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COMPARISON OF EQUATIONS FOR THE FDTD SOLUTION IN ANISOTROPIC AND DISPERSIVE MEDIA *

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I. Introduction

The finite-difference time-domain (FDTD) solution procedure developed by Yee [1], has in recent years been extended to dispersive and anisotropic media to handle materials such as magnetized ferrites and plasmas. The solution for dispersive media has been accomplished through a recursive update of a convolution integral in the constitutive relations for the fields [2], [3], [4], [5], by numerical solution of the differential equation form of the constitutive relations [6], [7] and using Z transforms [8]. The extension of the recursive convolution (RC) method to gyrotropic materials, which are both dispersive and anisotropic, was developed in [9] and [10] where it was applied to 1D problems. The anisotropic media results in coupling of field components and the need for averaging to obtain field components at locations where they are not directly available in the Yee formulation. The RC solution for gyrotropic media is reviewed in [4], and results validating the method are also given. The solution was developed for 3D Gyrotropic materials in [11] for a ferrite with biasing field in an arbitrary direction, and the issue of minimizing the storage added by the recursive convolution evaluation was also considered there. A piecewise linear RC method has also been developed that is more accurate than the pulse approximation considered here [12].

In published work on dispersive material there are some differences in the equations resulting from application of the RC method. The time derivative of the convolution integral can involve the derivative of the field, or integrating by parts can put the derivative on the susceptibility function. Reduction of these two results to discrete form leads to slightly different update equations. Also, the choice of the evaluation time and integration limit of the convolution integral can lead to differences in the discrete update equation. These different forms of the solution are compared here for accuracy and stability for time increments approaching the Courant limit. It is found that slightly greater accuracy and greater stability are obtained with the convolution evaluated at the time of the equation, a half step before the field being evaluated, using a pulse approximation of the integral ending in a half pulse. Modifications of this result lead to somewhat simpler but less stable equations. In the case that the susceptibility function starts at zero for time equal to zero the equations for anisotropic and dispersive material simplify greatly, requiring only the addition of the RC term to the normal FDTD equations, without further coupling of the field components. While 3D solutions are considered here, the accuracy and stability are demonstrated for the 1D problem of normal incidence on a slab of ferrite or plasma with biasing field in the direction of propagation, since simple analytic solutions are available for this problem.

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II. Equations for Gyrotropic Media

The solution for anisotropic and dispersive magnetic material will be considered here. The magnetic field update equation for such material is obtained by combining the Maxwell's equation

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \tag{1}$$

and the equation relating $\mathbf{B}$ and $\mathbf{H}$ in convolution form

$$\mathbf{B}(t) = \mu_0 \left[\mathbf{H}(t) + \int_0^t \bar{\bar{\chi}}_m(t-\tau) \cdot \mathbf{H}(\tau)\, d\tau \right] \tag{2}$$

where $\bar{\bar{\chi}}_m(t)$ is the susceptibility tensor representing the impulse response of the material. In the usual convention, equation (1) is solved together with the equation $\epsilon \partial \mathbf{E}/\partial t = \nabla \times \mathbf{H}$ with $\mathbf{E}$ evaluated at integral time steps $n\Delta t$, and $\mathbf{H}$ and $\mathbf{B}$ evaluated at half time steps $(n+1/2)\Delta t$. The time derivatives in Maxwell's equations are approximated with central differences so that $\mathbf{E}$ and $\mathbf{H}$ can be computed in a leap-frog scheme in time.

Equation (2) can be reduced to discrete form by assuming that $\mathbf{H}(t)$ is constant with value $\mathbf{H}^{n+1/2}$ for $n\Delta t < t < (n+1)\Delta t$ with the result

$$\begin{aligned}
\mathbf{B}^{n+1/2} =& \mu_0 \Bigg[\mathbf{H}^{n+1/2} + \sum_{i=0}^{n-1} \int_{i\Delta t}^{(i+1)\Delta t} \bar{\bar{\chi}}_m \left[(n+\tfrac{1}{2})\Delta t - \tau \right] d\tau \cdot \mathbf{H}^{i+1/2} \\
&+ \int_{n\Delta t}^{(n+1/2)\Delta t} \bar{\bar{\chi}}_m \left[(n+\tfrac{1}{2})\Delta t - \tau \right] d\tau \cdot \mathbf{H}^{n+1/2} \Bigg] \\
=& \mu_0 \Bigg[\mathbf{H}^{n+1/2} + \sum_{i=0}^{n-1} \int_{(n-i-1/2)\Delta t}^{(n-i+1/2)\Delta t} \bar{\bar{\chi}}_m(\tau')\, d\tau' \cdot \mathbf{H}^{i+1/2} + \int_0^{\frac{1}{2}\Delta t} \bar{\bar{\chi}}_m(\tau')\, d\tau' \cdot \mathbf{H}^{n+1/2} \Bigg].
\end{aligned}$$

If the integrals over $\bar{\bar{\chi}}_m(\tau')$ are also approximated by sums of pulses the result for $\mathbf{B}^{n+1/2}$ is

$$\mathbf{B}^{n+1/2} = \mu_0 \left[\left(\bar{\bar{\mathbf{I}}} + \tfrac{1}{2}\Delta t\, \bar{\bar{\chi}}_m(0) \right) \cdot \mathbf{H}^{n+1/2} + \Delta t \sum_{i=0}^{n-1} \bar{\bar{\chi}}_m \left[(n-i)\Delta t \right] \cdot \mathbf{H}^{i+1/2} \right].$$

With a similar representation for $\mathbf{B}^{n-1/2}$ the central difference approximation of $\partial \mathbf{B}(t)/\partial t$ in equation (1) leads to the update equation

$$\mathbf{H}^{n+1/2} = \left[\bar{\bar{\mathbf{I}}} + \frac{\Delta t}{2} \bar{\bar{\chi}}_m(0) \right]^{-1} \cdot \left\{ \left[\bar{\bar{\mathbf{I}}} - \frac{\Delta t}{2} \bar{\bar{\chi}}_m(0) \right] \cdot \mathbf{H}^{n-1/2} - \Delta t \Psi^n - \frac{\Delta t}{\mu_0} \nabla \times \mathbf{E}^n \right\} \tag{3}$$

where

$$\Psi^n = \sum_{i=0}^{n-1} \left[\bar{\bar{\chi}}_m[(n-i)\Delta t] - \bar{\bar{\chi}}_m[(n-i-1)\Delta t] \right] \cdot \mathbf{H}^{i+1/2}. \tag{4}$$

In the recursive convolution solution the elements of the susceptibility tensor are sums of exponentials, $\chi_{ij} = \sum_\ell a_{ij\ell} e^{b_\ell t}$, in which case Ψ^n in (4) can be updated by a simple recursion relation

$$\Psi_\ell^{n+1} = e^{b_\ell \Delta t} \Psi_\ell^n + (e^{b_\ell \Delta t} - 1)\, \bar{\bar{\mathbf{a}}}_\ell \cdot \mathbf{H}^{n+1/2}, \qquad \Psi_\ell^0 = 0 \tag{5}$$

and $\Psi^{n+1} = \sum_\ell \Psi_\ell^{n+1}$.

A result differing somewhat from equation (3) can be obtained by evaluating the derivative of equation (2) as

$$\frac{\partial}{\partial t}\mathbf{B}(t) = \mu_0 \left[\frac{\partial}{\partial t}\mathbf{H}(t) + \bar{\bar{\chi}}(t)\cdot\mathbf{H}(0) + \int_0^t \bar{\bar{\chi}}(\tau)\cdot\frac{\partial}{\partial t}\mathbf{H}(t-\tau)\,d\tau\right] \tag{6}$$

or, integrating by parts, as

$$\frac{\partial}{\partial t}\mathbf{B}(t) = \mu_0 \left[\frac{\partial}{\partial t}\mathbf{H}(t) + \bar{\bar{\chi}}(0)\cdot\mathbf{H}(t) + \int_0^t \frac{d}{dt}\bar{\bar{\chi}}(t-\tau)\cdot\mathbf{H}(\tau)\,d\tau\right]. \tag{7}$$

Converting equation (6) to discrete form with a pulse approximation of the integral leads to a result close to that of equation (3). Starting with equation (7) and representing $\partial\mathbf{H}(t)/\partial t$ with a central difference and $\bar{\bar{\chi}}(0)\cdot\mathbf{H}(n\Delta t)$ with an average of $\mathbf{H}$ from times $(n-1/2)\Delta t$ and $(n+1/2)\Delta t$ leads to an equation similar to (3), but with Ψ^n replaced by $\Delta t\,\Psi'^n$ where the prime indicates a derivative and

$$\Psi'^n = \sum_{i=0}^{n-1} \bar{\bar{\chi}}'_m[(n-i-\tfrac{1}{2})\Delta t]\cdot\mathbf{H}^{i+1/2}.$$

Somewhat different results can also be obtained for equation (3) from different interpretations of the pulse approximation of the integral, or in the equation derived from equation (7) by using the value at the forward or back time step rather than the average for $\mathbf{H}(n\Delta t)$. In fact, adding the same small quantity to both square-bracketed terms in (3) results in a second-order change in the product multiplying $\mathbf{H}^{n-1/2}$ and relatively small changes in the solution. Hence equation (3) can be reduced to

$$\mathbf{H}^{n+1/2} = \left[\bar{\bar{\mathbf{I}}} + \Delta t\bar{\bar{\chi}}_m(0)\right]^{-1}\cdot\left(\mathbf{H}^{n-1/2} - \Delta t\Psi^n - \frac{\Delta t}{\mu_0}\nabla\times\mathbf{E}^n\right) \tag{8}$$

by adding $(\Delta t/2)\bar{\bar{\chi}}(0)$ to both coefficients, or by subtracting the same quantity

$$\mathbf{H}^{n+1/2} = \left[\bar{\bar{\mathbf{I}}} - \Delta t\bar{\bar{\chi}}_m(0)\right]\cdot\mathbf{H}^{n-1/2} - \Delta t\Psi^n - \frac{\Delta t}{\mu_0}\nabla\times\mathbf{E}^n. \tag{9}$$

These modifications of equation (3) result in small errors when the elements of $\Delta t\bar{\bar{\chi}}(0)$ have magnitudes much less than one, as demonstrated in the next section, but the solutions show increased late-time instability when Δt is near the Courant limit. Equation (9) seems to be a nicer form for solving, but it still mixes values of $\mathbf{H}^{n-1/2}$ in averaging for field components in the product with the tensor. The evaluation of a single vector component of the product of a tensor and $\nabla\times\mathbf{E}$ involves 36 field components in the Yee cell, but this can be reduced to 20 components by combining and canceling terms.

When $\bar{\bar{\chi}}(0) = 0$, which occurs in materials such as Lorentz dielectrics, equations (3), (8) and (9) all reduce to the simpler and easier to use form

$$\mathbf{H}^{n+1/2} = \mathbf{H}^{n-1/2} - \Delta t\Psi^n - \frac{\Delta t}{\mu_0}\nabla\times\mathbf{E}^n. \tag{10}$$

The above discussion has assumed total fields. The generalization of equation (3) for separate incident and scattered fields is

$$\mathbf{H}_s^{n+1/2} = \left[\bar{\bar{\mathbf{I}}} + \frac{\Delta t}{2}\bar{\bar{\chi}}_m(0)\right]^{-1} \cdot \left\{ \left[\bar{\bar{\mathbf{I}}} - \frac{\Delta t}{2}\bar{\bar{\chi}}_m(0)\right] \cdot \mathbf{H}_s^{n-1/2} - \Delta t \boldsymbol{\Psi}_s^n \right.$$
$$\left. - \Delta t \frac{d}{dt}\int_0^t \bar{\bar{\chi}}_m(\tau) \cdot \mathbf{H}_i(t-\tau)\, d\tau \right|_{t=n\Delta t} - \frac{\Delta t}{\mu_0}\nabla \times \mathbf{E}_s^n \Big\}$$

where $\mathbf{H}_i$ is the incident field, which satisfies Maxwell's equations for free space everywhere in the problem space, and

$$\boldsymbol{\Psi}_s^n = \sum_{i=0}^{n-1}\left[\bar{\bar{\chi}}_m[(n-i)\Delta t] - \bar{\bar{\chi}}_m[(n-i-1)\Delta t]\right] \cdot \mathbf{H}_s^{i+1/2}.$$

The convolution integral with $\mathbf{H}_i$ can be evaluated analytically for some special incident field functions [5], or otherwise is evaluated numerically along with the scattered field.

III. Results

The FDTD solution for gyrotropic media was validated for a plane wave normally incident on a ferrite slab with the biasing magnetic field in the direction of propagation, along the z axis. This problem was also solved in [4] and [9], and is chosen because the reflection and transmission coefficients for the slab are available in simple analytic form. In this case a 3D code was written, using the equations from the preceding section. Codes solving equations (3), (8) and (9) were compared for accuracy and stability with time increments approaching the Courant limit. Since a plane wave propagating through the ferrite in the direction of the biasing field splits into right-hand and left-hand circularly polarized waves with different propagation constants, the problem space was terminated in even-symmetry boundary conditions in both x and y boundary planes. In the direction of propagation z the problem space was made large enough to gate out reflections, thus eliminating the boundary conditions as a source of error.

The components of the susceptibility tensor $\bar{\bar{\chi}}(t)$ for the ferrite are

$$\chi_{11}(t) = \chi_{22}(t) = \mathrm{Re}\left\{\frac{\omega_m}{\alpha + j}\exp\left[\frac{-\omega_0 t}{\alpha + j}\right]\right\} U(t)$$

$$\chi_{12}(t) = -\chi_{21}(t) = \mathrm{Re}\left\{\frac{j\omega_m}{\alpha + j}\exp\left[\frac{-\omega_0 t}{\alpha + j}\right]\right\} U(t).$$

where $U(t)$ is the unit step function. The parameters of the ferrite modeled here were

$$\omega_0 = (2\pi) \cdot 20 \times 10^9 \text{ rad/s}$$
$$\omega_m = (2\pi) \cdot 10 \times 10^9 \text{ rad/s}$$
$$\alpha = 0.1$$

Results from solving equation (3) for this ferrite with $\Delta x = 75(10^{-6})$ m and $\Delta t = \Delta x/(2c\sqrt{3})$ with 6000 time steps are shown in Figure 1. The source was a Gaussian-pulse plane wave with full-width-half-max equal to 0.001 m. Since the solution is uniform in the x and y directions the problem was solved with 3 cells in x and y and 4000 cells in z to eliminate the radiating boundaries, and the ferrite filled 50 cells (k = 2000 through 2049) for a thickness of 0.00375 m. The reflection and transmission coefficients were obtained by numerical deconvolution of the reflected fields, with right and left-hand circular polarizations obtained as

$$R_{\mathrm{rcp}}(\omega) = R_x(\omega) + jR_y(\omega)$$

and

$$R_{\mathrm{lcp}}(\omega) = R_x(\omega) - jR_y(\omega).$$

In Figure 1 the magnitudes of the numerically determined reflection and transmission coefficients for left-hand polarization are compared with the exact results, and the relative errors in the complex quantities are also shown. The error increases with frequency due mainly to dispersion in the FDTD mesh. Although these results were obtained with a 3D code for which the Courant limit is $\Delta t \leq \Delta x/c\sqrt{3}$ the problem is actually 1D. As a result Δt can be extended to the 1D Courant limit of $\Delta x/c$, and the resulting dispersion errors in solving equation (3) are reduced by one to two orders of magnitude above about 200 GHz.

The errors from solving the simpler equations (8) and (9) are compared with the errors from equation (3) in Figure 2 for $\Delta t = \Delta x/2c\sqrt{3}$. Below about 100 GHz equations (8) and (9) yield slightly higher error than (3). Above 100 GHz the results of equation (8) have slightly lower error than (3) for reflected field while all errors become the same for transmitted field. Solving equations (8) or (9) at the Courant limit of $\Delta t = \Delta x/c$ resulted in a rapid blowup in the ferrite, as shown in Figure 3. Equation (3) also became unstable at late time with $\Delta t = \Delta x/c$, but the instability did not become significant until after about 4000 time steps, which was late enough to get useful results. The fields plotted in Figure 3 were at 60 cells in front of the ferrite slab. When the solution of equation (8) was stopped at t = 0.15 ns the field in the ferrite was over 10^{13}. With $\Delta t = \Delta x/c\sqrt{3}$ equation (8) still showed an instability at a reduced rate, as shown in Figure 4. No significant instability was seen in equation (3) at this Δt. Equation (9) showed a stability close to that of equation (8). The equation derived with using equation (7), which was used in [13], yielded about a factor of two lower error than equation (3) below 50 GHz and the errors were identical at higher frequencies. Stability of this equation was also similar to (3).

IV. Conclusion

The recursive-convolution solution for anisotropic and dispersive media was seen to yield accurate results for reflection from ferrite slabs up to a frequency limit set by the sampling interval. Depending on the application, the results shown might be considered usable up to about 300 GHz, which corresponds to about 13 cells per wavelength. Results at still higher frequencies might be usable when a time delay or frequency shift due to dispersion can be tolerated.

Several different forms of the update equations were considered which can result from different approximations in reducing the continuous equations to discrete form. Equation (3) and the alternate form derived with equation (7), and used in [13], result from direct application of the pulse approximations of the fields and susceptibilities, and differ only in the way that the time derivative of the convolution integral is approximated. Equations (8) and (9) are similar to equation (3) with the limit of the convolution interval shifted by a half time step. These equations correspond to the different forms for a conductive medium when the field multiplying the conductivity is taken as the forward or back value in time or the average. Equation (3) was found to be more stable than (8) or (9), and slightly more accurate at low frequencies. Since equation (9) does not involve a tensor multiplying the curl operation it could be considerably faster to evaluate.

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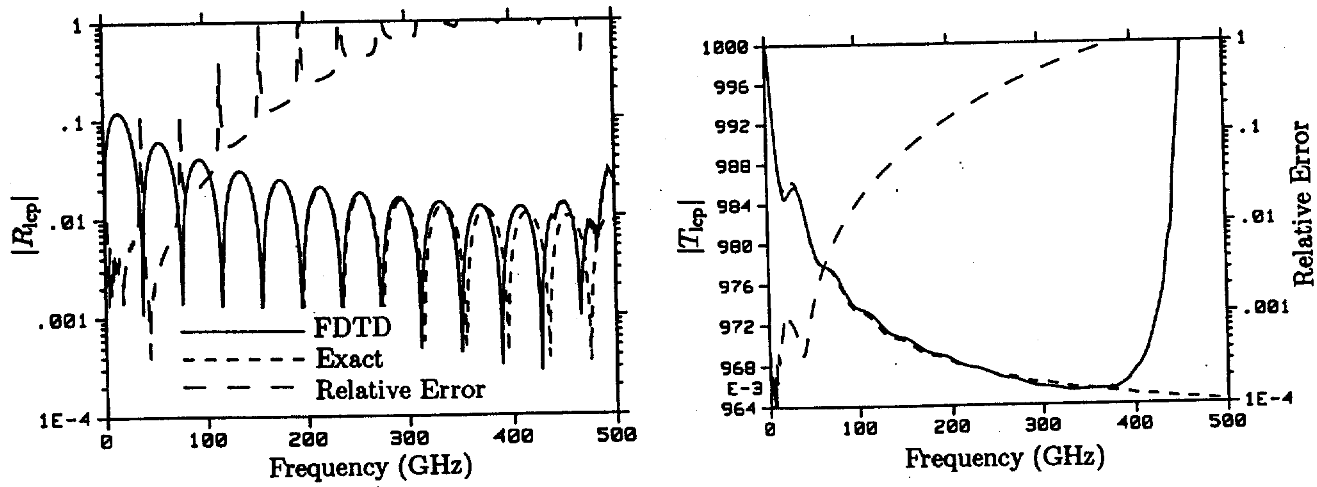

Fig. 1. Magnitudes of the left-hand polarized reflection and transmission coefficients for normal incidence of a plane wave on a ferrite slab from the FDTD solution are compared with the exact solutions. The relative errors in the complex quantities are also shown.

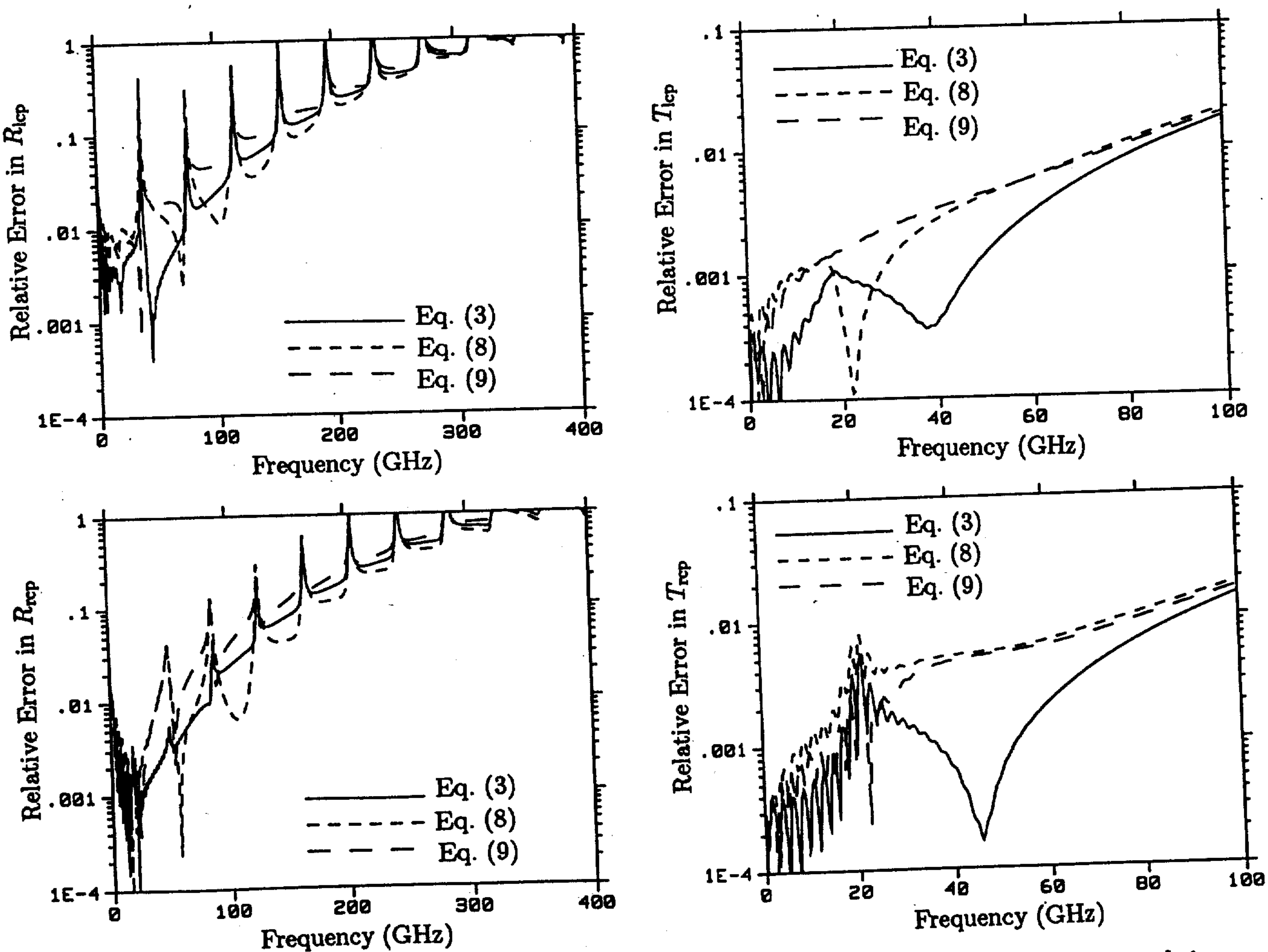

Fig. 2. Relative errors in the complex reflection and transmission coefficients from solving equations (3), (8) and (8) for normal incidence on a ferrite slab with $\Delta t = \Delta x/2c\sqrt{3}$.

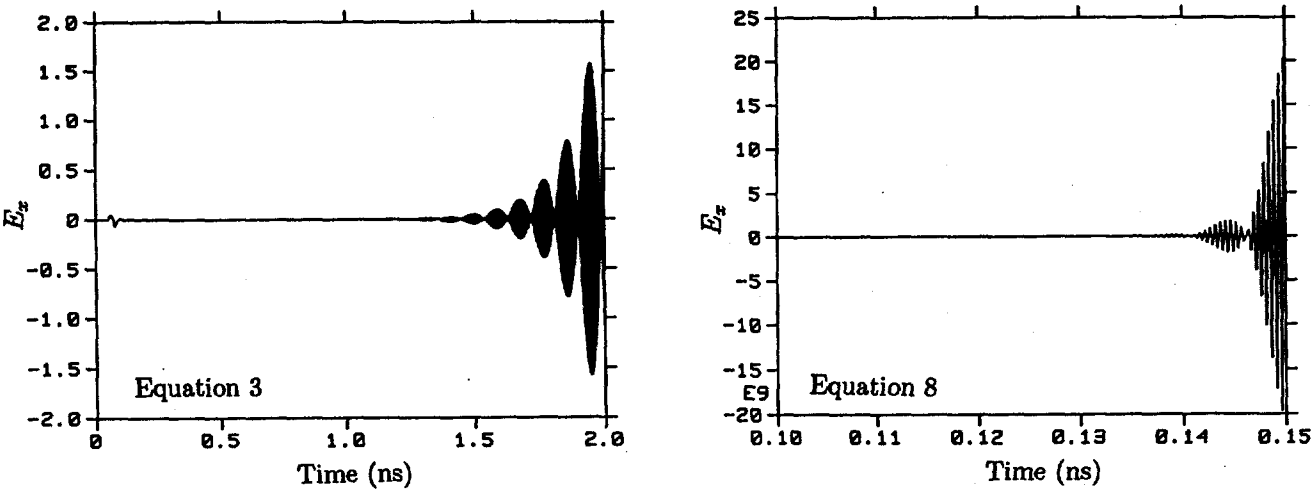

Fig. 3. Electric field in front of the ferrite slab showing the difference in stability of equations (3) and (8) solved at the 1D Courant limit of $\Delta t = \Delta x/c$.

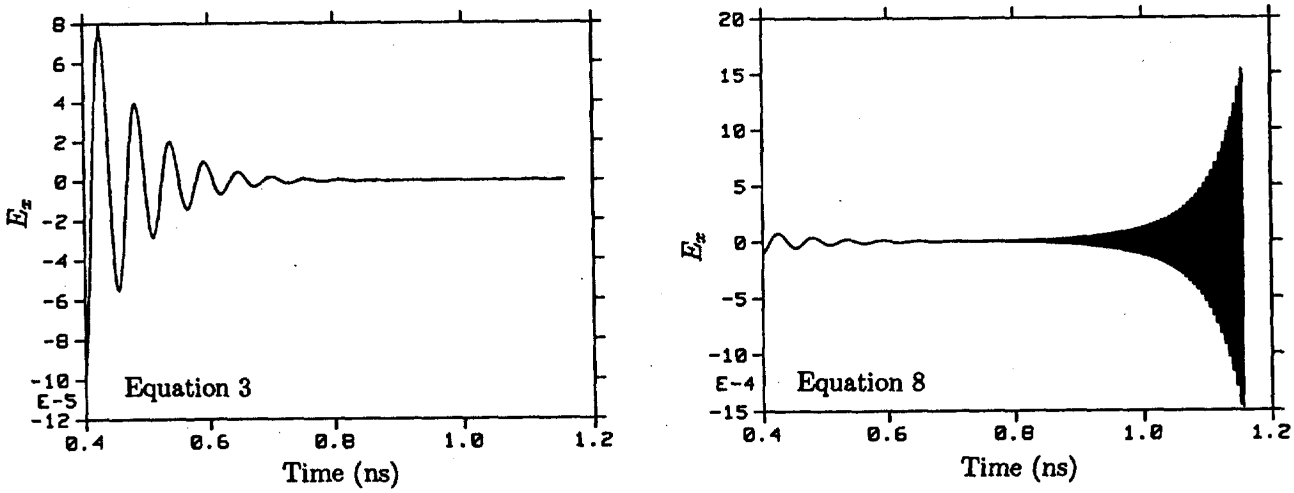

Fig. 4. Electric field in front of the ferrite slab showing the difference in stability of equations (3) and (8) solved at a time step of $\Delta t = \Delta x/c\sqrt{3}$.

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