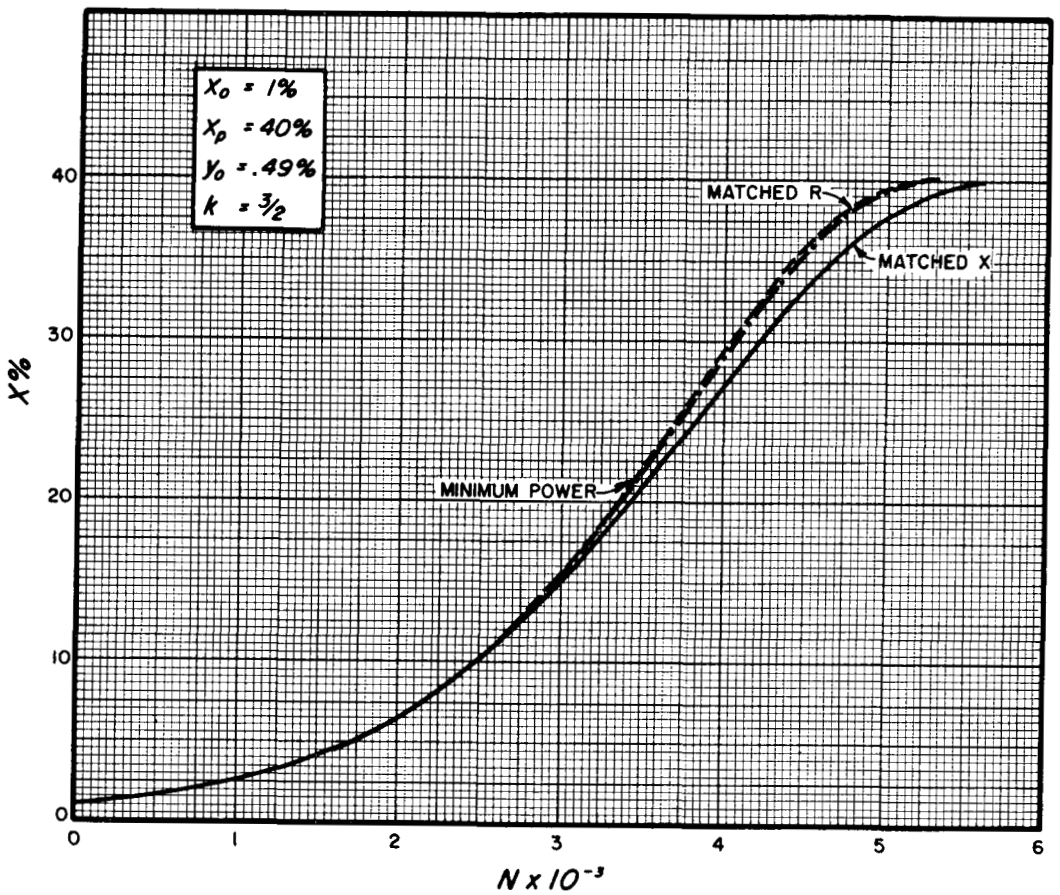
Figure 11



MIDDLE ISOTOPE PROBLEM

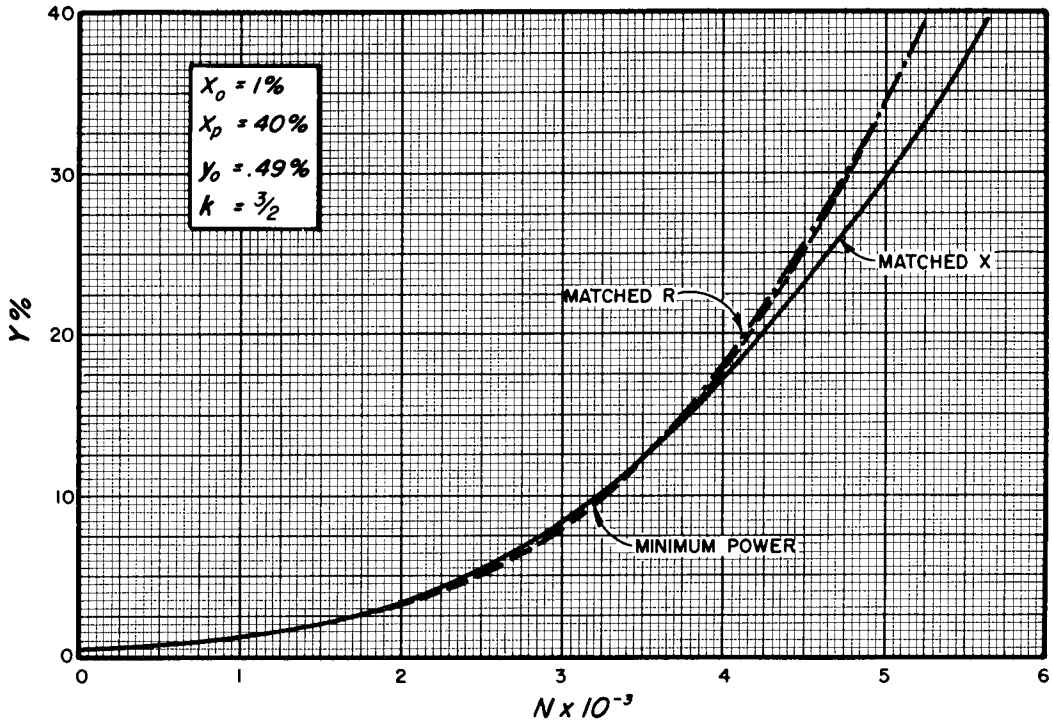
NUMBER OF STAGES vs BOTTOM CONCENTRATION OF LIGHT ISOTOPE

Figure 12



MIDDLE ISOTOPE PROBLEM
MIDDLE ISOTOPE GRADIENT
X CONCENTRATION vs STAGE NUMBER

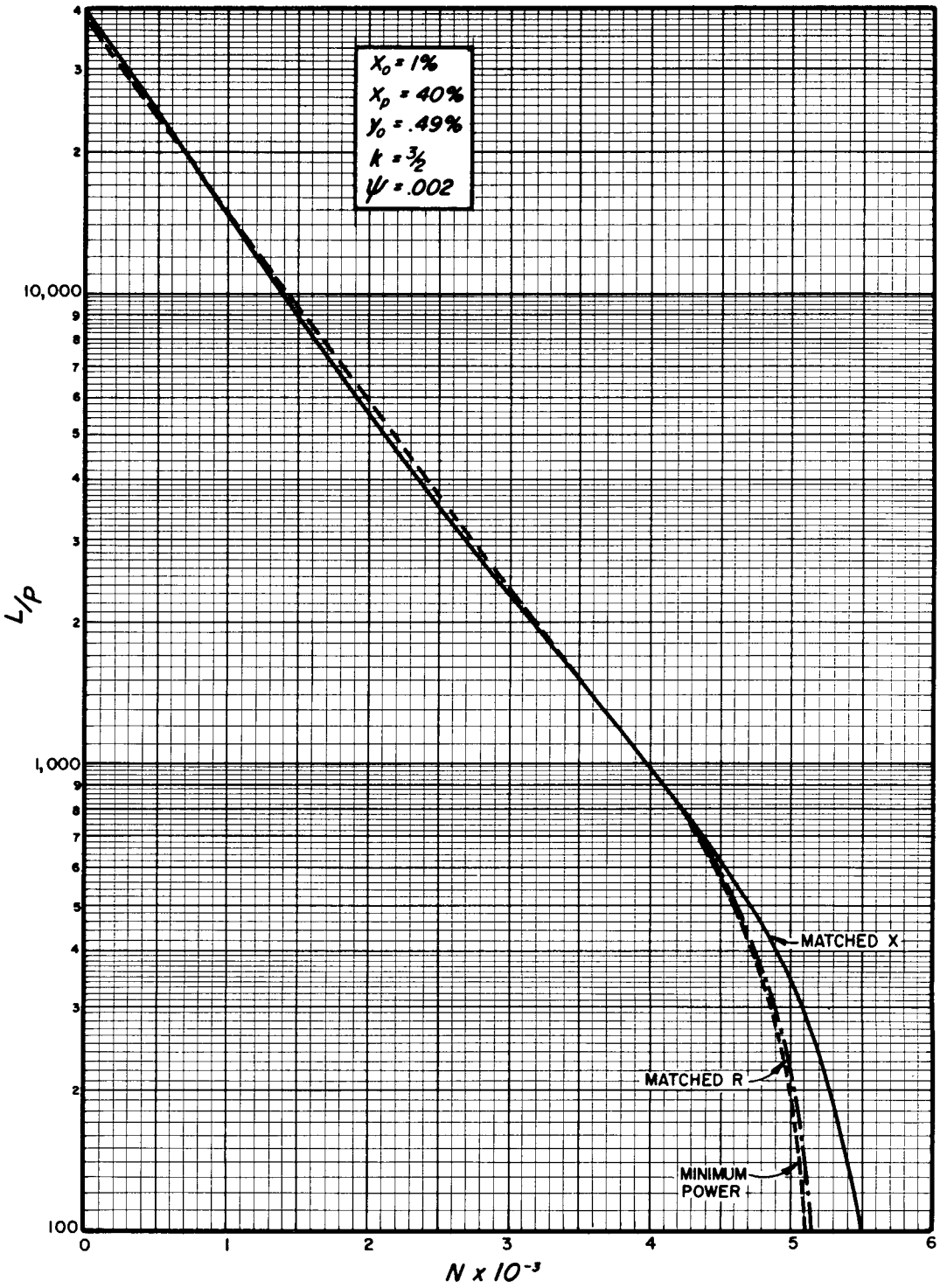
Figure 13



MIDDLE ISOTOPE PROBLEM

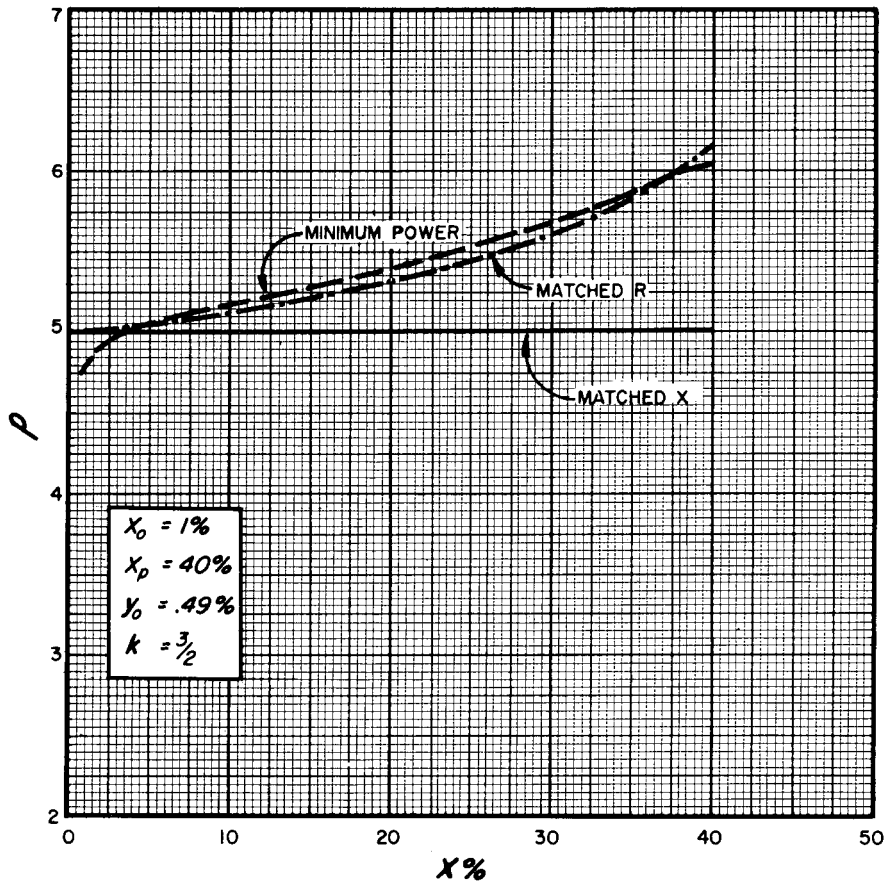
LIGHT ISOTOPE GRADIENT Y CONCENTRATION vs STAGE NUMBER

Figure 14



MIDDLE ISOTOPE PROBLEM
FLOW vs STAGE NUMBER FOR MATCHED X, MATCHED R, AND MINIMUM POWER CASCADES

Figure 15



MIDDLE ISOTOPE PROBLEM

$$\rho = \frac{dx}{dn}\,(\text{OPERATING}) \Big/ \frac{dx}{dn}\,(\text{TOTAL REFLUX})$$

Figure 16

MATHEMATICAL DEVELOPMENT OF

CONSIDERED MINIMUM FLOW CASCADES

Material balance equations and equilibrium relations are combined to yield enrichment equations in a gaseous diffusion cascade. Equilibrium relations are analagous to vapor phase-liquid phase equilibrium relations in a distillation tray. These two considerations do not suffice to specify completely the cascade design, since the energy input per stage or interstage flow rate must also be set.

In a cascade connected as in Figure 1, it is possible to specify either the interstage flow rate or some relation between the two streams which are mixed at the inlet of each stage. The concept of a match function as set forth in Ref. (1) leads to a specification of the latter kind. Matched mol fraction cascades and matched abundance ratio cascades are specified in terms of match functions.

In a binary cascade the concepts of match function, value function and minimum power operation are all related, i.e., the proper match function is the mol fraction (or abundance ratio). Cascades so constructed admit value functions and the flow taper so specified is the minimum power flow taper.

In Ref. (1) it is shown that value functions applicable to multicomponent cascades are obtained from matched abundance ratio cascades. One purpose of this study is to find out if matched R cascades have the same minimum power property as their binary counterpart. Although the matched x cascade does not provide a value function, it too has the ideal cascade as its binary counterpart and because of this, it too was considered relative to the minimum power problem.

The curves presented demonstrate that neither the matched R cascade nor the matched x cascade yields the optimum flow taper, and show some conditions under which these cascades provide suitable approximations to the minimum power cascade.

The end conditions specified greatly affect the solution obtained. In the problem studied, the reservoir concentrations and the product concentration of the desired isotope are kept fixed. These results differ from those which would be obtained by fixing all product concentrations and one reservoir concentration. A crude analogue which demonstrates some of the differences between the two above stated problems is shown in Figure 18.

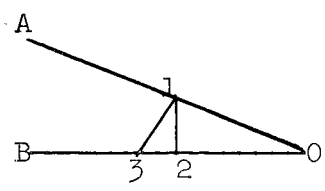


Figure 18

In the figure, line (3,1) is the shortest distance between point 3 and the line (0,A) and line (1,2) is the shortest distance between point 1 and the line (0,B). This example shows that the choice of which end point of an extremal is to be fixed may change the solution to the problem.

The problem studied in this paper may be described as follows:

Out of all of the possible flow tapers L(n) which will provide a solution to equations (1) and (2) choose that flow taper for which the total flow as measured by

$$V_c^* = \int_0^N L(n)dn$$

is the least.

Given any admissible flow taper, i.e., one which permits the solution of equations (1) and (2) to be obtained, a value for the integral V_c^* may be calculated. The problem is to choose from the class of admissible flow tapers that one for which the integral V_c^* is least.

The calculation scheme is as follows.

(1) Find any admissible flow taper,

(2) Calculate V_c^*,

(3) Improve the flow taper,

(4) Calculate V_c^*.

The successful application of this scheme requires first that an admissible flow taper be specified and second, that a method be found which will systematically improve the flow taper. Steps (3) and (4) may then be repeated until no further improvement in V_c^* is obtained.[1] The initial flow taper is found by using the matched x cascade equations and the principles of the calculus of variations are applied to the problem of improving the flow taper.

The numbers V_c^* are scaled for convenience and the measure of total work used is

$$V_c = \frac{1}{P} \left(\frac{\psi_1}{2}\right)^2 V_c^*.$$

---

1. It can be shown that V_c^* is bounded below by its value for the appropriate binary cascade. This assures the convergence of any decreasing sequence obtained by the calculation.

The problem studied may be expressed as follows: find numbers N and y_p and a function L(n) such that

$$\left(\frac{1}{P}\right)\left(\frac{\psi_1}{2}\right)^2 \int_0^N L(n)dn = V_c$$

is a minimum subject to the constraints (1) and (2). The function L(n) is also restricted to be continuous* and such that

$$\frac{dx}{dn} > 0.$$

The second restriction is physically plausible since the cascade is required to enrich the isotope whose concentration is denoted by x. This restriction permits the problem to be transformed so as to eliminate the parameter n.

With the following definitions:

(1) $g(x,y) = \psi_1 x(1-x) - \psi_2 xy,$

(2) $h(x,y) = \psi_2 y(1-y) - \psi_1 xy,$

(3) $\rho = \frac{1}{g}\left(g - \frac{(x_p - x)P}{L}\right),$

and the appropriate change of variables, the problem may be written as follows. Find a number y_p and a function $\rho(x)$ such that

$$V_c = \int_{x_c}^{x_p} \frac{x_p - x}{\rho(1 - \rho)g^2}\, dx \tag{3}$$

is a minimum subject to

$$\frac{dy}{dx} = \frac{1}{\rho}\left(\frac{h}{g} - \frac{y_p - y}{x_p - x}\right) + \frac{y_p - y}{x_p - x}, \tag{4}$$

$$y(x_o) = y_o.$$

It can easily be shown that for $y_o = 0$, the answer to this problem is

$$y_p = 0 \text{ and } \rho(x) = \frac{1}{2}.$$

---

* The continuity of L(n) and of dy/dx can be demonstrated by using the Weierstrass-Erdmann corner conditions. Cf. Bliss "Lectures on the Calculus of Variations" Chicago 1946, in particular Chapter 1, Section 6.

The result produces the ideal binary cascade. The value $\rho = 1/2$ implies that the mixed streams at the stage links have the same x concentration, i.e., no mixing loss in the desired isotope is incurred.

With the definitions

$$f(x,y,x_p,y_p,\rho) = \frac{1}{\rho}\left(\frac{h}{g} - \frac{y_p - y}{x_p - x}\right) + \frac{y_p - y}{x_p - x} \tag{5}$$

and

$$G(x,x_p,y,\rho) = \frac{x_p - x}{\rho(1 - \rho)\, g^2}, \tag{6}$$

the problem may be written:

$$\begin{aligned} V_c &= \int_{x_o}^{x_p} G\, dx, \\ \frac{dy}{dx} &= f \text{ with } y(x_o) = y_o. \end{aligned} \tag{7}$$

The range of interest of $\rho(x)$ is:

$0 < \rho < 1$ since

$\rho = 0$ implies $\frac{dx}{dn} = 0$ and

$\rho = 1$ implies $L(n) = \infty$.

The first variation of V_c is calculated by considering variations in ρ of the specific form,

$$\rho^* = \rho + \epsilon\mu, \tag{8}$$

where μ is to be a continuous function of x.

From (7) and (8) one obtains:

$$\Delta V_c = \left.\frac{\partial V_c}{\partial \epsilon}\right|_{\epsilon=0} = \int_{x_o}^{x_p} \left(\frac{\partial G}{\partial \rho}\mu + \frac{\partial G}{\partial y}\left.\frac{\partial y^*}{\partial \epsilon}\right|_{\epsilon=0}\right) dx, \tag{9}$$

with the conditions

$$\left.\frac{\partial}{\partial \epsilon}\frac{dy^*}{dx}\right|_{\epsilon=0} = f_\rho \mu + f_y \left.\frac{\partial y^*}{\partial \epsilon}\right|_{\epsilon=0} + f_{y_p} \left.\frac{\partial y_p^*}{\partial \epsilon}\right|_{\epsilon=0} \tag{10}$$

and

$$\left.\frac{\partial y^*}{\partial \epsilon}\right|_{\epsilon=0}(x_o) = 0 \tag{11}$$

where $\frac{dy^*}{dx} = f(x,y^*,s_p,y_p^*,\rho^*)$ with $y^*(x_o) = y_o$.

Equation (11) is not a condition on the class of admissible variations $\mu(x)$ since it is only the requirement that the bottom concentration y_o remain fixed.

If the order of differentiation be interchanged in (10) and a function $\phi(x)$ be defined by

$$\phi(x) = \left.\frac{\partial y^*}{\partial \epsilon}\right|_{\epsilon=0}, \tag{12}$$

then (10) and (11) may be written as follows:

$$^*\frac{d\phi}{dx} - f_y\phi = f_\rho\mu + f_{y_p}\phi_p, \tag{13}$$

with

$$\phi(x_p) = \phi_p \text{ and } \phi(x_o) = 0. \tag{14}$$

Equation (13) is the linear variational equation for (7). The validity of the interchange operation depends upon the continuity of both $\frac{dy}{dx}$ and $\mu(x)$.

Definition (12) is used in (9) to get

$$\Delta V_c = \int_{x_o}^{x_p} (G_y\phi + G_\rho\mu)\, dx, \tag{15}$$

and by defining,

$$D(x) = \exp\left[-\int_{x_o}^{x_p} f_y dt\right] \text{ and } D(x_p) = D_p \tag{16}$$

one finds the solution to (13) to be

$$\phi(x) = \frac{\phi p}{D(x)} D_p - \int_x^{x_p} D(t)\, f_{y_p}(t)dt - \frac{1}{D(x)} \int_x^{x_p} \mu D f_\rho dt. \tag{17}$$

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* Unless otherwise noted, the superscripts stand for partial derivatives, e.g., $f_{y_p} = \frac{\partial f}{\partial y_p}$.

Use (17) in (15) to get

$$\Delta V_c = \phi_p \int_{x_o}^{x_p} \frac{G_y}{D} \left[D_p - \int_x^{x_p} D(t) f_{y_p}(t) dt \right] dx$$

$$- \int_{x_o}^{x_p} \frac{G_y}{D} \int_x^{x_p} \mu D f_\rho dt dx + \int_{x_o}^{x_p} \mu G_\rho dx. \tag{18}$$

If the order of integration in the second integral of (18) is interchanged and the terms are collected, the first variation may be written:

$$\Delta V_c = \phi_p \int_{x_o}^{x_p} \frac{G_y}{D} \left[D_p - \int_x^{x_p} D(t) f_{y_p}(t) dt \right] dx$$

$$+ \int_{x_o}^{x_p} \mu \left[G_p - f_\rho D \int_{x_o}^{x} \frac{G_y}{D} dt \right] dx \tag{19}$$

Evaluate (17) at $x = x_o$ noting that $D(x_o) = 1$ to obtain:

$$\phi(x_o) = 0 = \phi_p \left[D_p - \int_{x_o}^{x_p} D f_{y_p} dt \right] - \int_{x_o}^{x_p} \mu D F_\rho dt. \tag{20}$$

Define,

$$K = \frac{\int_{x_o}^{x_p} \frac{G_y}{D} \left[D_p - \int_x^{x_p} D f_{y_p} dt \right] dx}{D_p - \int_{x_o}^{x_p} D f_{y_p} dx}$$

and combine (19) with (20) to eliminate ϕ_p with the result:

$$\Delta V_c = \int_{x_o}^{x_p} \mu \left[G_\rho - f_\rho D \left(\int_{x_o}^{x} \frac{G_y}{D} dt - K \right) \right] dx = 0. \tag{21}$$

The only restriction placed on μ is that of continuity and by the usual methods of the calculus of variations, the requirement that $\Delta V_c = 0$ implies that

$$G_\rho - f_\rho D \left(\int_{x_o}^{x} \frac{G_y}{D} dt - K \right) = 0. \tag{22}$$

Define,

$$H = \frac{h}{g} - \frac{y_p - y}{x_p - x} \tag{23}$$

so that

$$f_\rho = -\frac{H}{\rho^2}$$

and note that

$$G_\rho = -\frac{x_p - x}{[\rho(1-\rho)]^2 g^2}(1-2\rho).$$

Substitute these expressions into (22) and with some rearrangement, obtain the result:

$$\frac{1 - 2\rho}{(1-\rho)^2} = \frac{g^2 HD}{x_p - x}\left[\int_{x_o}^{x} \frac{G_y}{D}\, dt - K\right] \tag{24}$$

This is the equation whose numerical solution is obtained as follows:

(1) Assume $\rho(x) = \frac{1}{2}$,

(2) Calculate $y(x)$ using (7),

(3) Calculate the right hand side of (24) including K,

(4) Calculate V_c using equation (3),

(5) Solve for a new $\rho(x)$ using (24),

(6) Return to step (2) until no further change in V_c is obtained. All of the integrations and the solutions of equation (7) are obtained numerically.

It can be shown that equation (24) is a first integral of the Euler equation for the problem with ρ eliminated. This conventional Euler equation for y has a markedly non-linear character and offers difficulty at the point $x = x_p$ where a boundary condition must be applied.

APPENDIX

Limits on Enrichment Obtainable in a Matched R Cascade

It is possible to obtain any product purity less than 100 percent with a matched R cascade when the reservoir contains a binary mixture. If the contaminant is lighter than the desired isotope then there is a limit to the obtainable product purity. This limit is due to the peaking of the desired middle isotope. This enrichment would be limited for the same reason no matter what the flow distribution was in the cascade.

Under some conditions there is a limit to the obtainable enrichment in a matched R cascade when the contaminant is a middle isotope. This limitation is characteristic of the matched R cascade, i.e., the matched x cascade does not have this limitation.

To examine this, let x_o and x_p be fixed and determine the maximum value of y_o for which the matched R enrichment equations have a solution.

With

$$S_p = \frac{y_p}{1 - x_p - y_p}; \quad S_o = \frac{y_o}{1 - x_o - y_o};$$

and

$$R_p = \frac{x_p}{1 - x_p - y_p}; \quad R_o = \frac{x_o}{1 - x_o - y_o};$$

as shown in Ref. (1), the enrichment equations are:

$$S_p = \frac{R_p - R_o}{R_o^{2k}\,(R_o^{2k-1} - R_p^{2k-1})}\,(S_o)\,(2k-1) \text{ for } 2k-1 \neq 0 \qquad \text{(i)}$$

and

$$S_p = \frac{R_p - R_o}{\ln(R_p/R_o)}\,\frac{S_o}{R_o} \text{ for } 2k = 1.^{*} \qquad \text{(ii)}$$

Using the fact that in a matched R cascade

$$\frac{dR}{dn} = \frac{\psi_1}{2}\,R,$$

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* Note that these are the enrichment equations for a cascade operating over an infinite reservoir with a production rate of one and that the enrichment limits obtained are specifically for this type of operation.

one can find that

$$R_o/R_p = e^{-M} \text{ where } M = \frac{\psi_1}{2} N \qquad \text{(iii)}$$

and N is the total number of stages in the cascade. Equation (iii) may be substituted in (i) and (ii) to get

$$\frac{S_p}{R_p} = \frac{S_o}{R_o} (2k-1) \frac{1 - e^{-M}}{1 - (e^{-M})^{2k-1}} \text{ for } 2k \neq 1 \qquad \text{(iv)}$$

and

$$\frac{S_p}{R_p} = \frac{S_o}{R_o} \frac{1 - e^{-M}}{M} \text{ for } 2k=1. \qquad \text{(v)}$$

The case when $2k - 1 < 0$ may be considered by fixing $S_o/R_o = y_o/x_o$ and letting $M \rightarrow \infty$. In this case, $\lim_{M \rightarrow \infty} S_p/R_p = 0$. This means that in this case x_p can be brought arbitrarily close to 100 percent and thus no matter what the reservoir concentrations are, the product concentration is not limited below 100 percent.

On physical grounds, one would expect this to be true for $0 < k < 1$, since it is the light isotope desired in this case. However, for the matched R cascade, this conclusion is false. Consider (iv) for $1/2 < k < 1$ and note that

$$\lim_{M \rightarrow \infty} \frac{S_p}{R_p} = \frac{S_o}{R_o} (2k-1)$$

or that

$$\lim_{M \rightarrow \infty} \frac{y_p}{x_p} = \frac{y_o}{x_o} (2k-1).$$

For any fixed x_p and any cascade we must have

$$\lim_{M \rightarrow \infty} \frac{y_p}{x_p} \leq \frac{1 - x_p}{x_p}.$$

Equation (iv) may be written as

$$\frac{y_p}{x_p} = \frac{y_o}{x_o} (2k-1) \frac{1 - e^{-M}}{1 - (e^{-M})^{2k-1}} \qquad \text{(vi)}$$

and examination of this equation shows that for

$$\frac{y_o}{x_o}(2k-1) < \frac{1 - x_p}{x_p}$$

there exists a finite M and a $y_p < 1 - x_p$ such that equation (vi) has a solution. This, in turn, implies that for fixed x_p,

$$\lim_{M \to \infty} \frac{y_o}{x_o} = \frac{1 - x_p}{x_p}\left(\frac{1}{2k-1}\right) \qquad \text{(vii)}$$

In the light isotope problem studied in this paper, i.e., $x_o = 1\%$, $x_p = 60\%$ and $k = 2/3$, formula (vii) gives the result

$$\lim_{M \to \infty} y_o = 2\%$$

as is shown on the plots. The points on the asymptote were calculated using the same numerical techniques as were used in the minimum power calculation and provide some indication of the degree of accuracy obtained throughout the calculation.

The case $k > 1$ provides two distinct limits to the obtainable enrichment. These are

(1) The limit defined by equation (vii),

(2) The limit obtained in a matched R cascade for which $\frac{dx}{dn} > 0$ for all n. This limit is lower than that specified in (1).

To obtain limit (2), note that equation (1) in the body of the report is

$$\frac{dx}{dn} = \psi_1 x(1-x) - \psi_2 xy - \frac{P}{L}(x_p - x) = g - \frac{P}{L}(x_p - x)$$

The no peaking condition requires that $x_p \geq x$ and thus that $0 \leq \frac{dx}{dn} \leq g = \psi_1 \frac{x}{1-x}\left(1 - k\frac{y}{1-x}\right)$. Thus we must have $\left(1-k\frac{y}{1-x}\right) \geq 0$ for all n with the only possible point of equality at the point $x = x_p$, $y = y_p$. From this it follows that $\frac{y}{1-x} \leq \frac{1}{k}$ and from the definition of S we have

$$\frac{y}{1-x-y} = S = \frac{\frac{y}{1-x}}{1-\frac{y}{1-x}} \text{ and } \frac{y}{1-x} = \frac{S}{1+S}.$$

From this it follows that $\frac{S}{1+S} \leq \frac{1}{k}$ for all n, which is a necessary condition on S if the condition $\frac{dx}{dn} \geq 0$ is to be met throughout the cascade.

Equation (iv) may be written in the form

$$S = R \frac{S_p}{R_p} \frac{1}{2k-1} \frac{1 - (e^{-M})^{2k-1}}{1 - e^{-M}}$$

from which it follows that

$$\frac{ds}{dR} = \frac{S}{R} + R \frac{S_p}{R_p} \frac{1}{2k-1} \frac{d}{dM} \left(\frac{1 - (e^{-M})^{2k-1}}{1 - e^{-M}}\right) \frac{dM}{dR}. \qquad \text{(viii)}$$

By definition for a matched R cascade

$$\frac{dR}{dM} = \frac{dR}{dn} \frac{dn}{dM} = \frac{2}{\psi_1} \frac{dR}{dn} = \frac{2}{\psi_1} \frac{\psi_1}{2} R = R \qquad \text{(ix)}$$

and the substitution of (ix) into (viii) yields

$$\frac{dS}{dR} = \frac{S}{R} + \frac{S_p}{R_p} \frac{1}{2k-1} \frac{d}{dM} \left(\frac{1 - (e^{-M})^{2k-1}}{1 - e^{-M}}\right). \qquad \text{(x)}$$

Examination of equation (x) shows that for $2k - 1 > 1$

$$\frac{dS}{dR} > 0.$$

This, in turn, implies that a necessary condition on the problem is

$$\frac{S_p}{1+S_p} \leq \frac{1}{k}. \qquad \text{(xi)}$$

The following analysis will establish for fixed x_p and x_o the maximum value of y_o for which no peaking of the x gradient will occur in a matched R cascade. Once the relationship between y_o and x_p is established, it may be used in reverse to determine the limit on x_p under the specified conditions.

Expression (xi) shows that the appropriate limit for S_p is $S_p = \frac{1}{k-1}$, and with x_p fixed we have

$$x_p = \frac{R_p}{1+R_p+S_p} = \frac{R_p}{(1+R_p) + (\frac{1}{k-1})} = \frac{R_p}{R_p + \frac{k}{k-1}}$$

or

$$R_p = \frac{k}{k-1} \left(\frac{x_p}{1-x_p}\right).$$

Thus

$$\lim_{M \to \infty} \frac{S_p}{R_p} = \frac{1}{k-1} \left(\frac{1-x_p}{x_p}\right) \left(\frac{k-1}{k}\right) = \frac{1 - x_p}{x_p} \frac{1}{k} = \lim_{M \to \infty} y_o \left(\frac{2k-1}{x_o}\right)$$

or

$$\lim_{M \to \infty} y_o = \left(\frac{1-x_p}{x_p}\right) \left(\frac{1}{k}\right) \left(\frac{1}{2k-1}\right) x_o.$$

In the middle isotope case studied, i.e., $x_p = 40\%$, $x_o = 1\%$, and $k = 3/2$, this result is

$$\lim_{M \to \infty} y_o = \left(\frac{3}{2}\right) \left(\frac{1}{3/2}\right) \left(\frac{1}{2}\right) = .5\%$$

as shown on the plots.

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