Semi-orthogonal Wavelets for Elliptic Variational Problems

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Abstract

In this paper, we give a construction of wavelets which are (a) semi-orthogonal with respect to an arbitrary elliptic bilinear form $a(\cdot,\cdot)$ on the Sobolev space $H_0^1((0,L))$ and (b) continuous and piecewise linear on an arbitrary partition of $[0,L]$. We illustrate this construction using the model problem

$$-\epsilon^2 u'' + u = f$$
$$u(0) = u(L) = 0.$$

We also construct a-orthogonal Battle-Lemarié type wavelets which fully diagonalize the Galerkin discretized matrix for the model problem with domain $\mathbb{R}$.

Finally, we describe a hybrid basis consisting of a combination of elements from the semi-orthogonal wavelet basis and the hierarchical Schauder basis. Numerical experiments indicate that this basis leads to robust, scalable Galerkin discretizations of the model problem which remain well-conditioned independent of ϵ, L, and the refinement level K.

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DISCLAIMER

1 Introduction

In this section, we review some basic theory about Galerkin discretizations of elliptic variational problems and their relationship to the Riesz bounds of the underlying basis (cf. [2, 5]).

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. Let $a(\cdot,\cdot)$ be a symmetric coercive continuous bilinear form on $\mathcal{H}$, that is a is a symmetric bilinear form such that

$$C\|v\|_{\mathcal{H}}^2 \leq a(v,v) \leq D\|v\|_{\mathcal{H}}^2$$

for some positive constants C and D. Define $\|\cdot\|_E := \sqrt{a(\cdot,\cdot)}$ to be the energy norm generated by a. The coercivity and continuity of a imply that the energy norm is equivalent to the norm associated with $\mathcal{H}$.

Let $\mathcal{H}'$ ($\cong \mathcal{H}$) denote the dual of $\mathcal{H}$. Consider the elliptic variational problem:

$$\begin{array}{c}\text{Given } F \in \mathcal{H}', \text{ find } u \in \mathcal{H} \text{ such that} \\ a(u,v) = F(v), \quad \forall v \in \mathcal{H}.\end{array} \tag{1}$$

Let $\mathcal{V}$ be a finite dimensional subspace of $\mathcal{H}$. Then the Galerkin approximate solution $u_{\mathcal{V}}$ is the unique solution of (1) with $\mathcal{H}$ replaced by $\mathcal{V}$. Let $\Phi = (\phi^1,\ldots,\phi^N)^T$ be a basis for $\mathcal{V}$. (Throughout this paper, a basis will be arranged as a column vector.) Then $u_{\mathcal{V}} = c^T\Phi$ can be found by solving the linear system

$$a(\Phi,\Phi)c = F(\Phi), \tag{2}$$

where $a(\Phi,\Phi)$ is the $N \times N$ matrix $\big(a(\phi^i,\phi^j)\big)$ and $F(\Phi)$ is the column vector $(F(\phi^1),\ldots,F(\phi^N))^T$.

For large N, it is usually impractical to solve the linear system (2) using direct methods. When the matrix $A^\Phi := a(\Phi,\Phi)$ is well-conditioned, the system can be efficiently solved using iterative methods. We say that $\underline{\alpha}$ (respectively $\overline{\alpha}$) is a lower (upper) Riesz bound for the basis Φ with respect to $\|\cdot\|_E$ if

$$\underline{\alpha}\, c^Tc \leq \|c^T\Phi\|_E^2 \leq \overline{\alpha}\, c^Tc. \tag{3}$$

Define $\underline{\alpha}_\Phi$ ($\overline{\alpha}_\Phi$) to be the largest (smallest) lower (upper) Riesz bound for Φ with respect to $\|\cdot\|_E$. Observe that

$$\|c^T\Phi\|_E^2 = c^TA^\Phi c.$$

Since A^Φ is symmetric and positive definite we have

$$\begin{aligned} \|A^\Phi\|_2 &= \max_c \frac{c^TA^\Phi c}{c^Tc} = \overline{\alpha}_\Phi \\ \|(A^\Phi)^{-1}\|_2^{-1} &= \min_c \frac{c^TA^\Phi c}{c^Tc} = \underline{\alpha}_\Phi. \end{aligned}$$

Therefore, the spectral condition number of A^Φ, $\mathrm{cond}(A^\Phi)$, is related to the Riesz bounds for Φ in the following way:

$$\mathrm{cond}(A^\Phi) = \overline{\alpha}_\Phi / \underline{\alpha}_\Phi. \tag{4}$$

Suppose Ψ is another basis for $\mathcal{V}$ and suppose W is the nonsingular $N \times N$ matrix such that

$$\Psi = W^T \Phi.$$

Then $u_{\mathcal{V}} = d^T \Psi$ may also be found by solving

$$a(\Psi, \Psi)d = F(\Psi). \tag{5}$$

Note that

$$A^\Psi = a(\Psi, \Psi) = W^T a(\Phi, \Phi) W. \tag{6}$$

Thus the linear system (5) resulting from (2) by a change of basis can also be considered to arise from (2) by preconditioning with W.

2 Multiscale Transformations

Suppose

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_k \subset \cdots$$

is a one-sided sequence of nested finite-dimensional subspaces of $\mathcal{H}$ such that $\overline{\bigcup \mathcal{V}_k} = \mathcal{H}$. Define $\mathcal{W}_0 := \mathcal{V}_0$ and, for $k \geq 1$, choose $\mathcal{W}_k$ in $\mathcal{V}_k$ so that

$$\mathcal{V}_k = \mathcal{V}_{k-1} \dot{+} \mathcal{W}_k \tag{7}$$

where $\dot{+}$ denotes the direct sum. Let Φ_k be a basis for $\mathcal{V}_k$ and let ψ_k be a basis for $\mathcal{W}_k$ (we choose $\psi_0 = \Phi_0$). Then

$$\Psi_k := \begin{pmatrix} \psi_0 \\ \vdots \\ \psi_k \end{pmatrix}$$

is also a basis for $\mathcal{V}_k$. Let W_k be the multiscale transformation such that

$$\Psi_k = W_k^T \Phi_k,$$

and let T_k be the two-scale transformation such that

$$\begin{pmatrix} \Phi_{k-1} \\ \psi_k \end{pmatrix} = T_k^T \Phi_k.$$

Observe that

$$W_k = T_k \begin{pmatrix} T_{k-1} & 0 \\ 0 & I_{k-1} \end{pmatrix} \cdots \begin{pmatrix} T_1 & 0 \\ 0 & I_1 \end{pmatrix}$$

where I_j is the $n \times n$ identity matrix with $n = \mathrm{card}(\psi_j)$.

Fix K and let $\Psi = \Psi_K$, $\Phi = \Phi_K$, and $W = W_K$. We assume that (a) multiplication by W can be implemented with a fast algorithm (this is the case for compactly supported wavelet bases), (b) A^Ψ is well-conditioned, and (c) $F(\Phi)$ can be easily approximated. Algorithm A summarizes the solution of the discretized problem given in (2) using the multiscale transform W.

Algorithm A:

- Approximate F^{Φ}.
- Calculate $F^{\Psi} = W^T F^{\Phi}$.
- Solve $A^{\Psi} d = F^{\Psi}$.
- $c = Wd$.

It is interesting to note that this algorithm does not use the decomposition matrix W^{-1}.

3 Wavelet Construction

Let $(X_k)_{k\geq 0}$ be a given sequence of nested knot sequences on $[0,1]$ satisfying

- $X_k = (x_k^j)_{0\leq j\leq N_k}$
- $0 = x_k^0 < \cdots < x_k^j < \cdots < x_k^{N_k} = L$
- $x_{k+1}^{2j} = x_k^j$

Let ϕ_k^j be the piecewise linear continuous function with knot sequence X_k such that $\phi_k^j(x_k^{j'}) = \delta_{j,j'}$. Let $\Phi_k = (\phi_k^1, \ldots, \phi_k^{N_k-1})^T$, then Φ_k is a nodal basis for $\mathcal{V}_k$ which is the usual finite element space of piecewise linear continuous functions on $[0, L]$ with knot sequence X_k.

We next describe two choices for $\mathcal{W}_k$.

3.1 Hierarchical Schauder Basis

One simple choice for $\mathcal{W}_k$ satisfying (7) is the well known Schauder basis (cf. [7, 9, 10])

$$\psi_k^j := \phi_k^{2j-1}, \quad j = 1, \ldots, N_k$$

illustrated in Figure 1.

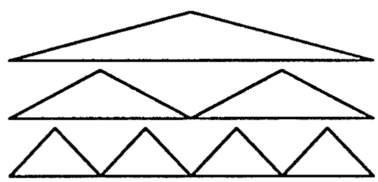

Figure 1: Schauder Basis $\Psi_2 = (\psi_0^T \ \ \psi_1^T \ \ \psi_2^T)^T$ with dimension 7 on a uniform partition.

Next we construct the two-scale transformation for the Schauder basis. Denote the length of the subinterval $[x_k^{j-1}, x_k^j]$ by

$$\Delta_k^j := x_k^j - x_k^{j-1}.$$

The function values $h_k^{j,j'}$ and $g_k^{j,j'}$ for ϕ_{k-1}^j and ψ_k^j respectively at the knot $x_k^{j'}$ are given by

$$h_k^{j,j'} = \begin{cases} \frac{\Delta_k^{j'}}{\Delta_{k-1}^j}, & j' = 2j-1 \\ 1, & j' = 2j \\ \frac{\Delta_k^{j'+1}}{\Delta_{k-1}^j}, & j' = 2j+1 \\ 0, & \text{otherwise} \end{cases}$$

$$g_k^{j,j'} = \delta_{j',2j-1}.$$

Then

$$\phi_{k-1}^j = \sum_{j'} h_k^{j,j'} \phi_k^{j'}$$

$$\psi_k^j = \sum_{j'} g_k^{j,j'} \phi_k^{j'}.$$

Now let H_k be the $(N_k-1)\times(N_{k-1}-1)$ matrix $H_k = (h_k^{j,j'})_{j',j}$ and let G_k be the $(N_k-1)\times(N_{k-1})$ matrix $G_k = (g_k^{j,j'})_{j',j}$. Thus, the two-scale transformation for the Schauder basis is given by

$$T_k = (H_k \;\; G_k).$$

3.2 Semi-orthogonal Sombrero Wavelets

Here we choose $\mathcal{W}_k$ to be the orthogonal complement of $\mathcal{V}_{k-1}$ in $\mathcal{V}_k$ with respect to the scalar product $a(\cdot,\cdot)$, that is

$$\mathcal{W}_k := \mathcal{V}_k \cap \mathcal{V}_{k-1}^{\perp_a}.$$

Regardless of the choice of basis ψ_k for $\mathcal{W}_k$, the matrix A^{Ψ_k} is then decoupled between levels so that it is a block diagonal matrix:

$$A^{\Psi_k} = \mathrm{diag}(A^{\psi_0}, A^{\psi_1}, \ldots, A^{\psi_k}).$$

We next give a procedure for constructing a local basis of wavelets for $\mathcal{W}_k$. Let

$$B := B(k) = a(\Phi_{k-1}, \Phi_k)$$

where we supress the k dependence when the choice for k is unambiguous. Note that

$$\mathcal{W}_k = \{g^T \Phi_k \mid g \in \ker B\}.$$

We will use certain subblocks of B in our construction. To this end we define R_{i_1,i_2} to be the $(i_2-i_1+1)\times(N_k-1)$ matrix whose i-th row is the $(i-i_1+1)$-th row of the $(N_k-1)\times(N_k-1)$ identity matrix. Then the $[i_1,i_2]\times[j_1,j_2]$ block of B is given by

$$B^{i_1,i_2}_{j_1,j_2} := R_{i_1,i_2} B R^T_{j_1,j_2}.$$

Let

$$C_n := \begin{cases} B^{1,n}_{1,2n-1} & \text{for } n = 2,3, \\ B^{n-3,n}_{2n-5,2n-1} & \text{for } 3 \le n \le N_{k-1}-1 \\ B^{n-3,n-1}_{2n-5,2n-1} & \text{for } n = N_{k-1} \end{cases}$$

For $4 \le n < N_{k-1}$, the matrix C_n is a 4×5 matrix which generically has a kernel of dimension one. This kernel then corresponds to a wavelet with support contained in $[x_k^{2n-6}, x_k^{2n}] = [x_{k-1}^{n-3}, x_{k-1}^{n}]$. More generally, we define the following procedure for constructing a local basis for $\mathcal{W}_k$.

Let $K_n := \ker(C_n)$ and, for $n \ge 3$, let K_n^0 denote the subspace of K_n consisting of the elements in $w \in K_n$ whose last two components are both zero. In the generic case, K_n^0 is the trivial subspace. Let

$$\Phi_k^n := \begin{cases} (\phi_k^1, \phi_k^2, \phi_k^3)^T, & n = 2 \\ (\phi_k^{2n-5}, \ldots, \phi_k^{2n-1})^T, & 3 \le n \le N_{k-1}. \end{cases} \tag{8}$$

Algorithm B:

- Let w_3 denote a basis for C_3 and set $\psi_k^3 = \{w^T \Phi_k^3 \mid w \in w_3\}$.

- For $n = 3, \ldots, N_{k-1}$, do
  - Choose K_n^1 so that $K_n = K_n^0 \dot{+} K_n^1$ and choose a basis w_n for K_n^1.
  - Set $\psi_k^n = \{w^T \Phi_k^n \mid w \in w_n\}$.
- $\psi_k = \bigcup_{n=3}^{N_{k-1}} \psi_k^n$.

We next give a sufficient condition that the above procedure produces a basis for $\mathcal{W}_k$. For $4 \le n < N_{k-1}$, we note that C_n has the following block form

$$C_n = \begin{pmatrix} D_n & E_n \\ 0 & F_n \end{pmatrix} \tag{9}$$

where D_n is 3×3, E_n is 3×2, and F_n is 1×2.

Lemma 1 *Let ψ_k be the set produced by Algorithm B. Suppose*

$$\text{range } D_n \supset E_n(\ker F_n) \quad (4 \le n \le N_{k-1}). \tag{10}$$

Then ψ_k is a basis for $\mathcal{W}_k := \mathcal{V}_k \cap \mathcal{V}_{k-1}^{\perp_a}$.

Proof:
Let $S^{n_1,n_2} := \{f \in W_k \mid \operatorname{supp}(f) \subset [x_k^{2n_1}, x_k^{2n_2}]\}$. Let $B_n := B^{1,n}_{1,2n-1}$. Note that $f \in S^{0,n}$ if and only if $f = y^T(\phi_k^1, \ldots, \phi_k^{2n-1})^T$ for some $y \in \ker B_n$ and that $f \in S^{n-3,n}$ if and only if $f = y^T \Phi_k^n$ for some $y \in \ker C_n$. Hence, the proof will be complete if we can show that

$$S^{0,n} = S^{0,n-1} + S^{n-3,n}, \qquad (4 \le n \le N_{k-1}). \tag{11}$$

Observe that

$$B_n = \begin{pmatrix} B_{n-1} & E_n \\ 0 & F_n \end{pmatrix} \tag{12}$$

for $n \geq 4$.

Suppose $v \in \ker F_n$, then by (10) there is some $u \in \ker C_n$ such that $(u^4, u^5)^T = v$. Suppose $y \in \ker B_n$. From (12) it is clear that $w := (y_{2n-2}, y_{2n-1})^T \in \ker F_n$ and hence there is some $u \in \ker C_n$ such that the last two components of y agree with the last two components of u. We then obtain

$$\ker B_n = P_1(\ker B_{n-1}) + P_2(\ker C_n) \tag{13}$$

where P_1 is the padding operator that takes a vector v of length $2n-3$ to one of length $2n-1$ by appending two zeros to v and P_2 is the padding operator that takes a vector v of length 5 to one of length $2n-1$ by prepending $2n-6$ zeros to v.

Then (11) follows from (13) and the proof is complete. ■

4 Uniform Partition

In this section we give the construction of piecewise-linear wavelets on a uniform partition which are semi-orthogonal with respect to the bilinear form associated with the following model problem:

$$\begin{aligned} -\epsilon^2 u'' + u &= f \\ u(0) = u(L) &= 0. \end{aligned} \tag{14}$$

Let $\langle \cdot, \cdot \rangle$ be the usual inner product on $L_2([0, L])$ and let $\mathcal{H}$ be the Sobolev space $\mathcal{H}_0^1((0, L))$ for some $L \in \mathbb{R}^+$. We assume f is such that $F := \langle f, \cdot \rangle$ is in $\mathcal{H}'$. Then the weak formulation becomes: Find $u \in \mathcal{H}$ so that

$$a(u, v) := \epsilon^2 \langle u', v' \rangle + \langle u, v \rangle = \langle f, v \rangle \quad \forall v \in \mathcal{H}. \tag{15}$$

Moreover, suppose (X_k) is a uniform partition of $[0, L]$. In order to avoid special cases associated with boundary wavelets, we let $L = 4$ and $N_0 = 4$:

$$x_k^j = \frac{j}{2^k}, \quad j = 0, \ldots, N_k = 2^k L.$$

Let $\phi(x) = (1 - |x-1|)_+$ and define

$$\phi_k^j := 2^{k/2}\phi(2^k \cdot - j).$$

(Here we have chosen a different normalization of ϕ_k^j than the normalization used in the non-uniform case.) Then

$$\phi_k^j = \sum_{l=-1}^{1} h_l \phi_{k+1}^{j-l}$$

where $h_{-1} = h_1 = \frac{1}{2\sqrt{2}}$, and $h_0 = \frac{1}{\sqrt{2}}$.

As in the previous section, let Φ_k^n be defined by (8). Because of the differentiation in the scalar product a, the ϵ in the model problem is scaled differently at each level resulting in a level dependent parameter ϵ_k given by

$$\epsilon_k := 2^k \epsilon.$$

In this case, C_n is independent of $4 \le n \le N_{k-1} - 1$ and its kernel is the space spanned by the vector

$$w = (24\epsilon_k^2 - 1, 6, 48\epsilon_k^2 - 10, 6, 24\epsilon_k^2 - 1)^T \tag{16}$$

The kernel of C_3 (respectively, $C_{N_{k-1}}$) contains w plus an additional vector w_L (respectively, w_R) given below:

$$w_L = (9 + 72\epsilon_k^2, -6, 1 - 24\epsilon_k^2, 0, 0)^T$$

and

$$w_R = (0, 0, 1 - 24\epsilon_k^2, -6, 9 + 72\epsilon_k^2)^T.$$

Then we let $\psi_k^1 = w_L^T \Phi_k^3$, $\psi_k^j = w^T \Phi_k^{j+1}$ for $2 \le j \le N_{k-1} - 1$ and $\psi_k^{N_{k-1}} = w_R^T \Phi_k^{N_{k-1}}$.

The wavelet ψ is shown in Figure 2 for selected ϵ and for a larger set in Figure 3. Another more general construction of semi-orthogonal wavelets on a uniform grid was given in [3, 4].

4.1 Unbounded Domain: Riesz bounds and Battle-Lemarié type wavelets

We next consider the simpler choice of domain $\mathbb{R}$. In this case we can calculate the Riesz bounds for the wavelet bases ψ_k for $\mathcal{W}_k$ using Fourier transform techniques. For $\theta \in L_2(\mathbb{R})$ we define the Grammian symbol E_θ (with respect to the scalar product $a(\cdot,\cdot)$) by

$$E_\theta(\omega) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} a(\theta, \theta(\cdot - n)) e^{in\omega} \tag{17}$$

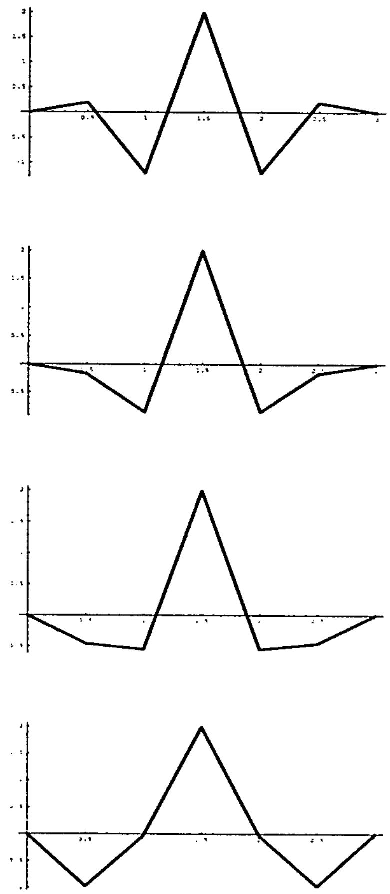

Figure 2: Semi-orthogonal Sombrero for $\epsilon = 0, 0.3, 0.5, 3$ respectively on a uniform grid

It is a standard result (see [8], for example) that the Riesz bounds $\underline{\alpha}_\Theta$ and $\overline{\alpha}_\Theta$ for the infinite basis $\Theta = (\theta(\cdot - n))_{n\in\mathbb{Z}}$ with respect to $a(\cdot,\cdot)$ are equal to the essential infimum and essential supremum of E_Θ, respectively. The L_2-condition number of the infinite matrix (A^Θ) is then the ratio $\overline{\alpha}_\Theta/\underline{\alpha}_\Theta$.

In the case of our model problem with the sombrero wavelets ψ_k we get

$$E_{\psi_k}(\omega) = \alpha_0 + 2\alpha_1 \cos(w) + 2\alpha_2 \cos(2w)$$

where

$$\begin{aligned}
\alpha_0 &= 12(3 + 122\epsilon_k^2 + 480\epsilon_k^4 + 1152\epsilon_k^6) \\
\alpha_1 &= 20/3 + 384\epsilon_k^2 - 2304\epsilon_k^4 - 9216\epsilon_k^6 \\
\alpha_2 &= (2/3)(1 - 24\epsilon_k^2)^2(-1 + 6\epsilon_k^2).
\end{aligned}$$

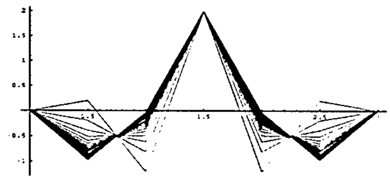

Figure 3: Semi-orthogonal Sombrero for ϵ between 0 and 3

It is an elementary, but tedious, exercise to verify that

$$\frac{\max_\omega E_{\psi_k}(\omega)}{\min_\omega E_{\psi_k}(\omega)} = \begin{cases} 4\frac{(1+12\epsilon_k^2)^3}{9+432\epsilon_k^2} & \text{for } 0 \le \epsilon_k \le 0.33 \\ \\ 4\frac{9+432\epsilon_k^2}{(1+12\epsilon_k^2)^3} & \text{for } 0.36 \le \epsilon_k < \infty \end{cases}$$

and that

$$\frac{\max_\omega E_{\psi_k}(\omega)}{\min_\omega E_{\psi_k}(\omega)} \le 1.2 \quad \text{for } 0.33 \le \epsilon_k \le 0.36. \tag{18}$$

Since A^{Ψ_k} is block diagonal, A^{Ψ_k} can be preconditioned with a simple diagonal preconditioner so that the resulting A^{Ψ_k} satisfies

$$\mathrm{cond}(A^{\Psi_k}) = \max_{j\le k} \mathrm{cond}(A^{\psi_j}).$$

Then (18) shows that $\mathrm{cond}(A^{\Psi_k})$ is uniformly bounded for $0 \le \epsilon_k \le \epsilon^*$ for any fixed ϵ^*. For instance, we get the following:

$$\mathrm{cond}(A^{\Psi_k}) < \begin{cases} 2.4 & \text{for } \epsilon_k < .5 \\ 271 & \text{for } \epsilon_k < 2 \\ 1330 & \text{for } \epsilon_k < 3 \end{cases}$$

For the unbounded domain case we use the following well known Fourier trick (cf. [8]) to construct an a-orthogonal basis for $\mathcal{W}_k$. Let $(\xi_\ell)_{\ell\in\mathbb{Z}}$ denote the Fourier coefficients of $\sqrt{1/E_{\psi_k}}$ and define

$${}^{BL}\psi_k^j := \sum_\ell \xi_\ell \psi_k^{j+\ell}.$$

In the case $\epsilon = 0$, we get the usual Battle-Lemarié wavelets. In this case, A^{Ψ_k} is the identity matrix. It is interesting to observe that ${}^{BL}\psi_0^0$ appears to converge pointwise to the Schauder wavelet ${}^h\psi_0^0 = \phi_1^1$ as ϵ goes to infinity. The wavelet ${}^{BL}\psi$ is shown in Figure 4 for selected ϵ.

4.2 Hybrid Basis

Our goal is to achieve a robust, fully scalable algorithm which is uniformly $\mathcal{O}(N_K)$ independent of the size of the problem L, the maximum refinement

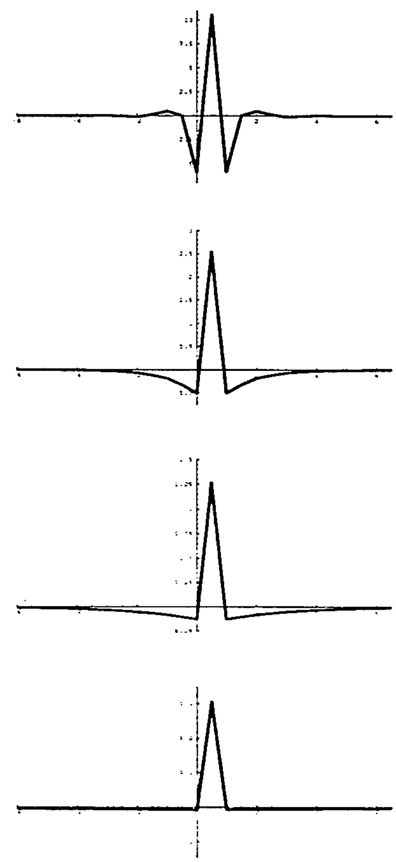

Figure 4: Battle-Lemarié type wavelets for $\epsilon = 0, 1, \sqrt{5}, 10$ respectively

level K, and the parameter ϵ. In this section we assume that our bases are normalized in the a-norm. This corresponds to a preconditioning of the form $D^{-1/2}AD^{-1/2}$ where D is the diagonal of A. We let ${}^{h}\Psi_k$ denote the normalized Schauder basis described in Section 3.1 and ${}^{s}\Psi_k$ the normalized Sombrero basis described in Section 4.

For the model problem, the semi-orthogonal basis is ill-conditioned for large ϵ and well-conditioned for small ϵ. One approach we have explored numerically is to use the hybrid basis

$$ {}^{sh}\Psi_K := \begin{pmatrix} {}^{s}\Psi_{\tilde{k}} \\ {}^{h}\psi_{\tilde{k}+1} \\ \vdots \\ {}^{h}\psi_K \end{pmatrix} $$

where $\tilde{k}$ is chosen so that $\epsilon_{\tilde{k}} = \mathcal{O}(1)$. The resulting discretized matrix $A^{{}^{sh}\Psi_K}$

is illustrated in Figure 5. Our numerical experiments indicate that the hybrid basis achieves the above mentioned goals for the model problem.

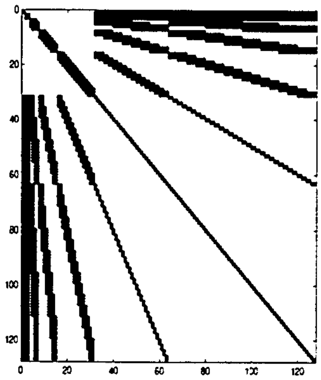

Figure 5: $A^{{}^{sh}\Psi_K}$ consisting of 4 levels with the semi-orthogonal basis combined with 2 additional levels with the Schauder basis.

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