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Informal Report

CALCULATED FREQUENCIES AND PEAK FIELDS OF TE_{01n} AND TE_{11n} MODES IN CYLINDRICAL CAVITIES

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Abstract

In this report some formulas are collected which are often used in evaluating the rf properties of cylindrical cavities. In two tables the values of frequency, geometry factor and peak field for TE_{01n} and TE_{11n} modes are given.

Cylindrical cavities are often used to determine the rf-superconducting properties of materials. In order to evaluate the wanted quantities, e.g., surface resistance and peak fields, one needs some theoretical numbers. These numbers and related formulas are collected in this report.[1]

1. Field Components in $TE_{\nu mn}$ Modes

$$\left.\begin{aligned} E_r &= j\,\frac{k}{k_r}\,\frac{\nu\,J_\nu(k_r r)}{k_r r}\,\sin k_z z\,\sin \nu\theta \\ E_\theta &= j\,\frac{k}{k_r}\,J_\nu'(k_r r)\,\sin k_z z\,\cos \nu\theta \\ E_z &= 0 \end{aligned}\right\} \qquad (1)$$

$$\left.\begin{aligned} H_r &= \frac{k_z}{k_r}\,J_\nu'(k_r r)\,\cos k_z z\,\cos \nu\theta \\ H_\theta &= -\frac{k_z}{k_r}\,\frac{\nu\,J_\nu(k_r r)}{k_r r}\,\cos k_z z\,\sin \nu\theta \\ H_z &= J_\nu(k_r r)\,\sin k_z z\,\cos \nu\theta \quad . \end{aligned}\right\} \qquad (2)$$

The symbols used in Eqs. (1) and (2) have the following meaning: E_r, E_θ, E_z and H_r, H_θ, H_z are the components of the electric and magnetic fields, respectively, with the cylindrical coordinates r, θ, z as shown in Fig. 1.

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1. See, for instance, H.G. Unger, Elektromagnetische Wellen (F. Vieweg u. Sohn, Braunschweig, 1967).

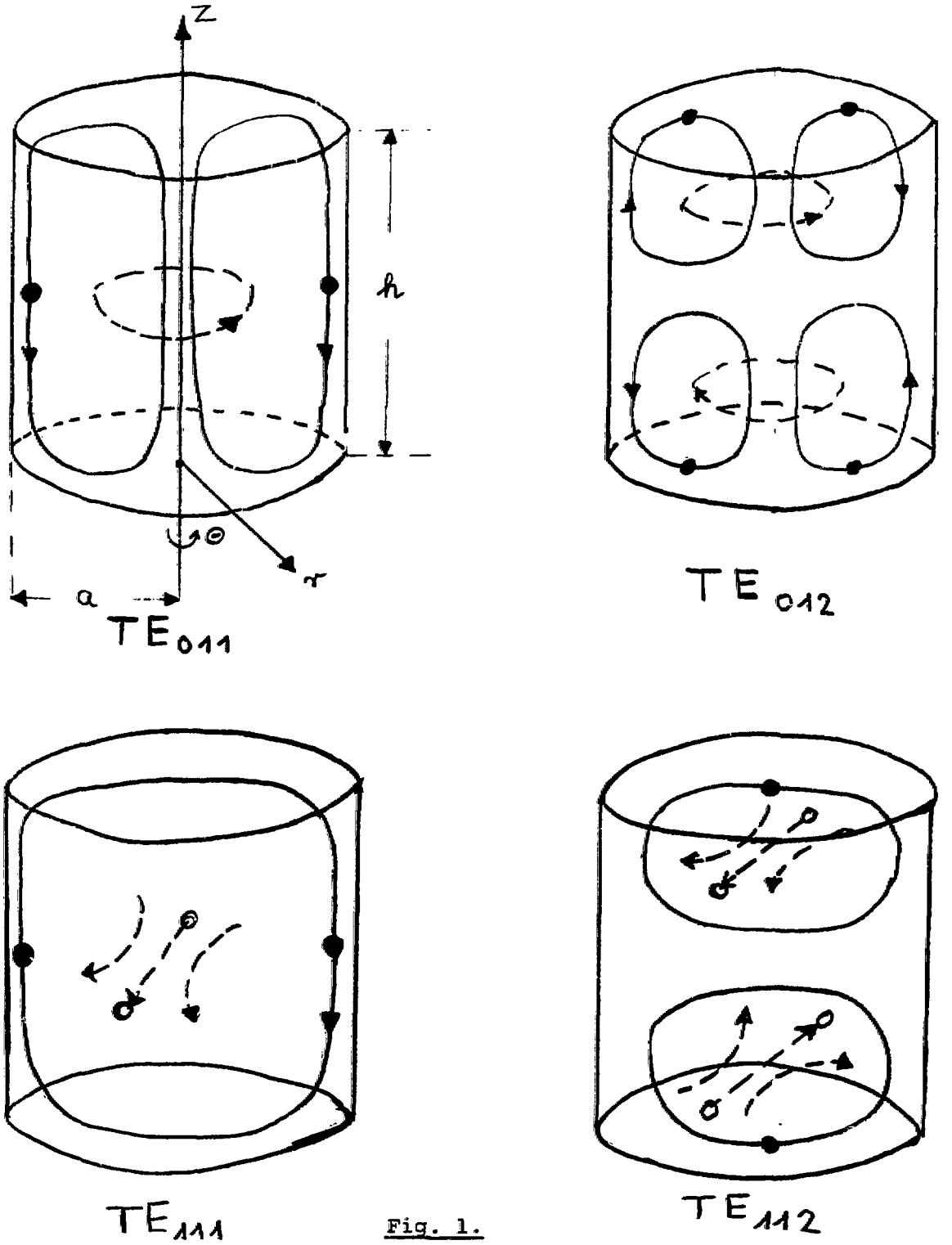

Fig. 1.

Field configuration and peak fields in several modes

—— magnetic field —●— location of peak field

– – – electric field -●- " " " "

J_ν and J'_ν are Bessel functions and their derivatives. The remaining quantities k, k_r, k_z are determined by the boundary conditions to be

$$k_r a = j'_{\nu m} \quad , \tag{3}$$

$j'_{\nu m}$ being the first zero of $J'_{\nu m}$,

$$k_z a = n\pi \frac{a}{h} \quad , \tag{4}$$

$$k^2 = k_r^2 + k_z^2 \quad ,$$

$$ka = j'_{\nu m} \sqrt{1 + \left(\frac{n\pi a}{j'_{\nu m} h}\right)^2} \quad , \tag{5}$$

and

$$\frac{k}{k_r} = \sqrt{1 + \left(\frac{n\pi a}{j'_{\nu m} h}\right)^2} \quad . \tag{6}$$

As $\omega = 2\pi f = kc$, the resonant frequency is derived from Eq. (5) to be

$$f = \frac{kc}{2\pi} \quad . \tag{7}$$

2. Geometry Factor $Q \cdot r_s$

The Q-value is defined as

$$Q_o = \frac{kW}{P} = \frac{k \cdot \frac{1}{2} \int \vec{H}\, \vec{H}^* \, dV}{r_s \int H_{\parallel} H_{\parallel}^* \, dS} \tag{8}$$

with W = energy stored and P = power lost in the cavity. $H_{\parallel}$ means the component of $\vec{H}$ parallel to the surface. r_s is the surface resistance, a material constant.

Q_o can be written as

$$Q_o = \frac{G}{r_s} \quad . \tag{9}$$

G is a factor which depends only on the geometry. For $TE_{\nu mn}$ modes one gets:

$$G = Q_o \cdot r_s = \frac{1}{2} \frac{(ka)^3 \left(1 - \frac{\nu^2}{j'^2_{\nu m}}\right)}{\nu^2 \left(\frac{k_r}{k_z}\right)^2 + j'^2_{\nu m} \left[1 + \frac{2a}{h}\left(\frac{k_r}{k_z}\right)^2 \left(1 - \frac{\nu^2}{j'^2_{\nu m}}\right)\right]} \tag{10}$$

In order to get G in ohms, the above result has to be multiplied by $Z_o = \sqrt{\mu_o/\varepsilon_o} = 376.73$ ohms.

3. <u>Peak Fields</u>

Using Eq. (8) one can write

$$k^2 P Q_o = \frac{1}{2} k^3 \int \vec{H}\,\vec{H}^* \, dV = \frac{1}{2} k^3 \int \vec{E}\,\vec{E}^* \, dV$$

$$k^2 P Q_o = \frac{1}{2} \frac{k^3 \int \vec{H}\,\vec{H}^* \, dV}{\hat{H}^2} \cdot \hat{H}^2 = G_B^{-2} \cdot \hat{H}^2 \quad . \tag{11}$$

In order to determine the peak magnetic field $\hat{H}$ from a measurement of Q_o and P, one therefore needs the factor G_B,

$$\hat{H} = G_B \cdot k \cdot \sqrt{PQ_o} \quad . \tag{12}$$

To obtain the value in convenient units it is better to use

$$\hat{B}[\text{gauss}] = C_B \cdot f[\text{GHz}] \cdot \sqrt{P[\text{watt}] \cdot Q_o[\text{millions}]} \tag{13}$$

with

$$C_B = G_B \cdot 13.569 \quad . \tag{14}$$

To evaluate G_B using Eq. (11) one needs the stored energy and the peak fields $\hat{H}$. The stored energy is given by the following expression, which is achieved by integrating Eqs. (1):

$$k^2 W = \frac{1}{2} k^3 \int \vec{E}\,\vec{E}^* \, dV = \frac{\pi}{8} \frac{h}{a} \left(\frac{k}{k_r}\right)^5 j'_{\nu m} \left(j'^2_{\nu m} - \nu^2\right) J^2_\nu(j'_{\nu m}) \times \begin{cases} 2 \; ; & \nu = 0 \\ 1 \; ; & \nu = 1 \end{cases} \quad . \tag{15}$$

In order to find $\hat{H}$ one has to look at the field components in Eqs. (1) and find the maximum amplitude. It can be seen that there are maximums at the cylinder wall (r = a) and other maximums at the top and bottom end plates.

By putting numbers into Eqs. (1) one finds that for both TE_{011} and TE_{111} modes the maximum H-field appears at $r = a$ and $Z = h/2$. Therefore, $\hat{H} = H_z\ (a, h/2)$. For $n > 1$ $\hat{H}$ shows to be H_r at $Z = 0$ and h and the radius $r = 0$ at which $J_\nu'(k_r r)$ becomes maximum.

One also shows that for all TE_{01n} modes the electric field on the surface is always zero and that for all TE_{11n} modes the peak electric field appears at the cylinder wall and is given by E_r at $r = a$ and $Z = h/2n$. The ratio $\hat{E}/\hat{H}$ in usual units is derived by

$$\hat{E}/\hat{H}\left[\frac{MV/m}{gauss}\right] = \hat{E}/\hat{H} \cdot 0.02998 \quad . \tag{16}$$

Figure 1 shows sketches of several field distributions.

The numbers calculated from the above formulas are given in Table I for the TE_{01n} modes and in Table II for the TE_{11n} modes.

Some Bessel function values often used during these calculations are:

$j_{01}' = 3.831706$ $\qquad$ $J_0(j_{01}') = -0.402759$

$j_{11}' = 1.84118$ $\qquad$ $J_1(j_{11}') = 0.58187$

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TABLE I. Calculated frequencies, geometry factors and peak fields of TE_{01n} modes in a cylindrical cavity of radius a = 69.30 mm and length h = 136.366 mm.

n	Frequency [GHz]	Geometry Factor [Ω]	G_B	C_B
1	2.858	780	0.08780	1.1913
2	3.434	930	0.07320	0.9933
3	4.223	1142	0.06491	0.8808
4	5.128	1382	0.05233	0.7230
5	6.097	1631	0.04321	0.5863
6	7.103	1916	0.03538	0.4801
7	8.134	2193	0.02942	0.3992
8	9.181	2426	0.02484	0.3371
9	10.239	2728	0.02128	0.2887
10	11.304	3045	0.01846	0.2504
11	12.376	3334	0.01619	0.2197

TABLE II. Calculated frequencies, geometry factors and peak fields of TE_{11n} modes in a cylindrical cavity of radius a = 69.30 mm and length h = 136.366 mm.

n	Frequency [GHz]	Geometry Factor [Ω]	G_B	C_B	$\hat{E}/\hat{H}$ [MV/m/gauss]
1	1.679	322	0.2362	3.205	0.02156
2	2.541	485	0.1251	1.698	0.02186
3	3.538	675	0.08203	1.113	0.02029
4	4.583	873	0.05727	0.7772	0.01971
5	5.649	1076	0.04244	0.5759	0.01944
6	6.726	1281	0.03291	0.4467	0.01929
7	7.810	1488	0.02643	0.3586	0.01920
8	8.898	1694	0.02180	0.2958	0.01914
9	9.989	1902	0.01837	0.2492	0.01910
10	11.082	2110	0.01570	0.2136	0.01907
11	12.177	2318	0.01368	0.1857	0.01905