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On the Theory of Internal Kink Oscillations

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Abstract

In this paper we derive a time evolution equation for internal kink oscillations which is valid for both stable and unstable plasma regimes, and incorporates the nonlinear response of an energetic particle population. A linear analysis reveals a parallel between (i) the time evolution of the spatial derivative of the internal kink radial displacement and (ii) the time evolution of the perturbed particle distribution function in the field of an electrostatic wave (Landau problem). We show that diamagnetic drift effects make the asymptotic decay of internal kink perturbations in a stable plasma algebraic rather than exponential. However, under certain conditions the stable root of the dispersion relation can dominate the response of the on-axis displacement for a significant period of time. The form of the evolution equation naturally allows one to include a nonlinear, fully toroidal treatment of energetic particles into the theory of internal kink oscillations.

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I. Introduction

Two important instabilities of axisymmetric, current-carrying toroidal plasmas – sawtooth relaxations [1] and fishbone oscillations [2] – are associated with the instability of internal kink modes. These are waves with toroidal mode number $n = 1$ and dominant poloidal mode number $m = 1$, such that the radial displacement is nearly constant in the central plasma region where the safety factor is less than unity; *i.e.*, where $q(r) < 1$. This radial displacement rapidly drops to zero in a narrow layer centered at the $q = 1$ surface.

Sawteeth are MHD events determined by properties of the bulk plasma, which occur in both Ohmic and auxiliary-heated plasma discharges. Fishbones, however, involve the resonant interaction of the MHD response with a fast ion population, and are commonly observed in present tokamak experiments when the plasma is heated by neutral beam injection and/or by ion cyclotron radio frequency waves. There is also the possibility that fishbones may be excited by fusion-product alpha particles.

This paper addresses two aspects of the theory of the internal kink; namely (i) the derivation of a new, compact evolution equation for the internal kink including the nonlinear dynamics of fast ions, and (ii) an elucidation of the detailed structure of internal kink oscillations for stable as well as unstable plasma parameters.

In standard linear MHD theory, the radial profile of the internal kink displacement is readily obtained for unstable conditions; *i.e.*, when the associated potential energy functional, δW, is negative. The situation is similar when the energy functional is generalized to include kinetic effects due to fast ions. In this case a normal mode is found when the system is unstable [3-4]. However, in the stable case the usual normal mode solution appears to fail. Nonetheless, we show that the stable root of the dispersion relation is physically significant.

In the first part of this work we derive a reduced evolution equation for the fishbone instability in the limit that the background plasma behaves as a linear internal kink. In the background response we include diamagnetic effects associated with thermal ions, so that the reduced equation is applicable to the full range of collisionless fishbone instability regimes discussed in the literature, both when diamagnetic terms are important [3], and when they are not [4]. However, this evolution equation does not take into account the effect of fluid nonlinearities, and as a consequence, represents only a first step towards a complete nonlinear theory of the fishbone instability.

A detailed mathematical analysis of the reduced equation, which takes the form of an initial value problem for the on-axis plasma displacement, then enables us to extend the standard linear theory to include the description of stable eigenmodes by means of an appropriate analytic continuation. In doing so, we show that the internal kink response is in fact a general type of collective plasma oscillation which has both a physical and mathematical similarity to Landau's description of electron plasma oscillations [5].

The paper is organized in the following way. In Sec. II, we derive an integral equation for the on-axis kink displacement, and then compute the long-time asymptotic response of the solution in Sec. III. Sec. IV presents an explicit calculation of the time-dependent radial profile for the case where diamagnetic and fast ion effects are negligible. A general connection with the mathematical structure of the Landau problem is made in Sec. V. Finally, in Sec. VI, we give a summary of the most important results. Two Appendices are included, and outline the essential mathematical steps required to justify results quoted in the main body of the paper.

II. Reduced Equation of Motion

We begin by writing the collisionless equation of motion for the bulk plasma displacement $\boldsymbol{\xi}$:

$$\rho D_{tt}\,\boldsymbol{\xi} = \frac{1}{c}\left(\delta\mathbf{j}\times\mathbf{B} + \mathbf{j}\times\delta\mathbf{B}\right) - \nabla\delta p_{\rm core} - \nabla\cdot\delta\mathbb{P}_{\rm hot}\ , \tag{1}$$

where ρ is the mass density and D_{tt} is a differential operator that describes the effect of inertia and finite Larmor radius. $\mathbf{j}$ and $\delta\mathbf{j}$ are the equilibrium and perturbed plasma currents, $\mathbf{B}$ and $\delta\mathbf{B}$ are the equilibrium and perturbed magnetic fields, and $\delta p_{\rm core}$ is the perturbed (isotropic) plasma pressure. The quantities $\delta\mathbf{j}$, $\delta\mathbf{B}$ and $\delta p_{\rm core}$ are taken to be linear in $\boldsymbol{\xi}$. The only nonlinear term which we retain in this analysis is connected with $\delta\mathbb{P}_{\rm hot}$ – the perturbed hot particle pressure tensor.

It is convenient to decompose the $n = 1$ displacement into a sum over complex poloidal components, $\boldsymbol{\xi}^{(m)}$, according to

$$\boldsymbol{\xi}(\mathbf{r},t) = \sum_m \boldsymbol{\xi}^{(m)}(r,t)e^{i(\varphi - m\theta)} + \text{c.c.}\ , \tag{2}$$

where φ and θ are the poloidal and toroidal angles respectively, and r is a radial flux coordinate. Since only the radial component of $\boldsymbol{\xi}^{(1)}$ will arise in subsequent calculations, we will denote this simply by ξ. The limiting form of the radial displacement is the familiar internal kink step function

$$\xi(r,t) \longrightarrow \begin{cases} \xi_0(t)\,, & \text{if } r < r_1; \\ 0\,, & \text{if } r > r_1, \end{cases} \tag{3}$$

where r_1 is the radius of the $q = 1$ surface. This form of the displacement requires that $\varepsilon_1 \equiv r_1/R_0 \ll 1$, where R_0 is the major radius. We also assume that $1 - q(0) > 0$. From the equation of motion, Eq. (1), one can then employ standard

techniques to derive an equation for the radial displacement ξ in a narrow layer about $r = r_1$.

$$\frac{d}{dr}\left[r^3 B_0^2\left(\frac{M}{{v_{\rm A}}^2}D_{tt} + k_\parallel^2\right)\frac{d}{dr}\,\xi(r,t)\right] = 0\ . \tag{4}$$

The time-derivative operator which appears above is

$$D_{tt} \equiv \frac{\partial^2}{\partial t^2} + i\omega_{*\rm i}\frac{\partial}{\partial t}\ , \qquad \text{where} \quad \omega_{*\rm i} \equiv -\frac{1}{r_1\rho\omega_{\rm ci}}\frac{dp_{\rm core}}{dr}\Big]_{r=r_1} \tag{5}$$

is the ion diamagnetic frequency and ω_{ci} is the ion-cyclotron frequency. In Eq. (4), B_0 is the magnetic field on axis, M is an ion inertia enhancement factor [6] (which can be greater than unity for low frequency modes), $v_A = B_0/\sqrt{4\pi\rho}$ is the Alfvén speed and $k_\parallel$ is the parallel wavenumber. Eq. (4) can be integrated once in r to give

$$\left(\frac{\partial^2}{\partial\tau^2} + i\Omega_*\frac{\partial}{\partial\tau} + x^2\right)\frac{\partial}{\partial x}\xi(x,\tau) = S(\tau) , \tag{6}$$

where we have introduced the layer variable $x \equiv (r - r_1)/r_1$, the normalized time $\tau \equiv \omega_A t$, and the normalized frequency $\Omega_* \equiv \omega_{*i}/\omega_A$ such that $\omega_A \equiv v_A s/\sqrt{M}R_0$ (for definiteness, we specify $\Omega_* \geq 0$). The parallel wavevector has been expanded in the usual way according to $k_\parallel = sx/R$, with $s \equiv d(\ln q)/d(\ln r)$ the magnetic shear. The function $S(\tau)$ is an integration constant which depends only on the on-axis displacement, $\xi_0(\tau) \equiv \xi(-\infty,\tau)$. It can be shown that

$$S(\tau) = -\frac{\lambda_H}{\pi}\xi_0(\tau) - \frac{1}{\pi}\Lambda_K[\xi_0](\tau) , \tag{7}$$

where the coefficient λ_H is directly related to the minimized MHD potential energy [7-8]

$$|\xi_0(t)|^2\lambda_H \equiv -\frac{2}{(s\varepsilon_1 B_0)^2}\frac{\delta W_{\rm MHD}}{R_0} . \tag{8}$$

In $S(\tau)$, the quantity $\Lambda_K[\xi_0](\tau)$ represents the fast particle dynamics. The square brackets indicate that Λ_K is a time-dependent functional of the on-axis displacement, $\xi_0(\tau)$. One can also define Λ_K in terms of an equivalent fast particle energy, $\delta W_{\rm hot}$ – which, unlike $\delta W_{\rm MHD}$, is not in general a self-adjoint form:

$$\xi_0^*(\tau)\Lambda_K[\xi_0](\tau) \equiv -\frac{2}{(s\varepsilon_1 B_0)^2}\frac{\delta W_{\rm hot}(t)}{R_0} . \tag{9}$$

An explicit form for $\delta W_{\rm hot}$, which is readily found in the literature [9], is

$$\delta W_{\rm hot} = \frac{1}{2}\int d\Gamma\left(mv_\parallel^2 + \mu B\right)\delta f\ \boldsymbol{\kappa}\cdot\boldsymbol{\xi}^* , \tag{10}$$

where $\Gamma = d^3x\,d^3v$ is the phase space volume element, $\boldsymbol{\kappa} = \mathbf{b}\cdot\nabla\mathbf{b}$ is the magnetic curvature vector, $\mathbf{b} \equiv \mathbf{B}/B$, and μ is the magnetic moment. In Eq. (10), $\delta f = f - f_0$ is the deviation of the full, nonlinear energetic particle distribution, f, from its equilibrium value, f_0.

The task which remains is to find a solution of the layer equation, Eq. (6), which satisfies the boundary conditions $\xi(x,\tau) \to \xi_0(\tau)$ as $x \to -\infty$, and $\xi(x,\tau) \to 0$ as $x \to \infty$. In order to reduce the problem to a single equation for $\xi_0(\tau)$, we first solve Eq. (6) for $\partial\xi/\partial x$ subject to the initial conditions

$$\frac{\partial}{\partial x}\xi(x,\tau)\big]_{\tau=0} = F(x) \quad , \qquad \frac{\partial}{\partial \tau}\frac{\partial}{\partial x}\xi(x,\tau)\big]_{\tau=0} = G(x) \quad ; \tag{11}$$

where $F(x)$ and $G(x)$ are localized functions of x such that

$$\int_{-\infty}^{\infty} dx\, F(x) = -\xi_0(0)\ , \qquad \int_{-\infty}^{\infty} dx\, G(x) = -\dot{\xi}_0(0)\ . \tag{12}$$

The solution of the Eq. (6), subject to the initial conditions, Eq, (12), is

$$\begin{aligned} \frac{\partial \xi(x,\tau)}{\partial x} = {} & e^{-i\Omega_*\tau/2}\left\{F(x)\cos(X\tau) + \left[i\frac{\Omega_*}{2}F(x) + G(x)\right]\frac{\sin(X\tau)}{X}\right\} \\ & - \frac{1}{\pi}\int_0^\tau d\tau'\ [\lambda_H\xi_0(\tau') + \Lambda_K(\tau')]\ \frac{\sin[X(\tau-\tau')]}{X} e^{-i\Omega_*(\tau-\tau')/2}\ , \end{aligned} \tag{13}$$

where $X(x) \equiv \sqrt{(\Omega_*/2)^2 + x^2}$. We then integrate $\partial\xi/\partial x$ over all x to find

$$\xi_0(\tau) = Q(\tau) + \int_0^\tau d\tau'\ [\lambda_H\xi_0(\tau') + \Lambda_K(\tau')]\, J_0\left[\frac{\Omega_*}{2}(\tau-\tau')\right] e^{-i\Omega_*(\tau-\tau')/2}\ , \tag{14}$$

where

$$Q(\tau) \equiv -e^{-i\Omega_*\tau/2}\int_{-\infty}^{\infty} dx\ \left\{F(x)\cos(X\tau) + \left[\frac{i\Omega_*}{2}F(x) + G(x)\right]\frac{\sin(X\tau)}{X}\right\}\ . \tag{15}$$

Eq. (14) is a nonlinear integral equation which determines the on-axis displacement, $\xi_0(\tau)$. While it can be analyzed in the present form, it may also be convenient to cast it into an alternate form such that the fast ion term appears outside the integral. This can be done by means of the transformation described in Appendix A, with the result

$$\underbrace{\mathcal{I}[\xi_0](\tau)}_{\text{linear operator}} = \underbrace{\Lambda_K[\xi_0](\tau)}_{\substack{\text{nonlinear source}\\ \text{from fast particles}}} + \underbrace{\widehat{Q}(\tau)}_{\substack{\text{linear source due}\\ \text{to initial conditions}}}\ , \tag{16}$$

where $\mathcal{I}$ is the *linear* integro-differential operator:

$$\mathcal{I}[\xi_0] \equiv \dot{\xi}_0(\tau) + \left[\frac{i\Omega_*}{2} - \lambda_H\right]\xi_0(\tau) + \left(\frac{\Omega_*}{2}\right)^2\int_0^\tau d\tau'\xi_0(\tau')K\left[\frac{\Omega_*}{2}(\tau-\tau')\right]\ . \tag{17}$$

The kernel in Eq. (17) is $K(x) \equiv \exp(-ix)J_1(x)/x$. At each time step, the energetic particle current $\Lambda_K(\tau)$, defined in Eq. (9), must be computed by solving a kinetic equation for δf. The amplitude $\xi_0(\tau)$ can then be advanced in time according to a suitable finite-differencing of Eq. (17). This scheme has been implemented in a numerical code to calculate the fishbone response [10] where energetic particles are treated nonlinearly.

III. Long-Time Limit of Linear Oscillations

We now discuss the asymptotic behavior of $\xi_0(\tau)$ for large τ in the linear approximation, i.e., when Λ_K is a linear functional of ξ_0. We begin by taking the Laplace transform of Eq. (14) for ξ_0, and then using a known expression (6.611.1 of Ref. [11]) for the Laplace transform of the Bessel function

$$\int_0^\infty d\tau\, e^{-\alpha\tau} J_0(\beta\tau) = \frac{1}{\sqrt{\alpha^2+\beta^2}} \,.$$

After some rearrangement, we find

$$\left[\sqrt{\omega(\omega-\Omega_*)} - i\lambda_H\right] \mathcal{L}[\xi_0] = \sqrt{\omega(\omega-\Omega_*)}\mathcal{L}[Q] + i\mathcal{L}[\Lambda_K] \,, \tag{18}$$

where $\mathcal{L}$ is the integral operator

$$\mathcal{L}[\xi_0] \equiv \int_0^\infty d\tau\, e^{i\omega\tau}\xi_0(\tau) \,. \tag{19}$$

In the linear approximation, the fast particle term reduces to

$$\mathcal{L}[\Lambda_K] \longrightarrow \lambda_K(\omega)\mathcal{L}[\xi_0] + g(\omega) \,, \tag{20}$$

where $g(\omega)$ is a term associated with the initial perturbation of the hot particle distribution function, δf, which is not induced by ξ_0. In this paper, we simply set $g(\omega) = 0$, which means that the response connected with g is ignored, although in principle this response can be calculated as a seperate contribution to ξ_0 – in the same manner as we calculate the contribution from $Q(\tau)$. Also, the definition of λ_K is consistent with Ref. [12-13]. With this result, the transformed integral equation becomes simply $D(\omega)\mathcal{L}[\xi_0] = \sqrt{\omega(\omega-\Omega_*)}\mathcal{L}[Q]$, where D is the dispersion function

$$D(\omega) = \sqrt{\omega(\omega-\Omega_*)} - i\left[\lambda_H + \lambda_K(\omega)\right] \,. \tag{21}$$

The causality principle requires $D(\omega)$ to be analytic in the upper half plane with the branch of the square root determined by the condition

$$\mathrm{Im}\,\sqrt{\omega(\omega-\Omega_*)} > 0 \qquad \text{for} \qquad \mathrm{Im}\,\omega > 0 \,. \tag{22}$$

The inverse transformation to the time domain gives the following integral representation for the on-axis displacement:

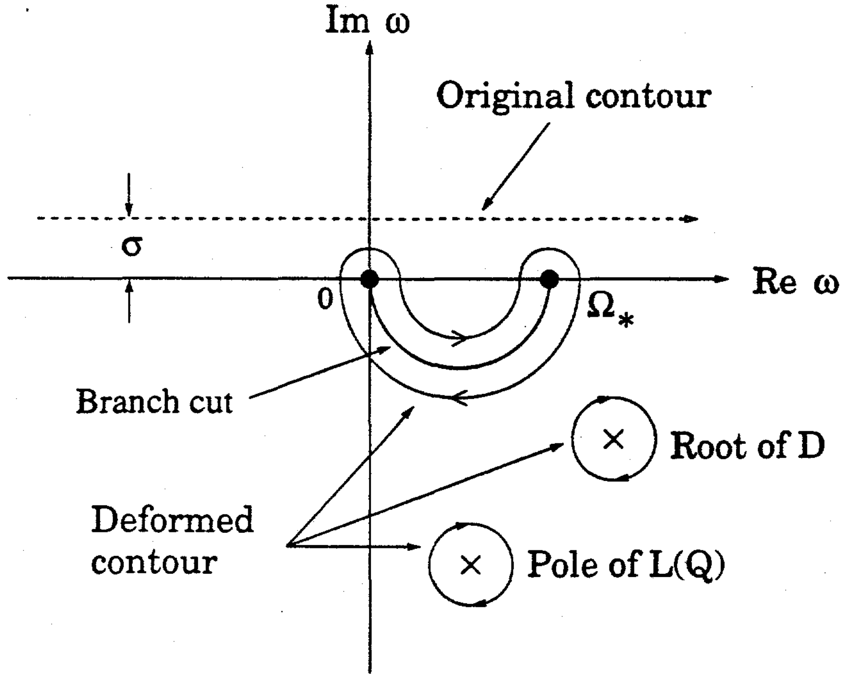

FIG. 1. Structure of the complex ω-plane showing the original Laplace inversion contour, and the same contour deformed into a loop integral encircling the branch cut connecting $\omega = 0$ and $\omega = \Omega_*$, and around the poles associated with (i) a stable root of $D(\omega)$, and (ii) a pole of $\mathcal{L}[Q]$.

$$\xi_0(\tau) = \frac{1}{2\pi}\int_{-\infty+i\sigma}^{\infty+i\sigma} d\omega\, e^{-i\omega\tau}\frac{\sqrt{\omega(\omega-\Omega_*)}\mathcal{L}[Q]}{D(\omega)} \ . \tag{23}$$

The inversion integral requires σ positive and chosen so that the contour lies *above* all poles of the integrand (zeros of $D(\omega)$ and poles of $\mathcal{L}[Q]$).

To evaluate $\xi_0(\tau)$ in the large-τ limit, it is convenient to close the contour in the lower half-plane and shrink it around the branch cut and poles as indicated in Fig. 1. Upon doing so, we can rewrite Eq. (23) as

$$\begin{aligned}\xi_0(\tau) = \oint \frac{d\omega}{2\pi} e^{-i\omega\tau}\frac{\sqrt{\omega(\omega-\Omega_*)}\mathcal{L}[Q]}{D(\omega)} \\ - i\sum_n e^{-i\omega_n\tau}\,\mathrm{Res}\left[\frac{\sqrt{\omega(\omega-\Omega_*)}\mathcal{L}[Q]}{D(\omega)}\right]_{\omega=\omega_n} \ .\end{aligned} \tag{24}$$

The first term is the loop integral which encircles the branch cut connecting the branch points at $\omega = 0$ and $\omega = \Omega_*$, while the second represents the contributions which arise from the poles of $\mathcal{L}[Q]/D(\omega)$. To simplify the discussion, we assume that the singularities of $\mathcal{L}[Q]/D(\omega)$ are poles. In the general case, however, a more complicated analytic structure is possible.

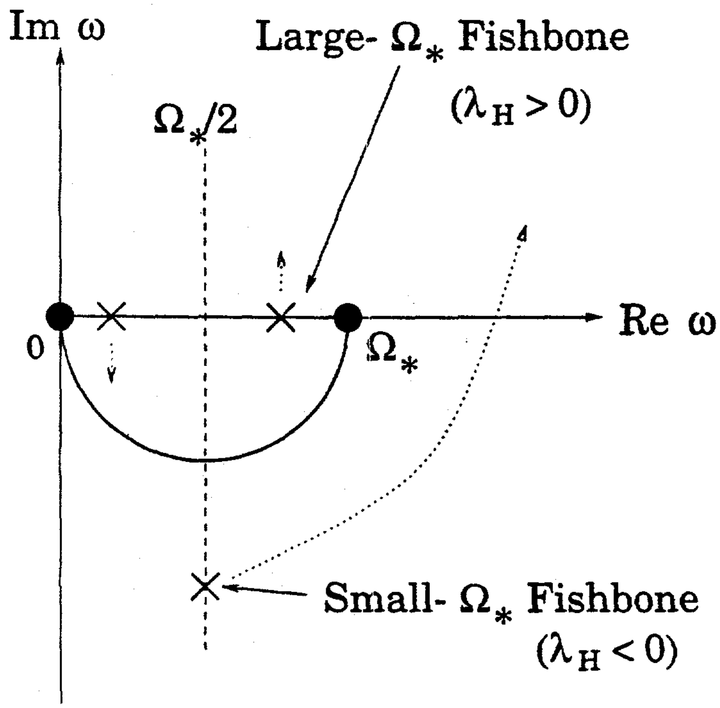

FIG. 2. Location of the zeros of $D(\omega)$ for the limiting cases of small and large Ω_*. In the small-Ω_* case, when the plasma is MHD stable ($\lambda_H < 0$) and λ_K is small, a single root lies in the lower half plane. This mode is eventually destabilized as λ_K increases in magnitude. In the large-Ω_* case, two roots exist on the real axis for $\lambda_K = 0$. The root near $\omega = \Omega_*$ is destabilized by fast ions while the root near $\omega = 0$ is stabilized. The dotted curves indicate the path of each root as fast ion pressure increases.

The semicircular branch cut in Fig. 1 is convenient to simultaneously treat two important limiting cases: that of small Ω_*, when diamagnetic effects play a minor role; and that of large Ω_*, such that the MHD drive from positive λ_H is stabilized by diamagnetic effects. These two cases correspond to two qualitatively different regimes of the fishbone instability. In the first regime, for which $\Omega_* \ll \lambda_H$ and the plasma is MHD stable ($\lambda_H < 0$), the root of $D(\omega)$ occurs at $\omega \sim \Omega_*/2 + i\lambda_H$ for zero fast ion pressure, as shown in Fig. 2. As the fast ion pressure increases from zero, the root moves upward and eventually crosses the real axis – indicating the onset of the fishbone instability. In the second regime, where diamagnetic effects are strong enough to stabilize the internal kink ($0 < \lambda_H < \Omega_*/2$), one finds two roots on the real axis in the absence of fast ions. The root which sits close to $\omega = \Omega_*$ is the diamagnetic fishbone root, and is destablized by a fast ion population (since for low frequencies, we typically have $\mathrm{Re}\,\lambda_K, \mathrm{Im}\,\lambda_K < 0$). This is also illustrated in Fig. 2.

An asymptotic evaluation of the loop integral in Eq. (24), detailed in Appendix B, shows that it generally decays in time as $\tau^{-3/2}$ for nonzero Ω_*. The corresponding contribution to the asymptotic form of $\xi_0(\tau)$ is

$$\xi_0(\tau) \sim \frac{\sqrt{\Omega_*}}{\tau^{3/2}} \left[e^{-3\pi i/4} A(0) + e^{i(3\pi/4-\Omega_*\tau)} A(\Omega_*)\right] \quad \text{as} \quad \tau \to \infty\,, \tag{25}$$

where the factor $A(\omega)$ is calculated in Appendix B.

When the system is linearly unstable, the exponentially growing contribution to $\xi_0(\tau)$ from the eigenvalue ω_n with $\operatorname{Im}\omega_n > 0$ is the dominant one, although the unstable root may be so close to the real axis that the transient contribution from the branch cuts can compete with it over an extended period of time. In a stable system, the cut contribution, Eq. (25), is the dominant one in the limit $\tau \to \infty$ since every pole, ω_n, gives rise to an exponentially decaying term. However, under certain conditions, the pole contributions may persist for long times. In order that the first term on the right hand side of Eq. (24) reduce to the asymptotic form given by Eq. (25), the condition

$$\frac{\Omega_*}{\tau} \ll \min\left\{x_0^2, |\lambda_{\rm H} + \lambda_{\rm K}|^2, \Omega_*^2\right\} \tag{26}$$

must be satisfied. Above, x_0 is the width of the initial profile (see also Sec. IV). We then conclude that neither the stable eigenvalue nor the poles of $\mathcal{L}[Q]$ generally determine the ultimate decay rate of $\xi_0(\tau)$, since the decay for nonzero Ω_* is a power law rather than exponential. In the limit $\Omega_* \to 0$, the branch points merge and the cut disappears. The decay then becomes exponential, with the rate determined by the pole with the least negative imaginary part (see Fig. 1). A final exception occurs when a pole with $\operatorname{Im}\omega_n = 0$ exists. Then, it is apparent that the mode is marginally stable and oscillatory with real frequency ω_n.

IV. Radial Displacement for Zero Diamagnetic Drift

In the limit $\Omega_* \to 0$, the branch cut contribution to Eq. (24) vanishes and the decay rate becomes purely exponential. We can explicitly illustrate this property of the solution for the simple case $\Omega_* = \Lambda_K = 0$ by constructing an analytic solution for $\xi(x,\tau)$ given the specific initial profiles

$$F(x) = -\frac{x_0}{\pi}\frac{\xi_0(0)}{(x-a)^2 + x_0^2} \qquad \text{and} \qquad G(x) = 0 \, . \tag{27}$$

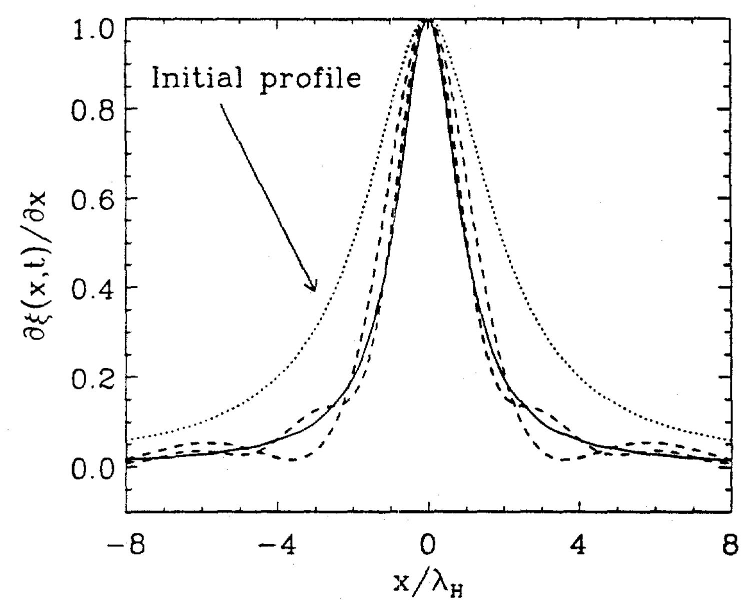

FIG. 3a. Profile of the radial derivative of the displacement for $z_0 = 2.0$ at times $\tau = [0, 1, 2, 4]$. In this case, the initial profile (dotted curve) evolves to a narrower eigenmode profile. The latest time is shown as a solid curve.

This choice satisfies the required constraint imposed by Eq. (12), and gives a "forcing" function Q which decays exponentially in time at a rate proportional to the initial profile width, x_0:

$$Q(\tau) = \xi_0(0)\, e^{-x_0\tau} \cos(a\tau) \, . \tag{28}$$

Now, we introduce the dimensionless width $z_0 \equiv x_0/\lambda_H$ and then solve Eq. (14) with Q given in Eq. (27). The result is

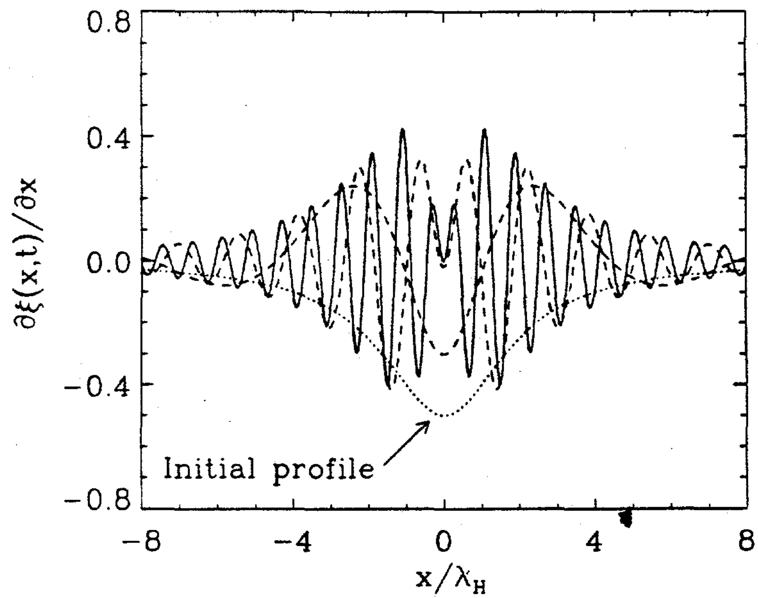

FIG. 3b. Profile of the radial derivative of the displacement for $z_0 = -2.0$ (*i.e.*, $\lambda_{\rm H} < 0$) at times $\tau = [0, 1, 4, 8]$. Here, the initial profile (dotted curve) decays to an oscillatory structure, with the latest time shown as a solid curve.

$$\begin{aligned}\xi_0(\tau) = &\frac{\xi_0(0)}{a^2 + (1 + z_0)^2}\Big[(1 + z_0)e^{\lambda_{\rm H}\tau} \\ &+ \left(a^2 + z_0(1 + z_0)\right) e^{-|z_0\lambda_{\rm H}|\tau}\cos(a\tau) + ae^{-|z_0\lambda_{\rm H}|\tau}\sin(a\tau)\Big] \,.\end{aligned} \tag{29}$$

In the stable case, $\lambda_{\rm H} < 0$, there are two possible decay rates. When $-1 < z_0 < 0$ (and taking x_0 strictly positive) the decay is exponential with oscillations at frequency a, while for $z_0 < -1$ the decay is purely exponential. Let us take the solution further, but first simplify to the symmetric case $a = 0$. Then the on-axis displacement becomes

$$\xi_0(\tau) = \xi_0(0)\left[\frac{e^{\lambda_{\rm H}\tau} + z_0 e^{-|z_0\lambda_{\rm H}|\tau}}{1 + z_0}\right] \,. \tag{30}$$

Substituting this result into Eq. (13), and evaluating the time integrals explicitly, yields the following expression for the radial profile:

$$
-\frac{\pi(1+z_0)}{\xi_0(0)}\frac{\partial\xi(z,\tau)}{\partial z} = \underbrace{\frac{e^{\lambda_H\tau}}{1+z^2}}_{\text{normal mode}} + \underbrace{\frac{z_0 e^{-|z_0\lambda_H|\tau}}{z_0^2+z^2}}_{\text{decaying transient}}
+ \underbrace{\frac{z^2(z_0^2-1)}{(1+z^2)(z_0^2+z^2)}\left[\cos(z\lambda_H\tau)+\frac{\sin(z\lambda_H\tau)}{z}\right]}_{\text{oscillating transient}} \tag{31}
$$

The interpretation of the above result is straightforward. The normal mode, or collective, response is exponential in time and independent of the initial perturbation. It grows/decays when λ_H is positive/negative. For the unstable case, this term quickly dominates the others and gives rise to the standard arctangent normal mode solution for the internal kink, with a width determined by λ_H (see Fig. 3a). The second term is an exponential transient which *always* decays. The radial shape of this transient can be wider ($z_0 < 1$) or narrower ($z_0 > 1$) than that of the collective response. The final term is a complicated oscillatory function which exhibits *nonseparable* dependence on space and time (see Fig. 3b). When integrated over x, the contribution to the on-axis displacement from this term decays exponentially in time.

That these results are consistent with the frequency-space analysis of the previous section can be seen by an explicit computation of the function $\mathcal{L}[Q]$. Using the causal x-space contour defined in Appendix B, it is easy to deduce

$$
\mathcal{L}[Q] = -\frac{\xi_0(0)}{\omega + ix_0} \,. \tag{32}
$$

This indicates clearly that the pole of $\mathcal{L}[Q]$ is responsible for the second term in the numerator of Eq. (29).

Although we have taken $\Omega_* = 0$ in this section, it is true that when Ω_* is small but nonzero, the above results apply when $\tau < 1/\Omega_*$.

V. Connection with the Landau Problem

We now discuss a parallel between the linear internal kink problem and the Landau damping problem for an electrostatic plasma wave [5]. The analogous quantities in these two problems are, respectively, the radial profile of the displacement gradient in the transition layer and the perturbed particle distribution function in velocity space near the Landau resonance

$$\frac{\partial\xi(r,t)}{\partial r} \longleftrightarrow \delta f(v,t)\ , \tag{33}$$

where the spatial profile of δf is sinusoidal. We note that both problems involve a continuum of localized perturbations (local Alfvén modes for $\partial\xi(r,t)/\partial r$ and Van Kampen modes [14] for δf). We also note that, in the unstable system, both $\partial\xi(r,t)/\partial r$ and $\delta f(v,t)$ have a Lorentzian-type structure near the resonance point, with the width of the resonance determined by the linear growth rate. The exponential time dependence factors out in the unstable case, since the dominant part of the perturbation shapes into a collective eigenmode in either real space (for the internal kink problem) or in velocity space (for the electrostatic problem). The magnitudes of the on-axis displacement and the perturbed electrostatic potential, $\delta\phi$, satisfy

$$\xi_0(t) = -\int dr\,\frac{\partial\xi(r,t)}{\partial r} \qquad \text{and} \qquad \delta\phi(t) \propto \int dv\,\delta f(v,t) \tag{34}$$

respectively. These grow at a rate determined by the unstable eigenvalue.

To obtain an eigenvalue in the case of a stable system, it is necessary to deform the integration contours in Eq. (33) into the complex plane. If this is not done, one will come to the conclusion that an eigenmode does not exist. The accurate conclusion, when analytic continuation via specification of a "causal coutour" is made, is that the eigenmode does *not* exist on the real axes but rather in the complex r- and v-planes. Such a causal contour for the internal kink is defined explicitly in Appendix B.

Unlike the case for an unstable perturbation, the dominant component of a stable perturbation, which is associated with the initial conditions and not the stable eigenvalue, cannot in general be factorized. This component evolves into an undulating profile with ever decreasing scale in position (internal kink) or velocity (electrostatic mode) – by consequence of phase mixing in the continuous spectrum. This phase mixing causes damping of the integrated quantities in Eq. (33). For stable internal kink perturbations this damping generally follows a power law for nonzero Ω_*.

VI. Summary

In this work we have studied the internal kink mode evolution problem by solving an initial value problem for the plasma displacement. The solution, in the form of a reduced integral equation for the on-axis displacement, is applicable to both stable and unstable plasma conditions, and includes nonlinear currents produced by energetic particles. In obtaining our reduced equation, we had to contend with the common belief that the roots of the analytic dispersion relation are meaningful only when the linear system is unstable. We have carefully analyzed the initial-value problem to show rigorously that the stable roots of the dispersion relation also have physical meaning. Indeed, when diamagnetic effects are negligible and the spatial width of the initial perturbation is larger than the natural width of collective modes, the root of the dispersion relation determines the dominant time-asymptotic response of the on-axis displacement. In the general case for finite diamagnetic frequency, a stable disturbance is shown to decay asymptotically as $\tau^{-3/2}$, although the exponentially decaying collective mode can dominate for a long transient time when its damping is weak. This feature suggests that if the effect of the energetic particles changes in time so as to make the system stable (for example, through flattening of the distribution function), the decay of the fishbone burst may be slower than exponential.

The model we have developed is also sufficiently general to allow for computation of the fast-ion dynamics in a fully toroidal manner – including arbitrary equilibrium orbits. A numerical simulation code has also been developed to track the nonlinear energetic particle dynamics [10], and results of this calculation will be presented in a forthcoming publication.

VII. Acknowledgements

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Appendix A: Alternative Form of the Evolution Equation

In this appendix we outline the formal procedure which was used to derive Eqs. (16) and (17) of the main text. Multiplying Eq. (18) by $-i$ and inverting leaves

$$\mathcal{L}^{-1}\left\{\left[-i\sqrt{\omega(\omega-\Omega_*)}-\lambda_{\rm H}\right]\mathcal{L}[\xi_0]\right\}=\mathcal{L}^{-1}\left\{-i\sqrt{\omega(\omega-\Omega_*)}\mathcal{L}[Q]\right\}+\Lambda_{\rm K}[\xi_0](\tau)\ . \tag{A1}$$

The operator $\mathcal{L}^{-1}$ is the inverse Laplace transform,

$$\mathcal{L}^{-1}[f]\equiv\frac{1}{2\pi}\int_{-\infty+i\sigma}^{\infty+i\sigma}d\omega\, e^{-i\omega\tau}f(\omega)\ , \tag{A2}$$

with $\sigma>0$ chosen so that the contour lies above any singularities of the integrand. Although the term associated with the source $Q(\tau)$ can be worked out explicitly, it is not necessary for the present purpose to do so. Instead, we simply define a new source term

$$\widehat{Q}(\tau)\equiv -i\mathcal{L}^{-1}\left\{\sqrt{\omega(\omega-\Omega_*)}\mathcal{L}[Q]\right\}\ . \tag{A3}$$

With this definition, we are left with Eq. (16) of the main text:

$$\underbrace{\mathcal{I}[\xi_0](\tau)}_{\text{linear operator}}\quad=\quad\underbrace{\Lambda_{\rm K}[\xi_0](\tau)}_{\substack{\text{nonlinear source}\\ \text{from fast particles}}}\quad+\quad\underbrace{\widehat{Q}(\tau)}_{\substack{\text{linear source due}\\ \text{to initial perturbation}}}\ , \tag{16}$$

such that

$$\begin{aligned}\mathcal{I}[\xi_0]&=\mathcal{L}^{-1}\left\{\left[-i\sqrt{\omega(\omega-\Omega_*)}-\lambda_{\rm H}\right]\mathcal{L}[\xi_0]\right\}\ ,\\ &=-\lambda_{\rm H}\xi_0(\tau)+\mathcal{L}^{-1}\left\{-i\sqrt{\omega(\omega-\Omega_*)}\mathcal{L}[\xi_0]\right\}\ ,\\ &=-\lambda_{\rm H}\xi_0(\tau)+\int_0^\tau d\tau' Z(\tau-\tau')\xi_0(\tau')\ .\end{aligned} \tag{A4}$$

The function $Z(\tau)$ is just the inverse transform of the square root, namely

$$Z(\tau)=\mathcal{L}^{-1}\left[-i\sqrt{\omega(\omega-\Omega_*)}\right]\ . \tag{A5}$$

This can be computed analytically if we rewrite the square root as

$$-i\sqrt{\omega(\omega-\Omega_*)} = -i\omega + \frac{i\Omega_*}{2} + z(\omega) \ , \qquad (A6)$$

so that $z(\omega) \to 0$ as $\omega \to \infty$. The first two terms on the right hand side of Eq. (A6) are evidently the Laplace transform of a δ-function and its derivative. To evaluate the third term, we shrink the original contour around the branch cut $[0, \Omega_*]$ that is chosen to lie along the real axis. This gives

$$\begin{aligned}
\oint \frac{d\omega}{2\pi} e^{-i\omega\tau} z(\omega) &= -i \oint \frac{d\omega}{2\pi} e^{-i\omega\tau} \sqrt{\omega(\omega-\Omega_*)} \\
&= \frac{2}{\pi} e^{-i\Omega_*\tau/2} \int_0^{\Omega_*/2} dx \, \cos(x\tau) \sqrt{(\Omega_*/2)^2 - x^2} \qquad (A7) \\
&= \frac{\Omega_*}{2\tau} e^{-i\Omega_*\tau/2} J_1\left(\frac{\Omega_*\tau}{2}\right) \ ,
\end{aligned}$$

where Eq. 3.752.2 of Ref. [11] has been used to evaluate the integral. Finally, we obtain

$$Z(\tau) = \delta'(\tau - 0) + \frac{i\Omega_*}{2}\delta(\tau - 0) + \left(\frac{\Omega_*}{2}\right)^2 K\left[\frac{\Omega_*\tau}{2}\right] \ , \qquad (A8)$$

where $K(x) \equiv \exp(-ix)J_1(x)/x$. These results give $\mathcal{I}[\xi_0]$ as written in Eq. (17) of Section II.

The form of Eq. (A5) is also suggestive with regard to generalization of the time evolution operator $\mathcal{I}$ to cases which include a more detailed treatment of inner layer physics. This generalization is conceptually straightforward. For example, when resistivity, η, is included in the inner layer equations, a more complicated function of frequency replaces the square root function:

$$Z(\tau) \longrightarrow \mathcal{L}^{-1}\left[G(\omega, \Omega_*, \eta)\right] \ , \qquad (A9)$$

where G can be deduced from, for example, Eq. (70) of Ref. [13]. The net effect is to make the kernel, K, in Eq. (A8) η-dependent.

Appendix B: Laplace Transform of the Source

In this appendix, we indicate how to obtain the asymptotic expansion of the loop integral in Eq. (24) in the limit $\tau \to \infty$. First an explicit expression for $\mathcal{L}[Q]$ in terms of the functions F and G is obtained by transforming $Q(\tau)$ as defined in Eq. (15):

$$\begin{aligned}\mathcal{L}[Q] &\equiv \int_0^\infty d\tau\, e^{i\omega\tau} Q(\tau) \\ &= -\int_{-\infty}^\infty dx \int_0^\infty d\tau\, e^{i(\omega-\Omega_*/2)\tau}\left[\left(G(x) + \frac{i\Omega_*}{2}F(x) + F(x)\frac{\partial}{\partial\tau}\right)\right. \\ &\qquad\qquad \left. \times \frac{\sin\tau\sqrt{x^2+(\Omega_*/2)^2}}{\sqrt{x^2+(\Omega_*/2)^2}}\right] . \end{aligned} \tag{B1}$$

It is convenient to integrate by parts the term containing a partial time derivative. The result is

$$\mathcal{L}[Q] = -\int_{-\infty}^\infty dx \int_0^\infty d\tau e^{i(\omega-\Omega_*/2)\tau} H(x,\omega)\frac{\sin\tau\sqrt{x^2+(\Omega_*/2)^2}}{\sqrt{x^2+(\Omega_*/2)^2}} , \tag{B2}$$

where we have introduced the new profile function

$$H(x,\omega) \equiv i(\Omega_* - \omega)F(x) + G(x) . \tag{B3}$$

Performing the τ-integration in Eq. (B2) gives a simple result valid for all ω in the upper-half plane:

$$\mathcal{L}[Q] = -\int_{-\infty}^\infty dx \frac{H(x,\omega)}{\omega(\Omega_* - \omega) + x^2} . \tag{B4}$$

When $\mathrm{Im}\,\omega$ becomes negative, we must remember to deform the contour off the real x-axis to ensure causality, as shown in Fig. 4.

The large-τ asymptotic expansion of ξ_0 in Eq. (24) is dominated by the endpoint contributions at $\omega = 0$ and $\omega = \Omega_*$, where at each endpoint a similar integral will arise. The asymptotic behavior of each endpoint integral is

$$\int_0^\alpha dz\, e^{-z\tau}\sqrt{z} \sim \frac{\sqrt{\pi}}{\tau^{3/2}} \qquad \text{as} \quad \tau \to \infty , \tag{B5}$$

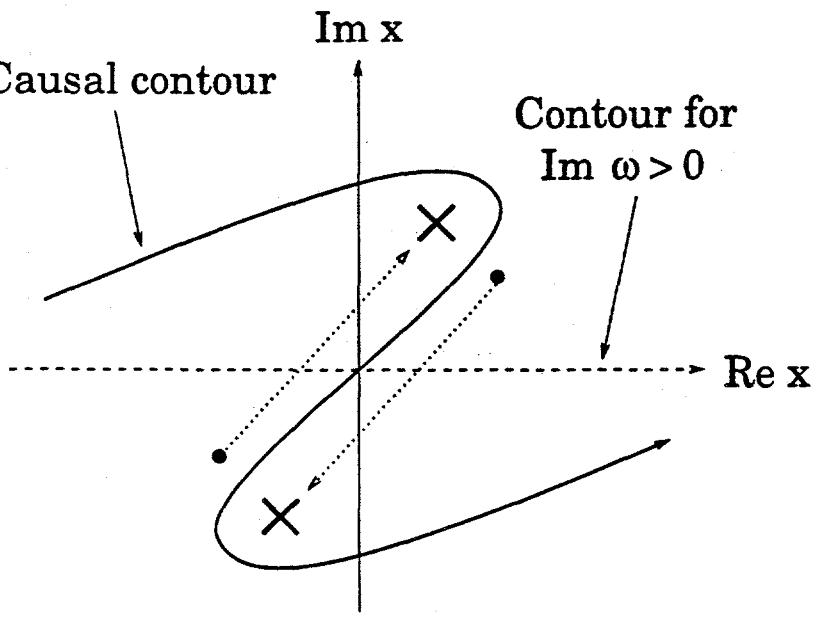

FIG. 4. Structure of the complex x-plane showing the deformation of the real x-axis integration contour required to ensure causality when the ω-contour falls in the lower-half ω-plane. The poles are solutions of $x^2 = \omega(\omega - \Omega_*)$; they must remain on the same side of the integration contour to ensure causality.

where α is a small, positive number. In addition, we must expand Eq. (B4) in the neighborhood of each branch point, and retain the terms which do not vanish as $\omega(\Omega_* - \omega) \to 0$. This can be carried out directly by use of the separation

$$\int_{-\infty}^{\infty} dx \, \frac{H(x,\omega)}{\omega(\Omega_* - \omega) + x^2} = \int_{-\infty}^{\infty} dx \, \frac{H(0,\omega)}{\omega(\Omega_* - \omega) + x^2} + \mathcal{P}\int_{-\infty}^{\infty} dx \, \frac{H(x,\omega) - H(0,\omega)}{\omega(\Omega_* - \omega) + x^2} \, . \tag{B6}$$

The first integral on the right hand side of this equation can be done explicitly. The second is taken in the principle value sense and so remains finite in the limit $\omega(\Omega_* - \omega) \to 0$. The required expansion for $\mathcal{L}[Q]$ is thus

$$\mathcal{L}[Q] \sim -\frac{i\pi H(0,\omega)}{\sqrt{\omega(\omega - \Omega_*)}} - \mathcal{P}\int_{-\infty}^{\infty} \frac{dx}{x} \frac{\partial H}{\partial x} \qquad \text{as} \qquad \omega(\Omega_* - \omega) \to 0 \, , \tag{B7}$$

where an integration by parts has been used to simplify the principal value integral. For illustrative purposes, it is useful to evaluate Eq. (B7) for the Lorentzian profile of Sec. IV (with $\Omega_* = a = 0$) to show that the result is identical to the first two terms in the Taylor expansion of Eq. (32) as $\omega \to 0$.

Finally, when Eq. (B7) is substituted into Eq. (24) and the branch point contributions are evaluated using Eq. (B5), we arrive at Eq. (25) of the main text, which is reproduced below:

$$\xi_0(\tau) \sim \frac{\sqrt{\Omega_*}}{\tau^{3/2}} \left(e^{-3\pi i/4} A(0) + e^{i(3\pi/4 - \Omega_* \tau)} A(\Omega_*) \right) . \tag{22}$$

The function $A(\omega)$ is defined as

$$A(\omega) = \sqrt{\frac{1}{4\pi}} \left(\frac{\pi H(0,\omega)}{[\lambda_{\mathrm{H}} + \lambda_{\mathrm{K}}(\omega)]^2} + \frac{1}{\lambda_{\mathrm{H}} + \lambda_{\mathrm{K}}(\omega)} \mathcal{P} \int_{-\infty}^{\infty} \frac{dx}{x} \frac{\partial H(x,\omega)}{\partial x} \right) . \tag{B8}$$