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Constraints on the form factors for $K \to \pi\ell\nu$ and implications for $|V_{us}|$

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Rigorous bounds are established for the expansion coefficients governing the shape of semileptonic $K \to \pi$ form factors. The constraints enforced by experimental data from $\tau \to K\pi\nu$ eliminate uncertainties associated with model parameterizations in the determination of $|V_{us}|$. The results support the validity of a powerful expansion that can be applied to other semileptonic transitions.

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I. INTRODUCTION

Semileptonic $K \to \pi\ell\nu$ ($K_{\ell 3}$) decays provide the most precise determination of the Cabibbo-Kobayashi-Maskawa (CKM) matrix element $|V_{us}|$ [1]. Presently, the dominant uncertainty in the experimentally determined quantity $|V_{us}F_+(0)|$ arises from the unknown shape of the hadronic form factor $F_+(q^2)$, as a function of momentum transfer q [2, 3, 4, 5]. The data can be fit to both simplified pole models and series expansions, but the uncertainty inherent in these simplifications is difficult to estimate, and the difference between the resulting $|V_{us}F_+(0)|$ determinations represents a systematic error. Constraining this shape from first principles and providing rigorous error estimates is an important problem.

The remaining (and presently, dominant) error in the $K_{\ell 3}$ determination of $|V_{us}|$ arises from the normalization, $F_+(0)$. $K \to \pi\ell\nu$ data (specifically $K_{\mu 3}$) also constrain the shape of the scalar form factor, $F_0(q^2)$. This can potentially reduce dominant errors in the theory normalization, either by avoiding an extrapolation from zero recoil in lattice determinations [6], or by fixing low-energy constants of chiral perturbation theory [7]. At present, the comparison between theory and experiment, and between different experiments, is complicated by uncertainties in the form factor parameterization. Again, extracting as much information as possible from the experimental data without model assumptions is an important task.

This paper establishes rigorous bounds for the expansion coefficients appearing in a general parameterization of the semileptonic $K \to \pi$ form factors. The framework for the analysis is based on familiar arguments invoking analyticity and crossing symmetry [8]. However, in contrast to conventional arguments appealing to unitarity and the evaluation of an operator product expansion (OPE), bounds for the vector form factor are derived directly from experimental data for hadronic tau decays, $\tau \to K\pi\nu$. The result is more stringent than can be obtained from the OPE analysis, and applies to a more general class of parameterizations.

A. Analyticity and convergence

Form factors are defined as usual by the matrix element of the weak vector current $V^\mu \equiv \bar{u}\gamma^\mu s$: ($q \equiv p - p'$, $\Delta_{K\pi} \equiv m_K^2 - m_\pi^2$)

$$\begin{aligned} &\langle \pi^+(p')|V^\mu|\bar{K}^0(p)\rangle \\ &= F_+(q^2)\left(p^\mu + p'^\mu - \frac{\Delta_{K\pi}}{q^2}q^\mu\right) + F_0(q^2)\frac{\Delta_{K\pi}}{q^2}q^\mu\,. \end{aligned} \tag{1}$$

$F_+(t = q^2)$ and $F_0(t)$ can be extended to analytic functions throughout the complex t plane, except along a branch cut on the positive real axis starting at $K\pi$ production threshold $t = t_+$ [$t_\pm \equiv (m_K \pm m_\pi)^2$]. The cut plane is mapped onto the unit circle by

$$t \to z(t, t_0) \equiv \frac{\sqrt{t_+ - t} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - t} + \sqrt{t_+ - t_0}}\,, \tag{2}$$

where $t_0 \in (-\infty, t_+)$ is the point mapping onto $z = 0$. [22] The form factors are analytic in $|z| < 1$, and so may be expanded in a convergent power series:

$$F(t) = \frac{1}{\phi(t, t_0, Q^2)}\sum_{k=0}^{\infty} a_k(t_0, Q^2)\, z(t, t_0)^k\,, \tag{3}$$

where ϕ is an as-yet arbitrary function analytic in $|z| < 1$, which may depend on one or more parameters, denoted generically (and for reasons to become clear) by Q^2.

The function $z(t, t_0)$ sums an infinite number of terms, transforming the original series, naively an expansion involving $t/t_+ \lesssim 0.3$, into a series with a much smaller expansion parameter. For example, the choice $t_0 = t_+(1 - \sqrt{1 - t_-/t_+})$ minimizes the maximum value of z occurring in the semileptonic region, and for this choice $|z(t, t_0)| \lesssim 0.047$. The function ϕ and the number t_0 may be regarded as defining a "scheme" for the expansion. The expansion parameter z and coefficients a_k are then "scheme-dependent" quantities, with the scheme dependence dropping out in physical observables such as $F(t)$.

Neglecting terms beyond z^N in (3) introduces a relative normalization error $\Delta F/F = \mathcal{O}(z^{N+1})$. Similarly, the error on the relative slope involves terms of order $(N+1)z^N$. Since $|z|^2 \lesssim 2 \times 10^{-3}$ and $|z|^3 \lesssim 1 \times 10^{-4}$, simple power counting in z yields strong constraints on the impact of higher-order terms in the expansion, provided that the coefficients a_k/a_0 are well-behaved.