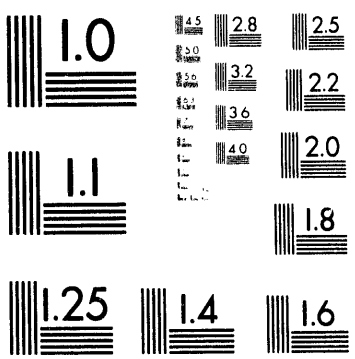
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AN EQUAL-ORDER APPROXIMATE PROJECTION FEM

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In this short paper we introduce and test a technique for solving the incompressible Navier-Stokes equations using bilinear basis functions for velocity and pressure.

INTRODUCTION

The equations of principal interest are

$$\partial \underline{u}/\partial t + \underline{u}\cdot\nabla\underline{u} + \nabla P = \upsilon\nabla^2\underline{u} \text{ and } \nabla\cdot\underline{u} = 0 \quad \text{in} \quad \Omega, \tag{1), (2}$$

$$\text{with } \underline{u} = \underline{w}(t) \text{ on } \Gamma \text{ and } \int_\Gamma \underline{n}\cdot\underline{w} = 0\ , \tag{3}$$

$$\text{and } \underline{u} = \underline{u}_0 \text{ with } \nabla\cdot\underline{u}_0 = 0 \text{ in } \Omega \text{ at } t = 0 \text{ and with } \underline{n}\cdot\underline{u}_0 = \underline{n}\cdot\underline{w}(0) \text{ on } \Gamma. \tag{4}$$

Since these "primitive equations" are usually considered as too difficult to solve as stated (fully-coupled), the following pressure Poisson equation (PPE) formulation is of much interest: Using $\partial(\nabla\cdot\underline{u})/\partial t = 0$ in (1) leads to the PPE:

$$\nabla^2 P = \nabla\cdot(\nu\nabla^2\underline{u} - \underline{u}\cdot\nabla\underline{u}) \equiv \nabla\cdot\underline{a} \quad \text{in} \quad \Omega\ , \partial P/\partial n = \underline{n}\cdot\left(\nu\nabla^2\underline{u} - \underline{u}\cdot\nabla\underline{u} - \frac{\partial\underline{w}}{\partial t}\right) \text{ on } \Gamma. \tag{5), (6}$$

The PPE (derived) formulation is (1) and (3)–(6): i.e., (2) is omitted because it is implied [See, e.g., Gresho (1991a)].

The weak (Galerkin) form of the PPE formulation is

$$\int \underline{v}\cdot\left(\frac{\partial\underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) + \upsilon(\nabla\underline{u})^T:\nabla\underline{v} - P\nabla\cdot\underline{v} = 0 \quad \forall\,\underline{v}\in\underline{H}_O^1 \tag{7}$$

$$\text{and} \int \nabla\varphi\cdot\nabla P = \int \underline{a}\cdot\nabla\varphi - \int_\Gamma \varphi\underline{n}\cdot\partial\underline{w}/\partial t \quad \forall\varphi\in H_0^1. \tag{8}$$

Remarks:

1. The above PPE formulation assures that $\nabla\cdot\underline{u} = 0$ in a very weak (and vague) sense in that *if* (7) and (8) $\Rightarrow$ (1) and (3)–(6) then $\nabla\cdot\underline{u} = 0$. I.e., *if* our weak solution is also a classical solution, then $\underline{u}$ will be (strongly) div-free. These are big if's.

2. The PPE formulation has more solutions than the $\underline{u} - P$ formulation—and the extras are spurious, a particular example being that $\nabla \cdot \underline{u}_0$ need not vanish in order that PPE solutions exist—*any* $\underline{u}_0$ 'works'. [See Gresho (1992) for others.]

If we discretize (1)–(4) via the GFEM, we arrive at the following Index 2 DAE system:

$$M\dot{u} + N(u)u + CP = Ku + f, \quad u(0) = u_0 \text{ and } C^T u = g \ , \text{ with } C^T u_0 = g_0 \qquad (9), (10)$$

where f and g correspond to BC data. Note that (9) also corresponds to the finite-dimensional version of (7). Converting this to a lower index (easier) problem is achieved by inserting $\dot{u}$ into the *time-differentiated* version of (10): the resulting Index 1 DAE system is (9) and

$$\left(C^T M^{-1} C\right) P = C^T M^{-1}\left(Ku + f - N(u)u\right) - \dot{g}. \qquad (11)$$

As with the PPE formulation, the Index 1 formulation uncouples the pressure from the acceleration (its "raison d´etre"). Also as with the continuous formulation, violation of $C^T u_0 = g_0$ results in an ill-posed Index 2 problem but the Index 1 version—having more solutions—is not ill-posed. It is just not *right;* the resulting discrete velocity field will not satisfy $C^T u = g$. [It *will* satisfy $C^T u = g + C^T u_0 - g_0$; see Gresho (1991b).].

For comparison with what follows below, we now introduce the projection matrix, $\wp$:

$$\wp \equiv I - M^{-1} C\left(C^T M^{-1} C\right)^{-1} C^T, \qquad (12)$$

which is a 'formal' construction only. $\wp$ has the following properties:

1. $\wp^2 = \wp$; it satisfies the *definition* of a projection.
2. If w is an arbitrary velocity field, then $\wp w$ is its projection to the (discretely) div-free subspace; i.e. $u \equiv \wp w$ satisfies $C^T u = 0$. In fact, $C^T \wp \equiv 0$.
3. The eigenvalues of $\wp$ are either zero or 1, and its norm is unity. All discretely divergence-free vector fields are eigenvectors of $\wp$ with eigenvalue unity.

In terms of $\wp$, the Index 1 problem can be 'condensed' to an Index 0 problem (i.e., to a set of ODE's):

$$\dot{u} = \wp\left[M^{-1}\left(Ku + f - N(u)u\right)\right] + M^{-1} C\left(C^T M^{-1} C\right)^{-1} \dot{g} \ , \qquad (13)$$

which clearly satisfies $C^T \dot{u} = \dot{g}$; (only) an initially div-free velocity field will remain div-free.

Suppose though that we use instead (7) and (8) to generate our GFEM equations? The result is (9) and

$$LP = h \ , \text{ where } L_{ij} \equiv \int \nabla \varphi_i \cdot \nabla \varphi_j \qquad (14), (15)$$

is the 'conventional' GFEM Laplacian and φ_k is the bilinear basis function associated with node k. The RHS vector is

$$h_i \equiv \int \nabla \varphi_i \cdot \left(\nu \nabla^2 \underline{u}^h - \underline{u}^h \cdot \nabla \underline{u}^h\right) - \int_\Gamma \varphi_i \underline{n} \cdot \partial \underline{w} / \partial t \qquad (16)$$

from the finite-dimensional form of (8). Clearly some remedial action is required if C^0 basis functions are to be employed (for which $\nabla^2 \underline{u}^h$ is not well-defined); for one type of response, see Sohn and Heinrich (1990). For a 'rationalization', note that $\upsilon \int \nabla \varphi \cdot \nabla^2 \underline{u} = \upsilon \int \varphi \underline{n} \cdot \nabla^2 \underline{u}$ when $\nabla \cdot \underline{u} = 0$ and thus the *omission* of the viscous term in (16) can *only* 'affect things' near Γ—an especially valid approximation for large Reynolds number (small υ). (See Hassanzadeh *et al.* 1994) In our formulation, an approximation to (14)–(16) is actually only required at $t = 0$—to estimate the *initial* pressure field; for $t > 0$, the pressure is determined by the (approximate) projection method—described below.

We, and many others, have generated codes based on the Index 1 formulation of (9) and (11) using the Q_1P_0 element; i.e., the quadrilateral element with bilinear basis functions (2D version) for velocity and piecewise-constant basis functions for P. But, with the exception of our ad hoc procedure in Gresho and Chan (1990), all Q_1P_0 Index 1 codes 'required' the also ad hoc (and more deleterious) approximation of *mass lumping* (because M^{-1} is otherwise dense), which introduces a *serious loss* of *accuracy* for the bilinear element when the flow is advection-dominated (vortex shedding, for example). Additional bad features of this element are:

1. It suffers from the "Bent-Element Blues"—(see Gresho and Leone , 1984); the CP approximation to ∇P is not very accurate when the elements are bent pretty hard (are far from rectangular).
2. The staggered mesh 'bookkeeping' is not fun.
3. The Laplacian matrix $C^T M_L^{-1} C$ is 'awkward' and is suspected to converge 'too slowly' when iterative solvers are used. (M_L is the lumped version of M.)
4. It fails the LBB test! I.e., the velocity and pressure spaces are not quite compatible.
5. The (necessary) integration by parts of the ∇P term causes 'problems' in 'flow-through' domains wherein a homogeneous normal natural boundary condition $\left(\nu\, \partial u_n/\partial n - P = 0\right)$ is employed and a significant 'body force' is present [see Sani and Gresho (1994)].

THEORY

So, to get back 'in vogue', we now describe *our* attempt at creating a *stabilized* equal-order element; viz Q_1Q_1, which overcomes at least flaws (1), (2), and (5) above. But (3) and (4) still 'shoot it down' *and* mass lumping still seems required. To implement a consistent mass Q_1Q_1 element *and* to pass 'LBB' (which Q_1Q_1 fails in spades), we invoke an "approximate projection" [Almgren *et al.* (1993), and Dvinsky and Dukowicz (1993)] in much the same way that many before us have done, viz, we replace the 'bad' Laplacian in (11) by the good (conventional GFEM) Laplacian, (15). Actually, the procedure is closer to the following:

1. Replace $-P\nabla\cdot\underline{v}$ in (7) by $\underline{v}\cdot\nabla P$
   which implies $\nu\, \partial\underline{u}/\partial n = \underline{0}$ as an OBC which is generally an ill-posed boundary condition (see Sani and Gresho 1994). But we get away with this crime because we will no longer require a 'stringent' version (even discretely) of $\nabla\cdot\underline{u} = 0$.
2. Generate the following Index 2 DAE's:
   $$M\dot{u} + N(u)u + GP = Ku + f,\ \ u(0) = u_0 \text{ and } Du = g,\ \ Du_0 = g_0 \qquad (17)\text{–}(20)$$
   where no longer are G and D transposes of each other.
3. Solve the DAE's via the following approximate projection method: Given u_n and P_n,
   (*i*) Solve $M\frac{(\tilde{u}_{n+1}-u_n)}{\Delta t} + N(u_n)u_n + GP_n = \frac{1}{2}K(\tilde{u}_{n+1}+u_n) + f_n$ for $\tilde{u}_{n+1}$. (21)
   (*ii*) Project $\tilde{u}_{n+1}$ to the approximately discretely div-free subspace as follows:
   (1) Solve $\Delta t L(P_{n+1}-P_n) = D(\tilde{u}_{n+1}-u_n) - (g_{n+1}-g_n)$ for ΔP. (22)
   (2) Compute the projected velocity from $u_{n+1} = \tilde{u}_{n+1} - \Delta t\, M_L^{-1} G(P_{n+1}-P_n)$ (23)
   (3) Update P and go to (*i*)

The approximate projection above is derived as follows: Given $\tilde{u}_{n+1}$, solve

$$M_L\frac{(u_{n+1}-\tilde{u}_{n+1})}{\Delta t} + G(P_{n+1}-P_n) = 0 \text{ and } Du_{n+1} = g_{n+1} + (Du_n - g_n), \qquad (24)\text{–}(25)$$

which is called 'projecting the difference'—a required 'trick' obtained from J. Bell (LLNL, personal communication). Note that, because of the next trick, below, $Du = g$ is never quite achieved in this method.

These equations imply the following 'PPE' for the pressure update:

$$\Delta t \left(DM_L^{-1}G\right)\left(P_{n+1} - P_n\right) = D\left(\tilde{u}_{n+1} - u_n\right) - \left(g_{n+1} - g_n\right), \tag{26}$$

which is replaced by (22) $\left(D M_L^{-1} G \to L\right)$ and is the final "trick" referred to above—a replacement made necessary because $DM_L^{-1}G$ has too many spurious pressure modes (fails LBB) and made 'interesting' because iterative solvers should like L better (i.e., require fewer iterations).

Remarks:

If $Du_n - g_n$ is omitted from the RHS of (22), the results are disappointing in that (at least) a steady pressure cannot be attained even when the velocity becomes steady.

Now comes the 'hard part'—analyzing the resulting algorithm to show that it is actually *useful*. And we admit up front that we have not yet been totally successful. But, being more like engineers than mathematicians (mathematical engineers?), we tested it in the computational laboratory anyway. (And it works. This happens often with engineers—but definitely not always.) If we study (21), (22), and (23) for $\Delta t \to 0$ we get, with $F_n \equiv f_n + Ku_n - N(u_n)u_n - GP_n$,

(*i*) $\tilde{u}_{n+1} = u_n + \Delta t \;\; M^{-1} F_n + O\left(\Delta t^2\right)$

(*ii*) $L\left(P_{n+1} - P_n\right) = DM^{-1} F_n - \dot{g}_n + O(\Delta t)$

(*iii*) $u_{n+1} = \tilde{u}_{n+1} - \Delta t M_L^{-1} \;\; GL^{-1}\left\{DM^{-1} F_n - \dot{g}_n\right\} + \;\; O\left(\Delta t^2\right)$

$\equiv \mathrm{u_n} + \Delta t\left[\wp_a M^{-1} F_n + M_L^{-1} G L^{-1} \dot{g}_n\right] + O\left(\Delta t^2\right)$

(*iv*) $Du_{n+1} = Du_n + \Delta t\left[EL\left(P_{n+1} - P_n\right) + \dot{g}_n\right] + O\left(\Delta t^2\right)$

where $E \equiv I - DM_L^{-1}GL^{-1}$ and $E^n D = D\wp_a{}^n$

(*v*) Thus $\Delta t \to 0 \Rightarrow \dot{u} = \wp_a M^{-1}[f + Ku - N(u)u - GP] + M_L^{-1} G L^{-1} \dot{g}$

(*vi*) $D\dot{u} = D\wp_a M^{-1} F + DM_L^{-1} G L^{-1} \dot{g} = EDM^{-1}[f + Ku - N(u)u - GP] + \left(DM_L^{-1}G\right)L^{-1}\dot{g}$

and the *problem* is to *reconcile* (*iv*) with (*vi*); the former $\Rightarrow D\dot{u} = EL \cdot \lim_{\Delta t \to 0} \left(P_{n+1} - P_n\right) + \dot{g}$ and the two are only in accord if $P_{n+1} - P_n \to 0$ as $\Delta t \to 0$, *and* if

$$\left(DM^{-1}G\right)P = DM^{-1}\left[f + Ku - N(u)u\right] - \dot{g} \;, \tag{27}$$

which is just the PPE that we'd *like* the pressure to obey! And if this is true, then (*v*) above yields

$$\dot{u} = M^{-1}\left[f + Ku - GP - N(u)u\right] \;, \tag{28}$$

also the desired result. (Note the mysterious disappearance of the L-matrix.) *But*—we have not been able to show that $P_{n+1} - P_n = O(\Delta t)$. In fact, somewhat the converse *seems* to be true; viz, starting from the initial pressure, P_0, obtained from

$$LP_0 = DM^{-1}\left[f_0 + Ku_0 - N(u_0)u_0\right] - \dot{g}_0 \;, \tag{29}$$

proceeding through a few steps, and generalizing via induction leads to

$$L\left(P_{n+1} - P_n\right) = E^{n+1} LP_0 + O(\Delta t) \text{ and to} \tag{30}$$

$$Du_{n+1} = Du_n + \Delta t\left[E^{n+2} LP_0 + \dot{g}_n\right] + O\left(\Delta t^2\right) . \tag{31}$$

Thus, at least initially, $P_{n+1} - P_n = O(1)$ in Δt and, for large n, we recover to $P_{n+1} - P_n = O(\Delta t)$ iff $E^n \to 0$. But since that E has eigenvectors with unit eigenvalues whenever the initial vector has any non-zero projection onto the null space of G, E^n may not $\to 0$. It seems that *only* if LP_0 (and $\therefore P_0$) has no spurious pressure node components can we assert that the algorithm can 'recover'—and even then the early (small n) results are likely to be poor...

But the resulting code seems to perform much better than these gloomy results would indicate—as we shall show.

NUMERICAL RESULTS

We have successfully (for the most part) compared Q_1Q_1 against our workhorse Q_1P_0 code for lid-driven cavities and vortex sheddng past circular cylinders. Here we show a sample of results for flow past an airfoil at $\mathrm{Re} = 10^4$ (based on chord, c) at a small (1.2°) angle of attack. The particular airfoil, "a NACA 16 thickness form with maximum thickness $t_0/c = 8.84\%$ and maximum camber 2.576% with a beveled anti-singing trailing edge"—E. H. Lurie (1993), was tested in a *water* tunnel (*incompressible* flow!) at MIT's Ocean Engineering Department as part of a Navy/ARPA program on unsteady fluid dynamics. But our laminar flow simulations were not meant to be compared with their measured data at $\mathrm{Re} > 10^6$ because we do not yet have a believable turbulence model. Maybe later.

We designed a 'truth' mesh of ~40,000 elements and a test mesh of ~6400 elements. The fine mesh was run at $\mathrm{Re} = 10^3$, 10^4, 10^5, and 10^6 using Q_1P_0 and our 'projection 2' algorithm (Gresho and Chan 1990). The results were not believable at $\mathrm{Re} = 10^6$ (chaotic), semi-so at $\mathrm{Re} = 10^5$ (nearly periodic), and believable at $\mathrm{Re} \le 10^4$. $\mathrm{Re} = 10^3$ (only) produced a steady-state result. The coarse mesh for $\mathrm{Re} = 10^3$ was then run with Q_1P_0 and the new Q_1Q_1; all three results were acceptably close to each other.

We now focus on the $\mathrm{Re} = 10^4$ case and compare Q_1P_0 on the two meshes with Q_1Q_1 on the coarse mesh. Qualitatively, the flow is one of periodic vortex formation and shedding of vortices at the trailing edge. Nowhere else is there any interesting dynamics. Fig. 1 shows snapshots of vectors, streamfunction, and vorticity from the coarse mesh (vectors are interpolated via a coarser-yet mesh graphics package.). Fig. 2 shows time histories, starting from potential flow, of the x-component of velocity at a node just above $(\sim .002c)$ the trailing edge—and the pressure there, as well as the RMS-norm of the discrete divergence for Q_1Q_1. The agreement of Q_1Q_1 with Q_1P_0 on the coarse mesh is quite close—the range of u's during vortex shedding being $\sim -.07$ to $+.16$ for Q_1P_0 (2c) and $\sim -.08$ to $+.15$ for Q_1Q_1 (2e). (The fine mesh results (2a) show $\sim -.10$ to $+.20$, somewhat stronger, but still reasonably close). The corresponding pressure histories are (Dank Gott!) of a similar quality; the coarse mesh results ranged from $\sim -.17$ to $-.31$ for Q_1P_0 (2d) and $\sim -.16$ to $-.27$ for Q_1Q_1 (2f), compared to the 'true' results (2b) of $\sim -.15$ to $-.33$. (In all cases, the inlet velocity is 1.0, as is that on the top and bottom boundaries (tow tank BC's). The OBC's were: 'natural'; i.e., $\upsilon\, \partial u/\partial x - P = 0 = \upsilon\, \partial v/\partial x$ for Q_1P_0 and $\upsilon\, \partial u/\partial x = \upsilon\, \partial v/\partial y = P = 0$ for Q_1Q_1, the latter being imposed as an *essential* BC in the PPE.).

CONCLUSIONS

While interesting and viable (apparently—in spite of the shortcomings in our analysis), the new Q_1Q_1 has not yet displaced 'old faithful' (Q_1P_0), largely because the PPE solution—via DSCG—was little cheaper (10–15% fewer iterations). Perhaps it would look better using multi-grid.

For a simpler application of this Q_1Q_1 stabilization technique using forward Euler time marching, see Gresho and Chan (1994).

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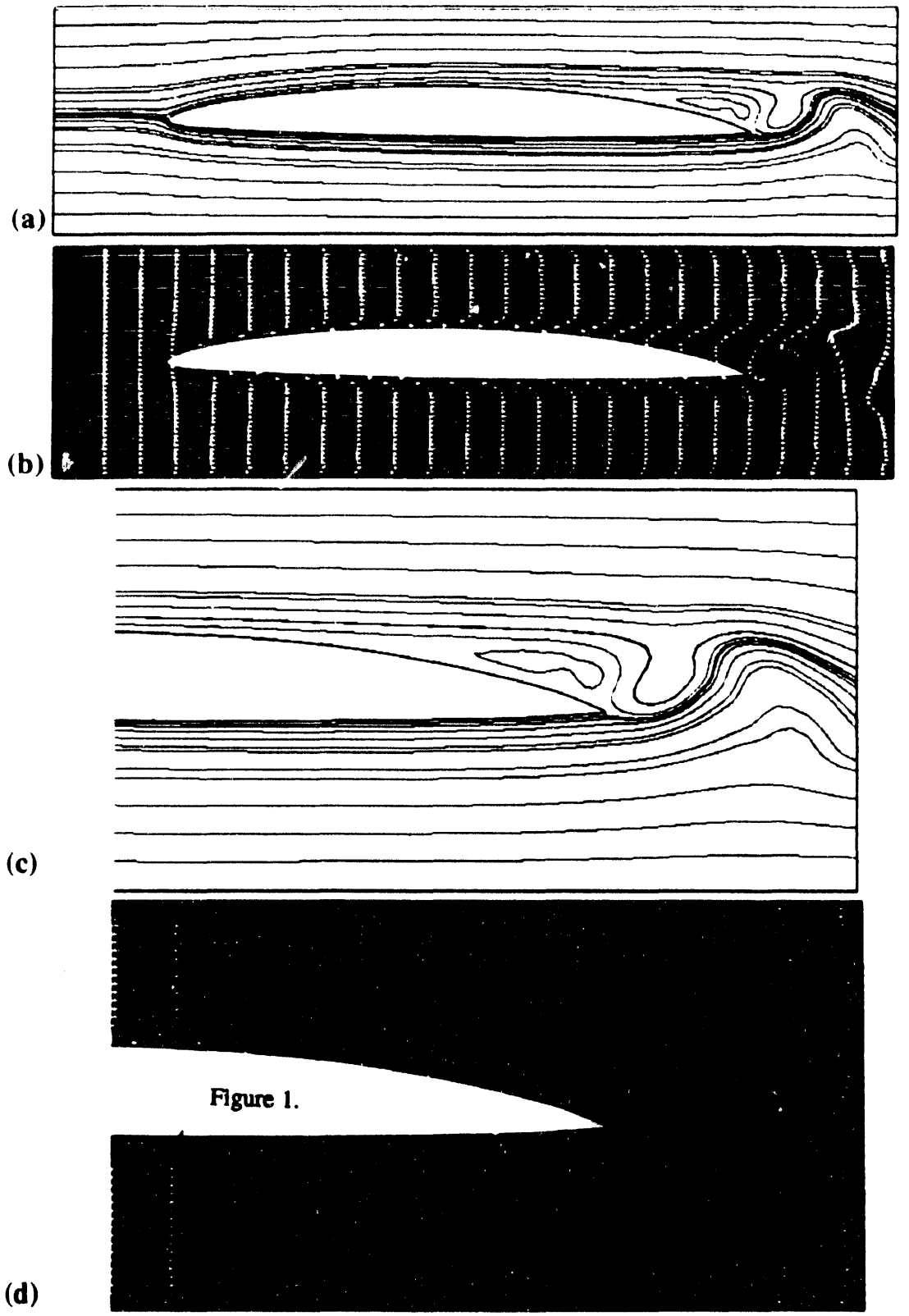

Figure 1.

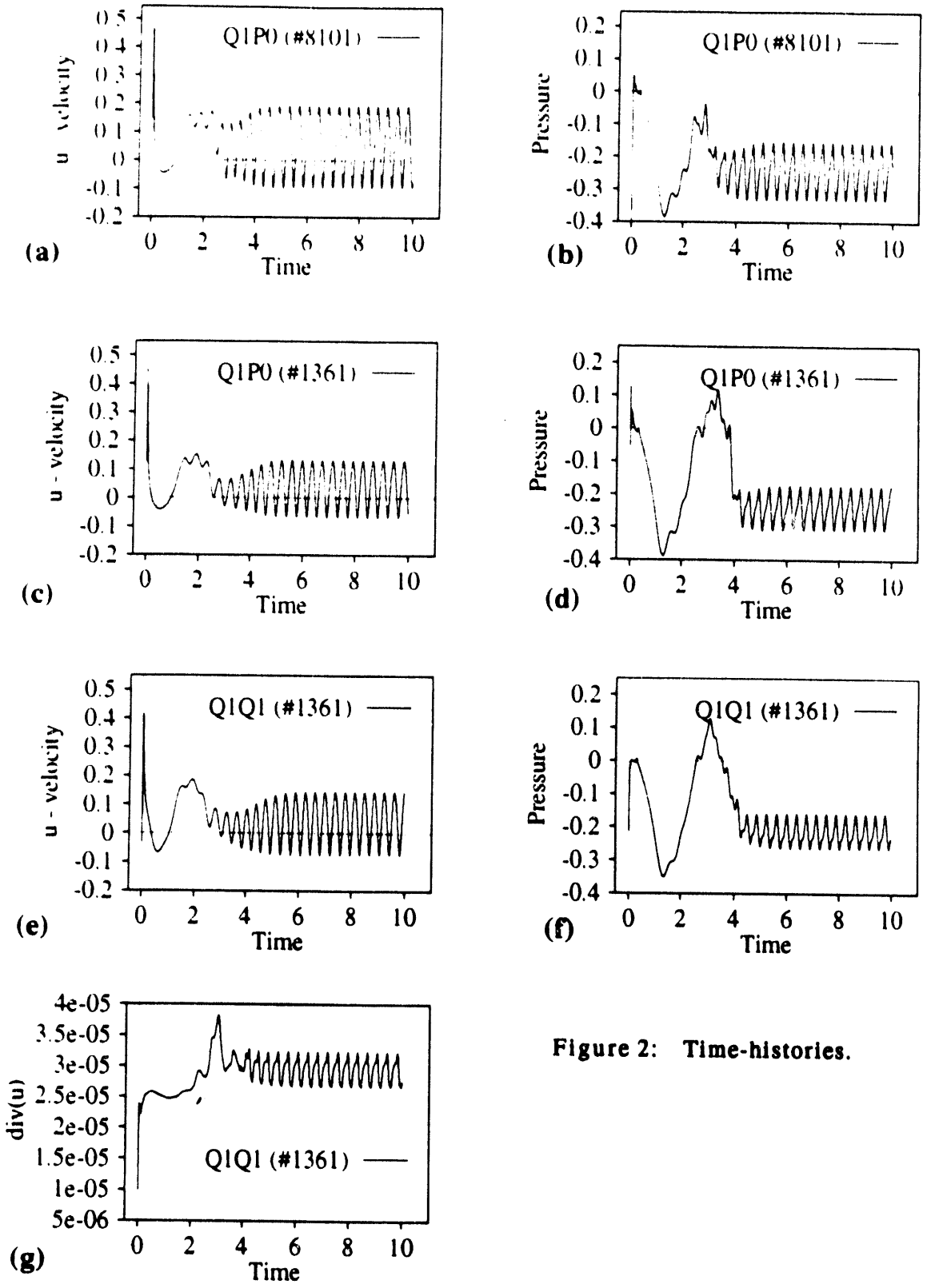

Figure 2: Time-histories.

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