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Radiation Transport Between Concentric Spheres

Steven W. Haan

August 8, 1994

Lawrence Livermore National Laboratory

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Radiation Transport Between Concentric Spheres*

Steven W. Haan
Lawrence Livermore National Laboratory
Livermore, CA

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This is a note originally distributed in 1983. I am re-releasing it now, with a couple of words changed, so that it can be used for test problems, distributed more openly, and so forth. One could argue that it should be published, but I do not have time to reshape it into something I would regard as suitable for journal publication. A different derivation of the same result is being published in an appendix in D.W. Phillion and S.M. Pollaine, "Dynamical Compensation of Irradiation Nonuniformities in a Spherical Hohlraum Illuminated with Tetrahedral Symmetry by Laser Beams," submitted to Phys. Plasmas.

Consider two concentric spheres with a prescribed temperature distribution on the inside of the outer sphere. What is the incoming flux distribution on the inner sphere?

This problem is fundamental to hohlraum design, since the radiation on the smaller sphere becomes more symmetric as its radius decreases. Green[1] calculated the limiting case of zero inner radius, and Kershaw[2] has done the P_1 case for arbitrary radii. Garrison[3] has done higher P_n for a slightly different, simpler problem (in my notation below, he has $\cos\theta_L \equiv 1$). I will present numerical results for P_1 through P_6 for arbitrary radii.

Let the spheres' radii be R_s and R_L. Use a coordinate system with the z axis at the point where the flux onto the small sphere is to be calculated (see Fig. 1). In these coordinates, the temperature on the outer sphere is written $T(\theta, \varphi)$. An arbitrary T can be expanded as

$$\sigma T(\theta,\varphi)^4 = \sigma T_o^4 \sum_{\ell=o}^{\infty} \sum_{m=-\ell}^{\ell} \varepsilon_{\ell m}\, e^{im\phi}\, P_\ell^m(\cos\theta) \tag{1}$$

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where $\varepsilon_{oo} = 1$.

We are usually interested in the case where T is symmetric about some axis. Let $\hat{\Omega}_o$ in direction θ_o, φ_o be a unit vector along the axis of symmetry, and let $\hat{\Omega}$ be a unit vector in direction θ, φ. Expand the radiation pattern in P_ℓ moments about its own axis of symmetry:

$$\sigma T(\theta,\varphi)^4 = \sigma T_o^4 \sum_{\ell=o}^{\infty} t_\ell P_\ell\left(\hat{\Omega}\cdot\hat{\Omega}_o\right) \quad , \tag{2}$$

where $t_o = 1$. The addition theorem for spherical harmonics says that

$$P_\ell\left(\hat{\Omega}\cdot\hat{\Omega}_o\right) = \sum_{m=-\ell}^{\ell} (-1)^m P_\ell^{-m}(\cos\theta_o)\, P_\ell^m(\cos\theta)\, e^{im(\varphi-\varphi_o)} \tag{3}$$

Thus, for this special case the coefficients in Eq. (1) are given by

$$\varepsilon_{\ell m} = e^{-im\phi_o} (-1)^m P_\ell^{-m}(\cos\theta_o)\, t_\ell \tag{4}$$

Referring again to Fig. 1, let θ_s, θ_L, and r be defined as shown. Let θ_{max} be the θ for which $\theta_s = \pi/2$. The flux onto the pole of the small sphere is

$$F = \int_o^{2\pi} d\varphi \int_{\cos\theta_{max}}^{1} d(\cos\theta)\, R_\ell^2\, \frac{\sigma T^4}{\pi}\, \frac{\cos\theta_s \cos\theta_L}{r^2} \tag{5}$$

Substitute the P_ℓ^m expansion, Eq. (1). The φ integral of $e^{im\varphi}$ vanishes unless m = 0, and so

$$F = 2\sigma T_o^4 \sum_{\ell=o}^{\infty} \varepsilon_{\ell 0} \int_{\cos\theta_{max}}^{1} d(\cos\theta)\, R_\ell^2\, \frac{\cos\theta_s \cos\theta_L}{r^2}\, P_\ell(\cos\theta) \tag{6}$$

$$F = 2\sigma T_o^4 \sum_{\ell=o}^{\infty} \varepsilon_{\ell 0} \int_{\cos\theta_{max}}^{1} d(\cos\theta) R_\ell^2 \frac{\cos\theta_s \cos\theta_L}{r^2} P_\ell (\cos\theta) \tag{6}$$

The law of cosines on the r, R_S, r_ℓ triangle in Fig. 1 implies that

$$\cos\theta = \frac{R_L^2 + R_s^2 - r^2}{2R_s R_L} \tag{7}$$

$$\cos\theta_s = \frac{R_L^2 - R_s^2 - r^2}{2R_s r} \tag{8}$$

$$\cos\theta_L = \frac{R_L^2 - R_s^2 + r^2}{2R_L r} \tag{9}$$

Using these, and defining η and z as

$$\eta = {}^{R_S}/_{R_L} \tag{10}$$

$$z = (r / R_L)^2 , \tag{11}$$

it is easy to show that

$$F = \sigma T_o^4 \sum_{\ell=o}^{\infty} \varepsilon_{\ell o} \, \mathrm{f}_\ell (\eta) \tag{12}$$

where

$$f_\ell(\eta) = \frac{1}{4\eta^2} \int_{(1-\eta)^2}^{(1-\eta)(1+\eta)} dz \left[\frac{(1-\eta^2)^2}{z^2} - 1 \right] P_\ell \left(\frac{1+\eta^2 - z}{2\eta} \right) . \tag{13}$$

For the case where T is given by Eq. (2), the flux on the inner sphere is

$$F = \sigma T_o^4 \sum_{\ell=o}^{\infty} t_\ell \, P_\ell\,(\cos\theta_o)\, f_\ell\,(\eta) \quad . \tag{14}$$

A procedure for determining f_ℓ is the following. Define

$$I_m(\eta) = \frac{1}{4\eta^2} \int_{(1-\eta)^2}^{1-\eta^2} dz \left[\frac{(1-\eta^2)^2}{z^2} - 1 \right] z^m \tag{15}$$

These are easily evaluated to be (m≥2)

$$\mathrm{I_o}\,(\eta) = 1 \tag{16}$$

$$I_1(\eta) = \frac{\left(1-\eta^2\right)^2}{4\eta^2} \ln\left(\frac{1+\eta}{1-\eta}\right) - \frac{(1-\eta)^2}{2\eta} \tag{17}$$

$$\begin{aligned} I_m = \frac{(1-\eta)^{m+1}}{4\eta^2} & \left\{ \frac{1}{m-1} \left[(1+\eta)^{m+1} - (1-n)^2\,(1+\eta)^{m-1} \right] \right. \\ & \left. - \frac{1}{m+1} \left[(1+\eta)^{m+1} - (1-n)^{m+1} \right] \right\}. \end{aligned} \tag{18}$$

Next define

$$S_m(\eta) = \frac{1}{4\eta^2} \int_{(1-\eta)^2}^{1-\eta^2} dz \left[\frac{\left(1-\eta^2\right)^2}{z^2} - 1 \right] \left(\frac{1+\eta^2 - z}{2\eta} \right)^m . \tag{19}$$

Note that

$$S_m(\eta) = \frac{1}{(2\eta)^m} \sum_{n=0}^{m} \binom{m}{n} \left(1+\eta^2\right)^n \quad (-1)^{m-n} \; I_{m-n}(\eta) \tag{20}$$

Also, comparing Eqs. (13) and (19), you can see that f_ℓ is easily written in terms of S_m; for example, since

$$P_4(\mathrm{x}) = \frac{1}{8}\left[35\ \mathrm{x}^{\,4} - 30\ \mathrm{x}^{\,2} + 3\right] \quad , \tag{21}$$

f_4 is

$$\mathrm{f}_4 = \frac{1}{8}\left[35\ S_4 - 30\ S_2 + 3\right] \quad . \tag{22}$$

One could now substitute I_m from Eqs. (16)-(18) in Eq. (20), and substitute the result into the analogues of Eq. (22). However, the resulting expressions are not illuminating and it is easy to make mistakes. So I just used the computer: for each η, first I use Eqs. (16)-(19) to evaluate I_m, $m \geq 6$; then I calculate S_m from Eq. (20); finally, I form the appropriate linear combinations to give f_ℓ. This relies on considerable computer accuracy for η near 0 and 1 since cancellations are important. I had to use double precision to get f_6 at small η.

The results are plotted in Fig. 2 and tabulated in Table 1. What I see as significant observations are:

(i) f_2 increases quite rapidly with η, reaching 0.5 already by $\eta = 0.3$. So a 0.2 mm capsule in a 1 mm hohlraum feels less flux smoothing than one might have thought, and experiences a 50% decrease in P_2 flux if the ablation surface moves in.

(ii) The coefficients f_4 and f_6 change sign at $\eta \approx 0.2$. Since this is in the ballpark of our capsule-to-case scale factors, and since intrinsic P_4 and P_6 are usually present, this may be important.

(iii) Both f_4 and f_6 are non-negligible at $\eta \geq 0.1$. We should definitely try to design our hohlraums so that the large-radius P_4 and P_6 moments are as small as possible.

REFERENCES

1. J. Green, "Radiation Flux Smoothing," R&D Associates Report x80-1637 (1982).
2. D. Kershaw, private communication (1982).
3. J. C. Garrison, private communication (1975).

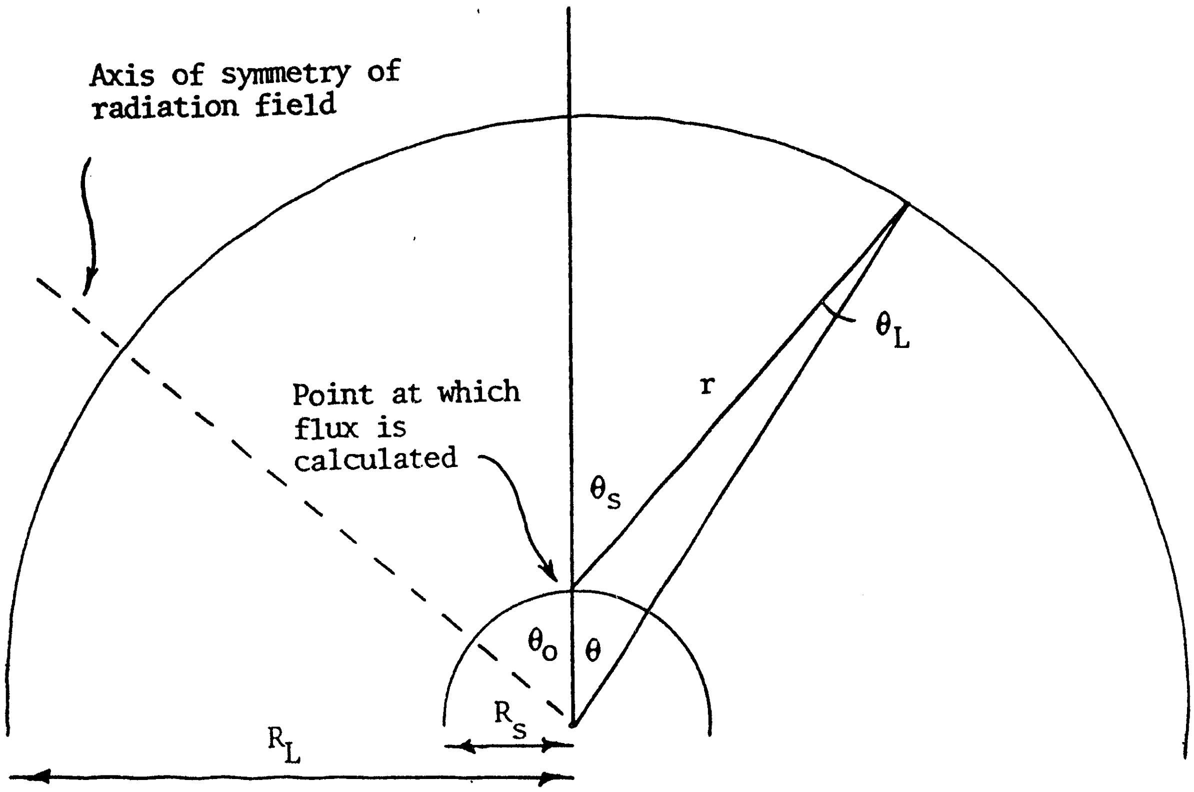

Figure 1. Details of geometrical arrangement.

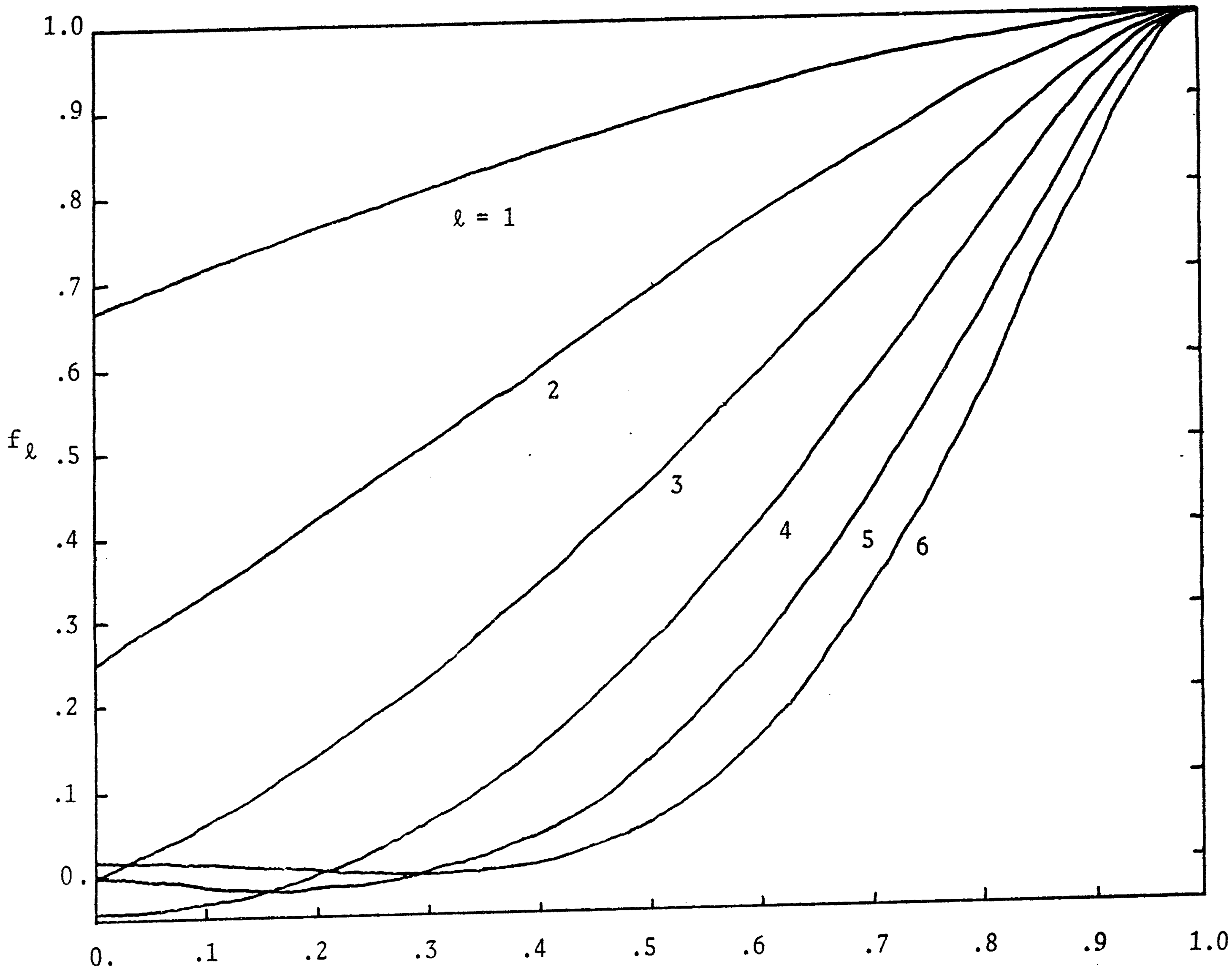

Figure 2. Amplitudes of spherical harmonic moments vs. the ratio of the radii.

ETA	F1	F2	F3	F4	F5	F6
0.01000	0.67165	0.25802	0.00509	-0.04158	-0.00156	---
0.02000	0.67661	0.26610	0.01037	-0.04133	-0.00311	0.01553
0.03000	0.68155	0.27422	0.01582	-0.04089	-0.00455	0.01526
0.04000	0.68645	0.28239	0.02146	-0.04027	-0.00616	0.01497
0.05000	0.69133	0.29060	0.02726	-0.03946	-0.00764	0.01460
0.06000	0.69519	0.29885	0.03329	-0.03845	-0.00908	0.01416
0.07000	0.70101	0.30715	0.03947	-0.03724	-0.01046	0.01364
0.08000	0.70581	0.31548	0.04584	-0.03583	-0.01178	0.01305
0.09000	0.71059	0.32386	0.05239	-0.03420	-0.01304	0.01240
0.10000	0.71533	0.33227	0.05911	-0.03235	-0.01421	0.01168
0.11000	0.72005	0.34072	0.06602	-0.03028	-0.01529	0.01091
0.12000	0.72474	0.34920	0.07310	-0.02798	-0.01627	0.01009
0.13000	0.72941	0.35772	0.08035	-0.02545	-0.01714	0.00922
0.14000	0.73405	0.36627	0.08780	-0.02268	-0.01789	0.00832
0.15000	0.73866	0.37485	0.09542	-0.01966	-0.01850	0.00738
0.16000	0.74324	0.38346	0.10321	-0.01640	-0.01897	0.00642
0.17000	0.74780	0.39210	0.11117	-0.01289	-0.01929	0.00545
0.18000	0.75233	0.40076	0.11930	-0.00912	-0.01945	0.00448
0.19000	0.75683	0.40945	0.12761	-0.00509	-0.01942	0.00351
0.20000	0.76130	0.41816	0.13608	-0.00079	-0.01921	0.00256
0.21000	0.76575	0.42689	0.14472	0.00377	-0.01881	0.00163
0.22000	0.77017	0.43565	0.15353	0.00861	-0.01819	0.00075
0.23000	0.77456	0.44442	0.16250	0.01372	-0.01735	-0.00008
0.24000	0.77892	0.45321	0.17164	0.01911	-0.01628	-0.00085
0.25000	0.78326	0.46202	0.18094	0.02479	-0.01497	-0.00155
0.26000	0.78756	0.47084	0.19040	0.03074	-0.01341	-0.00215
0.27000	0.79184	0.47967	0.20001	0.03699	-0.01157	-0.00264
0.28000	0.79609	0.48851	0.20978	0.04353	-0.00946	-0.00301
0.29000	0.80031	0.49737	0.21970	0.05036	-0.00707	-0.00325
0.30000	0.80451	0.50623	0.22978	0.05749	-0.00437	-0.00334
0.31000	0.80867	0.51510	0.24000	0.06491	-0.00137	-0.00325
0.32000	0.81281	0.52398	0.25037	0.07263	0.00196	-0.00298
0.33000	0.81691	0.53286	0.26038	0.08065	0.00562	-0.00251
0.34000	0.82099	0.54174	0.27153	0.08899	0.00961	-0.00182
0.35000	0.82504	0.55062	0.28232	0.09761	0.01397	-0.00089
0.36000	0.82905	0.55949	0.29325	0.10654	0.01858	0.00029
0.37000	0.83304	0.56837	0.30430	0.11577	0.02376	0.00175
0.38000	0.83700	0.57724	0.31549	0.12531	0.02923	0.00351
0.39000	0.84092	0.58610	0.32681	0.13516	0.03509	0.00557
0.40000	0.84482	0.59496	0.33824	0.14531	0.04134	0.00797
0.41000	0.84868	0.60380	0.34980	0.15576	0.04801	0.01072
0.42000	0.85252	0.61263	0.36147	0.16652	0.05509	0.01384
0.43000	0.85632	0.62145	0.37325	0.17757	0.06260	0.01735
0.44000	0.86009	0.63025	0.38515	0.18893	0.07055	0.02127
0.45000	0.86383	0.63904	0.39715	0.20059	0.07893	0.02562
0.46000	0.86753	0.64780	0.40924	0.21254	0.08776	0.03042
0.47000	0.87121	0.65654	0.42144	0.22479	0.09704	0.03569
0.48000	0.87485	0.66526	0.43373	0.23732	0.10678	0.04144
0.49000	0.87846	0.67395	0.44610	0.25015	0.11699	0.04771
0.50000	0.88203	0.68261	0.45856	0.26326	0.12765	0.05449
0.51000	0.88557	0.69125	0.47110	0.27665	0.13881	0.06182
0.52000	0.88908	0.69985	0.48371	0.29031	0.15043	0.06972
0.53000	0.89255	0.70841	0.49639	0.30425	0.16254	0.07819
0.54000	0.89598	0.71694	0.50914	0.31845	0.17512	0.08725
0.55000	0.89939	0.72543	0.52194	0.33291	0.18819	0.09693
0.56000	0.90275	0.73388	0.53479	0.34763	0.20174	0.10724
0.57000	0.90608	0.74228	0.54770	0.36259	0.21578	0.11818
0.58000	0.90937	0.75064	0.56064	0.37780	0.23030	0.12978
0.59000	0.91262	0.75894	0.57362	0.39323	0.24529	0.14205

Table 1. Amplitudes of the P_ℓ moments. (continued on following page)

0.60000	0.91584	0.76719	0.58662	0.40889	0.26076	0.15499
0.61000	0.91902	0.77539	0.59965	0.42477	0.27671	0.16862
0.62000	0.92216	0.78353	0.61269	0.44065	0.29312	0.18295
0.63000	0.92526	0.79161	0.62574	0.45713	0.30999	0.19799
0.64000	0.92832	0.79962	0.63879	0.47359	0.32732	0.21373
0.65000	0.93133	0.80757	0.65183	0.49023	0.34509	0.23018
0.66000	0.93431	0.81545	0.66486	0.50703	0.36329	0.24735
0.67000	0.93724	0.82325	0.67787	0.52398	0.38191	0.26524
0.68000	0.94014	0.83098	0.69084	0.54107	0.40095	0.28383
0.69000	0.94298	0.83862	0.70377	0.55827	0.42038	0.30314
0.70000	0.94578	0.84618	0.71665	0.57559	0.44018	0.32314
0.71000	0.94854	0.85365	0.72947	0.59299	0.46035	0.34383
0.72000	0.95125	0.86103	0.74223	0.61047	0.48085	0.36520
0.73000	0.95391	0.86831	0.75490	0.62800	0.50167	0.38723
0.74000	0.95652	0.87549	0.76748	0.64556	0.52279	0.40989
0.75000	0.95909	0.88256	0.77995	0.66315	0.54417	0.43318
0.76000	0.96160	0.88952	0.79231	0.68072	0.56578	0.45705
0.77000	0.96406	0.89637	0.80455	0.69827	0.58761	0.48147
0.78000	0.96646	0.90309	0.81664	0.71576	0.60960	0.50642
0.79000	0.96881	0.90968	0.82857	0.73317	0.63173	0.53184
0.80000	0.97110	0.91614	0.84034	0.75048	0.65395	0.55768
0.81000	0.97334	0.92246	0.85191	0.76765	0.67622	0.58391
0.82000	0.97551	0.92863	0.86329	0.78465	0.69349	0.61045
0.83000	0.97762	0.93465	0.87444	0.80145	0.72071	0.63723
0.84000	0.97967	0.94050	0.88535	0.81801	0.74282	0.66419
0.85000	0.98164	0.94619	0.89600	0.83429	0.76476	0.69124
0.86000	0.98355	0.95168	0.90636	0.85025	0.78646	0.71828
0.87000	0.98539	0.95699	0.91641	0.86584	0.80786	0.74521
0.88000	0.98714	0.96210	0.92613	0.88102	0.82885	0.77192
0.89000	0.98882	0.96699	0.93549	0.89572	0.84937	0.79827
0.90000	0.99042	0.97164	0.94445	0.90989	0.86930	0.82411
0.91000	0.99192	0.97606	0.95297	0.92347	0.88854	0.84929
0.92000	0.99333	0.98021	0.96103	0.93637	0.90696	0.87361
0.93000	0.99464	0.98407	0.96856	0.94851	0.92442	0.89687
0.94000	0.99585	0.98763	0.97552	0.95979	0.94076	0.91881
0.95000	0.99693	0.99085	0.98185	0.97010	0.95579	0.93915
0.96000	0.99789	0.99369	0.98747	0.97929	0.96927	0.95754
0.97000	0.99870	0.99612	0.99227	0.98719	0.98094	0.97355
0.98000	0.99935	0.99806	0.99613	0.99357	0.99040	0.98663
0.99000	0.99981	0.99942	0.99884	0.99807	0.99711	0.99596

Table 1. Amplitudes of the P_ℓ moments. (see previous page)

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